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Z Cui, J Wu

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THE SELBERG-DELANGE METHOD IN SHORT INTERVALS WITH AN APPLICATION

Z. CUI & J. WU

ABSTRACT. In this paper, we establish a general mean value result of arithmetic functions over short intervals with the Selberg-Delange method. As an application, we generalize the Deshouillers-Dress-Tenenbaum's arcsin law on divisors to the short interval case.

1. INTRODUCTION

Many number theoretic problems lead to the study of the mean values of arithmetic functions. Between 1954 and 1971, Selberg [8] and Delange [2, 3] developed a quite general method using the analytic properties of the Dirichlet series associated to the arithmetic function. This is nowadays known as the *Selberg-Delange method*. We refer the readers to [10, Chapter II.5] for an excellent exposition of this theory.

Let $f(n)$ be an arithmetic function and denote its corresponding Dirichlet series by

$$(1.1) \quad \mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}.$$

Suppose that $\mathcal{F}(s)$ admits the factorization

$$\mathcal{F}(s) = \mathcal{G}(s; z)\zeta(s)^z$$

for $\Re s > 1$, where $\zeta(s)$ is the Riemann ζ -function and $z \in \mathbb{C}$. Under some suitable assumptions on $\mathcal{G}(s; z)$, we may apply the Selberg-Delange method to establish a very precise asymptotic formula for the summatory function

$$S_f(x) := \sum_{n \leq x} f(n).$$

See [10, Theorem II.5.3]. In 2008, Hanrot, Tenenbaum & Wu [5] further extended this method to investigate the mean value of $f(n)$ over the friable integers:

$$S_f(x, y) := \sum_{\substack{n \leq x \\ P(n) \leq y}} f(n),$$

where $P(n)$ is the largest prime factor of n with the convention $P(1) = 1$. In particular, suppose $\zeta_{\mathbb{K}}(s)$ is the Dedekind ζ -function of the number field $\mathbb{K}$ and

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$\kappa_j \in \mathbb{R}$ such that $\kappa_1 + \cdots + \kappa_r > 0$. If $\mathcal{F}(s)$ factors into

$$\mathcal{F}(s) = \mathcal{G}(s; z) \prod_{1 \leq j \leq r} \zeta_{\mathbb{K}_j}(s)^{\kappa_j}$$

for $\Re s > 1$, then together with the saddle-point method in [9], it is established (cf. [5, Théorème 1.2]) a very precise asymptotic formula for $S_f(x, y)$ in wide ranges of x and y . It is worth while to note that f is not assumed to be multiplicative albeit it is a Dirichlet convolution.

In this paper, we extend the Selberg-Delange method to handle the sum $\sum f(n)$ where n ranges over a short interval and give an application. We shall proceed along the same line of argument as in [10, Chapter II.5]. Let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $M > 0$ be some constants. A Dirichlet series $\mathcal{F}(s)$ defined as in (1.1) is said to be of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$ if the following conditions are verified:

(a) for any $\varepsilon > 0$ we have

$$(1.2) \quad |f(n)| \ll_{\varepsilon} n^{\varepsilon} \quad (n \geq 1);$$

(b) we have

$$\sum_{n=1}^{\infty} |f(n)| n^{-\sigma} \ll (\sigma - 1)^{-\alpha} \quad (\sigma > 1);$$

(c) the Dirichlet series

$$(1.3) \quad \mathcal{G}(s; \kappa, w) := \mathcal{F}(s) \zeta(s)^{-\kappa} \zeta(2s)^w$$

is analytically continued to a holomorphic function in (some open set containing) $\Re s \geq \frac{1}{2}$ and, in this region, $\mathcal{G}(s; \kappa, w)$ satisfies the bound

$$(1.4) \quad |\mathcal{G}(s; \kappa, w)| \leq M(|\tau| + 1)^{\max\{\delta(1-\sigma), 0\}} \log^A(|\tau| + 1) \quad (s = \sigma + i\tau)$$

uniformly for $0 < \kappa \leq B$ and $|w| \leq B$.

