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SIGN CHANGES OF FOURIER COEFFICIENTS OF MODULAR FORMS OF HALF INTEGRAL WEIGHT, 2

Y.-J. JIANG, Y.-K. LAU, G.-S. LÜ, E. ROYER & J. WU

ABSTRACT. In this paper, we investigate the sign changes of Fourier coefficients of half-integral weight Hecke eigenforms and give two quantitative results on the number of sign changes.

1. INTRODUCTION

The study of sign-changes of Fourier coefficients of automorphic forms is recently very active. For modular (Hecke eigen-)forms of integral weight, the consequent result from Matomäki and Radziwill [14] is exceptionally charming, where the multiplicative properties of the Fourier coefficients play a substantial role. However, the modular forms of half-integral weight do not share the same kind of multiplicativity, and many problems deserve delving.

Let \( \ell \geq 2 \) be a positive integer, and denote by \( \mathcal{S}_{\ell+1/2} \) the set of all cusp forms of weight \( \ell + 1/2 \) for the congruence subgroup \( \Gamma_0(4) \). Consider the coefficients in the Fourier expansion of a complete Hecke eigenform \( f \in \mathcal{S}_{\ell+1/2} \) at \( \infty \),

\[
(1.1) \quad f(z) = \sum_{n \geq 1} \lambda_f(n)n^{\ell/2-1/4}e(nz) \quad (z \in \mathcal{H}),
\]

where \( e(z) = e^{2\pi i z} \) and \( \mathcal{H} \) is the Poincaré upper half plane. A specific question is the number of sign-changes when all \( \lambda_f(n) \) are real. We interlude with the meaning of sign-changes of a sequence.

Let \( N \) be a subset of \( \mathbb{N} \) endowed with the ordering of integers. The sets of squarefree integers or arithmetic progressions are basic examples. Given a real sequence \( \{a_n\}_{n \in \mathbb{N}} \), a sign-change is realized via a closed and bounded interval \( [i, j] \subset (0, \infty) \) such that

(i) its end-points \( i, j \) lie in \( \mathbb{N} \) and satisfy \( a_i a_j < 0 \), and

(ii) \( a_n = 0 \) for all \( n \in (i, j) \cap \mathbb{N} \).

The sequence \( \{a_n\}_{n \in \mathbb{N}} \) is said to have a sign-change in the interval \( I \) if \( I \) contains one such interval \( [i, j] \). Besides, the number of sign-changes of \( \{a_n\}_{n \in \mathbb{N}} \) in \( [1, x] \), denoted by \( C_N(x) \), is meant to be the number of intervals \( [i, j] \) contained in \( [1, x] \).

Let \( \mathcal{S} \) be the set of squarefree numbers. Hulse, Kiral, Kuan & Lim [6] proved that the sequence \( \{\lambda_f(t)\}_{t \in \mathbb{N}} \) has an infinity of sign-changes. A quantitative version is given in Lau, Royer & Wu [13, Theorem 4], which says \( C_N(x) \gg x^{(1-4\vartheta)/5-\varepsilon} \) where \( C_N(x) \) denotes the number of sign-changes of \( \{\lambda_f(t)\}_{n \in \mathbb{N}} \) in \( [1, x] \) and the constant \( \vartheta \) is determined by (3.5) below. Conjecturally \( \vartheta = \varepsilon \) but it is still hard to guess the tight lower bound.
On the other hand, Meher & Murty [15] studied the sign-change problem for Hecke eigenforms \( f \) in Kohnen plus subspace of \( \mathfrak{S}_{\ell+1/2} \). A form \( f \) in the plus space has its Fourier coefficients supported at integers \( n \equiv 0 \) or \((-1)^\ell \pmod{4}\), i.e. \( f \) has the Fourier expansion at \( \infty \) of the form

\[
f(z) = \sum_{(-1)^\ell n \equiv 0,1 \pmod{4}} \lambda_f(n) n^{\ell/2 - 1/4} e^{2\pi i n z}.
\]

When \( f \) is a Hecke eigenform in the plus space and its coefficients \( \lambda_f(n) \) are all real, Meher & Murty proved in [15, Theorem 2] that \( \{\lambda_f(n)\}_{n \in \mathbb{N}} \) has a sign-change in the short interval \((x, x + x^{43/70+\varepsilon}]\) for any \( \varepsilon > 0 \) and for all sufficiently large \( x \geq x_0(\varepsilon) \). An immediate consequence is \( C_\ell^0(x) \gg x^{27/70-\varepsilon} \). This work naturally motivates the sign-change problem for arithmetic progressions.

In this paper, we furnish progress, based on our work in [10], in the above problems for complete Hecke eigenforms \( f \in \mathfrak{S}_{\ell+1/2} \). Firstly for the case \( N = b \), we sharpen the lower bound for \( C_\ell^0(x) \).

**Theorem 1.** Let \( \ell \geq 2 \) be an integer and \( f \in \mathfrak{S}_{\ell+1/2} \) a complete Hecke eigenform such that its Fourier coefficients are real. Let \( \vartheta \) be defined as in (3.5) below, and \( \vartheta \) any number satisfying

\[
0 < \vartheta < \min\left(\frac{1-2\varrho}{3}, \frac{1}{4}\right).
\]

Then

\[
C_\ell^0(x) \gg_{f, \vartheta} x^{\vartheta}
\]

for all \( x \geq x_0(f, \vartheta) \), where the constant \( x_0(f, \vartheta) \) and the implied constant depend on \( f \) and \( \vartheta \) only.

**Remark 1.** In particular, Conrey & Iwaniec [2] gives \( \varrho = \frac{1}{6} + \varepsilon \) which leads to

\[
C_\ell^0(x) \gg_{f, \varepsilon} x^{2/9-\varepsilon}
\]

for all \( x \geq x_0(f, \varepsilon) \), improving the exponent \( 1/15 - \varepsilon \) in [13].

Secondly we generalize the case of \( N = \mathbb{N} \) in Meher & Murty [15] to arithmetic progressions. Let \( Q \geq 1 \) be an integer, and \( a = 0 \) or \( a \in \mathbb{N} \) with \((a, Q) = 1\). Define

\[
\mathcal{A} = \mathcal{A}_{a, Q} := \{n \in \mathbb{N} : n \equiv a \pmod{Q}\}.
\]

We study the sign-changes of \( \{\lambda_f(n)\}_{n \in \mathcal{A}} \) and sharpen the exponent \( \frac{43}{70} + \varepsilon \) of Meher & Murty’s result to \( \frac{1}{2} \), which in turn gives the better lower bound \( C_\ell^{\mathcal{A}}(x) \gg x^{1/2} \).

**Theorem 2.** Assume the same conditions for \( f \) and \( \varrho \) in Theorem 1. Let \( Q \geq 1 \) be odd and \( \mathcal{A} = \mathcal{A}_{a, Q} \) defined as in (1.3). Suppose one of the following condition holds:

1° \( Q = 1 \);

2° \( a = 0 \) and \( Q = \prod_{p|Q} p^{\alpha_p} \) where all \( \alpha_p \) are odd;

3° \((a, Q) = 1 \) and \( Q = \prod_{p|Q} p^{\alpha_p} \) where all \( \alpha_p \) are \( \geq 2 \).

Then there are positive constants \( c_0 = c_0(f, Q) \) and \( x_0 = x_0(f, Q) \) such that the sequence \( \{\lambda_f(n)\}_{n \in \mathcal{A}} \) has at least one sign change in the interval \((x, x + c_0 x^{1/2}]\) for all \( x \geq x_0 \). In particular, we have

\[
C_\ell^{\mathcal{A}}(x) \gg_{f, Q} x^{1/2}
\]

for all \( x \geq x_0 \).
2. Methodologies

Let $\lambda_f(n)$ be the coefficients as in (1.1) and $N$ a subset of $\mathbb{N}$. Define

\[(2.1) \quad S_N^f(x) := \sum_{n \leq x, \ n \in N} \lambda_f(n).\]

A typical approach for the sign-change detection exploits the oscillation exhibited in the mean $S^f_N(x)$, while to locate the sign-change, the mean over short intervals, i.e. $S^f_N(x + h) - S^f_N(x)$ for small $h$, will be a good device. Suppose a sign-change is found in the interval $[x, x + h]$ for every $x$ large enough. Then it follows immediately that the number of sign-changes in $[1, x]$ is at least $x/h + O(1)$ (and hence $\gg x/h$). A standard way to study $S^f_N(x)$ is via the Dirichlet series. But for various $N$, we get different degree of its analytic information.

