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In this paper I present an analytic framework for generality in textbook proving tasks that involve functions. The framework is discussed in relation to results obtained when analysing tasks in integral calculus. The results show that the frameworks' categories are easily distinguishable if the functions are explicitly described. The results are also promising regarding the possibility to clarify differences between textbooks. The analysed sections exemplify that there is not necessarily a correlation between the number of general proving tasks and the opportunities for students to engage in reasoning about arbitrary functions. Limitations and possible refinements of the framework are also discussed.

Keywords: Mathematical proof, mathematics textbook, upper secondary school, undergraduate mathematics, integral calculus.

INTRODUCTION

Research on the teaching and learning of proof often involves the distinction between specific and general arguments and properties. For instance, students' tendency to take specific cases as sufficient justification of general properties is well documented (Harel & Sowder, 2007). The distinction between specific and general has also been used to study how mathematics textbooks support proof-related activities and learning (e.g., Stylianides, 2008; Thompson, Senk, & Johnson, 2012).

In an ongoing study, we use the analytical framework of Thompson and colleagues (2012) to investigate Swedish and Finnish upper secondary textbooks. One part of the analysis consists of determining whether or not textbook tasks provide opportunities for general reasoning. When tasks involve functions, this distinction is not always obvious. Various combinations of dependent variables, independent variables and other parameters mean significant differences in the 'degree of generality' between tasks. This suggests that such a textbook analysis would benefit from a more fine-grained classification of generality. In this paper I will address this issue by discussing a tentative generality framework for proving tasks based on the 'size' of the set of functions that the tasks call for a proof about. I will refer to this as the *function generality framework*. By 'proving task', I mean a textbook exercise explicitly asking the student to prove or show a mathematical property.

The function generality framework is an answer to the first of three questions stated below, around which this paper is focused. By applying it to proving tasks in Swedish and Finnish textbooks, some results relating to the other two questions will be obtained. The questions are: (1) How can 'degree of generality' in proving tasks involving functions be framed? (2) What analytical difficulties arise when proving tasks are classified according to function generality? (3) What can classification according to function generality reveal about textbooks that a 'specific-or-general' classification cannot?

Some initial results concerning the analysed textbooks will also be discussed.

BACKGROUND

One characteristic feature of a mathematical proof is that it usually provides a valid justification for a *general* property. However, numerous studies (many of which are referred to in Harel and Sowder (2007)) show that students on most educational levels, even at university (e.g., Hemmi, 2008; Weber, 2001), have limited understanding of this aspect of proof. Typically, students justify general statements with specific examples, view counter-examples as exceptions, believe that counter-examples might exist even if there is a general proof etc. In the literature, this has been referred to as *empirical response* (Bell, 1976), *pragmatic justification* (Balacheff, 1988) and *empirical proof scheme* (Harel & Sowder, 1998). Central to all these frameworks is some kind of distinction between the general and the specific.

Even though no curriculum program is self-enacting, research has stressed the wide use of textbooks in classrooms and how they are crucial links between national curricula and teaching practice (e.g., Stein, Remillard, & Smith, 2007). In line with this research, mathematics textbooks can be seen as potential sources for opportunities to learn. Hence, textbooks' treatment of reasoning and proving is an important object of study. Historically, such studies are rare (Hanna & de Bruyn, 1999), but in the past decade a number of studies with this focus have been published (Nordström & Löfwall, 2005; Stylianides, 2008; Thompson et al., 2012). In an analysis of an American reform-based curriculum for middle school, it was found that 40% of the textbook tasks were designed to engage students in reasoning and proving, but only 12% of these offered opportunities to provide general proofs (Stylianides, 2008). Thompson and colleagues (2012) report on an extensive study of US textbooks for upper secondary school, concerning opportunities offered for students to engage in proof-related reasoning within the topics of exponents, logarithms and polynomials. Their study showed that about 50% of the properties stated in narratives were given some kind of justification; 30% with a general argument (i.e. a proof) and 20% with a specific case. About 5% of all exercises were considered proof-related, half of them of a general kind and half a specific kind. Approximately 1% of all exercises urged the student to develop a general argument.

