The genesis of proof in ancient Greece: The pedagogical implications of a Husserlian reading

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In this essay, we present a reading of the genesis of proof in ancient Greece through the lenses of Husserl's transcendental phenomenology. We argue that the Husserlian perspective acts as the epistemological bedrock upon which a didactical framework that fosters the students' need for proof may be built. Importantly, we posit that this framework allows for the students' developing internal need for organising the corpus of mathematical knowledge within a deductively derived structure.

Keywords: Proof, geometry, Husserl, phenomenology.

THE STUDENTS' NEED FOR PROOF

The notion of proof is at the crux of modern mathematics, constituting the backbone of the axiomatic system implied by Euclid. Mathematics educators have investigated the phenomena related to proof, considering amongst others different protagonists (including students, teachers, mathematicians), their conceptions of proof and its functions, their cognitive and affective proving products and processes (Boero, 2007; Moutsios-Rentzos & Kalozoumi-Paizi, 2014; Reid & Knipping, 2010).

Though researchers have documented various functions of proof (including verification, explanation, systemisation; Hanna, 2000), the students seem not to share these conceptions. For example, high-school students appear to consider proof as means for establishing verification and to a lesser extent for explaining and communicating (Healy & Hoyles, 2000). Moreover, mathematics undergraduates would employ mathematical proof as an exam-appropriate answer, but they may choose a 'softer' argument (example, figure etc) to convince themselves (Moutsios-Rentzos & Simpson, 2011): they produce a proof to meet the externally-set requirements of a task, but their internal need for proof seems not to necessarily be in line with a fully-fledged conception of proof. The students need a reason to produce a proof (Balacheff, 1991), which may be externally or internally referenced (Moutsios-Rentzos & Simpson, 2011).

Zaslavsky, Nickerson, Stylianides, Kidron and Winicki-Landman (2012) discussed the mathematical and pedagogical aspects about the need for proof, differentiating internal needs amongst: *certainty* (verification of the truth of a statement), *causality* (why a statement is true), *computation* (quantification of definitions, properties or relationships through algebraic symbolism), *communication* (formulation and formalisation in conveying ideas), *structure* (logical re-organisation of knowledge).

Everyday activities utilising the notion of inquiry are suggested as possible means for fostering the students' developing these aspects of internal need for proof. Though existing didactical frameworks may be employed to help the students to develop internal need for proof, we argue that a Husserlian reading of the genesis of proof in ancient Greece may provide the epistemological backbone of a didactical framework that would foster the students' developing all aspects of internal need for proof, notably 'structure'. The realistic mathematics education research paradigm (Streefland, 1991) appears to be a suitable framework, since a problematic situation that is perceived as 'real' for the students is actively re-organised by the students with the teachers' guidance. The re-organisation of the situation results in the 're-invention' of the required mathematical tools that, constructed as a response to a 'real' need, are meaningful for the students. The process of mathematisation of the 'real' situation allows the incorporation of the constructed mathematical ideas within the existing mathematical world, but it does not explicitly address 'structure'. The new mathematical constructs need to derive from existing mathematical knowledge, but this necessarily implies (at best) only a local mathematical structure and certainly there is no 'real' need for attempting to re-organise the re-invented mathematical tools within a global mathematical structure (such as an axiomatic system). Additional requirements have to be activated for a student to develop the internal need for the logical re-organisation of the re-invented mathematical tools. From a different perspective, Radford (2003) emphasised the sociocultural aspects of mathematical thinking, suggesting a semiotic-cultural approach to highlight the subjective nature of the meaning constructed through semiotic activities. Meaning is constructed by subjects within specific sociocultural context and, thus, proof is meaningful for a student who experiences a specific sociocultural reality. Though we acknowledge the importance of the socio-semiotic dimension, Radford's research was not focused on the students' development of an internal need for proof.