For $N = \flat$, i.e. the case of squarefree integers, we only get an analytic continuation of the Dirichlet series

\[(2.2) \quad L^\flat_f(s) := \sum_{\flat t \geq 1} \lambda_f(t) t^{-s}\]

in the half-plane $\Re s > \frac{1}{2}$, where $\sum_{t \geq 1}^\flat$ ranges over squarefree integers $t \geq 1$. As illustrated in [13], it turns out that the weighted mean is more effective. Thus, to prove Theorem 1, we first derive (2.3) below,

\[(2.3) \quad \sum_{x \leq t \leq x + h} \lambda_f(t) \min \left\{ \log \left( \frac{x + h}{t} \right), \log \left( \frac{x}{t} \right) \right\} \ll \epsilon h^{\frac{3}{2}} x^\epsilon.\]

The better exponent $\frac{1}{2}$ (versus $\frac{3}{4}$ in [13]) of $h$ is a key for the improvement. Another key is to have a mean square formula with better $O$-term. In [13], we showed that

\[
\sum_{X < n \leq 2X} |\lambda_f(n)|^2 = D_f X + O_{\epsilon}(X^{\beta + \epsilon}).
\]

with $\beta = \frac{3}{4} + \vartheta$. Here we sharpen it to $\beta = \frac{3}{4}$ in Lemma 4.1 and then conclude Theorem 1 with argument in [13]. This will be done in Section 4.

Next for $N = \mathcal{A}$ (see (1.3)), we shall provide a truncated Voronoi formula for $S^\mathcal{A}_f(x)$ in Section 6. This result is itself interesting since the Voronoi formula is an vital tool for many applications, see [7], [11] for example. Then we complete the proof of Theorem 2 with the method of Heath-Brown and Tsang [5]. However the congruence condition underlying $\mathcal{A}$ gives rise to new (but interesting) difficulties. To transform the congruence, additive characters of modulus $d|Q$ will be invoked and then two consequences follow: the summands in the Voronoi formula are intertwined with Kloosterman-Salié sums, and the frequencies in the cosines are of the form $\sqrt{n/d}$. We need to select a suitable frequency for amplification with a pair of non-vanishing Salié sum and Fourier coefficient in the associated summand. The implementation is successful when $Q$ fulfills the conditions in Theorem 2, which will be elucidated in Sections 7 & 8. It is worthwhile to remark that the mean square result of $\lambda_f(n)$ is not needed for the method in [5].
A cusp form $f \in \mathcal{S}_{\ell+1/2}$ has Fourier expansions at the three inequivalent cusps $\infty, -\frac{1}{2}, 0$ of $\Gamma_0(4)$, which are respectively given by (1.1), and (3.1), (3.2) below:

$$g(z) := 2^{\ell+1/2}(-8z+1)^{-(\ell+1/2)}f \left( \frac{4z}{-8z+1} \right)$$

(3.1)

and

$$h(z) := (-i2z)^{-(\ell+1/2)}f \left( \frac{-1}{4z} \right) = \sum_{n \geq 1} \lambda_g(n) n^{\ell/2-1/4} e(nz).$$

(3.2)

Following the argument in [13, Section 2.2], we have

$$\sum_{n \leq x} |\lambda_f(n)|^2 \sim x \quad (\text{for all three cases } f = f, g, h).$$

(3.3)

When $f$ is a complete Hecke eigenform, we know from [10] that $g$ and $h$ are Hecke eigenforms of $T_p^2$ for all odd prime $p$. A consequence is, cf. [10, Lemma 3.2 with $Q = \{2\}$]: for all odd $m \geq 1$, all squarefree $t$ and $j \geq 0$,

$$\lambda_f(2^j t) = 0 \Rightarrow \lambda_f(2^j tm^2) = 0 \quad (f = f, g, h).$$

(3.4)

In addition, we have the following pointwise estimate, see [10, Lemma 3.3].

**Lemma 3.1.** Let $f$ be a complete Hecke eigenform, $g$ and $h$ be defined as above. For any integer $m = tr^2$ where $t \geq 1$ is squarefree, we have

$$\lambda_f(m) \ll f |\lambda_f(t)\tau(r)\tau^2 + |\lambda_f(t)|\tau(r)\tau^2 \ll f \varrho t^\varrho \tau(r)^2$$

for $f = f, g, h$ respectively, where $\tau(n)$ is the divisor function and $\varrho$ satisfies (3.5) below. The first implied $\ll$-constant depends only on $f$ and the second implied $\ll$-constant depends at most on $f$ and $\varrho$.

Here $\varrho$ denotes the exponent for which

$$\lambda_f(t) \ll_\varrho t^\varrho \quad \forall \ t \ \text{squarefree},$$

i.e. the bound towards the Ramanujan Conjecture for the half-integral weight Hecke eigenforms. The conjectural value is $\varrho = \varepsilon$. Conrey & Iwaniec [2] obtained $\varrho = \frac{1}{6} + \varepsilon$.

Let $d \geq 1$ be an integer and $(u, d) = 1$. Define the twisted $L$-function for $f$ by

$$L_f(s, u/d) = \sum_{m \geq 1} \frac{\lambda_f(m)e(mu/d)}{m^s} \quad (\Re s > 1)$$

(3.6)

and define similarly for $g$ and $h$. These twisted $L$-functions when attached with suitable factors may be expressed as integrals of $f$ along vertical geodesics, and extend to entire functions, cf. [6, (4.4)-(4.5)]. Moreover Hulse et al found the functional equation for $L_f(s, u/d)$, which is put in the following form

$$q_d^s L_\infty(s) L_f(s, u/d) = i^{-(\ell+1/2)} q_d^{1-s} L_\infty(1-s) \tilde{L}_f(1-s, v/d),$$

(3.7)
where \( uv \equiv 1 ( \mod d ) \) and \( L_\infty ( s ) := (2\pi)^{-s} \Gamma \left( s + \frac{\ell}{2} - \frac{1}{4} \right) \) is the gamma factor, cf. [6, Lemma 4.3] and [10]. The conductor \( q_d \) and the dual \( L \)-function \( \tilde{L}_f ( s, v/d ) \) are defined as follows:

(3.8) \[ q_d = d \text{ or } 2d \text{ according to } 4 \mid d \text{ or not}, \]

and

(3.9) \[ \tilde{L}_f ( s, v/d ) := \sum_{n \geq 1} \lambda(n; d) \varpi_d(n, v) n^{-s}, \]

where

<table>
<thead>
<tr>
<th>( \lambda(n; d) )</th>
<th>( \varpi_d(n, v) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 4 \mid d )</td>
<td>( \lambda_4(n) )</td>
</tr>
<tr>
<td>( 2 \parallel d )</td>
<td>( \lambda_6(n) )</td>
</tr>
<tr>
<td>( 2 \not\mid d )</td>
<td>( \lambda_8(n) )</td>
</tr>
</tbody>
</table>

with \( 4\bar{\mathbb{T}} \equiv 1 ( \mod d ) \).

In [6], Hulse et al applied \( L_4(s, u/d) \) to obtain the analytic properties of \( L_\ell^\circ(s) \), which was sharpened to the following result [10, Theorem 1].

**Lemma 3.2.** For a complete Hecke eigenform \( f \in \mathcal{S}_{\ell+1/2} \), the series \( L_\ell^\circ(s) \) extends analytically to a holomorphic function on \( \Re s > 1/2 \), and for any \( \varepsilon > 0 \),

(3.11) \[ L_\ell^\circ(s) \ll_{\ell, \varepsilon} (|\tau| + 1)^{1-\sigma + 2\varepsilon}, \quad \left( \frac{1}{2} + \varepsilon \leq \sigma \leq 1 + \varepsilon, \tau \in \mathbb{R} \right), \]

where the implied constant depends on \( f \) and \( \varepsilon \) only.

**Remark 2.** Using Lemma 3.2 in place of [13, Proposition 7], the estimate in (2.3) follows plainly from the same argument as in [13, Section 4.1], so we do not repeat here.

### 4. Proof of Theorem 1

We start with the following lemma where the \( O \)-term in (4.1) is smaller than [13, (14)].

**Lemma 4.1.** Let \( \ell \geq 2 \) be a positive integer and \( f \in \mathcal{S}_{\ell+1/2} \) be a complete Hecke eigenform. Then for any \( \varepsilon > 0 \) and all \( x \geq 2 \), we have

(4.1) \[ \sum_{n \leq x} |\lambda_f(n)|^2 = D_f x + O_{f, \varepsilon} \left( x^{3/4 + \varepsilon} \right), \]

where \( D_f \) is a positive constant depending on \( f \).