In order to state our result, it is necessary to introduce some more notation. From [10, Theorem II.5.1], * the function

$$Z(s; z) := \{(s-1)\zeta(s)\}^z \quad (z \in \mathbb{C})$$

is holomorphic in the disc $|s-1| < 1$, and admits the Taylor series expansion

$$Z(s; z) = \sum_{j=0}^{\infty} \frac{\gamma_j(z)}{j!} (s-1)^j,$$

where the $\gamma_j(z)$'s are entire functions of z and satisfy: for all $B > 0$ and $\varepsilon > 0$, the estimate

$$(1.5) \quad \frac{\gamma_j(z)}{j!} \ll_{B, \varepsilon} (1 + \varepsilon)^j \quad (j \geq 0, |z| \leq B).$$

Under our hypothesis, the function $\mathcal{G}(s; \kappa, w) \zeta(2s)^{-w} Z(s; \kappa)$ is holomorphic in the disc $|s-1| < \frac{1}{2}$ and

$$(1.6) \quad |\mathcal{G}(s; \kappa, w) \zeta(2s)^{-w} Z(s; \kappa)| \ll_{A, B, \delta, \varepsilon} M$$

*In [10], $Z(s; z)$ is defined as $s^{-1} \{(s-1)\zeta(s)\}^z$ but obviously the argument of the proof there works for our $Z(s; z)$.

for $|s - 1| \leq \frac{1}{2} + \varepsilon$, $0 < \kappa \leq B$ and $|w| \leq B$. Thus for $|s - 1| < \frac{1}{2}$, we can write

$$(1.7) \quad \mathcal{G}(s; \kappa, w) \zeta(2s)^{-w} Z(s; \kappa) = \sum_{\ell=0}^{\infty} g_{\ell}(\kappa, w) (s-1)^{\ell},$$

where

$$(1.8) \quad g_{\ell}(\kappa, w) := \frac{1}{\ell!} \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{\partial^{\ell-j} (\mathcal{G}(s; \kappa, w) \zeta(2s)^{-w})}{\partial s^{\ell-j}} \Big|_{s=1} \gamma_j(\kappa).$$

The following result is an analogue of Theorem II.5.3 of [10] for the mean value over short intervals.

Theorem 1.1. *Let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $M > 0$ be some constants. Suppose that*

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$$

is a Dirichlet series of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$. Then for any $\varepsilon > 0$, we have

$$(1.9) \quad \sum_{x < n \leq x+y} f(n) = y(\log x)^{\kappa-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_{\ell}(\kappa, w)}{(\log x)^{\ell}} + O(R_N(x, y)) \right\}$$

uniformly for

$$x \geq y \geq x^{\theta(\kappa, \delta) + \varepsilon} \geq 2, \quad N \geq 0, \quad 0 < \kappa \leq B, \quad |w| \leq B,$$

where

$$\begin{aligned} \lambda_{\ell}(\kappa, w) &:= \frac{g_{\ell}(\kappa, w)}{\Gamma(\kappa - \ell)}, \\ \theta(\kappa, \delta) &:= \frac{5\kappa + 15\delta + 21}{5\kappa + 15\delta + 36}, \\ R_N(x, y) &:= \frac{y}{x} \sum_{\ell=1}^{N+1} \frac{\ell |\lambda_{\ell-1}(\kappa, w)|}{(\log x)^{\ell}} + \frac{(c_1 N + 1)^{N+1}}{x^{1/2}} \\ &\quad + M \left\{ \left(\frac{c_1 N + 1}{\log x} \right)^{N+1} + e^{-c_2 (\log x / \log_2 x)^{1/3}} \right\} \end{aligned}$$

for some constants $c_1 > 0$, $c_2 > 0$. The implied constant in the O -term depends only on A, B, α, δ and ε .

The proof of Theorem 1.1 is rather similar to that of [10, Theorem II.5.3]. The main new ingredient we introduce is the contour of integration as in [7]. Thanks to the hypothesis (1.2), our proof seems slightly simpler.

As an application of (1.9), we generalize the Deshouillers-Dress-Tenenbaum's arc-sin law on divisors to the short interval case. For each positive integer n , denote by $\tau(n)$ the number of divisors of n and define the random variable D_n which takes

the value $(\log d)/\log n$, as d runs through the set of the $\tau(n)$ divisors of n , with the uniform probability $1/\tau(n)$. The distribution function F_n of D_n is given by

$$F_n(t) = \text{Prob}(D_n \leq t) = \frac{1}{\tau(n)} \sum_{d|n, d \leq n^t} 1 \quad (0 \leq t \leq 1).$$

It is clear that the sequence $\{F_n\}_{n \geq 1}$ does not converge pointwisely on $[0, 1]$. However Deshouillers, Dress & Tenenbaum ([4] or [10, Theorem II.6.7]) proved that its Cesàro mean converges uniformly to the arcsin law, more precisely,

$$(1.10) \quad \frac{1}{x} \sum_{n \leq x} F_n(t) = \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right)$$

uniformly for $x \geq 2$ and $0 \leq t \leq 1$. The error term in (1.10) is optimal. Very recently Basquin [1] considered the generalization of (1.10) for friable integers. Interestingly he showed that the limit law shifts from the arcsine law towards the Gaussian as $u := (\log x)/\log y \rightarrow \infty$.

