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ABSTRACT. We prove that any abelian surface admits a rank 2 Ulrich bundle.

Let $X \subset \mathbb{P}^N$ be a projective variety of dimension d over an algebraically closed field. An *Ulrich bundle* on X is a vector bundle E on X satisfying $H^*(X, E(-1)) = \ldots = H^*(X, E(-d)) = 0$. This notion was introduced in [**?**], where various other characterizations are given; let us just mention that it is equivalent to say that E admits a linear resolution as a $\mathcal{O}_{\mathbb{P}^N}$-module, or that the pushforward of E onto $\mathbb{P}^d$ by a general linear projection is a trivial bundle.

In [**?**] the authors ask whether every projective variety admits an Ulrich bundle. The answer is known only in a few cases: hypersurfaces and complete intersections [**?**], del Pezzo surfaces [**?**, Corollary 6.5]. The case of K3 surfaces is treated in [**?**]. In this short note we show that the existence of Ulrich bundles for abelian surfaces follows easily from Serre's construction:

Theorem 1. *Any abelian surface $X \subset \mathbb{P}^N$ carries a rank 2 Ulrich bundle.*

Proof : We put $\dim H^0(X, \mathcal{O}_X(1)) = n$. Let C be a smooth curve in $|\mathcal{O}_X(1)|$; we have $\mathcal{O}_C(1) \cong \omega_C$, and $g(C) = n+1$. We choose a subset $Z \subset C$ of n general points. Then Z has the *Cayley-Bacharach property* on X (see for instance [**?**], Theorem 5.1.1): for every $p \in Z$, any section of $H^0(X, \mathcal{O}_X(1))$ vanishing on $Z \smallsetminus \{p\}$ vanishes on Z. Indeed, the image V of the restriction map $H^0(X, \mathcal{O}_X(1)) \to H^0(C, \mathcal{O}_C(1))$ has dimension $n-1$, hence the only element of V vanishing on $n-1$ general points is zero; thus the only element of $|\mathcal{O}_X(1)|$ containing $Z \smallsetminus \{p\}$ is C.

By *loc. cit.*, there exists a rank 2 vector bundle E on X and an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} E \longrightarrow \mathcal{I}_Z(1) \to 0 \ . \tag{1}$$

Let η be a non-zero element of $\mathrm{Pic}^{\mathrm{o}}(X)$; then $h^0(\omega_C \otimes \eta) = n$, so $H^0(C, \omega_C \otimes \eta(-Z)) = 0$ since Z is general, and therefore $H^0(X, \mathcal{I}_Z\eta(1)) = 0$. Since $\chi(\mathcal{I}_Z\eta(1))) = 0$ we have also $H^1(X, \mathcal{I}_Z\eta(1)) = 0$; from the above exact sequence we conclude that $H^*(X, E \otimes \eta) = 0$.

The zero locus of the section s of E is Z; since $\det E_{|C} = \mathcal{O}_C(1) = \omega_C$, we get an exact sequence

$$0 \to \mathcal{O}_C(Z) \xrightarrow{s_{|C}} E_{|C} \longrightarrow \omega_C(-Z) \to 0 \ . \tag{2}$$

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As above the cohomology of $\omega_C \otimes \eta(-Z)$ and $\eta(Z)$ vanishes, hence $H^*(C,(E\otimes\eta)_{|C})=0$. Now from the exact sequence

$$0 \to E(-1) \to E \to E_{|C} \to 0$$

we conclude that $H^*(X,E\otimes\eta(-1))=H^*(X,E\otimes\eta)=0$, hence $F:=E\otimes\eta(1)$ is an Ulrich bundle. □

Remarks.— 1) There is no Ulrich line bundle on a general abelian surface X. Indeed a line bundle M on X with $\chi(M)=0$ satisfies $c_1(M)^2=0$ by Riemann-Roch; since X is general we have $\mathrm{NS}(X)=\mathbb{Z}$, hence M is algebraically equivalent to $\mathcal{O}_X$. Thus if L is a Ulrich line bundle $L(-1)$ and $L(-2)$ must be algebraically equivalent to $\mathcal{O}_X$, a contradiction.

On the other hand, some particular abelian surfaces do carry a Ulrich line bundle. Let $(A,\mathcal{O}_A(1))$, $(B,\mathcal{O}_B(1))$ be two polarized elliptic curves, and let α,β be non-zero elements of $\mathrm{Pic}^{\mathrm{o}}(A)$ and $\mathrm{Pic}^{\mathrm{o}}(B)$. Put $X=A\times B$ and $\mathcal{O}_X(1)=\mathcal{O}_A(1)\boxtimes\mathcal{O}_B(1)$. Then $\alpha(1)\boxtimes\beta(2)$ is a Ulrich line bundle for $(X,\mathcal{O}_X(1))$.

2) It follows from the exact sequence (2) that $E_{|C}$ is semi-stable, hence E, and consequently F, are semi-stable (actually any Ulrich bundle is semi-stable, see [**?**, Proposition 2.12]). Moreover if F is not stable, there is a line bundle $L\subset E$ with $(L\cdot C)=n$, so that $L_{|C}$ must be isomorphic to $\mathcal{O}_C(Z)$ or $\omega_C(-Z)$. But we have $2=\dim\mathrm{Pic}^{\mathrm{o}}(X)<\dim\mathrm{Pic}^{\mathrm{o}}(C)=n+1$, so for Z general $\mathcal{O}_C(Z)$ and $\omega_C(-Z)$ do not belong to the image of the restriction map $\mathrm{Pic}(X)\to\mathrm{Pic}(C)$. Therefore F is stable.

3) We have constructed the vector bundle E from a curve $C\in|\mathcal{O}_X(1)|$, a subset Z of C and an extension class in $\mathrm{Ext}^1(\mathcal{I}_Z(1),\mathcal{O}_X)$. This space is dual to $H^1(X,\mathcal{I}_Z(1))$; from the exact sequence $0\to\mathcal{I}_Z(1)\to\mathcal{O}_X(1)\to\mathcal{O}_Z(1)\to 0$ we get $h^1(\mathcal{I}_Z(1))=h^0(\mathcal{I}_Z(1))=1$, thus the extension class is unique up to a scalar. It is not difficult to prove that $H^0(X,E)=\mathbb{C}s$; hence E determines $Z=Z(s)$ and the curve C, so it depends on $\dim|C|+\mathrm{Card}(Z)=2n-1$ parameters. To get a Ulrich bundle we put $F=E\otimes\eta(1)$ with $\eta\in\mathrm{Pic}^{\mathrm{o}}(X)$; the line bundle η is determined up to 2-torsion by $\det F=\eta^2(3)$. Thus our construction depends on $2n+1$ parameters.

On the other hand, the moduli space of stable rank 2 vector bundles with the same Chern classes as F is smooth of dimension $2n+2$ [**?**]; the Ulrich bundles form a Zariski open subset $\mathcal{M}_U$ of this moduli space. Therefore our construction gives a hypersurface in $\mathcal{M}_U$.

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