**Proof.** We choose two smooth compactly supported functions \( w_+ \) such that

- \( w_-(x) = 1 \) for \( x \in [X + Y, 2X - Y] \), \( w_-(x) = 0 \) for \( x \geq 2X \) and \( x \leq X \);
- \( w_+(x) = 1 \) for \( x \in [X, 2X] \), \( w_+(x) = 0 \) for \( x \geq 2X + Y \) and \( x \leq X - Y \);
- \( w_\pm(j)(x) \ll_j Y^{-j} \) for all \( j \geq 0 \);
the Mellin transform of \( w(x) \) is
\[
\hat{w}_\pm(s) := \int_0^\infty w_\pm(x)x^{s-1} \, dx
\]
(4.2)
\[
\approx j \frac{Y}{X^{1-\sigma}} \left( \frac{X}{|s|Y} \right)^j \quad \forall \ j \geq 1;
\]
trivially \( \hat{w}_\pm(s) \ll X^\sigma \) and
(4.3)
\[
\hat{w}_\pm(1) = X + O(Y).
\]
Obviously we have
(4.4)
\[
\sum_n |\lambda_f(n)|^2 w_-(n) \ll \sum_{X < n \leq 2X} |\lambda_f(n)|^2 \leq \sum_n |\lambda_f(n)|^2 w_+(n).
\]
Let the Dirichlet series associated with \( |\lambda_f(n)|^2 \) be defined as (see e.g. [13, (11)])
\[
D(f \otimes \overline{f}, s) = \sum_{n=1}^\infty |\lambda_f(n)|^2 n^{-s}.
\]
By the Mellin inversion formula
\[
w_\pm(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \hat{w}_\pm(s)x^{-s} \, ds,
\]
we write
\[
\sum_n |\lambda_f(n)|^2 w_\pm(n) = \frac{1}{2\pi i} \int_{(2)} \hat{w}_\pm(s) D(f \otimes \overline{f}, s) \, ds.
\]
With the help of Cauchy’s residue theorem, we obtain that
(4.5)
\[
\sum_n |\lambda_f(n)|^2 w_\pm(n) = D_f \hat{w}_\pm(1) + \frac{1}{2\pi i} \int_{(\kappa)} \hat{w}_\pm(s) D(f \otimes \overline{f}, s) \, ds,
\]
where \( \frac{1}{2} < \kappa < 1 \) and \( D_f := \text{Res}_{s=1} D(f \otimes \overline{f}, s) \). By (4.3), (4.2) with \( j = 2 \) and the convexity bound [13, Proposition 7]
\[
D(f \otimes \overline{f}, s) \ll_{f,\varepsilon} (1 + |\tau|)^{2 \max(1-\sigma,0)+\varepsilon} \quad \left( \frac{1}{2} < \sigma \leq 3 \right),
\]
we derive
\[
\sum_n |\lambda_f(n)|^2 w_\pm(n) = D_f X + O_{f,\varepsilon} \left( Y + X^{1+\kappa}Y^{-1} \right).
\]
Taking \( \kappa = \frac{1}{2} + \varepsilon \) and \( Y = X^{3/4} \), and combining the obtained estimation with (4.4), we find that
\[
\sum_{X < n \leq 2X} |\lambda_f(n)|^2 = D_f X + O_{f,\varepsilon} \left( X^{3/4+\varepsilon} \right),
\]
which implies (4.1) after a dyadic summation. \( \Box \)

Now we return to prove the theorem. Take \( h = x^\eta \) where \( \eta > \frac{3}{4} \) is specified later. Lemma 4.1 gives
\[
(i) \quad C h \leq \sum_{x \leq n \leq x+h} |\lambda_f(n)|^2 \quad \text{and} \quad (ii) \quad \sum_{x/m^2 \leq t \leq (x+h)/m^2} |\lambda_f(n)|^2 \ll hm^{-3/2}
\]
for any \( m \leq \sqrt{x+h} \), where the positive constant \( C \) and the implied \( \ll \)-constant depend on \( f \) and \( \eta \) only. Combining (i) with Lemma 3.1 leads to

\[
Ch \leq \sum_{x \leq n \leq x+h} \lambda_f(n)^2 \leq C' \sum_{m \leq \sqrt{x+h}} \sum_{x/m^2 \leq t \leq (x+h)/m^2} \lambda_f(t)^2
\]

where \( \sum^b \) confines the running index over squarefree integers only and \( C' > 0 \) is a constant depending at most on \( f \). By (ii) and the fact \( \sum_{m \geq A} \tau(m)^4 m^{-3/2} \gg A^{-1/2+\varepsilon} \), we conclude that for a large enough constant \( A \),

\[
\sum_{m \leq A} \tau(m)^4 \sum_{x/m^2 \leq t \leq (x+h)/m^2} \lambda_f(t)^2 \gg \left\{ C/C' + O(A^{-1/2+\varepsilon}) \right\} h \gg h
\]

which is [13, (23)]. Thus, repeating the same argument (in [13, (24)-(26)]), we obtain [13, (26)] with a smaller admissible \( h = x^\eta \) (here \( \eta > \frac{3}{4} \) is required instead of \( \eta > \frac{3}{4} + \varpi \)).

Next we note that the new estimate (2.3) improves the upper bound \( h^{3/4}x^\varepsilon \) in [13, (21) of Section 4.2] to \( h^{1/2}x^\varepsilon \). Consequently, we get the new lower bound

\[
x^{-1-\varepsilon}h^2 + O(h^{1/2}x^\varepsilon)
\]

for [13, (27)]. The optimal choice of \( \eta \) is \( \frac{2}{3}(1+\varrho) + \varepsilon \), and together with the constraint \( \eta > \frac{3}{4} \), we choose

\[
\eta = \max \left\{ \frac{2}{3}(1+\varrho), \frac{3}{4} \right\} + \varepsilon.
\]

We complete the proof of Theorem 1 with the same argument in remaining part of [13, Section 4.2].

5. Preparation for the truncated Voronoi formula

Applying the additive character to replace the congruence condition, that is,

\[
Q^{-1} \sum_{d \mid Q} \sum_{u \equiv a (mod d)}^* e\left( \frac{u(n-a)}{d} \right) = \delta_{n \equiv a (mod Q)}
\]

where \( \delta_1 = 1 \) if \( * \) holds and 0 otherwise, we have

\[
S_f^A(x) := \sum_{n \leq x \atop n \equiv a (mod Q)} \lambda_f(n) = Q^{-1} \sum_{d \mid Q} S_f(x, a/d),
\]

where

\[
S_f(x, a/d) := \sum_{u \equiv a (mod d)}^* e\left( \frac{-au}{d} \right) \sum_{n \leq x} \lambda_f(n)e\left( \frac{nu}{d} \right).
\]

Here \( \sum_{u \equiv a (mod d)}^* \) denotes the sum over \( u \equiv a (mod d) \) with \( (u, d) = 1 \). The inner sum over \( n \) is clearly associated with \( L_f(s, u/d) \), thus we introduce the auxiliary function

\[
L_f(s, a/d) := \sum_{u \equiv a (mod d)} e\left( \frac{-au}{d} \right) L_f(s, u/d).
\]

The Dirichlet series associated to \( S_f^A(x) \),

\[
L_f(s, a, Q) := \sum_{n \geq 1 \atop n \equiv a (mod Q)} \lambda_f(n)n^{-s}
\]
is equal to

\[(5.5) \quad L_f(s, a, Q) = Q^{-1} \sum_{d|Q} \mathcal{L}_f(s, a/d). \]

Plainly \( \mathcal{L}_f(s, a/d) \) satisfies a functional equation by (3.7),

\[(5.6) \quad q_d \tilde{L}_f(s, a, d) = e^{-(s+1/2)} q_d^{1-s} L_f(1-s, a/d) \]

where \( \tilde{L}_f(s, v/d) \) is defined as in (3.9) and

\[\tilde{L}_f(s, a/d) = \sum_{u (\mod d)}^* e \left( -\frac{au}{d} \right) \tilde{L}_f(s, u/d) \quad (u/d \equiv 1 (\mod d)). \]

