Disparate arguments in mathematics classrooms

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Understanding students' classroom argumentation requires the analysis at multiple levels and from multiple perspectives. Using analyses of students' arguments when solving the Problem of Points we illustrate three perspectives: a classroom interactional perspective, a Stoffdidaktik task analysis (at the individual level), and a sociological theoretical perspective (at the community level). Each of these perspectives offers insights into students' argumentation in mathematical contexts, but no single one is adequate to completely describe the nature of students' argumentations, their underlying influences, and ways to support their development. Multiple perspectives and levels of analysis are required when researching classroom argumentation in particular and mathematics learning in general.

Keywords: Argumentation, classroom interaction, task design, social context.

In this paper we argue that understanding students' classroom argumentation requires the analysis at multiple levels and from multiple perspectives. Levels of analysis touched on here include individual, small group, whole class, school and community, but more are possible. Perspectives considered here include a classroom interactional perspective, a *Stoffdidaktik* task analysis (at the individual level), and a sociological theoretical perspective (at the community level).

We begin with a short description of a task which was used in our research and a brief description of some of our results. We then compare these results to those of Goizueta (2014; Goizueta, Mariotti, & Planas 2014) and offer his framework for analysis as the first perspective considered: an interactional perspective at the classroom level. Goizueta proposed the task in Spanish and we proposed it in German, and we will consider changes to the task that may have occurred in translation, using a second perspective from the work of Schupp (1986): a *Stoffdidaktik* task analysis at the individual level. We will then describe additional results from our research that suggest that additional perspectives are needed. An interesting parallel between our results and those obtained by Holland (1981) suggests a third perspective that applies to our results: a sociological theoretical perspective at the community level. We close with some final comments on the need for multiple perspectives and levels of analysis in researching classroom argumentation in particular and mathematics learning in general.

THE PROBLEM OF POINTS

Rott (2014) asked 14/15-year-old students at Gymnasium (grammar school/college prep school) in Bremen (Germany) to solve the classic Problem of Points in the context of their regular classroom mathematics lessons.

> Silke and Acun take turns flipping a coin. Silke scores a point if the coin comes up heads. Acun scores a point if the coin comes up tails. At the beginning of the game Silke and Acun each bet 3€. The first to score 8 points receives the 6€. When the score is 7 to 5 in Silke's favour, they have to interrupt their game. How can they now divide the money? (Our translation from Rott, 2014, p. 25)

Rott classifies the students' answers into five types:

> Proportional A total of 12 rounds were played (7 + 5) and there is 6€ to be won. So each round played scores obtained 0.50€. Accordingly, Silke wins 7 · 0.50€ = 3.50€. Acun receives 2.50€.

> Point Difference Silke requires only one throw to win. The difference between Silke and Acun is 2 points. So Silke gets 2€ more than Acun.

> Mistaken Reasoning Silke receives 7/8 of the 6€, Acun 5/8. However, that totals too much: 5.25€ + 3.75€ = 9€. So Silke gets 3.75€ and Acun 6€ − 3.75€ = 2.25€.

Point-Ratio The ratio of the scores so far is 7:5 = 1.4 So Silke gets 1.4 · 3€ = 4.20€. That leaves 1.80€ for Acun.

Winner Silke has more points.

(p. 40, our translation)

INTERACTION AT THE CLASSROOM LEVEL: RATIONAL BEHAVIOUR AND THE DIDACTIC CONTRACT

Some of these types of answers correspond to answers obtained by Goizueta (as reported in Goizueta, Mariotti, & Planas, 2014) and they can be accounted for by an analysis from the same theoretical perspective. Goizueta et al. analyse Goizueta's data using Habermas' construct of rational behaviour, in which accepting a validity claim amounts to accepting that certain conditions for validity have been fulfilled. This means that the criteria for validity have also been accepted. They describe three dimensions in students' argumentations:

> According to Habermas' construct of rational behavior and its adaptation by Boero et al. (2010), in the students' argumentative practices we can distinguish an epistemic dimension (inherent in the epistemologically constrained construction and control of propositions, justifications and validations), a teleological dimension (inherent in the strategic decision-making processes embedded in the goal-oriented classroom environment) and a communicative dimension (inherent in the selection of suitable registers and semiotic means to communicate within the given mathematical culture). (pp. 169–170)

They also make use of Brousseau's (1997) concept of the didactic contract, the implicit rules of acceptable classroom behaviour that govern teachers' and students' actions, and Douek's (2007) notion of a reference corpus, knowledge which is assumed to be unquestionable and shared and hence available as a basis for argumentation.

The five types of answers observed by Rott and listed above share a common characteristic which is of special interest to us here. They are all *mathematical*. Goizueta et al. account for this characteristic in the answers Goizueta recorded by reference to the didactic contract.

> When Vasi reminds the group of the need to resort to calculation, we recognize a constraint imposed by the didactic contract: any possible correct answer must be mathematics-related. (p. 172)

> According to the teleological and communicational dimensions, the didactic contract-related need to provide a mathematics-related answer, acting as a necessary normative validity condition, is what drives the students' efforts towards the construction of a first mathematical model. (pp. 174–175)

In short, from the perspective adopted by Goizueta et al. the students' choice to give mathematical answers to the question is accounted for by the didactic contract, which operates at the level of the classroom.

STOFFDIDAKTIK TASK ANALYSIS: LANGUAGE AT THE INDIVIDUAL LEVEL

The five types of answers listed above leave out one type that Rott observed: Money Back, of which an example is "The game was not completed. Both have a chance to win. The game was interrupted, without either reaching 8 points. The game is a tie." (2014, p. 40). This type of answer differs from the others in that it is not mathematical, instead making reference to everyday practices in game playing. Only two answers of this type were observed in the Gymnasium class, but its occurrence requires an explanation as it is not accounted for by Goizueta and colleagues' analysis.

The work of Schupp (1986) provides a different perspective that suggests a reason for the occurrence of such an answer. Schupp considers different wordings of the Problem of Points and suggests how students' answers might vary in relation to different wordings. He distinguishes three fundamentally different views of the problem: 1) situative; 2) quantitative; 3) stochastic and proposes different formulations of the problem to communicate these different views. The formulations all begin in the same way: "Two players are flipping a coin ... Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5. ..."

Situative view: What now?

One possibly ending to the task formulation is very open, asking “What now?” or “What happens next?” Schupp suggests that such a formulation invites a “situative view” of the problem resulting in answers drawing on everyday experiences of game playing. For example, students might answer: “Why not continue at another time?”, “Why not annul the game and give both players their money back?” “Toss the coin one more time to decide the winner.” All these answers seem plausible but they are not mathematical, as Schupp points out (p. 217).

Quantitative view: Fair division; how can this be accomplished?

Another possible ending is “The two players decide to split the money fairly. How can this be done?” (p. 218, our translation). Such a formulation invites a mathematical answer based on ratios. Interestingly, such answers were among the first given to the problem by mathematicians such as Pacioli. Many students in Goizueta’s and Rott’s research also seem to take this view.

Stochastic view: Dividing the pot according to probability

A third formulation considered by Schupp makes it clear that a “fair” division must take into account the players chances of winning when the game is interrupted, for example, by stating “Before dividing they agree that a ‘fair’ division must be done according to the probability each have at that moment of winning the game. How then must the pot be split?” (p. 218) This formulation of the problem motivates solutions like those given historically by Pascal and Fermat. In contrast to other approaches the focus is on the rounds that have not been played.

The tasks as given

How do the formulations of the task used by Goizueta and Rott fit into Schupp’s categories? Both Goizueta’s Spanish formulation and Rott’s German formulation, like Schupp’s, begin “Two players are flipping a coin ... Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5.” It is in the way the final question is asked that they differ.

