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Abstract: A continuous-time switched linear system with deterministic disturbances is considered in this paper. The distinguishability of the linear systems used to model the switched system is studied. Parity residuals of the linear systems are used to study this property. In this context, a notion of strict residual distinguishability is introduced. The paper develops two mains results: a necessary and sufficient condition for characterizing strict residual distinguishability and an index for quantifying the degree of strict residual distinguishability of the linear systems.

Keywords: Switched system, distinguishability, parity residuals, deterministic disturbances

1. INTRODUCTION

A Switched Linear System (SLS) is an hybrid dynamical system which can present different operating modes. Each mode is modeled by a linear system and a switching signal which indicates at each time the active mode.

This paper is concerned with the distinguishability of the linear systems that describe the operating modes of a SLS. This property is the ability to identify at each time the operating mode, using the input/output data of the SLS. Therefore, it plays a crucial role when studying the observability of switched systems as underlined in Gómez-Gutiérrez et al. (2010), Lou and Si (2009) and in Babaali and Pappas (2004). Moreover, for switched systems with healthy and faulty operating modes (Motchon et al., 2013), the distinguishability between the healthy modes and the faulty ones is necessary for the detectability of the faults and the distinguishability between the faulty modes is necessary for fault diagnosability.

In Cocquempot et al. (2004) and Motchon et al. (2013), parity residual signals which constitute fault indicators are used to study the property of distinguishability. The determination of the condition for distinguish the linear systems through their parity residuals is addressed in these works. A necessary and sufficient algebraic condition for distinguishability is established in Cocquempot et al. (2004). In Motchon et al. (2013), it is shown that two controllable and observable linear systems verify this algebraic condition if and only if their state space representation are not equivalent.

However, it should be noted that the model of the SLS considered in Cocquempot et al. (2004) and Motchon et al. (2013) does not include disturbances and the notion of distinguishability through parity residual introduced in these works depends implicitly of the choice of the initial state vectors of the linear systems used to model the SLS.

Therefore, this paper introduces the notion of strict residual distinguishability which takes into account the disturbances of the SLS and which is independent of the choice of the initial state vectors of the linear systems describing the modes of the SLS.

The characterization of the notion of strict residual distinguishability is the objective of this work. Our main results are twofold. First, we establish a necessary and sufficient condition for strict residual distinguishability. Second, we define an index for quantifying the degree of residual distinguishability of the linear systems of the SLS. With this index, we have found an answer to the problem posed in Motchon et al. (2013) concerning the definition of a degree of distinguishability between the linear systems. Furthermore, this index is used for studying the effect of the disturbances on the property of strict residual distinguishability. The outline of this paper is as follows.

Section 2 gives the definition of strict residual distinguishability. Before introducing this notion, the model of the SLS under consideration is first described and the method for obtaining the parity residuals of the linear systems which model the modes of the SLS is recalled. Section 3 gives a necessary and sufficient condition (NSC) for strict residual distinguishability. From this NSC, the index of strict residual distinguishability is defined. Section 4 includes the explicit formula of this index. An analysis of the effects of deterministic disturbances on the property of

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strict residual distinguishability is presented in Section 5. Finally, conclusions and perspectives are highlighted in Section 6.

2. STRICT RESIDUAL DISTINGUISHABILITY

This section aims to introduce the notion of strict residual distinguishability of the linear systems of a continuous-time SLS subject to disturbances.

2.1 State equations of the switched linear system

The continuous-time switched linear system considered in this paper is described by the following equations :

$$S \quad \begin{cases} \dot{x}(t) = A_{\sigma(t)}\, x(t) + B_{\sigma(t)}\, u(t) + w(t), \\ y(t) = C_{\sigma(t)}\, x(t) + D_{\sigma(t)}\, u(t), \\ x(0) = x^o, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^l$ the control input vector, $y(t) \in \mathbb{R}^m$ the output vector and $w(t) \in \mathbb{R}^n$ the deterministic disturbance vector. The piecewise constant function $\sigma\colon t \mapsto \sigma(t) \in \mathscr{Q} := \{1, 2, \cdots, N\}$ stands for the switched signal which indicates at each instant the active operating mode of S. For every $q \in \mathscr{Q}$, A_q, B_q, C_q and D_q are four matrices with appropriate dimensions. S is assumed to remain in only one of its modes during the time interval $[0\,; T]$. We denote by $\mathscr{U} \subseteq \mathscr{C}^\infty\left([0\,;T], \mathbb{R}^l\right)$ the set of control input u of S and by $\mathscr{W} \subseteq \mathscr{C}^\infty\left([0\,;T], \mathbb{R}^n\right)$ the domain of the deterministic disturbances w of S. The linear systems S_q, $q \in \mathscr{Q}$ describing the continuous dynamics of S have the following form:

$$S_q \quad \begin{cases} \dot{x}_q(t) = A_q\, x_q(t) + B_q\, u(t) + w(t), \\ y_q(t) = C_q\, x_q(t) + D_q\, u(t), \\ x_q(0) = x_q^o. \end{cases} \tag{2}$$

2.2 Parity residuals of the linear systems S_q

Parity residual signals have been originally designed as faults indicators. For a given system S_q, the parity residual reflects the consistency of its available data (measured inputs and outputs) with the behaviour given by the model in (2). The usual requirement that these signals must satisfy (Frank, 1990; Chow and Willsky, 1984) is to be zero-valued function in the operating conditions such that the actual mode is q and $w(t) = 0$, for every $t \geq 0$ and not identically zero in the other cases. The method for obtaining the parity residual of S_q is recalled as follows.