Overall, in this essay we address the fundamental question: *What are the didactical principles constituting an epistemologically coherent framework that may foster the students' developing a fully-fledged need for proof?*

THE GENESIS OF PROOF IN ANCIENT GREECE

Katz (2009) notes that the notion of proof appeared in ancient Greece. Many of the mathematical results were already known, in the same way that something is known in the sensory-perceived world: as rules that held true for all the till then considered cases. With Greek mathematics things changed, including "objects whose existence cannot be visualised and which cannot be physically realised" (Grabiner, 2012, p. 152). Moreover, the mathematical ideas were re-organised to form a primitive proof-based version of an axiomatic system. A multiplicity of factors formed a complexity within which proof appeared to be 'natural'. But which were those factors?

The sociocultural context of the ancient Greek city (*polis*) appears to be the crucial factor that enabled the change of perspective about the issues that proof addresses in mathematics. Polis was the result of a transformation from monarchy to democracy. Employing the case of ancient Athens as an exemplar, we find that the legislation of Solon and crucially of Cleisthenes changed the social structure of Athens, resulting to a radically transformed lived social reality. The Athenians were administratively organised in ten geographical regions that purposefully did not correspond to the traditional *phyla* ('families'), in order to shuffle the traditional, blood-centred, immediate social circle of the individual. Thus, the new immediate social family was based not only on blood relations, but also on a *purposefully arbitrary* geographical proximity. Moreover, each of the ten new regions was the ruling region for a tenth of the solar year. This meant that each ruling month was not a lunar month. The time that a region was in power was *not* measured with reference to nature, but according to a *purposefully arbitrary* chosen fraction of the solar year. 'Arbitrary' is emphasised, because the number of the new regions could be anything that would ensure the *un-settlement* of the old structure. Furthermore, Solon's changes produced a hierarchy of citizens, according to specific analogies forming a harmony (2/1, 3/2, 4/3). Cleisthenes' reform reduced all these relationships to a single analogy, the simplest possible: 1/1. All the citizens were equal with respect to access and power within the polis, regardless of their profession, family name or wealth.

Within the polis all the important aspects of life assumed a public character. The 'significant' private obtains its 'significant' status by becoming object of the community. For example, murder was not a private matter to be resolved amongst individuals. It is a public matter open to the actions of the community which focus on the 'objective', verbally described characteristics of the *situation*, rather than on *who* was involved in the incident. In order for 'justice' to be reached, the community had to be convinced of what happened, to construct a shared logos. Note that *logos* in Greek has a multiplicity of meanings including oral speech, reasoning and ratio (and relationship in general). The common logos emerged as the ruling power of the city, forming a differentiated from ethics law; the ethically acceptable may or may not be lawful. Justice became a matter of a social, non-metaphysical, construction. The citizens of the polis were characterised as such by actively participating in the common matters. The Athenian *idiot* ('private') was the person who either lacked the reasoning skills or chose not to contribute in the public affairs. The citizen was a 'subject' to the logos, to the verbal communication and co-construction of the common, argued meaning. Language and the arguments employed were at the crux of this process. Through language the private meanings were communicated and through convergences and divergences the shared public meaning emerged.

Cleisthenes' changes towards the equality of the citizens within the public affairs allowed the transcendental notion of power to obtain an anthropological character: the numerical majority was right, true and responsible and the minority had to accept it. The ruling power was not divinely-given, nor inherited, but lied within the *countable community*. The shared logos, the thesis voted by the citizens was within the reach of every citizen-subject, as long as it was accepted as such by the majority. This conceptual lift from the subjectively described to the objectively defined by a simple number, by a numerical relationship, allowed for the city itself to obtain a transcendental aspect, to exist regardless of who were its citizens. Its infrastructure transcended the people who represented it. In this way, the polis achieved its supertemporal continuity. Thus, the subject was at the same time unique and the same, one and many, important and insignificant. Heraclitus stressed that "although *logos* is common to all, most people live as if they had a wisdom of their own" and that "having listened not to me but to the logos it is wise to agree that one is all". It should be clarified that Heraclitus wording for 'agree' is *homo-logo* (common logos), indicating that agreement is a result of a shared logos. Hence, common logos implies all private understandings and reasonings are in agreement with (*homo-logia*), in a relationship with, the public *logos*. Notice that the shared logos does not imply the disappearing of personal identity (Vernant, 1983), as the self becomes a multiplicity of higher mental internalised social relationships. Vygotsky (1978) notes that the external social processes are closely linked with the internal psychological processes so that in "their own private sphere, human beings retain the functions of social interaction" (p. 164). Thus, the argument became a dominant social instrument.