Goizueta’s formulation ends, “¿Cómo deben repartirse el dinero? Justifiquen su respuesta.” (Goizueta, personal communication). The phrase “¿Cómo deben repartirse el dinero?” (How should they split the money?) refers only to dividing the money, and not to the concepts of fairness and probability. However, the word “deben” (should) suggests that there is a single correct answer to this question. Also, the requirement to “Justifiquen su respuesta.” (Justify your answer) in the context of a mathematics class could suggest that mathematical methods of justification are expected. Hence, while this formulation is not as clearly promoting a quantitative view as Schupp’s example of invoking “fairness”, it is not surprising that a quantitative view was adopted by all the students in Goizueta’s study.

Rott’s formulation ends, “Wie können sie nun das Geld aufteilen? Begründe deine Antwort!” (Rott, 2014, p. 25). The phrase “Wie können sie nun das Geld aufteilen?” refers only to dividing the money, and not to the concepts of fairness and probability. And the word “können” (can) is more open than the alternative “sollen” (should). However, Rott’s formulation is not so open as Schupp’s examples of situational view formulations as there is an explicit mention of dividing: “aufteilen” and a requirement to justify the answer “Begründe deine Antwort!”. This places the formulation somewhere in between the situational and the quantitative which would account for answers of both kinds occurring in the Gymnasium class. The task formulation gave students the option of choosing between taking a situational or a quantitative view, and while most took a quantitative view some did not, perhaps reflecting personal preferences at the individual level.

A SOCIOLOGICAL PERSPECTIVE AT THE COMMUNITY LEVEL: RESULTS FROM THE OBERSCHULE

Rott did not only propose the Problem of Points in the Gymnasium class. She also proposed it in another class, in an Oberschule (comprehensive or mixed school). There the results were strikingly different. No students gave mathematical answers of the five types listed above. Most answered that Silke and Acun should get their money back. The only other type of answer Rott observed was a group that discussed extensively a variation on the Point-Difference argument, in which the difference of each players’ winnings from their original bet of 3€ should be equal to the point difference between the players. As one member of the group, Melanie, put it:

> If the score were six five, would then, they would then — ah, if it were six five, I would say that she gets two Euro, because that, he would get two Euro and she four Euro, but it is actually seven five. One point less. Then I would make it five Euro and one Euro. (Rott, 2014, p. 56, our translation)

If the score were 6:5 the point difference would be one, so Silke would win 1€ more than she bet, that is 4€, and Acun would win 1€ less than he bet, 2€. As the score is actually 7:5, the point difference is two, so Silke wins 2€ more than she bet, 5€, and Acun wins 2€ less than he bet, 1€. Melanie's group did not accept this argument, and at the very end of their work together, they almost persuaded her to accept the answer given by the standard Point-Difference argument, but she objected, and their final answer is a sort of compromise without any justification.

125 Melanie: Let's make it Silke four and Acun two.
126 Ines: [???]
127 Melanie: But then, look, just think, then Acun has two Euro, no? And she has, she has three
128 Euro, only one euro of Acun, although she has ‖two‖ points more.
129 Anne: ‖let's make it‖ one fifty for him and four-fifty for her. -
130 Ines: That's also an idea. Then both are somehow on the same wavelength, so then the two are
131 actually equal.
132 Melanie: Okay, do that. (p. 55)

Earlier Rott asked Melanie how she arrived at her answer of 5€ from Silke and 1€ for Acun.

92 D. Rott: How did you get that Silke gets five euros and Acun one Euro?
93 Melanie: (Points to the worksheet) So there it was, yes seven to five.
94 D. Rott: Yes.
95 Melanie: And if the score would be seven to six, Silke would get four and Acun
96 two, but it is actually seven to five, so Silke gets more and Acun one Euro.
97 D. Rott: And what did you calculate exactly, to get that?
98 Melanie: Nothing
99 D. Rott: Just by intuition?
100 Melanie: Yes. (p. 57)

Rott calls this groups argument "Intuition" in light of Melanie's comment about not calculating, and the group's quick conclusion to divide the money 4.50€ to 1.50€. While this group's arguments included mathematical elements, which is unusual for the Oberschule groups, their final answer does not have a mathematical basis.