By successive derivations and substitutions of the equation in (2), it is straightforward to verify by recurrence that

$$Y_q^{[s]}(t) - \mathbb{T}_q^{[s]}\, U^{[s]}(t) = \mathbb{O}_q^{[s]}\, x_q(t) + \overline{\mathbb{T}}_q^{[s]}\, W^{[s]}(t) \tag{3}$$

where

$$Z^{[s]}(t) = \begin{bmatrix} z(t) \\ z^{(1)}(t) \\ \vdots \\ z^{(s)}(t) \end{bmatrix}, z \in \{y_q, u, w\}\ ;\ \mathbb{O}_q^{[s]} = \begin{bmatrix} C_q \\ C_q\, A_q \\ \vdots \\ C_q\, A_q^s \end{bmatrix}$$

$$\mathbb{T}_q^{[s]} = \begin{bmatrix} D_q & 0_{m\times l} & \cdots & 0_{m\times l} & 0_{m\times l} \\ C_q\, B_q & D_q & \cdots & 0_{m\times l} & 0_{m\times l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_q\, A_q^{s-2}\, B_q & C_q\, A_q^{s-3}\, B_q & \cdots & D_q & 0_{m\times l} \\ C_q\, A_q^{s-1}\, B_q & C_q\, A_q^{s-2}\, B_q & \cdots & C_q\, B_q & D_q \end{bmatrix}$$

$$\overline{\mathbb{T}}_q^{[s]} = \begin{bmatrix} 0_{m\times n} & 0_{m\times n} & \cdots & 0_{m\times n} & 0_{m\times n} \\ C_q & 0_{m\times n} & \cdots & 0_{m\times n} & 0_{m\times n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_q\, A_q^{s-2} & C_q\, A_q^{s-3} & \cdots & 0_{m\times n} & 0_{m\times n} \\ C_q\, A_q^{s-1} & C_q\, A_q^{s-2} & \cdots & C_q & 0_{m\times n} \end{bmatrix}.$$

For the purpose of designing parity residual of S_q, when $w(t) = 0,\ \forall t \geq 0$, the unknown variables x_q are eliminated in the relation (3), by multiplying this equation by a parity matrix $\mathbb{O}_{q\perp}^{[s]}$ defined by:

$$\begin{cases} \mathbb{O}_{q\perp}^{[s]}\, \mathbb{O}_q^{[s]} = 0_{r_q\times n}, \\ \mathrm{Ker}\left(\mathbb{O}_{q\perp}^{[s]}\right) = \mathrm{Im}\left(\mathbb{O}_q^{[s]}\right), \end{cases} \tag{4}$$

where

$$r_q = m\,(s+1) - \mathrm{rank}\left(\mathbb{O}_q^{[s]}\right). \tag{5}$$

Now, by left multiplying the equation in (3) by a parity matrix $\mathbb{O}_{q\perp}^{[s]}$, one obtains:

$$\mathbb{O}_{q\perp}^{[s]}\left[Y_q^{[s]}(t) - \mathbb{T}_q^{[s]}\, U^{[s]}(t)\right] = \mathbb{O}_{q\perp}^{[s]}\, \overline{\mathbb{T}}_q^{[s]}\, W^{[s]}(t). \tag{6}$$

The left member of the equality in (6),

$$R_q^{[s]}(t, u, y_q) := \mathbb{O}_{q\perp}^{[s]}\left[Y_q^{[s]}(t) - \mathbb{T}_q^{[s]}\, U^{[s]}(t)\right] \tag{7}$$

is known as the computational form (Cocquempot et al., 2004) of the parity residual signal. It is zero in the absence of disturbances. The right member of the equality in (6),

$$\bar{R}_q(t, w) = \mathbb{O}_{q\perp}^{[s]}\, \overline{\mathbb{T}}_q^{[s]}\, W^{[s]}(t) \tag{8}$$

represents the evaluation form (Cocquempot et al., 2004) of the parity residual signal.

When $w(t) = 0$ the value of the residual signal given by (8) equals zero and then $R_q^{[s]}$ matches the condition of a residual signal. A more complete discussion concerning the choice of the order s of derivation for designing the parity residuals can be found in Ding (2008).

For simplicity, we assume that the residuals of all the systems S_q, $q \in \mathscr{Q}$ are generated at a same order s.

2.3 Strict residual distinguishability

Suppose the mode q is active during the time interval $[0\,;T]$. Then for every $t \in [0\,;T]$, $R_q^{[s]}(t, u, y) = R_q^{[s]}(t, u, y_q)$. Thus a necessary and sufficient condition for identifying the active operating mode from the data y and u, and the residual signal $R_q^{[s]}$ is that for every $p \neq q$, the effect of all the admissible output signals y_p of S_p on $R_q^{[s]}$ must be different from that of y_q on $R_q^{[s]}$. In other words, this condition is equivalent to

$$R_q^{[s]}(\cdot, u, y_p) - R_q^{[s]}(\cdot, u, y_q) \neq 0 \tag{9}$$

on $[0\,;T]$ where $R_q^{[s]}(\cdot, u, y_p)$ is obtained by substituting y_q by y_p in (7). Consequently, we are interested with the following problem:

Problem 1. For a control input u, in what situation does the condition (9) holds for every admissible output signal y_p of S_p.

The expression of $R_q^{[s]}(\cdot, u, y_p)$ is obtained by writing (3) for $q = p$ and substituting in the formula (7), $Y_q^{[s]}(t)$ by $Y_p^{[s]}(t)$. Then the left hand side of (9) can be written as follows:

$$R_q^{[s]}(t, u, y_p) - R_q^{[s]}(t, u, y) = \Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right](t) \tag{10}$$

where the function $\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]$ is defined on $\mathbb{R}_+$ by :

$$\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right](t) = \mathbb{O}_{q\perp}^{[s]}\left[\mathbb{O}_p^{[s]}\,x_p(t) + \mathbb{T}_{p\,q}^{[s]}\,U^{[s]}(t)\right] + \mathbb{O}_{q\perp}^{[s]}\,\overline{\mathbb{T}}_{p\,q}^{[s]}\,W^{[s]}(t) \tag{11}$$

with

$$\mathbb{T}_{p\,q}^{[s]} = \mathbb{T}_p^{[s]} - \mathbb{T}_q^{[s]} \quad ; \quad \overline{\mathbb{T}}_{p\,q}^{[s]} = \overline{\mathbb{T}}_p^{[s]} - \overline{\mathbb{T}}_q^{[s]}.$$

Consequently, Problem 1 is equivalent to determine when for a control input u, the function $\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]$ is not identically zero on $[0\,;T]$ for every $x_p^o \in \mathbb{R}^n$ and for every $w \in \mathscr{W}$. Notice that if (9) holds for every x_p^o and for every $w \in \mathscr{W}$, then one can distinguish S_q from S_p regardless of the system's initial state vector and the system's disturbances by using the residual function $R_q^{[s]}$. Thus to best focus on this problem of distinguishability, we introduce the following notion of "strict residual distinguishability".