When \( \Re s > 1 \), we may express \( \tilde{L}_f(s, a/d) \) as a Dirichlet series whose coefficients are products of \( \lambda(n; d) \) and the Kloosterman-Salié sums. Indeed, by (3.9), we have

\[(5.7) \quad \tilde{L}_f(s, a/d) = \sum_{n \geq 1} \lambda(n; d) K(a, n; d) n^{-s} \]

where (noting \( v = \overline{\pi} (\mod d) \)),

\[(5.8) \quad K(a, n; d) := \sum_{u (\mod d)}^* \overline{\pi} d(n, \overline{\pi}) e \left( -\frac{au}{d} \right). \]

By (3.10),

\[K(a, n; d) = \begin{cases} 
\sum_{u (\mod d)}^* \varepsilon_u^{2d+1} \left( \frac{d}{u} \right) e \left( -\frac{au + nu}{4d} \right) & \text{if } 4 \mid d, \\
\sum_{u (\mod d)}^* \varepsilon_u^{2d+1} \left( \frac{d}{u} \right) e \left( -\frac{4au + nu}{4d} \right) & \text{if } 2 \mid d, \\
i^{s+1/2} \varepsilon_d^{-(2\ell+1)} \sum_{u (\mod d)}^* \left( \frac{u}{d} \right) e \left( -\frac{au + 4nu}{d} \right) & \text{if } 2 \nmid d.
\end{cases} \]

**Lemma 5.1.** Let \( \tau(d) \) be the divisor function. We have

\[(5.9) \quad |K(a, n; d)| \ll (d, n)^{1/2} d^{1/2} \tau(d). \]

Moreover, for the case \( 2 \nmid d \), if there exists \( x \in \{a, n\} \) such that \( (x, d) = 1 \), then

\[(5.10) \quad K(a, n; d) = i^{s+1/2} \varepsilon_d^{-(2\ell+1)} d^{1/2} \left( \frac{x}{d} \right) \sum_{y^2 \equiv an (\mod d)} e \left( \frac{y}{d} \right). \]

**Proof.** We express \( K(a, n; d) \) in terms of Kloosterman-Salié sums (see Appendix for their definitions), as follows:

\[(5.11) \quad K(a, n; d) = \begin{cases} 
\overline{K}_{2\ell+1}(n, a; d) & \text{for } 4 \mid d, \\
n^{-1} K_{2\ell+1}(n, a; 4d) & \text{for } 2 \mid d, \\
i^{s+1/2} \varepsilon_d^{-(2\ell+1)} S(\overline{\pi} n, a; d) & \text{for } 2 \nmid d,
\end{cases} \]

where in the case of \( 2 \mid d \), the range of summation is enlarged to a reduced residue system (mod 4d). From (9.2) below, we have

\[(5.12) \quad |K(a, n; d)| \ll (d, n)^{1/2} d^{1/2} \tau(d). \]
The formula (5.10) follows from the result in [9, Lemma 4.9] for the Salie sum. □

**Lemma 5.2.** Let \( d \geq 1 \) and \( a \) be any integers. For any \( \varepsilon > 0 \), we have

\[
\mathcal{L}_f(\sigma + \imath \tau, a/d) \ll d^{3-\sigma)/2+2\varepsilon} (1 + |\tau|)^{1-\sigma+2\varepsilon} \quad (-\varepsilon \leq \sigma \leq 1 + \varepsilon, \tau \in \mathbb{R}),
\]

where the implied \( \ll \)-constant depends on \( \mathfrak{f} \) and \( \varepsilon \) only.

**Proof.** Let \( \Re s = 1 + \varepsilon \). By (3.3) and (3.6), we have trivially \( L_f(s, u/d) \ll \varepsilon \) and with (5.3), \( \mathcal{L}_f(s, a/d) \ll \varepsilon, d \). Next for \( \Re s = -\varepsilon \), we infer from (5.6) and (5.7) that

\[
\mathcal{L}_f(s, a/d) = i^{-(\ell+1/2)} q_d^{1-2s} \frac{L_\infty(1-s)}{L_\infty(s)} \sum_{n \geq 1} \lambda(n; d) K(a, n; d) n^{-s}.
\]

Thus, with (5.12) and Stirling’s formula, it follows that

\[
\mathcal{L}_f(-\varepsilon + i\tau, a/d) \ll (d^{3/2}(1 + |\tau|))^{1+\varepsilon} \sum_{n \geq 1} |\lambda(n; d)| (n, d)^{1/2} n^{-(1+\varepsilon)}
\]

\[
\ll (d^{3/2}(1 + |\tau|))^{1+\varepsilon}
\]

because \( |\lambda(n; d)| (n, d)^{1/2} \leq |\lambda(n; d)|^2 + (n, d) \), implying that the last summation is

\[
\ll \sum_{n \geq 1} |\lambda(n; d)|^2 n^{-(1+\varepsilon)} + \sum_{l|d} l^{-\varepsilon} \sum_{n \geq 1} n^{-(1+\varepsilon)} \ll \tau(d).
\]

A application of Phragmén–Lindelöf principle completes the proof. □

6. Truncated Voronoi formula

This section is devoted to the Voronoi formulas. In order for a simpler form for the result, let us set, with the notation (5.8),

\[
\phi_a(n, d) := \sqrt{q_d^{1-(\ell+1/2)}} K(a, n; d) \ll (n, d)^{1/2} \tau(d) d
\]

by (5.12), and trivially \( |\phi_a(n, d)| \leq \sqrt{2d^{3/2}} \). We have the following result.

**Theorem 3.** Let \( \ell \geq 2 \) be an integer and \( \mathfrak{f} \in S_{\ell+1/2} \) be an eigenform of all Hecke operators. Then for any \( \varepsilon > 0 \), we have

\[
S_{\mathfrak{f}}(x, a/d) = \frac{x^{1/4}}{\sqrt{\pi} \sqrt{2} \sqrt{\varepsilon}} \sum_{n \leq M} \lambda(n; d) \phi_a(n, d) \frac{n^{3/4}}{q_d} \cos \left( 4\pi \sqrt{\frac{n x}{q_d}} - \frac{\ell + 1/2}{2\pi} \right)
\]

\[
+ O_{\ell, \varepsilon} \left( x^\varepsilon d^2 (x^{1/2+\varepsilon} M^{-1/2} + M^\varepsilon) \right)
\]

uniformly for \( 2 \leq M \leq x \) and \( 1 \leq d \leq x^{1/2} \), where \( q \) is defined as in (3.5).

Moreover for \( 1 \leq Q \leq x^{1/2} \) and any integer \( a \),

\[
S_{\mathfrak{f}}^A(x) = \frac{x^{1/4}}{\sqrt{2\pi} Q} \sum_{d|Q} \sum_{n \leq M} \lambda(n; d) \phi_a(n, d) \frac{n^{3/4}}{q_d} \cos \left( 4\pi \sqrt{\frac{n x}{q_d}} - \frac{\ell + 1/2}{2\pi} \right)
\]

\[
+ O(\varepsilon^2 Q (x^{1/2+\varepsilon} M^{-1/2} + M^\varepsilon)).
\]

In particular, for \( Q \leq x^{1/2-\varepsilon} \) and any \( a \),

\[
S_{\mathfrak{f}}^A(x) \ll_{\ell, \varepsilon} Q^{1/3} x^{(1+\varepsilon)/3}. 
\]
Remark 3. It is shown in [15, Proposition 3.2] that $S_n \ll x^{2/5+\varepsilon}$, which is superseded by the particular case $\mathcal{A} = \mathbb{N}$ (and $Q = 1$) of (6.3) for $\varrho = 1/6 + \varepsilon$ is admissible.