Here we obtain an analogue of (1.10) for short intervals.

Theorem 1.2. *Let $\varepsilon > 0$ be an arbitrarily small positive constant. We have*

$$(1.11) \quad \frac{1}{y} \sum_{x < n \leq x+y} F_n(t) = \frac{2}{\pi} \arcsin \sqrt{t} + O_\varepsilon\left(\frac{1}{\sqrt{\log x}}\right)$$

uniformly for $0 \leq t \leq 1$, $x \geq 2$ and $x^{62/77+\varepsilon} \leq y \leq x$, where the implied constant depends only on ε . Further (1.11) with $y = x$ implies (1.10).

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2. PROOF OF THEOREM 1.1

Since $\mathcal{F}(s)$ is a Dirichlet series of type $\mathcal{P}(\kappa, \alpha, w, \delta, A, B, M)$, we can apply [10, Corollary II.2.2.1] with the choice of parameters $\sigma_a = 1$, $B(n) := n^\varepsilon$, $\alpha = \alpha$, $\sigma = 0$ to write

$$\sum_{x < n \leq x+y} f(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where $b := 1 + 2/\log x$ and $100 \leq T \leq x$ such that $\zeta(\sigma + iT) \neq 0$ for $0 < \sigma < 1$.

Let $\mathcal{L}$ be the boundary of the modified rectangle with the vertices $\frac{1}{2} \pm iT$ and $b \pm iT$, where

- the zeros of $\zeta(s)$ of the form $\frac{1}{2} + i\gamma$ with $|\gamma| < T$ are avoided by the semicircles of infinitely small radius lying to the right of the line $\Re s = \frac{1}{2}$,
- the zeros of $\zeta(s)$ of the form $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ and $|\gamma| < T$ are avoided by the horizontal cut drawn from the critical line inside this rectangle to $\rho = \beta + i\gamma$,
- the pole of $\zeta(2s)$ at the point $s = \frac{1}{2}$ is avoided by two arcs $\mathcal{L}_3$ and $\mathcal{L}_4$ with the radius $r := 1/\log x$,
- the pole of $\zeta(s)$ at the point $s = 1$ is avoided by the truncated Hankel contour Γ (its upper part is made up of an arc surrounding the point $s = 1$ with the radius $r := 1/\log x$ and a line segment joining $1 - r$ to $1/2 + r$).

Figure 1 – Contour $\mathcal{L}$

Clearly the function $\mathcal{F}(s)$ is analytic inside $\mathcal{L}$. By the Cauchy residue theorem, we can write

$$(2.1) \quad \sum_{x < n \leq x+y} f(n) = I + I_1 + \cdots + I_6 + \sum_{\beta > \frac{1}{2}, |\gamma| < T} I_\rho + O_\varepsilon\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where

$$\begin{aligned} I &:= \frac{1}{2\pi i} \int_{\Gamma} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds, \\ I_\rho &:= \frac{1}{2\pi i} \int_{\Gamma_\rho} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds, \\ I_j &:= \frac{1}{2\pi i} \int_{\mathcal{L}_j} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} ds. \end{aligned}$$

A. Evaluation of I

Let $0 < c < \frac{1}{10}$ be a small constant. Since $\mathcal{G}(s; \kappa, w) \zeta(2s)^{-w} Z(s; \kappa)$ is holomorphic and $O(M)$ in the disc $|s-1| \leq c$, the Cauchy formula implies that

$$(2.2) \quad g_\ell(\kappa, w) \ll M c^{-\ell} \quad (\ell \geq 0, 0 < \kappa \leq B, |w| \leq B),$$

where $g_\ell(\kappa, w)$ is defined as in (1.8). From this and (1.7), it is easy to deduce that for any integer $N \geq 0$ and $|s-1| \leq \frac{1}{2}c$,

$$\mathcal{G}(s; \kappa, w) \zeta(2s)^{-w} Z(s; \kappa) = \sum_{\ell=0}^N g_\ell(\kappa, w) (s-1)^\ell + O(M(|s-1|/c)^{N+1}).$$

Thus we have

$$(2.3) \quad I = \sum_{\ell=0}^N g_\ell(\kappa, w) M_\ell(x, y) + O(M c^{-N} E_N(x, y)),$$

where

$$\begin{aligned} M_\ell(x, y) &:= \frac{1}{2\pi i} \int_{\Gamma} (s-1)^{\ell-\kappa} \frac{(x+y)^s - x^s}{s} ds, \\ E_N(x, y) &:= \int_{\Gamma} \left| (s-1)^{N+1-\kappa} \frac{(x+y)^s - x^s}{s} \right| |ds|. \end{aligned}$$