Definition 1. (Strict residual distinguishability). Let a control input $u \in \mathscr{U}$ be fixed.

(i) The system S_p is said to be strictly residual $(u, \mathscr{W})$-distinguishable from S_q on the time interval $[0\,;T]$ if for every $\left(x_p^o, w\right) \in \mathbb{R}^n \times \mathscr{W}$, the function $\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]$ is non-zero on $[0\,;T]$.

(ii) The systems S_p and S_q are said to be strictly residual $(u, \mathscr{W})$-distinguishable on the time interval $[0\,;T]$ if S_p is strictly residual $(u, \mathscr{W})$-distinguishable from S_q on $[0\,;T]$ or vice versa. If not, S_p and S_q are said to be not strictly residual $(u, \mathscr{W})$-distinguishable on $[0\,;T]$.

The definition of residual-distinguishability (Motchon et al., 2013) does not take into account the disturbances ($\mathscr{W}$ is restricted to $\{0\}$). It states that the disturbance-free models of S_p and S_q are residual-distinguishable if there exists a couple $\left(u, x_p^o\right)$ such that $\Psi_{q\,p}^{[s]}\left[x_p^o, u, 0\right]$ is not identically zero or if there exists a couple $\left(u, x_q^o\right)$ such that $\Psi_{p\,p}^{[s]}\left[x_q^o, u, 0\right]$ is not identically zero. Consequently, the notion of strict residual distinguishabillity given by Definition 1 is stronger than the notion of residual-distinguishability.

In the next section, we give a necessary and sufficient condition for strict residual distinguishability.

3. A NECESSARY AND SUFFICIENT CONDITION FOR STRICT RESIDUAL DISTINGUISHABILITY

Throughout the remainder of the paper, we adopt the following notation :

Notation 1.

- $\|\xi\|_2 = \xi^T\,\xi$: 2-norm of the vector ξ
- $\|\psi\|_{RMS} = \sqrt{\dfrac{1}{T}\displaystyle\int_0^T \|\psi(\tau)\|_2^2\,\mathrm{d}\tau}$: root mean square of the signal ψ

The main result of this section (Theorem 1) gives a necessary and a sufficient condition for strict residual $(u, \mathscr{W})$-distinguishability of S_p and S_q. This condition is based on checking the strict positivity of the real-valued function $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$ defined on $\mathscr{W}$ by

$$\boldsymbol{\Delta}_{p\,q}^{[s]}[u](w) = \min\left\{\boldsymbol{\Delta}_{p/q}^{[s]}[u](w)\ ,\ \boldsymbol{\Delta}_{q/p}^{[s]}[u](w)\right\} \tag{12}$$

where

$$\boldsymbol{\Delta}_{p/q}^{[s]}[u](w) = \min_{x_p^o \in \mathbb{R}^n}\left\|\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]\right\|_{RMS}^2 \tag{13}$$

and $\boldsymbol{\Delta}_{q/p}^{[s]}[u](w)$ is obtained by reverse p and q in the formula (13). The function $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$ represents the gap between the zero valued-function and the functions $\Psi_{q\,p}^{[s]}\left[u, x_p^o, w\right]$ of residuals differenciation. To characterize the strict residual distinguishability with the function $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$, we need to prove that the function $x_p^o \longmapsto \left\|\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]\right\|_{RMS}^2$ admits a minimum on $\mathbb{R}^n$. In Lemma 1 we establish that the existence of the minimum (13) is entirely controlled by the matrices $\mathbb{O}_{q\perp}^{[s]}\,\mathbb{O}_p^{[s]}$ and A_p. The proof of this lemma uses the formula (14) of the quadratic term $\left\|\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]\right\|_{RMS}^2$ and the Proposition 2 which gives three fundamental properties of the matrix $H_{q\,p}^{[s]}$ that appears in the formula (14) of $\left\|\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]\right\|_{RMS}^2$.

Proposition 1. Let $\left(u, x_p^o, w\right) \in \mathscr{U} \times \mathbb{R}^n \times \mathscr{W}$. Then

$$\left\|\Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]\right\|_{RMS}^2 = \left(x_p^o\right)^\top H_{q\,p}^{[s]}\,x_p^o + 2\,L_{q\,p}^{[s]}(u, w)\,x_p^o + K_{q\,p}^{[s]}(u, w) \tag{14}$$

where $H_{q\,p}^{[s]} \in \mathbb{R}^{n\times n}$, $L_{q\,p}^{[s]}(u, w) \in \mathbb{R}^{1\times n}$ and $K_{q\,p}^{[s]}(u, w) \in \mathbb{R}_+$ are defined by :

$$H_{q\,p}^{[s]} = \frac{1}{T}\int_0^T \mathrm{e}^{\tau\,A_p^\top}\left(\Lambda_{q\,p}^{[s]}\right)^\top \Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}\,\mathrm{d}\,\tau, \tag{15}$$

$$L_{q\,p}^{[s]}(u, w) = \frac{1}{T}\int_0^T \left(\widehat{\Psi}_{q\,p}^{[s]}[u, w](\tau)\right)^\top \Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}\,\mathrm{d}\,\tau, \tag{16}$$