Recall that the answers from the Gymnasium groups were mostly justified in mathematical ways, except for a few who said to give the money back. In the Oberschule, however, the situation is the opposite. Almost all the groups said to give the money back, with only one group providing an intuitive argument with some mathematical elements. We can be sure the formulation of the task was the same in the Gymnasium and the Oberschule as it was provided to the teachers by Rott. How then to account for the very different types of answers? To do so we will need further results from the two classes, as well as another perspective, at another level.

A second difference between the two classes is in the number of different answers students gave. In the Gymnasium it was expected and happened that students gave alternative answers, including that the players should get their money back. The seven groups offered a total of 11 answers, of six different types, while in the Oberschule the five groups gave one answer each, four answering that the players should get their money back. One could account for this by saying the didactic contract is different in the two classrooms. However, even when pushed to give alternative answers in the interview, the Oberschule students only gave answers based on everyday experience. The different preferences of the students, for mathematical answers in the Gymnasium and answers based on everyday experience in the Oberschule, combined with the ability of the Gymnasium students to give answers based on everyday experience as well as mathematical answers, is reminiscent of a study done by Holland (1981).

Holland gave 8 year old children the task of sorting photographs of familiar foods. She found that children from middle-class backgrounds tended to sort the photographs in terms of abstract properties (e.g., "animal/vegetable/dairy/cereal" or "from the sea/farm"). On the other hand, working-class children tended to sort the photographs in terms of their personal experiences (e.g., “things I eat for breakfast/ lunch/supper”). Furthermore, when asked to re-sort the photographs, the children from middle class backgrounds could do so, sorting them in terms of their personal experiences, while the working class children could not offer additional sortings. As this result is very similar to the differences between the Gymnasium students and the Oberschule students, it is worth considering the theoretical framework used by Holland and looking at Rott’s results at a sociological level.

A sociological perspective

The neighbourhoods of Bremen are classified by the government into different types. The Oberschule is located in a Group A neighbourhood, in which the proportion of families with immigrant backgrounds as well as the proportion of people on social assistance is above average. Specifically in the neighbourhood around the school, the proportion of families with immigrant backgrounds is 65–86% and many parents have low levels of education (Die Senatorin für Bildung, Wissenschaft und Gesundheit, 2012a, p. 55). The Gymnasium, on the other hand, is a private religious school located in a Group B neighbourhood in which the proportion of families with immigrant backgrounds is below average (15–30%). This does not mean that all the students in the school come from upper-middle-class homes, but it does mean that the majority do.

This difference suggests that a sociological perspective might be useful in understanding the different answers given by students in the two schools to the Problem of Points.

Sociological approaches fall roughly into two groups. One approach collects data on very large groups of people and uses statistical techniques to draw conclusions about the relative weight of various social factors in determining, for example, success in school mathematics. Such an approach is clearly not suitable in this case. A second approach involves applying well developed sociological theories to describe and analyse the actions of smaller groups of people. We have chosen one such theory, the work of Bernstein, to analyse the data presented here.

Holland makes use of several concepts from Bernstein’s work: restricted and elaborated orientations to meaning, realisation and recognition rules, and re-contextualisation.

> For Bernstein an orientation to meaning is created by inter-actional practices which act selectively on what is to be meant, and what form the realization of meaning takes in which contexts. The inter-actional practices in the family and school transmit recognition rules which mark contexts as requiring a specific text, and realization rules which regulate what meanings are to be offered and how these are to be made public. Bernstein argues that families in different social class locations are typified by different inter-actional practices which regulate different recognition and realization rules and generate an elaborated or restricted code ... (Bernstein, 1977) or ... an orientation to context independent meanings or to context dependent meanings. For some children then the re-contextualizing principle of the school will entail recognition and realization rules very different from those acquired in the family. (Holland, 1981, p. 2)