The frameworks used by Stylianides (2008) and Thompson and colleagues (2012) are similar to (or inspired by) those used by Bell (1976), Balacheff (1988) and Harel and Sowder (1998). Hence, they also distinguish between specific and general aspects of textbook content. In an ongoing study, we have used the framework of Thompson and colleagues (2012) to analyse Swedish and Finnish textbooks. Traditionally, deductive reasoning has primarily been studied in geometry courses, but more recently it has been suggested that reasoning and proof are important in all content areas (e.g. NCTM, 2000). Therefore, Thompson and colleagues (2012) choose to focus on algebraic topics instead of geometry. For the same reason, and to further complement their study, we have focused on calculus. In this broader study we analyse all parts of textbooks: expository sections, worked examples, exercise sets, review exercises etc. It is during this work that I have encountered differences in generality that I have found difficult to capture with the earlier frameworks, and which I therefore address in this paper.

When functions are involved in mathematical tasks, there are dependent as well as independent variables. A task like "Prove that $Dx^2 = 2x$" is *general* in the sense that the student is asked to prove something for all x, but *specific* in the sense that it only concerns one particular function. The inclusion of more or less arbitrary functions, parameter families of functions and other parameters also means (potential) differences in the 'degree of generality'. Tasks like "Prove that $De^{kx} = ke^{kx}$" and "Prove that $Df(kx) = kf'(x)$" are both general in the sense that the identities hold for all x and all k, but the second one is obviously more general than the first since it also holds for any differentiable function f. This difference in generality also implies different content focus; while the first focuses on properties of a certain function, the second focuses on a fundamental property of differentiation itself. These examples show a need for a more fine-grained framing of generality based on properties of the functions involved in the tasks.

I have also found several tasks formulated as "Show that ..." but that were not general in any sense and only required a routine calculation. Sometimes it was the other way around: theoretically and cognitively demanding tasks that from a mathematical point of view concerned proving but were formulated in words like "Why is it ..." or "Motivate why ...". This relates to findings regarding proofs being "invisible" in textbooks (Nordström & Löfwall, 2005). While other studies (e.g., Stylianides, 2008; Thompson et al., 2012) look for opportunities to engage in reasoning and proving in a broad sense, it is therefore important to also look specifically at proving tasks.

METHODOLOGY

Topic, context and textbooks

For the pilot study reported here, I have restricted the analysis to proving tasks in integral calculus. Like differential calculus, this topic is central in upper secondary school as well as in introductory courses at universities. However, the theory of integrals is more complicated and proofs are often omitted. There is a tendency that the underlying theory is not treated in detail in introductory courses at universities but rather postponed to intermediate courses, due to students' difficulties with a theoretical approach (Hemmi, 2006). Thus, it is a real challenge for upper secondary textbook authors to incorporate elements of reasoning and proving within this topic, and it is of research interest to study how this is done.

To get a reasonably rich set of data, four different textbook sources were chosen (see Table 1 below): two Swedish upper secondary textbook series (referred to as SW1 & SW2), one Swedish undergraduate textbook (SWU), and one Finnish textbook for upper secondary school (FI1). Publishers are unwilling to reveal their market share, but it is well-known that SW1 and its predecessors have long dominated the Swedish market. In 2011, more than 80% of those entering engineering programs at Örebro University reported having used these textbooks. The main reason for choosing SW2 was that, while SW1 is a traditional Swedish textbook, SW2 is a newer one with more reform-oriented intentions and a stated focus on reasoning. FI1 is the only Finnish textbook series available in Swedish for use in the Swedish-speaking parts of Finland. Its Finnish original is probably the most widely used textbook in Finland. Finally, for the sake of curiosity, and to get an indication of the usefulness of the function generality framework on introductory calculus texts for the university level, SWU was included; this is a Swedish single-variable calculus textbook that has been around for several decades.

Swedish upper secondary school is course-based. There are five mathematics courses, of which the first four are often a prerequisite for university studies in science and technology. Integral calculus is treated in Courses 3 and 4. The first three courses exist in different versions, depending on whether they are part of a vocational program (Track a), a program in the social sciences (Track b) or a program in science and technology (Track c). For this study, only textbooks for Track c were chosen. In Finland there is a short mathematics course serving as preparation for university studies in, for example, the humanities, and a long course serving as preparation for university education involving higher mathematics. The long course is divided into 13 parts, the first ten of which are mandatory. Part 10 is devoted to integral calculus only (but this topic is further developed in Parts 12 and 13). This study only includes Part 10.