and

$$K_{q\,p}^{[s]}(u, w) = \left\|\widehat{\Psi}_{q\,p}^{[s]}[u, w]\right\|_{RMS}^2 \tag{17}$$

with

$$\Lambda_{q\,p}^{[s]} := \mathbb{O}_{q\,\perp}^{[s]}\,\mathbb{O}^{[s]} \tag{18}$$

and

$$\widehat{\Psi}_{q\,p}^{[s]}\,[u,w]\,(t) \;=\; \Lambda_{q\,p}^{[s]} \int_0^t \mathrm{e}^{(t-\tau)A_p}\;[B_p\,u\,(\tau)+w\,(\tau)]\,\mathrm{d}\,\tau +$$
$$\mathbb{O}_{q\,\perp}^{[s]}\,\mathbb{T}_{p\,q}^{[s]}\,U^{[s]}\,(t)+\mathbb{O}_{q\,\perp}^{[s]}\,\overline{\mathbb{T}}_{p\,q}^{[s]}\,W^{[s]}\,(t)\,. \tag{19}$$

Proof. In the expression (11) of $\Psi_{q\,p}^{[s]}\left[x_p^o,u,w\right]$, by substituting $x_p\,(t)$ by

$$x_p\,(t)=\mathrm{e}^{t\,A_p}\,x_p^o+\int_0^t \mathrm{e}^{(t-\tau)\,A_p}\;[B_p\,u\,(\tau)+w\,(\tau)]\;\mathrm{d}\tau,$$

one obtains

$$\Psi_{q\,p}^{[s]}\left[x_p^o,u,w\right](t)=\Lambda_{q\,p}^{[s]}\,\mathrm{e}^{t\,A_p}\,x_p^o+\widehat{\Psi}_{q\,p}^{[s]}\,[u,w]\,(t)\,.$$

Thus, for every $\tau\in[0\,;T]$,

$$\left\|\Psi_{q\,p}^{[s]}\left[x_p^o,u,w\right](\tau)\right\|_2^2 \;=\; \left\|\Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}\,x_p^o\right\|_2^2+\left\|\widehat{\Psi}_{q\,p}^{[s]}\,[u,w]\,(\tau)\right\|_2^2$$
$$2\,\left(\widehat{\Psi}_{q\,p}^{[s]}\,[u,w]\,(\tau)\right)^{\top}\,\Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}\,x_p^o.$$

Finally, one obtains the formula (14) by integrating the previous equality.

Proposition 2. The matrix $H_{q\,p}^{[s]}$ defined by (15) has the following properties :

(i) $H_{q\,p}^{[s]}$ is symmetric and positive semidefinite.

(ii) $H_{q\,p}^{[s]}$ is positive definite if and only if the pair $\left(\Lambda_{q\,p}^{[s]},A_p\right)$ is observable i.e.

$$\operatorname{rank}\left(\begin{bmatrix}\Lambda_{q\,p}^{[s]}\\ \Lambda_{q\,p}^{[s]}\,A_p\\ \vdots\\ \Lambda_{q\,p}^{[s]}\,A_p^{n-1}\end{bmatrix}\right)=n. \tag{20}$$

(iii) If the matrix $\Lambda_{q\,p}^{[s]}$ defined by (18) is full column rank then $H_{q\,p}^{[s]}$ is positive definite.

Proof.

(i) As the matrix $\mathrm{e}^{\tau\,A_p^{\top}}\,\left(\Lambda_{q\,p}^{[s]}\right)^{\top}\,\Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}$ is symmetric for all $\tau\in[0\,;T]$, from (15) one obtains $\left(H_{q\,p}^{[s]}\right)^{\top}=H_{q\,p}^{[s]}$. Hence $H_{q\,p}^{[s]}$ is symmetric. Now we will prove that $H_{q\,p}^{[s]}$ is positive semidefinite. For every $\xi\in\mathbb{R}^n$, one has

$$\xi^T\,H_{q\,p}^{[s]}\,\xi=\frac{1}{T}\int_0^T\left\|\Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}\,\xi\right\|_2^2\mathrm{d}\tau\geq 0. \tag{21}$$

Hence the matrix $H_{q\,p}^{[s]}$ is positive semidefinite.

(ii) Let $\xi\in\mathbb{R}^n$. Since the function $\tau\longmapsto\left\|\Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}\,\xi\right\|_2^2$ is continuous and positive on $[0\,;T]$, it follows from (21) that $\xi^T\,H_{q\,p}^{[s]}\,\xi=0$ if and only if $\Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}\,\xi=0_{r_q\times n}$ for every $\tau\in[0\,;T]$. Moreover, as the function $\tau\longmapsto\Lambda_{q\,p}^{[s]}\,\mathrm{e}^{\tau\,A_p}\,\xi$ is identically zero on $[0\,;T]$ if and only if

$$\Lambda_{q\,p}^{[s]}\,A_p^k\xi=0_{r_q\times n},\quad k=0,1,\ldots,n-1,$$

we conclude that $H_{q\,p}^{[s]}$ is positive definite if and only if

$$\Lambda_{q\,p}^{[s]}\,A_p^k\xi=0_{r_q\times n},\quad k=0,1,\ldots,n-1,\implies\xi=0_n.$$

This conclude the proof of (ii).

(iii) Suppose $\Lambda_{q\,p}^{[s]}$ is full column rank. Then the pair $\left(\Lambda_{q\,p}^{[s]},A_p\right)$ is observable and we conclude from statement (ii) that $H_{q\,p}^{[s]}$ is positive definite.

Lemma 1. Let $(u,w)\in\mathscr{U}\times\mathscr{W}$. If the pair $\left(\Lambda_{q\,p}^{[s]},A_p\right)$ is observable or if the matrix $\Lambda_{q\,p}^{[s]}$ is full column rank then $\min_{x_p^o\in\mathbb{R}^n}\left\|\Psi_{q\,p}^{[s]}\left[x_p^o,u,w\right]\right\|_{RMS}^2$ exists.