**Proof.** Let $d \leq x^{1/2}$, $1 \leq M \leq x$ and $T > 1$ be chosen as

$$T^2 = q_d^{-2} 4\pi^2 (M + 1/2)x \gg 1.$$  

We apply the Perron formula (cf. [16, Corollary II.2.2.1]) to (5.3) with $\sigma_a = \alpha = 1$ and $B(n) = C_n n^\varepsilon$ to write

$$S_l(x, a/d) = \frac{1}{2\pi i} \int_{\mathcal{L}_1(s, a/d)} \frac{x^s}{s} ds + O_{l, \varepsilon} \left( \frac{dx^{1+\varepsilon}}{T} + d^{\varepsilon/2}x^\varepsilon \right).$$

We deform the line of integration to the contour $\mathcal{L}$ joining the points $\kappa - iT, -\varepsilon - iT, -\varepsilon + iT, \kappa + iT$. By Lemma 5.2, the integrals over the horizontal segments of $\mathcal{L}$ are $\ll x^{\varepsilon} (xT^{-1} + d^{3/2})$, and the pole of the integrand at $s = 0$ gives $\mathcal{L}_1(0, a/d) \ll d^{3/2+\varepsilon}$. By the functional equation (5.6), the integral over $\mathcal{L}_0$ equals

$$\frac{1}{2\pi i} \int_{\mathcal{L}_0} \mathcal{L}_1(s, a/d) \frac{x^s}{s} ds = q_d^{-1} \sum_{n \geq 1} \Omega(n; d) \varphi_1(n, d) \frac{1}{n} I_{\mathcal{L}_0} \left( \frac{2\pi \sqrt{n} x}{q_d} \right) + O \left( \frac{dx^{1+\varepsilon}}{T} + d^{3/2}x^\varepsilon \right).$$

By (5.7) and (6.1), we express (6.5) into

$$S_l(x, a/d) = \frac{\sqrt{q_d}}{2\pi i} \sum_{n \geq 1} \frac{\lambda(n; d) \varphi_1(n, d)}{n} I_{\mathcal{L}_0} \left( \frac{2\pi \sqrt{n} x}{q_d} \right) + O \left( \frac{dx^{1+\varepsilon}}{T} + d^{3/2}x^\varepsilon \right)$$

where

$$I_{\mathcal{L}_0}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_0} \frac{\Gamma(s)}{\Gamma(s + 1/2 - 1/4)} \frac{y^{2s}}{s} ds.$$

Next we apply the stationary phase method to bound $I_{\mathcal{L}_0}(y)$ for large $y$ and give an asymptotic expansion in terms of trigonometric functions for small $y$.

With Stirling’s formula, for $\tau > 0$, the integrand equals

$$e^{i\pi (\ell - 1)/2} y^{2\sigma \tau - 2\sigma \log(y/\tau)} \left( 1 + c_1 \tau^{-1} + O(\tau^{-2}) \right)$$

for any $|\tau| \geq 1$ and $|\sigma| \leq A$, where $c_1$ and $A > 0$ denote some suitable constants and the implied $O$-constant is independent of $\tau$ and $y$. Set $g(\tau) := 2\tau \log(y/\tau)$, then $g'(\tau) = 2\log(y/\tau)$. With the second mean value theorem for integrals (cf. [16, Theorem I.0.3]), we obtain for $y > T$ and $\sigma = -\varepsilon$,

$$\int_1^T y^{2\sigma \tau - 2\sigma e^{ig(\tau)}} \left( 1 + c_1 \tau^{-1} + O(\tau^{-2}) \right) d\tau \ll T^{2\varepsilon} y^{2\sigma} \left| \log \frac{y}{T} \right|^{-1} + T^{2\varepsilon - 1} y^{2\sigma},$$

and for $y < T$ and $\sigma = \frac{1}{2} + \varepsilon$,

$$\int_T^\infty y^{2\sigma \tau - 2\sigma e^{ig(\tau)}} \left( 1 + c_1 \tau^{-1} + O(\tau^{-2}) \right) d\tau \ll T^{-1-2\varepsilon} y^{2\sigma} \left| \log \frac{y}{T} \right|^{-1} + T^{-1-2\varepsilon} y^{2\sigma}.$$

For $n > M$, we infer by (6.7) that

$$I_{\mathcal{L}_0} \left( \frac{2\pi \sqrt{n} x}{q_d} \right) \ll_k \left( \frac{x}{\sqrt{n}} \right)^{2\varepsilon} \left( \left| \log \frac{n}{M + 1/2} \right|^{-1} + d(Mx)^{-1/2} \right).$$
By \( \lambda(n; d) \ll n^{\theta + \varepsilon} \) from Lemma 3.1 and \( |\phi_a(n, d)| \ll \sqrt{2d^{3/2}} \), it follows that

\[
\sqrt{q_d} \sum_{n > M} \frac{\lambda(n; d)\phi_a(n, d)}{n^{1+\varepsilon}} \left| \log \frac{n}{M + 1/2} \right|^{-1} \ll d^2 M^\theta \sum_{M < n < 2M} |n - (M + 1/2)|^{-1} \ll d^2 M^{\theta + \varepsilon}.
\]

Consequently we deduce that

\[
\frac{\sqrt{q_d}}{2\pi} \sum_{n > M} \frac{\lambda(n)\phi_a(n, d)}{n} I_{\mathcal{L}_v} \left( \frac{2\pi \sqrt{n(x)}}{qd} \right) \ll x^\varepsilon d^2 M^\theta + x^\varepsilon d^2 (Mx)^{-1/2}.
\]

For \( n \leq M \), we complete the path \( \mathcal{L}_v \) to the contour \( \mathcal{L}_v^* \) so as to apply \cite[Lemma 1]{1}, where \( \mathcal{L}_v^* \) is the positively oriented contour consisting of \( \mathcal{L}_v \), \( \mathcal{L}_v^\pm \) and \( \mathcal{L}_h^\pm \) with

\[
\mathcal{L}_v^\pm := \left[ \frac{1}{2} + \varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm i\infty \right], \quad \mathcal{L}_h^\pm := \left[ -\varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm iT \right].
\]

Correspondingly we denote by \( I_{\mathcal{L}_v^\pm} \) and \( I_{\mathcal{L}_h^\pm} \) the integrals over these segments. By (6.8), the integral over the vertical line segments \( \mathcal{L}_v^\pm \) is

\[
I_{\mathcal{L}_v^\pm} \ll x^\varepsilon \left( \frac{n}{M} \right)^{1/2} \left| \log \frac{n}{M + 1/2} \right|^{-1},
\]

while for the horizontal segments, \( I_{\mathcal{L}_h^\pm} \) contributes at most \( O((n/M)^\varepsilon) \). Thus

\[
\frac{\sqrt{q_d}}{2\pi} \sum_{n \leq M} \frac{\lambda(n; d)\phi_a(n, d)}{n} \left( I_{\mathcal{L}_v^\pm} + I_{\mathcal{L}_h^\pm} \right)
\]

\[
\ll x^\varepsilon d^2 M^{\theta-1/2} \sum_{M/2 \leq n \leq M} n^{-1/2} \left| \log \frac{M + 1/2}{M + 1/2 - n} \right|^{-1} \ll x^\varepsilon d^2 M^\theta.
\]

Inserting (6.10) and (6.9) into (6.6), we get from our choice of \( T \),

\[
S_1(x, a/d) = \frac{\sqrt{q_d}}{2\pi} \sum_{1 \leq n \leq M} \frac{\lambda(n; d)\phi_a(n, d)}{n} I_{\mathcal{L}_v^*} \left( \frac{2\pi \sqrt{n(x)}}{qd} \right) + O(x^\varepsilon d^2(x^{1/2+\varepsilon}M^{-1/2} + M^n)).
\]

Now all the poles of the integrand in

\[
I_{\mathcal{L}_v^*}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v^*} \frac{\Gamma(1 - s + \ell/2 - 1/4)\Gamma(s)}{\Gamma(s + \ell/2 - 1/4)\Gamma(s + 1)} y^{2s} ds
\]

lie on the right of the contour \( \mathcal{L}_v^* \). After a change of variable \( s \) into \( 1 - s \), we have

\[
I_{\mathcal{L}_v^*}(y) = \frac{1}{\pi} I_0\left( y^2 \right),
\]

with

\[
I_0(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v^*} \frac{\Gamma(s + (2\ell - 1)/4)\Gamma(1 - s)}{\Gamma(1 - s + (2\ell - 1)/4)\Gamma(2 - s)} y^{1-s} ds.
\]

Here \( \mathcal{L}_v \) consists of the line \( s = \frac{1}{2} - \varepsilon + iT \) with \( |\tau| \geq T \), together with three sides of the rectangle whose vertices are \( \frac{1}{2} - \varepsilon - iT, 1 + \varepsilon - iT, 1 + \varepsilon + iT \) and \( \frac{1}{2} - \varepsilon + iT \). Clearly our \( I_0 \) is a particular case of \( I_\rho \) defined in \cite[Lemma 1]{1}, corresponding to the choice of
parameters $A = \delta = N = \omega = \alpha_3 = 1$, $\beta_1 = \mu = (\ell - 2)/4$, $\rho = m = 0$, $a = -\frac{3}{4}$, $c_0 = \frac{1}{2}$, $h = 2$, $k_0 = - (\ell + 1)/2$. It hence follows that

$$I(x) = \left(\frac{2\pi}{q_d} \sqrt{nx} \right) = e' \left(\frac{2\pi}{q_d} \sqrt{nx} \right)^{1/4} \cos \left(4\pi \frac{\sqrt{nx}}{q_d} \sqrt{\frac{\ell + 1}{2}} + O(d^{1/2}(nx)^{-1/4}).
$$

The value of $e_0'$ [1, Lemma 1] is $1/\sqrt{\pi}$, and the main term in (6.2) follows from (6.12) and (6.11). With a simple checking, the $O$-term in (6.12) gives a term that will be absorbed in (6.11).