Method

First, all textbook sections specifically dealing with integral calculus were identified. All exercises in these sections were included in the analysis, as were the review exercises on integral calculus (which were typically placed at the end of the book). The only exception was exercises on continuous distributions, which were all omitted since only one of the textbooks treated probability theory within the sections on integrals. Concerning the unit of analysis, whenever an exercise was divided into an enumerated list of subtasks, each subtask was regarded as one task. This resulted in a total number of 1,739 textbook tasks to be analysed (see Table 1). Since the function generality framework is meant to be a tool for analysing the opportunities offered to students to associate 'proving' with general justifications, I next looked for tasks explicitly asking the student to 'show' or 'prove' something. Such tasks are referred to as *proving tasks*. In total there were 80 proving tasks, all of which could be interpreted as concerning functions.

In SW1, SW2 and SWU all proving tasks were formulated as "Show that …"; i.e., the word 'prove' was

	Publisher	Series	Book	Total no. of	
				tasks	proving tasks
SW1	Liber	Matematik 5000	3c & 4	371	21 (6%)
SW2	Sanoma utbildning	Origo	3c & 4	450	13 (3%)
FI1	Schildts	Ellips	10	529	30 (6%)
SWU	Matematik-centrum, Lund	Analys i en variabel	Exercises	379	16 (4%)
				1,739	80 (5%)

Table 1: Textbooks, tasks and proving tasks within sections on integral calculus

never used. In FI1 'prove' was used as often as 'show'. On three occasions, all in SW1, 'show' was used in a non-mathematical way, as in "Show in detail how you calculate the integral ..." (SW1, Book 3c, p. 185, ex. 3412a). I chose to include them in the analysis since they play a role in forming what students will associate with the word 'show'.

For every proving task, a detailed account was made of the function(s) it concerned. This included information on whether the task concerned specific functions, parameter families of functions (including the number of parameters) or more general non-parametric classes of functions. Notes were taken about the kind of elementary functions involved (polynomial, trigonometric function, exponential etc.) or, in the more general cases, what classes of functions were involved (continuous, periodic, odd etc.). It was also noted if a task contained additional parameters (not connected to the functions), or if it could be seen as general in some other sense.

Analytical framework

One way to determine whether a proving task offers opportunities for general reasoning is to determine whether the property to prove itself is general or specific. Therefore, all proving tasks were categorized as *case-specific* or *case-general* following the framework of Thompson and colleagues (2012). As mentioned earlier, the distinction between specific and general is sometimes difficult to make for tasks involving functions. The general principle used in this paper is that if one can think of a more specific case than what is stated in the task, without substituting the independent variable with a specific number, then the task is considered case-general. This means that the presence of an independent variable is not enough for a property concerning functions to be deemed case-general – either there need to be other parameters involved, or the property must concern a class of functions. Thus, for example, a proving task about e^{2x} is usually considered case-specific (unless other parameters are involved), whereas a proving task about a^x or e^{kx} is considered case-general (see the *Results* section for further examples). I believe this is in line with how Thompson and colleagues (2012) would have distinguished between specific and general cases. To further clarify this notion, consider the following properties, which were all found (explicitly or implicitly) among the analysed proving tasks:

a) $\int_1^4 \sqrt{x}dx = \frac{14}{3}$
(SW1, Book 3c, p. 197, ex. 12a)

b) $\int_1^a \frac{1}{x^2}dx$ never exceeds π
(SW2, Book 4, p. 158, ex. 4371c)

c) $F(x) = \frac{a^x}{\ln a}$ is a primitive to $f(x) = a^x$
(SW2, Book 3c, p. 159 ex. 5124)

d) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even
(FI1, p. 156, ex. F1)

Here (a) is case-specific, while (b)–(d) are all case-general due to the parameter . But, like the examples given in the *Background* section, there are differences in the 'degree of generality' between (b), (c) and (d). While (b) concerns one specific function, (c) holds for a one-parameter family of functions and (d) holds for all even functions, a class too large to be represented by use of a finite number of parameters. In order to capture these differences, I introduce the *function generality framework* with three subcategories: statements about a finite number of specific functions, like (b), will be called *non-general*; statements about parameter families of functions, like (c), will be called *finitely general* and measured by the number of parameters, as long as the number of parameters is finite; statements about more general sets of functions, like (d), which are too large to be represented by use of a finite number of parameters, will be called *infinitely general*.