Proof. Suppose the pair $\left(\Lambda_{q\,p}^{[s]},A_p\right)$ is observable or the matrix $\Lambda_{q\,p}^{[s]}$ is full column rank. The matrix $H_{q\,p}^{[s]}$ is symmetric and positive definite and we derive from the formula (14) that the function $x_p^o\longmapsto\left\|\Psi_{q\,p}^{[s]}\left[x_p^o,u,w\right]\right\|_{RMS}^2$ is coercive and strictly convex. Consequently, its admits a minimum on $\mathbb{R}^n$.

Example 1. Consider the systems S_p and S_q described as follows :

$$A_p=\begin{bmatrix}-2 & 2\\ 2 & -2.5\end{bmatrix}\;;\;B_p=\begin{bmatrix}1\\0\end{bmatrix}\;;\;C_p=[0\;\;1]\;;\;D_p=0$$

$$A_q=\begin{bmatrix}-1 & 1\\ 1 & 4\end{bmatrix}\;;\;B_q=B_p\,;\;C_q=C_p\,;\;D_q=D_p$$

The systems S_p and S_q represent two operating modes of an hydraulic two-tank system linearized around an equilibrium point. By a simple computation, it is easy to verify that

$$\mathbb{O}_p^{[2]}=\begin{bmatrix}0 & 1\\ 2 & -2.5\\ -9 & 10.25\end{bmatrix}\;;\;\mathbb{O}_q^{[2]}=\begin{bmatrix}0 & 1\\ 1 & 4\\ 3 & 17\end{bmatrix}\;;\;\left(\mathbb{O}_{q\,\perp}^{[2]}\right)^{\top}=\begin{bmatrix}-5\\-3\\1\end{bmatrix}$$

$$\mathbb{T}_{p\,q}^{[2]}=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}\quad;\quad\overline{\mathbb{T}}_{p\,q}^{[2]}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\1&-6.5&0&0&0&0\end{bmatrix}.$$

Thus, the matrix $\Lambda_{q\,p}^{[2]}=[-15.0\;\;12.75]$ is not of full column rank and the pair $\left(\Lambda_{q\,p}^{[2]},A_p\right)$ is observable because

$$\operatorname{rank}\left(\begin{bmatrix}\Lambda_{q\,p}^{[2]}\\ \Lambda_{q\,p}^{[2]}\,A_1\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}-15.0 & 12.75\\ 55.5 & -61.87\end{bmatrix}\right)=2.$$

Consequently, $\min_{x_p^o\in\mathbb{R}^n}\left\|\Psi_{q\,p}^{[s]}\left[x_p^o,u,w\right]\right\|_{RMS}^2$ exists. The formula of the function $\Psi_{q\,p}^{[2]}\left[x_p^o,u,w\right]$ is

$$\begin{aligned}\Psi_{q\,p}^{[2]}\left[x_p^o,u,w\right](t)\;=\;&u\,(t)+w_1\,(t)-6.5\,w_2\,(t)+\\ &-15\,x_{p\,1}\left(t,x_p^o,u,w\right)+\\ &12.75\,x_{p\,2}\left(t,x_p^o,u,w\right).\end{aligned}$$

where $x_{p\,k}$ and w_k, $k=1,2$ denote respectively the kth component of the state variable x_p and the disturbance w. The existence of the minimum of the root mean square of this function is illustrated by Figure 1. This numerical result is obtained for the control input $u\,(t)=\sin\,(0.5\,t)$ and for the disturbance $w\,(t)=\begin{bmatrix}\cos\,(-0.25\,t)\\ 0\end{bmatrix}$

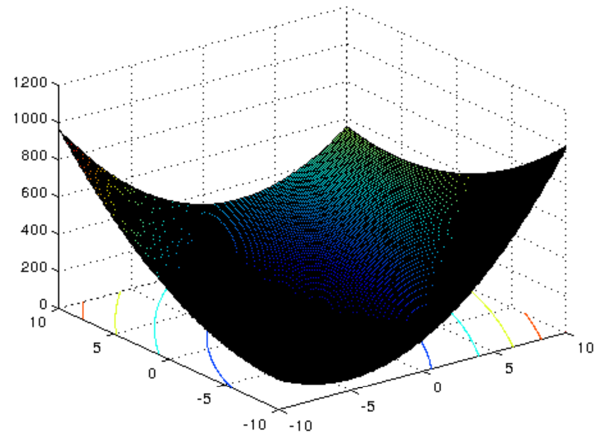

Fig. 1. Function $x_p^o \longmapsto \left\| \Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right] \right\|_{RMS}^2$ for $u(t) = \sin(0.5\,t)$, $w_1(t) = \cos(-0.25 * t)$ and $w_2(t) = 0$

Proposition 3. Let $u \in \mathscr{U}$ be fixed. If the pair $\left(\Lambda_{q\,p}^{[s]}, A_p\right)$ is observable or if the matrix $\Lambda_{q\,p}^{[s]}$ is full column rank then the following statements are equivalent :

- (i) S_p is strictly residual $(u, \mathscr{W})$-distinguishable from S_q on the time interval $[0\,;T]$.
- (ii) for every $w \in \mathscr{W}$, $\boldsymbol{\Delta}_{p/q}^{[s]}[u](w) > 0$, where the function $\boldsymbol{\Delta}_{p/q}^{[s]}[u]$ is defined by the relation (13).

Proof.