Within this sociocultural framework, the requirement of producing a proof for a mathematical statement seemed to naturally fit in. The mathematical community as part of the general community requires arguments that cannot be logically disputed. Such an argument could not be based on perception, which was philosophically treated at the time as false, changeable or unreliable. Nor could it be based on authority or affective linguistic tricks. The Sophists, the Eleates (notably Zeno) and the philosophy of Plato and Aristotle crucially determined Euclid's decision to organise old and new mathematical ideas in a deductive structure, within which each proposition derives from already proved or accepted as true ideas.

Moreover, within a social framework that the public is appreciated and the private is frowned upon, mathematical ideas had to be open to the community and not to be only for a certain social cast (the clergy or other). This required resorting to commonly lived experiences, which were inescapably bodily experiences masked as 'semi-abstract' ideas. This is reflected in the 'pseudo-axiomatic' character of Euclid's elements. The definitions, the common ideas, the axioms derived from the shared lived perceptual reality, which ensures the wider acceptance of the logos that draws upon such a structure, but clearly limits the breadth and depth of the mathematical structure. Nevertheless, Euclid's organisation enabled the synthesis of seemingly unrelated ideas, deriving from the same underlying ideas and reasoning (for example, the study of incommensurable magnitudes and the irrational numbers). Though Szabó (1978) claims that the notion of deductive proof did not meet any practical needs, we argue that it met the lived needs within the broader ancient Greek sociocultural context when transposed in the abstract-like Euclidean world. In this conceptual extension of the perceived reality, the logos and the argument are the only means for establishing the truth of a proposition.

Overall, we agree with Vernant (1975) who argued that the formation of the *polis* was the decisive event that allowed the shared logos to become the backbone of the social structure. We briefly discussed some of the factors that may have constituted this event: the *shared logos*; a *purposefully arbitrary administrative structure*; the *1/1 citizen relationship*; the *countable decisive power*; the *convincing the majority verbal argument*; the *reign of the public over the private*; the *quantification of power*; the *argument based on commonly experienced notions and ideas*; the inescapable *reign of the deductive over the inductive within an axiomatic-like system*. All these elements are some of the crucial events that posed the need for a deductive proof, rather than settling for an inductive or other argument.

ELEMENTS OF HUSSERL'S PHENOMENOLOGY

Husserl's phenomenology may be summarised in the phrase "back to 'the things themselves'" (Husserl, 2001, p. 168), implying the attempt to 'unearth' the sedimented relationships and the decisive factors, in order to mobilise the mental processes that constitute an ideality. Husserl's idealities crucially differ from the platonic ideas in that they are intentionally subjectively *constructed once within history*. Once objectified, they become atemporal, in the sense that every subsequent subjective knowing requires only the *reactivation* of this objectification. Language (oral or written) constitutes the means for the objectification of the subjective experiences, allowing their subsequent transcendental existence. The reactivation of objectification requires the subject to develop suitable intentionality, suitable "conscious relationship [...] to an object", (Sokolowski, 2000, p. 8). Such intentionality requires the suspension of the subjects' natural attitude, their "straightforward involvement of things and the world" (Audi, 1999, p. 405), implying that the objectification is not merely a psychological process, as it explicitly incorporates the relationship between the subject and the community.

Husserl contrasts the intersubjective experience of the communicated shared meaning with the transcendental subjectivity in which there is an awareness of a phenomenon that transcends the subjective perceptual experience: "a possible communicative subjectivity [...] through possible intersubjective acts of consciousness, it encloses together into a possible allness a multiplicity of individual transcendental subjects" (Husserl, 1974, p. 31). In order for such processes to be activated, Husserl's phenomenological reduction (*epoché*) is required. By bracketing out, suspending, natural attitude and by investigating the sedimented intentional history of the object, the phenomenological attitude is activated in order for the subject to "seek for its "constitutive origins" and its "intentional genesis" (Klein, 1940, p. 150). During epoché, the subjects' thinking is characterised by the subjects' intentionality and immanence to bring to the surface the sedimented already constructed and existing within the community knowledge.