Finally we set $M = Q^{4/3}x^{(1+4\rho)/3}$ and note from (6.1) that

$$\sum_{n \leq M} \frac{1}{n^{3/4}} \sum_{n \leq M} \frac{1}{n^{3/4}} \ll d^{1+\varepsilon} \sum_{n \leq M} \frac{1}{n^{3/4}} n^{-\varepsilon} + d^{1+\varepsilon} \sum_{n \leq M} (n, d)^{n^{-3/4},}
$$

which is $\ll x^d M^{1/4}$ with (3.3).

--

7. Preparation for the proof of Theorem 2

We consider odd $Q$ only, then $q_d = 2d$ and $\lambda(n; d) = \lambda_b(n)$ for all $d \mid Q$. The idea of proof is the same as in Heath-Brown & Tsang [5], however, some new technicality arises because of the new frequencies ($\sqrt{n}/q_d$ rather than $\sqrt{n}$). Consequently, instead of $\sqrt{T}$, we shall apply their argument to the frequency $\sqrt{m_0}/Q$ where $n_0 = 2^j f_0$ with $j \geq 0$ and $f_0$ squarefree, and simultaneously, require the coefficient $\lambda_b(n_0)\phi_{a}(n_0, Q)$ to be non-vanishing. We can guarantee the existence of $n_0$ under certain circumstances.

For convenience, let us recall our notation (specialized to this case $2 \nmid d$):

$$S_\ell(x) = \sum_{n \leq x} \lambda_\ell(n) \quad \text{and} \quad S_{\ell}(x, a/d) := \sum_{n \leq x} \lambda_\ell(n) R_d(n - a).$$

where $R_d(m) = \sum_{a \equiv(\mod Q)} e(\mu n/\ell d)$ is the Ramanujan sum. Their associated Dirichlet series are

$$L_{\ell}(s, a, Q) := \sum_{n \geq 1, a \equiv \ell (\mod Q)} \lambda_\ell(n) n^{-s} \quad \text{and} \quad L_{\ell}(s, a/d) := \sum_{n \geq 1} \lambda_\ell(n) R_d(n - a)^{-s}.$$

Moreover, $L_{\ell}(s, a, Q) = Q^{-1} \sum_{d|Q} L_{\ell}(s, a/d)$ and

$$(2d)^4 L_{\ell}(s) L_{\ell}(s, a/d) = i^{-(\ell+1)/2} (2d)^{-s} L_{\ell}(1 - s) \tilde{L}_{\ell}(1 - s, a/d)$$

where

$$\tilde{L}_{\ell}(s, a/d) := \sum_{n \geq 1} \lambda_b(n) K(a, n; d)^{-s}.$$

Lemma 7.1. Under the assumption that $\{\lambda_\ell(n)\}_{n \in \mathbb{N}}$ is a real sequence, for all $a, d$, the sequences $\{i^{-(\ell+1)/2} \lambda_b(n) K(a, n; d)\}_{n \in \mathbb{N}}$ are real.

Proof. Since the Ramanujan sum $R_d(m)$ is real-valued, $L_{\ell}(s, a/d)$ is real-valued for $s \in (1, \infty)$ under the given assumption. The holomorphicity of $L_{\ell}(s, a/d)$ implies that $L_{\ell}(s, a/d)$ is holomorphic. Thus $\tilde{L}_{\ell}(s, a/d) = L_{\ell}(s, a/d)$ on $\mathbb{C}$ (as they are equal on $(1, \infty)$). The lemma follows. \qed
Lemma 7.2. When the sequence \( \{\lambda_t(n)\}_{n \in \mathbb{A}} \) contains nonzero terms, the function \( \mathcal{L}_f(s, a/d) \) is non-identically zero for all \( d \mid Q \).

Proof. Suppose not, say, \( \mathcal{L}_f(s, a/d_0) \equiv 0 \). Then
\[
\sum_{n \equiv a \pmod{Q}} \lambda_t(n)n^{-s} = Q^{-1} \sum_{d \mid Q} \mathcal{L}_f(s, a/d) = \sum_{n \equiv a \pmod{Q}} n^{-s} \lambda_t(n)Q^{-1} \sum_{d \mid Q, d \neq d_0} R_d(n - a).
\]

With the standard formula for the Ramanujan sum, we infer that
\[
\delta_{n \equiv a \pmod{Q}} \lambda_t(n) = \lambda_t(n)Q^{-1} \sum_{d \mid Q, d \neq d_0} \mu(d) \delta(d/\delta) \quad \forall \ n \geq 1.
\]

Take \( n \equiv a \pmod{Q} \) such that \( \lambda_t(n) \neq 0 \). We obtain that
\[
Q - \phi(d_0) = \sum_{d \mid Q, d \neq d_0} \phi(d) = \sum_{d \mid Q, d \neq d_0} \mu(d) \delta(d/\delta) = Q.
\]

Contradiction arises. \( \square \)

Proposition 1. Let \( Q \geq 1 \) be odd and \( 0 < a < d \). Suppose \( n_0 = 2^j f_0 \) with \( f_0 \) squarefree and \( j \geq 0 \) is an integer such that
\[
(7.1) \quad \lambda_t(n_0)\phi_a(n_0, Q) \neq 0.
\]

Then there are constants \( c_0 = c_0(f, Q, n_0) \) and \( x_0 = x_0(f, Q, n_0) \) such that \( S_A^t(x) \) attains at least one sign change in the interval \([x, x + c_0\sqrt{x}]\) for all \( x \geq x_0 \).

Proof. Let \( \alpha \) a parameter determined later and \( T \) be any sufficiently large number. Set
\[
F_x(t + \alpha u) := \pi \sqrt{Q} \frac{S_A^t((Q(t + \alpha u))^2)}{\sqrt{t + \alpha u}} \quad (t \in [T, 2T], u \in [-1, 1]).
\]

By Theorem 3 with \( M = (QT)^2 \), we deduce that
\[
F_x(t + \alpha u) = \sum_{d \mid Q} \sum_{n \in \langle QT \rangle^2} \frac{\lambda_t(n)\phi_a(n, d)}{n^{S_A^t/4}} \cos \left( \pi(t + \alpha u)\frac{Q\sqrt{n}}{d} - \frac{\ell + 1}{2} \right)
\]
\[+ O((QT)^{\epsilon/2})].
\]

Let \( \tau = 1 \) or \(-1\), and define
\[
k_{\tau}(u) := (1 - |u|)(1 + \tau \cos(2\pi \alpha \sqrt{n_0} u)).
\]

Then as in the proof of [12, Lemma 3.2], for any \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \), the integral
\[
r_n = r_n(\alpha, \tau, t) := \int_{-1}^{1} k_{\tau}(u) \cos \left( 2\pi(t + \alpha u)\frac{Q\sqrt{n}}{d} - \frac{\ell + 1}{2} \right) du
\]
satisfies
\[
r_n = \delta_{Q\sqrt{n} = d\sqrt{n_0}} \cdot \frac{\tau}{2} \cos \left( 2\pi t\sqrt{n_0} - \frac{\ell + 1}{2} \right)
\]
\[+ O \left( \min \left( 1, \frac{1}{\alpha^2 n} \right) + \delta_{Q\sqrt{n} = d\sqrt{n_0}} \min \left( 1, \frac{1}{(\alpha_{n,d})^2} \right) \right),
\]
where \( \alpha_{n,d} = \alpha(Q\sqrt{n} - d\sqrt{n_0})/d \), \( \delta_{*} = 1 \) if \( * \) holds, or 0 otherwise. The \( O \)-constant is absolute.
Observe that $Q\sqrt{n} = d\sqrt{n_0}$ if and only if $2^j f_0 = (Q/d)^2 n$ which is equivalent to $n = 2^j f_0 = n_0$ and $d = Q$ since $f_0$ is squarefree and $Q/d$ is odd. Following from (7.2) and (7.4), the integral

$$J_r(t) = \int_{-1}^{1} F_j(t + \alpha u) k_r(u) \, du$$

can be written as

$$(7.3) \quad J_r(t) = \frac{\tau \lambda_b(n_0) \phi_a(n_0, Q)}{n_0^{3/4}} \cos \left( 2\pi t \sqrt{n_0} - \frac{\ell + 1}{2} \pi \right) + E + O(QQT)^{2\varepsilon-1/2+\varepsilon}$$

where

$$E \ll \frac{1}{\alpha^2} \sum_{d|Q} \sum_{n\leq QT^2} \frac{|\lambda_b(n)\phi_a(n, d)|}{n^{7/4}} + \sum_{d|Q} \frac{d^2}{\alpha^2} \sum_{n\leq QT^2} \frac{|\lambda_b(n)\phi_a(n, d)|}{n^{3/4}|Q\sqrt{n} - d\sqrt{n_0}|^2}.$$  