$(i) \implies (ii)$: Let $w \in \mathscr{W}$. There exists $\mathbf{x}_p^o$ such that

$$\boldsymbol{\Delta}_{p/q}^{[s]}[u](w) = \left\| \Psi_{q\,p}^{[s]}\left[\mathbf{x}_p^o, u, w\right] \right\|_{RMS}^2 .$$

Moreover, as S_p is strictly residual $(u, \mathscr{W})$-distinguishable from S_q on the time interval $[0\,;T]$, $\left\| \Psi_{q\,p}^{[s]}\left[\mathbf{x}_p^o, u, w\right] \right\|_{RMS}^2 \neq 0$. Consequently, $\boldsymbol{\Delta}_{p/q}^{[s]}[u](w) > 0$.

$(ii) \implies (i)$: The proof of this implication follows from the fact that for every $\left(x_p^o, w\right) \in \mathbb{R}^n \times \mathscr{W}$,

$$\left\| \Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right] \right\|_{RMS}^2 \geq \boldsymbol{\Delta}_{p/q}^{[s]}[u](w) .$$

Thus, the strict residual distinguishability of S_p and S_q can be characterized as follows:

Theorem 1. Let $u \in \mathscr{U}$ be fixed. If the pair $\left(\Lambda_{q\,p}^{[s]}, A_p\right)$ is observable or if the matrix $\Lambda_{q\,p}^{[s]}$ is full column rank then the following statements are equivalent :

- (i) S_p and S_q are strictly residual $(u, \mathscr{W})$-distinguishable on $\mathbb{R}^n \times \mathscr{W}$ during the interval times $[0\,;T]$.
- (ii) The function $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$ defined by equation (12) is strictly positive on $\mathscr{W}$ i.e

$$\boldsymbol{\Delta}_{p\,q}^{[s]}[u](\mathscr{W}) \subseteq \mathbb{R}_+^* \tag{22}$$

  where $\boldsymbol{\Delta}_{p\,q}^{[s]}[u](\mathscr{W})$ denotes the image of $\mathscr{W}$ under $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$.

Proof. The proof is an immediate consequence of Proposition 3.

The function $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$ is identically zero on $\mathscr{W}$ when S_p and S_q are not strictly residual $(u, \mathscr{W})$-input distinguishable and $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$ is different from zero otherwise. Thus, this function constitutes an index for quantifying the degree of strict residual distinguishability of S_p and S_q. Therefore, we introduce the following definition.

Definition 2. (index of distinguishability). The index of strict residual distinguishability of S_p with respect to S_q is the function $\boldsymbol{\Delta}_{p/q}^{[s]}[u]$ defined in (13) and $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$ defined in (12) is the index of strict residual distinguishability of S_p and S_q.

It should be noted that the condition (22) is not easy to verify in practice. Indeed, for any $w \in \mathscr{W}$, the relation (12) does not provide an expression of $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$ that depends explicitly of u and w. Thus, the next section is devoted to establish the explicit formula of $\boldsymbol{\Delta}_{p\,q}^{[s]}[u]$.

4. AN EXPLICIT FORMULA OF THE INDEX OF STRICT RESIDUAL DISTINGUISHABILITY

From the formula (14) of $\left\| \Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right] \right\|_{RMS}^2$, it is easy to see that

$$\frac{\partial \left\| \Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right] \right\|_{RMS}^2}{\partial x_p^o} = 2\,H_{q\,p}^{[s]}\,x_p^o + 2\,\left(L_{q\,p}^{[s]}(u, w)\right)^\top .$$

Consequently, for fixed values of u and w, the function $x_p^o \longmapsto \Psi_{q\,p}^{[s]}\left[x_p^o, u, w\right]$ attains its minimum at the point $\mathbf{x}_p^o$ solution of the following linear equation

$$H_{q\,p}^{[s]}\,x_p^o = -\left(L_{q\,p}^{[s]}(u, w)\right)^\top$$

Thus, if the pair $\left(\Lambda_{q\,p}^{[s]}, A_p\right)$ is observable or if $\Lambda_{q\,p}^{[s]}$ is full column rank then

$$\mathbf{x}_p^o = -\left(H_{q\,p}^{[s]}\right)^{-1}\left(L_{q\,p}^{[s]}(u, w)\right)^\top \tag{23}$$

and one obtains finally the following formula of the index $\boldsymbol{\Delta}_{p/q}^{[s]}[u]$:

Theorem 2. Let $u \in \mathscr{U}$ be fixed. If the pair $\left(\Lambda_{q\,p}^{[s]}, A_p\right)$ is observable or if the matrix $\Lambda_{q\,p}^{[s]}$ is full column rank then for every $w \in \mathscr{W}$,

$$\boldsymbol{\Delta}_{p/q}^{[s]}[u](w) = -L_{q\,p}^{[s]}(u, w)\left(H_{q\,p}^{[s]}\right)^{-1}\left(L_{q\,p}^{[s]}(u, w)\right)^\top + K_{q\,p}^{[s]}(u, w) \tag{24}$$

where $H_{q\,p}^{[s]}$, $L_{q\,p}^{[s]}(u, w)$ and $K_{q\,p}^{[s]}(u, w)$ are defined respectively by equations (15), (16) and (17).

Proof. The proof follows from the fact that

$$\boldsymbol{\Delta}_{p/q}^{[s]}[u](w) = \left\| \Psi_{q\,p}^{[s]}\left[\mathbf{x}_p^o, u, w\right] \right\|_{RMS}^2$$

where the explicit formula of $\mathbf{x}_p^o$ is given by the relation (23).

It should be noted that for the disturbance-free models of S_p and S_q, the formula of the index given by the relation (24) depends only on the known control input and some known matrices. Consequently, this index can be computed easily.