From the student perspective, the difference between non-general and finitely general proving tasks is that in the latter case the student needs to distinguish the independent variable from other variables, and to be able to handle parameters when manipulating function expressions. But in both cases there are expressions available for algebraic manipulation. In the case of infinite generality, though, the student needs to find suitable ways to represent and use the relevant property (like the property 'being even' in (d)). Thus, I believe the three categories of function generality to be of educational relevance, even though it is sometimes easier to prove an infinitely general statement than a non-general one.

The classification of tasks according to function generality can be done independently of the classification of tasks as case-specific or case-general. But since all case-specific tasks will be non-general, nothing is gained by applying the function generality framework to case-specific tasks. I therefore only apply this framework to case-general tasks; i.e. I see function generality as a way to divide case-general tasks into subcategories.

During the analysis I soon discovered that proving tasks often express a relation between different classes of functions. In such cases, the task was classified according to the 'largest' of these classes. Examples are given in the *Results* section.

In our broader study, mentioned earlier, a second Finnish textbook series is included and parts of the analysis have been done independently by two researchers. During this work we have discussed and compared our coding and resolved all differences.

Finally, even though Examples (a)–(d) are from integral calculus, the ambition is for the framing of generality described here to be applicable to any topic involving functions. Aspects of generality that might be unique to integral calculus will be touched upon in the discussion.

RESULTS

In this section I will present a representative sample of the analysed proving tasks belonging to the different framework categories, as well as tasks that highlight the strengths and weaknesses of the framework as an analytical tool. A summary of the number of proving tasks, by textbook series and generality, is shown in Table 2. For example, SW2 had 13 proving tasks: three case-specific and ten case-general. When these ten were analysed according to function generality, three were found to be non-general, six finitely general, and one infinitely general.

In SW1 and SW2 (but not in FI1 or SWU) I found proving tasks that were case-general, even though they were non-general in the sense that they only concerned specific functions. This was always due to additional parameters, typically as limits of integration, as in Example (b) in the framework section.

Proving tasks of finite generality mostly concerned one- or two-parameter families of functions. SW2 also contained two tasks with three-parameter families. In FI1, six out of 13 finitely general tasks had the constant of integration as its only parameter, as in "Prove the integration formula $\int \sin x\, dx = -\cos x + C$" (FI1, p. 29, ex. 259a).

In SW1 there was no proving task of infinite generality, while in SW2 there was one: "Show that if $f(x)$ is continuous in $a \le x \le b$ then $\int_a^b f(x)dx = -\int_b^a f(x)dx$" (SW2, Book 4, p. 146, ex. 4340). About a third of the proving tasks in FI1 were infinitely general. Half of these were similar to "Prove the integration formula $\int f'(x)e^{f(x)}\, dx = e^{f(x)} + C$" (FI1, p. 35, ex. 271b); i.e., they were essentially related to the chain rule.

During the classification according to function generality, only three tasks proved somewhat difficult to categorize, all of them in FI1. The first reads as follows: "Show that all primitive functions to $g(x) = x^2(5 + 4x^3)^2$ are strictly increasing" (FI1, p. 25, ex. 250). In this task only one specific function is explicitly given, but the statement concerns the one-parametric class of its primitive functions. Therefore, I classified this task as finitely general. However, the proof need not take into account any parameters, since it mainly rests on the fact that $g(x) > 0$.

A similar difficulty concerns ex. 460, p. 112 in FI1, where the student is asked to prove a general formula for the area bounded by a parabola and a straight line. No explicit formulas are given for the two curves. However, since lines and parabolas are graphs of first- and second-degree polynomials, i.e. two- and three-parameter families of functions, I classified this task as finitely general.

	Proving tasks	Case-specific	Case-general			
			Total	Non-general	Finitely general	Infinitely general
SW1	21	15	6	1	5	0
SW2	13	3	10	3	6	1
FI1	30	8	22	0	13	9
SWU	16	13	3	0	1	2

Table 2: Numbers of proving tasks of different case and function generality

The third example connects to the uniqueness of primitive functions, but is actually a result of differential calculus, often referred to as *Rolle's theorem*: "If $f'(x) = 0$ everywhere then f is constant" (FI1, p. 156, ex. F3). The conclusion of this theorem means that the class of functions the theorem concerns is one-parametric. But in the proof, f must be handled as an *arbitrary* function with the property $f'(x) = 0$. I therefore considered this proving task to be infinitely general.