Using the bounds $\phi_a(n, d) \ll d^{3/2}$ and $\lambda_b(n) \ll n^\varepsilon$, a little calculation gives

$$E \ll Q^3 n_0^{\varepsilon+1/4} \alpha^{-2}.$$  

Let $A_0 := |\lambda_b(n_0)\phi_a(n_0, Q)| n_0^{-3/4}$, which is $> 0$. Fix a sufficiently large $\alpha = \alpha(f, n_0, Q)$, so that $E < \frac{1}{8} A_0$, and then a sufficiently large $T_0 = T_0(f, n_0, Q, \alpha)$ such that the $O$-term $O(QQT)^{2\varepsilon-1/2+\varepsilon}$ is $\ll \frac{1}{2} A_0$ for all $T \geq T_0$. Now observe that for any $m \in \mathbb{N}$, the absolute value of the cosine factor is $1/\sqrt{2}$ if $t = t_m$ where

$$t_m := (m + \frac{1}{8}) n_0^{-1/2}.$$  

This implies $|J_r(t_m)| > \frac{1}{4}(\sqrt{2} - 1)A_0 > 0$ whenever $t_m > T_0 + \alpha$. Since $J_\pm(t_m)$ are of opposite signs and the kernel function $k_r$ is nonnegative, there is a pair of $t_m^\pm \in [t_m - \alpha, t_m + \alpha]$ for which $\pm F_j(t_m^\pm) > 0$. Equivalently, $S^4_A(y)$ attains a sign change in every interval of the form $[(Q(t_m - \alpha))^2, (Q(t_m + \alpha))^2]$ whose length is $\ll \alpha(Q^2 t_m) \ll f, Q, n_0 \sqrt{x}$ when $x = (Q t_m)^2$. Our result follows readily.

8. Proof of Theorem 2

In view of Proposition 1, the main task is to study the condition $\lambda_b(n_0)\phi_a(n_0, Q)$. Recall $\phi_a(n, Q) = \sqrt{2Q} e^{-i(\ell+1)/2} K(a, n; Q)$ by (6.1). Clearly, $\phi_a(n, 1) = \sqrt{2}$. In general, we have by Lemma 9.1 (2),

$$(8.1) \quad \phi_a(n, Q) = \sqrt{2Q} \varepsilon_Q^{-i(\ell+1)} \prod_{p \parallel Q} S(nQp, aQp; p^\alpha)$$

where $S(m, n; c)$ is defined as in (9.1), $Q_p = Q/p^\alpha$ and $\bar{x} x \equiv 1 \pmod{p^\alpha}$ for each term inside the product, $\forall \; p^\alpha \parallel Q$.

♠ Case 1. $Q = 1$. It suffices to find a squarefree $t$ and a $j \geq 0$ such that $\lambda_b(2^j t) \neq 0$. By Lemma 7.2, $L_1(s, 1)$ and thus $\tilde{L}_1(s, 1) = \sum_{n \geq 1} \lambda_b(n)n^{-s}$ are not identical to the zero function. Thus $\lambda_b(n) \neq 0$ for some $n \in \mathbb{N}$. Write $n = 2^j t m^2$ where $t$ is squarefree and $m$ is odd, $\lambda_b(2^j t) \neq 0$ from (3.4).
Case 2. $a = 0$ and $p^a \parallel Q$ implies $\alpha$ being odd. By Lemma 9.1 (2)-(3) and (8.1), $\phi_0(n, Q) = 0$ if $(n, Q) > 1$. Repeating the argument in Case 1, we get $\lambda_b(n)\phi_0(n, Q) \neq 0$ for some $n \in \mathbb{N}$. This $n$ has to be coprime with $Q$. Write $n = 2^itm^2$ with squarefree $t$ and odd $m$, then $\lambda_b(2^it) \neq 0$ (from $\lambda_b(2^itm^2) \neq 0$) and $\phi_0(2^it, Q) \neq 0$ because

$$S(hk, 0; Q) = \left(\frac{h}{Q}\right)S(k, 0; Q)$$

if $(h, Q) = 1$, from the definition of the Salie sum.

Case 3. $(a, Q) = 1$ and $p^2 \mid Q, \forall p|Q$. The argument is similar to the previous cases – firstly finding $n = 2^itm^2$, with squarefree $t$ and odd $m$, for which $\lambda_b(n)\phi_0(n, Q) \neq 0$. But now we need (5.10) to analyze the Salie sum, which gives

$$\phi_a(2^itm^2, Q) = \sqrt{2Q\varepsilon_a\left(\frac{a}{Q}\right)c_{a2^it}(m, Q)}$$

where

$$(8.2) c_b(m, d) = \sum_{y \equiv bm^2 (mod d), \ y^2 \equiv bm^2 (mod d)} e\left(\frac{y}{d}\right).$$

As in (8.1), we have the factorization

$$c_{a2^it}(m, Q) = \prod_{p^a \parallel Q} c_{Q, a2^it}(m, p^a)$$

and the lemma below assures $(m, Q) = 1$ and $\phi_a(2^it, Q) \neq 0$ when $\phi_a(2^itm^2, Q) \neq 0$. Hence this case is also complete.

Lemma 8.1. Let $b \in \mathbb{Z}$, $p$ an odd prime and $\alpha \geq 2$. Define $c_b(m, p^\alpha)$ as in (8.2). Then

(i) $c_b(m, p^\alpha) = 0$ if $p \mid m$, and
(ii) $c_b(1, p^\alpha) \neq 0$ if $c_b(m, p^\alpha) \neq 0$ with $p \nmid m$.

Proof. (i) Write $m = p^\beta m'$ where $p \nmid m'$.

• $\alpha = 2\gamma \leq 2\beta$. Then

$$c_b(m, p^\alpha) = \sum_{y \equiv 0 (mod p^\alpha)} e\left(\frac{y}{p^\alpha}\right) = \sum_{l \equiv 0 (mod p^\gamma)} e\left(\frac{l}{p^\gamma}\right) = 0.$$

• $\alpha = 2\gamma + 1 \leq 2\beta$. Then $y$ is of the form $y = lp^{\gamma+1}$, and as $\gamma \geq 1$,

$$c_b(m, p^\alpha) = \sum_{y \equiv 0 (mod p^\alpha)} e\left(\frac{y}{p^\alpha}\right) = \sum_{l \equiv 0 (mod p^\gamma)} e\left(\frac{l}{p^\gamma}\right) = 0.$$
\* \* 2. Then \( y = lp^\beta \) and thus
\[
c_b(m, p^\alpha) = \sum_{l^2 \equiv bm^2 \pmod{p^{\alpha-2\beta}}} \sum_{y \equiv p^\beta t \pmod{p^\beta}} e\left( \frac{y}{p^\beta} \right)
= \sum_{l^2 \equiv bm^2 \pmod{p^{\alpha-2\beta}}} \sum_{t \pmod{p^\beta}} e\left( \frac{l + lp^{\alpha-2\beta}}{p^{\alpha-\beta}} \right)
= \sum_{l^2 \equiv bm^2 \pmod{p^{\alpha-2\beta}}} e\left( \frac{l}{p^{\alpha-\beta}} \right) \sum_{t \pmod{p^\beta}} e\left( \frac{l}{p^\beta} \right) = 0.
\]

(ii) Suppose \( c_b(m, p^\alpha) \neq 0 \) where \( (m, p) = 1 \). We may assume \( p^2 \nmid b \), for otherwise, \( c_b(m, p^\alpha) = c_{b/p^2}(mp, p^\alpha) = 0 \) by (i). Also \( p \mid b \) cannot happen because, when \( \alpha \geq 2 \), \( p^2 \mid b \) and \( y^2 \equiv bm^2 \pmod{p^\beta} \) has solutions. Thus \( p \nmid b \).