Example 2. Consider the systems S_p and S_q of Example 1. The two systems are observed during the time interval $[0\,;10]$ i.e $T = 10$. For this value of T, it is easy to verify that

$$H_{qp}^{[s]} = \begin{bmatrix} 4.1 & -1.5 \\ -1.5 & 2.0 \end{bmatrix}.$$

For the control input $u(t) = \sin(0.5\,t)$, one has

$$L_{qp}^{[2]}(u,0) = [2.4 \;\; 2.0] \quad ; \quad K_{qp}^{[s]}(u,0) = 16.0$$

Thus the value of the index $\mathbf{\Delta}_{p/q}^{[s]}[u](0)$ that allows to quantify the degree of residual distinguishability of S_p with respect to S_q is

$$\mathbf{\Delta}_{p/q}^{[s]}[u](0) = 9.0 > 0.$$

Consequently, S_p and S_q are strictly residual $(u,\{0\})$-distinguishable on $[0\,;T]$. Consider the following domain of deterministic disturbances :

$$\mathscr{W} = \left\{ \begin{bmatrix} \gamma \, \cos(-0.25\,t) \\ 0 \end{bmatrix}, \quad \gamma \in \mathbb{R} \right\}. \tag{25}$$

For every $w \in \mathscr{W}$, one has

$$L_{qp}^{[2]}(u,w) = [3.4\,\gamma + 2.4 \;\; 2.4\,\gamma + 2.0]$$

and

$$K_{qp}^{[s]}(u,w) = 16.0\,\gamma^2 + 30.0\,\gamma + 16.0.$$

Consequently, the value of $\mathbf{\Delta}_{p/q}^{[s]}[u](w)$ is

$$\mathbf{\Delta}_{p/q}^{[s]}[u](w) = 3.7\,\gamma^2 + 11.0\,\gamma + 9.0. \tag{26}$$

Thus, it is easy to verify that $\mathbf{\Delta}_{p/q}^{[s]}[u](\mathscr{W}) \subseteq \mathbb{R}_+^\star$. We conclude that for the control input $u(t) = \sin(0.5\,t)$, the systems S_p and S_q are strictly residual $(u,\mathscr{W})$-distinguishable on the time interval $[0\,;10]$.

5. SOME EFFECTS OF DISTURBANCES ON STRICT RESIDUAL DISTINGUISHABILITY

In this part, we are interested by the following problem :

Problem 2. Given a control input u, characterize the domains $\mathscr{W}_u^-$ and $\mathscr{W}_u^+$ defined by :

$$\mathscr{W}_u^- = \left\{ w \in \mathscr{W} : \mathbf{\Delta}_{p/q}^{[s]}[u](w) \le \mathbf{\Delta}_{p/q}^{[s]}[u](0) \right\} \tag{27}$$

and

$$\mathscr{W}_u^+ = \left\{ w \in \mathscr{W} : \mathbf{\Delta}_{p/q}^{[s]}[u](w) > \mathbf{\Delta}_{p/q}^{[s]}[u](0) \right\}. \tag{28}$$

The domain $\mathscr{W}_u^-$ corresponds to the class of deterministic disturbances for which S_p is close to S_q (in terms of residual distinguishability) comparatively to their disturbance-free models. The space $\mathscr{W}_u^+$ represents the class of deterministic disturbances for which S_p is far from S_q comparatively to their disturbance-free models.

As illustrative example of domains $\mathscr{W}_u^-$ and $\mathscr{W}_u^+$, we consider again the systems S_p and S_q of Example 2.

Example 3. For $T = 10$, $u(t) = \sin(0.5\,t)$ and for $\mathscr{W}$ defined by the relation (25), it follows from (26) that for every $w \in \mathscr{W}$,

$$\mathbf{\Delta}_{p/q}^{[s]}[u](w) - \mathbf{\Delta}_{p/q}^{[s]}[u](0) = \gamma\,(3.7\,\gamma + 11.0).$$

Consequently, for the systems S_p and S_q of Example 2, one has

$$\mathscr{W}_u^- = \left\{ \begin{bmatrix} \gamma \, \cos(-0.25\,t) \\ 0 \end{bmatrix}, \quad \gamma \in [-\frac{11.0}{3.7}\,;0] \right\}$$

and

$$\mathscr{W}_u^+ = \left\{ \begin{bmatrix} \gamma \, \cos(-0.25\,t) \\ 0 \end{bmatrix}, \quad \gamma \in]-\infty\,;-\frac{11.0}{3.7}[\bigcup \mathbb{R}_+^* \right\}.$$

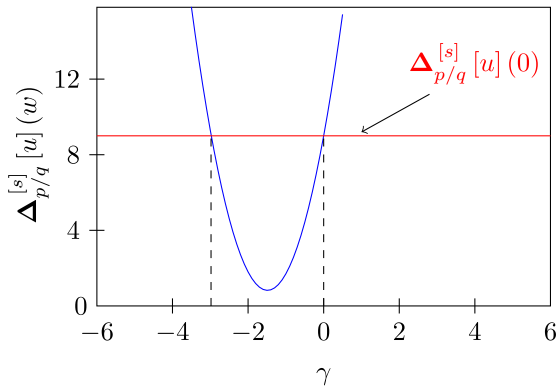

Fig. 2. Comparison of the index $\mathbf{\Delta}_{p/q}^{[s]}[u](0)$ and $\mathbf{\Delta}_{p/q}^{[s]}[u](w)$ for systems S_p and S_q of Example 2

In order to characterize the domains $\mathscr{W}_u^-$ and $\mathscr{W}_u^+$, we consider the following decomposition of the function $\widehat{\Psi}_{qp}^{[s]}[u,w]$:

$$\widehat{\Psi}_{qp}^{[s]}[u,w](t) = \widehat{\Psi}_{qp}^{[s]}[u,0](t) + F_{qp}^{[s]}[w](t) \tag{29}$$

where the function $F_{qp}^{[s]}[w]$ is defined on $\mathbb{R}_+$ by

$$F_{qp}^{[s]}[w](t) = \Lambda_{qp}^{[s]} \int_0^t \mathrm{e}^{(t-\tau)\,A_p}\, w(\tau)\, \mathrm{d}\tau + \mathbb{O}_{q\perp}^{[s]}\, \overline{\mathbb{T}}_{pq}^{[s]}\, W^{[s]}(t).$$