TOWARDS A DIDACTICAL FRAMEWORK

In what way can the aforementioned genesis of proof be read through a Husserlian perspective in order to inform a didactical framework that fosters a fully-fledged need for proof? Though 'replicating' history in the classroom is clearly not possible, an ancient idea "through an adaptive didactic work, may probably be redesigned and made compatible with modern curricula in the context of the elaboration of teaching sequences" (Radford, 1997, p. 32). We shall argue that the Husserlian perspective may help in determining the principles of the 'adaptive' work required.

In order to identify the ways that Husserl's views may inform a didactical framework, we should first consider the following: What is the students' natural attitude towards mathematics and learning in general? What is the role of language? Of technology? What is the perceived by the students' natural form of argumentation in mathematics? In everyday life? We do not claim that there are universally applicable answers to these questions. Each country, city, school unit, class have their special characteristics constituting a unique system (Moutsios-Rentzos, Kalavasis, & Sofos, 2013). Nevertheless, we shall describe some elements that we think characterise the lived reality in Greece. With respect to the students' natural attitude to mathematics, it appears that many students' consider mathematics to be beyond their lived reality, to be hard, boring or unnecessary (Brown, Brown, & Bibby, 2008). Healy (1999) argues that the current technologies prevent the students' minds from developing deductive reasoning, while it has a negative effect on their "ability to remain actively focussed on a task" (p. 201). Though such claims may sound too strong, the current sociocultural context is fast, based on inductive arguments and decisions, while the virtual social networking sites produce a multiplicity of realities within which the students act and interact (Moutsios-Rentzos et al., 2013). The role of language in this complex context appears to have radically transformed. The need for fast, usually factual, communication developed shorter versions of words, sentences, meanings. Such abbreviated forms hardly suffice when discussing mathematical objects. Thus, the verbal, logically complete argument identified as the main vehicle for establishing the need for proof appears to be in stark contrast with the contemporary linguistic habits. A further consequence of the steep rate of change is that even a local 'logos' or connotations may suffice for the arguer to accomplish his/her purpose. The shared memory is short and the lived present is even shorter; there is no real need for immanence. This fragmented, disjunctive, sociotemporarily fragile common 'logos' and practices do not bear any resemblance to the common logos experienced in ancient Athens. Moreover, the contemporary way of living is characterised by connectedness, by the existence of non-linear networks. The seemingly simplistic linear, deductive argument makes sense to be considered by the students as incompatible or even useless in such a connected, seemingly non-hierarchical reality, favouring other forms of argumentation (including inductive or abductive). Consequently, the shared 'logos' of the contemporary sociocultural students' reality, their natural attitude, seems to be far from the phenomenological attitude that lead to the genesis of proof. Which pedagogies may facilitate the students' experiencing the reactivation of the need for proof? We argue that an appropriate *epoché* should be cultivated to suspend the students' 'natural attitude' (for example, not to prove something obvious), allowing for the students' phenomenological attitude to reactivate their need for proof.

Drawing upon these and upon the realistic mathematics research paradigm (the epistemological bedrock of which is close to a Husserlian perspective), we provide a sketch of a didactical framework (with examples deriving from Moutsios-Rentzos, Spyrou, & Peteinara, 2014). First, the students should be familiar with the practical, everyday uses of mathematical ideas. Mathematics should be 'real' for the students, it should be 'discerned' in the lived world, as it can be practical, useful. This may require the teachers' drawing the students' attention to everyday situations that incorporate sedimented mathematical ideas. For this purpose, the starting point may be a problematic 'real' situation for the students that requires the re-invention of mathematical tools to be resolved. For example, the construction of a table requires a perpendicularity identification physical tool and the construction of such a tool may facilitate the students' re-invention of a mathematical tool (e.g. the Pythagorean Theorem). It should be stressed that the materials employed in the students' investigations are at the crux of the proposed framework, since they constitute the physical shared reference of each communication (see Moutsios-Rentzos, in press).