Children learn at home how to *recognise* contexts that require certain ways of making meaning, and how to *realise* those ways of making meaning. School is a context in which meaning is context independent. Children’s experiences are re-contextualised in school into abstractions (Bernstein, 1977). The middle class children in Holland’s study had both the recognition rules to see Holland’s task as calling for abstract categories, as well as the realisation rules needed to use abstract categories in classifying. The working class children did not. Bernstein (1977) characterises an orientation to abstract, context independent meanings as an elaborated orientation, and an orientation to context dependent meanings as a restricted orientation. From this perspective we can account for the differences between the argumentations of the Gymnasium students and the Oberschule students by suggesting that the out of school experiences of the students led some to develop elaborated orientations to meaning, including both recognition and realisation rules related to using mathematical arguments in school contexts, while others developed restricted orientations to meaning, lacking either the recognition or the realisation rules needed to produce mathematical argumentations in school contexts.

Cooper and Dunne (2000) describe difficulty recognising the border between everyday and mathematical contexts as the "boundary problem". They researched sixth grade and ninth grade students' solutions to test items of two types: "realistic" items in which a mathematical task is embedded in an everyday context, and "esoteric" items in which the task was decontextualised. Working class students performed less well than middle class students, especially on realistic items. However, in an interview situation in which the students were explicitly told to disregard the context, they were successful on items they had incorrectly answered in the test situation. This suggests that their difficulty did not stem from an inability to act appropriately. Rather, they faced the boundary problem; they lacked the recognition rules to distinguish between everyday and mathematical contexts. If this was also the difficulty faced by the Oberschule students, then a formulation of the task that more clearly invokes a mathematical or stochastic view of the problem might help them to recognise the type of argument expected. In addition, a reframing of the didactic contract to more explicitly require multiple solutions and mathematical solutions might also help to overcome the boundary problem.

Knipping (2012) discusses the value of sociological approaches to research on argumentation, and identifies decontextualised language as a key issue. She cites the work of Hasan (2001), who studied the opportunities children have to engage in this kind of discourse in their homes before they begin schooling. In some homes the talk of adults to small children remains tied to the context, related directly to the activities and objects that are present to the senses. In others there is already at this early age a fluid shifting between decontextualised and contextualised language, which Hasan refers to as a "con/textual shift". There is reason to believe that the division Hasan found, between home situations in which con/textual shifts occur often and those where they occur rarely, is influenced by social class. This could provide a mechanism to account for the occurrence of the boundary problem among the Oberschule students.

CONCLUSIONS

In this paper we have offered three different perspectives at three levels that help us account for students' responses to the Problem of Points: a classroom interactional perspective, a *Stoffdidaktik* task analysis at the individual level, and a sociological theoretical perspective at the community level. We have made no effort to integrate these perspectives into a single all-encompassing model. This is likely to be impossible given the different fundamental assumptions of the theories involved. Moreover, as Reid (1996) points out, multiple perspectives offering different interpretations are valuable, even if contradictory, as no single perspective can capture the complexity of human learning in social contexts. And as we take a closer or wider view of phenomena of interest, different theories become applicable. Learning can be viewed at many different levels, from the neurological to the ecological. Different theories and methodologies are appropriate for these different levels. "Discourses concerned with different phenomena (such as radical or social constructivism—or neurology, ecology, or biological evolution) can be simultaneously incommensurate with one another and appropriate to their particular research foci." (Davis & Simmt, 2003, p. 143).

Researching classroom argumentation must involve theories at at least three levels. Argumentation is conducted by human beings, and influences the thinking of human beings, and so perspectives at the level of individual human cognition are needed. But argumentation is also a necessarily social phenomenon which takes place among groups of people using language. Perspectives that focus on communication, language, and interactions between people in small groups are therefore needed. People and communication develop in larger social and cultural contexts that shape them and are shaped by them. Argumentation in mathematics classrooms cannot be considered independently of argumentation in mathematics and people in mathematics classrooms cannot be considered independently of their social and cultural backgrounds. Multiple perspectives at every level are needed.

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