From (29) and (16), we can rewrite $L_{qp}^{[s]}[u,w]$ as follows :

$$L_{qp}^{[s]}[u,w] = L_{qp}^{[s]}[u,0] + \frac{1}{T}\int_0^T \left(F_{qp}^{[s]}[w](\tau)\right)^\top \Lambda_{qp}^{[s]}\, \mathrm{e}^{\tau\,A_p}\, \mathrm{d}\tau. \tag{30}$$

By combining (29) and (17), we can rewrite $K_{qp}^{[s]}[u,w]$ as follows

$$K_{qp}^{[s]}[u,w] = \left\|\widehat{\Psi}_{qp}^{[s]}[u,0]\right\|_{RMS}^2 + \left\|F_{qp}^{[s]}[w]\right\|_{RMS}^2 + \frac{2}{T}\int_0^T \left(F_{qp}^{[s]}[w](\tau)\right)^\top \widehat{\Psi}_{qp}^{[s]}[u,0](\tau)\, \mathrm{d}\tau. \tag{31}$$

Now, from (31), (30) and (24), it is easy to verify that one can rewrite the index $\mathbf{\Delta}_{p/q}^{[s]}[u](w)$ as follows :

$$\mathbf{\Delta}_{p/q}^{[s]}[u](w) = \mathbf{\Delta}_{p/q}^{[s]}[u](0) + G_{qp}^{[s]}[u,w] \tag{32}$$

with

$$G_{q\,p}^{[s]}\left[u,w\right] = \left\| F_{q\,p}^{[s]}\left[w\right] \right\|_{RMS}^{2} + \frac{2}{T}\int_{0}^{T}\left(F_{q\,p}^{[s]}\left[w\right](\tau)\right)^{\top}\,\widehat{\Psi}_{q\,p}^{[s]}\left[u,0\right](\tau)\;\mathrm{d}\tau - \frac{2}{T}\,L_{q\,p}^{[s]}\left[u,0\right]\,\left(H_{q\,p}^{[s]}\right)^{-1}\int_{0}^{T}\mathrm{e}^{\tau\,A_p^{\top}}\,\left(\Lambda_{q\,p}^{[s]}\right)^{\top}\,F_{q\,p}^{[s]}\left[w\right](\tau)\;\mathrm{d}\tau - \frac{1}{T^2}\left\|\left(H_{q\,p}^{[s]}\right)^{-1/2}\int_{0}^{T}\mathrm{e}^{\tau\,A_p^{\top}}\,\left(\Lambda_{q\,p}^{[s]}\right)^{\top}\,F_{q\,p}^{[s]}\left[w\right](\tau)\,\mathrm{d}\tau\right\|_{2}^{2} \tag{33}$$

where $\left(H_{q\,p}^{[s]}\right)^{-1/2}$ denotes the square root of the matrix $\left(H_{q\,p}^{[s]}\right)^{-1}$.

Finally, we obtain the following characterization of $\mathscr{W}_u^-$ and $\mathscr{W}_u^+$:

Proposition 4. Let $u \in \mathscr{U}$ be fixed. If the pair $\left(\Lambda_{q\,p}^{[s]}, A_p\right)$ is observable or if the matrix $\Lambda_{q\,p}^{[s]}$ is full column rank then

$$\mathscr{W}_u^- = \left\{ w \in \mathscr{W} : G_{q\,p}^{[s]}\left[u,w\right] \leq 0 \right\} \tag{34}$$

and

$$\mathscr{W}_u^+ = \left\{ w \in \mathscr{W} : G_{q\,p}^{[s]}\left[u,w\right] > 0 \right\} \tag{35}$$

where $G_{q\,p}^{[s]}\left[u,w\right]$ is defined by the relation (33).

Proof. The proof follows from (27), (28) and (32).

6. CONCLUSION

This paper deals with the distinguishability of the linear systems describing the operating modes of a continuous-time switched linear system subject to disturbances. Parity residuals of the linear systems are used to study this property. A notion of strict residual distinguishability of the linear systems is introduced. A necessary and sufficient condition is given for testing the strict residual distinguishability of the systems. An index for quantifying the degree of strict residual distinguishability of two linear system is defined. The explicit formula of this index is calculated and this allowed a geometric characterization of the strict residual distinguishability.

REFERENCES

Babaali, M. and Pappas, G. (2004). Observability of switched linear systems in continuous time. *Hybrid Systems: Computation and Control (R. Alur and G. Pappas, eds.). Springers*, 48–63.

Chow, E. and Willsky, A. (1984). Analytical redundancy and the design of robust failure detection systems. *IEEE Transactions on Automatic Control*, 29(7), 603–614.

Cocquempot, V., Mezyani, T.E., and Staroswiecki, M. (2004). Fault detection and isolation for hybrid systems using structured parity residuals. In *Proceedings of the 5th Asian Control Conference*, volume 2, 1204–1212. Melbourne, Victoria, Australia.

Ding, S.X. (2008). *Model-based Fault Diagnosis Techniques: Design Schemes, Algorithms, and Tools.* Springer-Verlag.

Frank, P. (1990). Fault diagnosis in dynamic systems using analytical and knowledge based redundancy: A survey and some new results. *Automatica*, 26(3), 459–474.

Gómez-Gutiérrez, D., Ramirez-Prado, G., Ramírez-Treviño, A., and Ruiz-León, J. (2010). Observability of switched linear systems. *IEEE Transactions On Automatic Control*, 6(2), 127–135.

Lou, H. and Si, P. (2009). The distinguishability of linear control systems. *Nonlinear Analysis : Hybrid Systems*, 1, 21–38.

Motchon, K.M.D., Pekpe, K.M., Cassar, J.P., De Bièvre, S., and Cocquempot, V. (2013). Operating modes distinguishability condition in switching systems. In *Proceedings of the 52nd IEEE Conference on Decision and Control*, 79–84. Florence, Italy.