In line with our reading of the genesis of proof, the mathematical ideas should derive from some common (at least in the beginning) to human principles. For this purpose, the common to the human body sensory experiences of the world may be the bedrock upon which the shared logos may be professed. Though perceptually born, those common principles can, by the necessity of obtaining a shared meaning, be potentially stripped of their subjective nature. For example, the human body is evolutionally designed to identify verticality, which enables us to survive in a perceived as perpendicular to verticality (horizontal) world. The sensory experience of perpendicularity –in order to be potentially infinitely communicated– is required to be linguistically described with appropriate signs. The aforementioned perpendicularity identification physical tool may be initially constructed with reference to an independent from human activity, naturally existing, perpendicularity (e.g., the angle between the surface of the liquid and the string of the 'plumb-bob').

Furthermore, appropriate interventions may facilitate the students' conceptual shift in the semiotic registries employed in their *communicating* their embodied experiences. For this purpose, it is crucial for the students to realise the need for employing more symbolic and abstract semiotic registries in order to successfully resolve the situation *and* to communicate (and to convince) their argument about the validity of their solution to their classmates, to their teacher, to whomever whenever may face such a situation. For example, the students may construct a wooden triangular frame that visually fits the natural perpendicularity, but the teacher's guidance towards revealing what are the properties that the frame has that renders such a fit feasible may foster the employment of mathematical symbolism. For this purpose, the students may be guided to realise the constraints of the physical material in conveying the 'general' (rule, case, etc.) to a large (potentially infinite) audience.

Mathematical symbolism may help in realising that the mathematical ideas logico-deductively derive (through mathematisation processes) from the communicated, shared experiences, but they no longer (need to) exist within the experience. They (may) have a pragmatic reference, but only ideal essence. For example, the triangle the lengths of the sides of which are 3, 4 and 5 *units* is right-angled regardless of the physical magnitude of the unit, since $5^2 = 3^2 + 4^2$ holds true under the usual algebra.

Establishing a common linguistic expression (homo-logia) of the shared sedimented axiomatic system of some common ideal, yet anthropological, principles is a crucial step in transforming this system to an object upon which mental processes may be acted. In the proposed didactical framework, the students realise that the backbone of the axiomatic system derives from the physical constraints of the human body and as such cannot be absolute or 'given'. Hence, once the axiomatic framework has been objectified, it can itself be subjected to metacognitive investigations. For example, "What if ... we perceptually experience the surface we walk as the surface of a sphere?". Or, "What if ... the $5^2 = 3^2 + 4^2$ is not true?". Our reading of Husserl's phenomenology allows the students' questioning the very fabric upon which the situation is perceived, because the central role of language and communication allows the learners to realise (re-reveal) that the mathematics they experience everyday are only an instance of the infinite potential mathematics the mind can create. Within this potential, the students may come to realise that the mind games with the constituting common principles can be played only with conceptual tools, with reason (for example, algebraic geometry). The need for proof in these strange (to perception) worlds appear to be natural, since proof is the only means for evaluating the validity of a statement. At the same time, the lack of a means for establishing some perceptually derived intuition of the new structure facilitates the students' developing the need for proof as the gatekeeper of the structure itself.

Overall, we argue that the Husserlian reading of the genesis of proof in ancient Greece helped in identifying pedagogical principles – *a 'real'; problematic situation, embodied experiences, pre-scientific materials, language (oral or written), communication (argumentation) to self and others through different semiotic registries* – that form an epistemologically coherent didactical framework. Within this framework, the "divergences of the different levels of communication and experience are constantly re-negotiated in order to converge to a shared logos of condensed meanings and experiences" (Moutsios-Rentzos, in press), thus fostering the students' developing a fully-fledged need for proof (including 'structure').

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