DISCUSSION

In the introduction I posed three questions. The first was how the 'degree of generality' in proving tasks involving functions could be framed. To answer this question, I have described a framework based on the 'size' of the class of functions under consideration. Applying this to proving tasks in integral calculus in four sets of textbooks has made it clear that proving tasks of all three kinds (non-general, finitely general and infinitely general) exist, and that the classification is straightforward as long as the proving tasks are explicit regarding which functions they concern. This indicates that for a generality analysis of upper secondary textbook proving tasks, the categories of the function generality framework are relevant and well-defined.

The second question concerned analytical difficulties. The three concluding examples in the *Results* section indicate that my framing of generality is less suitable when the functions under consideration are not explicitly given. In such tasks, the first step in providing a proof is often to find a suitable representation of the functions involved. One might therefore expect students to experience them as more general than our classification shows. This dimension of generality is not captured by my framework. The third example shows the difficulty in measuring generality in terms of the size of the class of functions when the statement itself is about this size. It is reasonable to believe that such difficulties arise more often when theoretically oriented textbooks are analysed, regardless of whether or not the topic is integral calculus.

The third question concerned the usefulness of the framework. Let us first look at the differences between the textbooks, shown in Table 2. A larger part of the proving tasks are case-general in SW2 than in SW1, and the same holds if we focus on function generality. But if we compare SW2 and FI1, the function generality framework reveals differences that cannot be seen simply by checking case generality. The proportion of case-specific to case-general proving tasks is approximately the same (1:3) for these textbooks. But while a third of the case-general proving tasks in SW2 turn out to be non-general when it comes to function generality, FI1 has no such non-general tasks. In addition, nine out of 30 proving tasks in FI1 are infinitely general (concerns 'any' function), while there is only one such proving task in SW2. The fact that SW1 has no infinitely general proving tasks and SW2 has only one also means that they provide few (if any) opportunities to associate the imperative 'prove' with the providing of a general argument valid for 'any' function. Since proving tasks concerning specific functions or parameter families of functions turn the attention to features specific to these functions and not to properties of integration in itself, the absence of proving tasks of infinite generality also means fewer opportunities for reification (Sfard, 1991) of the integral concept. Such information about textbooks may be of importance to teachers in planning and choosing complementary materials so they will be able to offer students sufficient opportunities to learn the generality aspects of reasoning and proof.

Since the analysis presented here only includes sections on integral calculus, we cannot draw any general conclusions about the analysed textbooks. What is said above only applies to the exercise sets in the integral sections. But the point here is that the results show that the function generality framework has the potential to reveal textbook properties of educational importance that a categorization of proving tasks as case-specific or case-general cannot. It is reasonable to believe that this holds true for other mathematical topics as well. As a first step, we plan to widen the analysis to differential calculus and to include tasks that are proof-related in a broader sense.

There are of course other aspects of proving that are not captured with this framework, and situations in which this framing of generality may be misleading. One topic-specific aspect concerns the constant of integration. As mentioned in the *Results* section, half of the finitely general proving tasks in FI1 had this constant as their only parameter. In such tasks, this parameter is seldom an essential part of the proof. Hence, the number of finitely general proving tasks can be misleading without further analysis of the parameters of the tasks. Another aspect relates to findings in the university textbook SWU. I was surprised to find so few general proving tasks in this book. On the other hand, my impression was that while proving tasks in the upper secondary textbooks often required only routine calculation (direct use of standard formulas for differentiation and integration), the proving tasks in SWU were more non-routine. They often concerned inequalities, and the proofs required that functions be estimated. One way to put it is that authors of upper secondary textbooks seem to want to acquaint students with the word ‘show’ by using it when one could just as well have asked them to calculate. This tendency is not evident in the university text. The extent to which proving tasks actually require reasoning and not simply standard symbolic manipulation is not covered by my framework, but would certainly be an important element of textbook proving tasks to investigate further.

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