Now \( c_b(m, p^\alpha) \neq 0 \) implies the congruence \( y^2 \equiv bm^2 \pmod{p^\alpha} \) is soluble, and with \((m, p) = 1\), \( y^2 \equiv b \pmod{p^\alpha} \) has two solutions, say, \( \pm y_0 \) and \( p \nmid y_0 \). We see that
\[
\sum_{y^2 \equiv b \pmod{p^\alpha}} e\left( \frac{y}{p^\alpha} \right) = 2 \cos \left( \frac{2\pi y_0}{p^\alpha} \right) \neq 0
\]
because otherwise, \( y_0/p^\alpha = (2r+1)/4 \) for some \( r \in \mathbb{Z} \) or equivalently, \( 4y_0 = (2r+1)p^\alpha \) which contradicts to \( p \nmid y_0 \).

\[\square\]

9. Appendix

Let us denote, as in [8, Section 3], the Kloosterman-Salié sum by
\[
K_{2\ell+1}(m, n; c) := \sum_{d \pmod{c}} \varepsilon_d^{-(2\ell+1)} \left( \frac{c}{d} \right) e\left( \frac{md + nd}{c} \right)
\]
and
\[
(9.1) \quad S(m, n; c) := \sum_{x \pmod{c}} \left( \frac{x}{c} \right) e\left( \frac{mx + nx}{c} \right),
\]
where \( c \in \mathbb{N} \) and \( m, n \in \mathbb{Z} \). Then we have the following estimate,
\[
(9.2) \quad |K_{2\ell+1}(n, m; d)| \quad \text{and} \quad |S(m, n; d)| \leq d^{1/2} \tau(d) \tau(n) \leq m^{1/2}
\]
where \( \tau(n) \) is the divisor function. This follows from the well-known Weil’s bound for Kloosterman sums and the following lemma.

Lemma 9.1. We have the following results:

(a) Let \( c = qr \) with \( r \equiv 0 \pmod{4} \) and \( (q, r) = 1 \). Then
\[
K_{2\ell+1}(m, n; c) = K_{2\ell+2-q}(m\overline{q}, n\overline{q}; r) S(m\overline{r}, n\overline{r}; q)
\]
where \( q \overline{q} \equiv 1 \pmod{r} \) and \( r \overline{r} \equiv 1 \pmod{q} \).

(b) Let \( q \) be odd, \( q = uv \) with \( (u, v) = 1 \). Then
\[
S(m, n; q) = S(mu, nv; v) S(m\overline{v}, n\overline{v}; u)
\]
where \( u \overline{v} \equiv 1 \pmod{v} \) and \( v \overline{v} \equiv 1 \pmod{u} \).
(c) For an odd prime $p$ and odd $\alpha$, if $p \mid m$, then $S(m, 0; p^\alpha) = 0$.
(d) If $(c, 2) = 1$, then $|S(m, n; c)| \leq (m, n, c)^{1/2}e^{1/2\tau(c)}$.
(e) Let $4|r|2^\infty$. Then $|K_{2\ell+1}(m, n; r)| \leq (m, n, r)^{1/2r^{1/2}(r)}$.

Proof. (a) See [8, p. 390, Lemma 2].
(b) See [8, p. 390, Lemma 3].
(c) By definition, for odd $\alpha$, we have

$$S(m, 0; p^\alpha) = \sum_{x \equiv \alpha \pmod{p^\alpha}} \left( \frac{x}{p} \right) e\left( \frac{mx}{p^\alpha} \right).$$

When $\alpha = 1$, $S(m, 0; p^\alpha) = \sum_{x \equiv \alpha \pmod{p^\alpha}} \left( \frac{x}{p} \right) = 0$ as $p \mid m$. Suppose $\alpha \geq 3$. Putting $x = lp + v$, we get

$$\sum_{t \equiv \alpha \pmod{p^{\alpha-1}}} e\left( \frac{ml}{p^{\alpha-1}} \right) \sum_{v \equiv \alpha \pmod{p}} \left( \frac{v}{p} \right) e\left( \frac{mv}{p} \right) = 0.$$  

(d) Iwaniec [9, Section 4.6] handled the case $(c, 2n) = 1$, and thus $(c, 2m) = 1$ too by symmetry. Together with (b), it suffice to deal with $p \mid (m, n)$ and $c$ is a power of $p$.
Consider $S := S(p^\alpha m, p^\alpha b; p^{t+\alpha})$ where $b \equiv 0$, $p \nmid mn$, $a, t \geq 1$ and $a + t$ is odd. (The case that $a + t$ is even is done with the classical Kloosterman sum.) Clearly,

$$S = \sum_{d \equiv \alpha \pmod{p^{t+\alpha}}} \left( \frac{d}{p} \right) e\left( \frac{md + p^\alpha nd}{p^t} \right) = \left( \frac{m}{p} \right) \sum_{d \equiv \alpha \pmod{p^{t+\alpha}}} \left( \frac{d}{p} \right) e\left( \frac{d + p^\alpha mnd}{p^t} \right).$$

Mimicking Iwaniec’s proof in [8, p. 67] (in fact attributed to Sarnak), we consider

$$F(x) = \sum_{d \equiv \alpha \pmod{p^{t+\alpha}}} \left( \frac{d}{p} \right) e\left( \frac{x^2d + p^\alpha mnd}{p^t} \right).$$

and its Fourier transform

$$\hat{F}(y) = \sum_{x \equiv \alpha \pmod{p^t}} F(x) e\left( -\frac{xy}{p^t} \right).$$

As in [8, p. 67], we obtain $\hat{F}(y) = g(1, p^\alpha)G_t(4mp^b - y^2)$ where

$$G_t(4mp^b - y^2) = \sum_{d \equiv \alpha \pmod{p^{t+\alpha}}} \left( \frac{d}{p} \right)^{t+1} e\left( \frac{d(4mp^b - y^2)}{p^t} \right).$$

Case 1: $t$ is odd. Then

$$G_t(4mp^b - y^2) = \sum_{d \equiv \alpha \pmod{p^{t+\alpha}}} e\left( \frac{d(4mp^b - y^2)}{p^t} \right)$$

$$= \sum_{r=0,1} (-1)^r p^a \sum_{d \equiv \alpha \pmod{p^{t-r}}} e\left( \frac{d(4mp^b - y^2)}{p^{t-r}} \right).$$

Since

$$\sum_{d \equiv \alpha \pmod{p^{t-r}}} e\left( \frac{d(4mp^b - y^2)}{p^{t-r}} \right) = p^{t-r} \delta_{y^2 \equiv 4mp^b \pmod{p^{t-r}}},$$
we conclude
\[ \hat{F}(y) = g(1, p^t) \sum_{r=0,1} (-1)^{r} p^{a+t-r} \delta_{y^2 \equiv 4mnp^b (\text{mod} \ p^t)} \]
and
\[ F(x) = p^{-t} \sum_{y \pmod{p^t}} \hat{F}(y) e \left( \frac{xy}{p^t} \right) = g(1, p^t) \sum_{r=0,1} (-1)^{r} p^{a-r} \sum_{y \pmod{p^t}} \frac{e \left( \frac{xy}{p^t} \right)}{y^2 \equiv 4mnp^b (\text{mod} \ p^t)}. \]

As \(|g(1, p^t)| \leq p^{t/2}\) by [9, (4.43)], we see that \(|F(1)| \leq 2p^{a+t/2}\).

**Case 2: \(t\) is even.**

Then
\[ G_t(4mnp^b - y^2) = \sum_{d \pmod{p^{a+t}}} \left( \frac{d}{p} \right) e \left( \frac{d(4mnp^b - y^2)}{p^t} \right) \]
\[ = \sum_{u \pmod{p^{a+t-1}}} e \left( \frac{u(4mnp^b - y^2)}{p^{t-1}} \right) \sum_{v \pmod{p}} \left( \frac{v}{p} \right) e \left( \frac{v(4mnp^b - y^2)}{p^{t-1}} \right). \]
The first sum does not vanish only when \(y^2 \equiv 4mn \ (\text{mod} \ p^{t-1})\), but in this case, the second sum equals zero. i.e. \(G_t(4mnp^b - y^2) = 0\). So \(\hat{F}(y) = g(1, p^t)G_t(4mnp^b - y^2) = 0\), implying \(F(x) = 0\).

(e) Refer to [4], cf. [3, Section 14].

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