Firstly we evaluate $M_\ell(x, y)$. By using the formula

$$(2.4) \quad \frac{(x+y)^s - x^s}{s} = \int_x^{x+y} t^{s-1} dt$$

and Corollary II.5.2.1 of [10], we can write

$$\begin{aligned} M_\ell(x, y) &= \int_x^{x+y} \left(\frac{1}{2\pi i} \int_{\Gamma} (s-1)^{\ell-\kappa} t^{s-1} ds \right) dt \\ &= \int_x^{x+y} (\log t)^{\kappa-1-\ell} \left\{ \frac{1}{\Gamma(\kappa-\ell)} + O\left(\frac{(c_1 \ell + 1)^\ell}{t^{1/2}}\right) \right\} dt, \end{aligned}$$

where we have used the following inequality

$$47^{|\kappa-\ell|}\Gamma(1+|\kappa-\ell|) \ll_B (c_1\ell+1)^\ell \quad (\ell \geq 0, 0 < \kappa \leq B).$$

The constant c_1 and the implied constant depend at most on B . On the other hand, it is easy to see that, for $0 < \kappa \leq B$,

$$\begin{aligned} \int_x^{x+y} (\log t)^{\kappa-1-\ell} dt &= \int_0^y \log^{\kappa-1-\ell}(x+t) dt \\ &= y(\log x)^{\kappa-1-\ell} \left\{ 1 + O_B\left(\frac{(\ell+1)y}{x \log x}\right) \right\}. \end{aligned}$$

Inserting this into the preceding formula, we obtain

$$(2.5) \quad M_\ell(x, y) = y(\log x)^{\kappa-1-\ell} \left\{ \frac{1}{\Gamma(\kappa-\ell)} + O_B\left(\frac{(\ell+1)y}{\Gamma(\kappa-\ell)x \log x} + \frac{(c_1\ell+1)^\ell}{x^{1/2}}\right) \right\}$$

for $\ell \geq 0$ and $0 < \kappa \leq B$.

Next we estimate $E_N(x, y)$. In view of the trivial inequality

$$(2.6) \quad \left| \frac{(x+y)^s - x^s}{s} \right| \ll yx^{\sigma-1},$$

we deduce that

$$\begin{aligned} (2.7) \quad E_N(x, y) &\ll \int_{1/2+1/\log x}^{1-1/\log x} (1-\sigma)^{N+1-\kappa} x^{\sigma-1} y d\sigma + \frac{y}{(\log x)^{N+2-\kappa}} \\ &\ll \frac{y}{(\log x)^{N+2-\kappa}} \left(\int_{1/2}^{\infty} t^{N+1-\kappa} e^{-t} dt + 1 \right) \\ &\ll y(\log x)^{\kappa-1} \left(\frac{c_1 N + 1}{\log x} \right)^{N+1} \end{aligned}$$

uniformly for $x \geq y \geq 2$, $N \geq 0$ and $0 < \kappa \leq B$, where the constant $c_1 > 0$ and the implied constant depend only on B .

Inserting (2.5) and (2.7) into (2.3) and using (2.2), we find that

$$(2.8) \quad I = y(\log x)^{\kappa-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_\ell(\kappa, w)}{(\log x)^\ell} + O_B(E_N^*(x, y)) \right\},$$

where

$$E_N^*(x, y) := \frac{y}{x} \sum_{\ell=1}^{N+1} \frac{\ell |\lambda_{\ell-1}(\kappa, w)|}{(\log x)^\ell} + \frac{(c_1 N + 1)^{N+1}}{x^{1/2}} + M \left(\frac{c_1 N + 1}{\log x} \right)^{N+1}.$$

B. Estimations of I_3 and I_4

For $s = \frac{1}{2} + \frac{e^{i\theta}}{\log x}$ with $0 < |\theta| \leq \frac{\pi}{2}$, we have trivially

$$\mathcal{F}(s) \ll (\log x)^{|\Re w|+A}, \quad \left| \frac{(x+y)^s - x^s}{s} \right| \ll x^{1/2}.$$

Thus

$$(2.9) \quad |I_3| + |I_4| \ll x^{1/2} (\log x)^{|\Re w|+A-1} \quad (x \geq 3).$$

C. *Estimations of I_1 and I_6*

It is well known that

$$(2.10) \quad |\zeta(\sigma + i\tau)| \ll |\tau|^{(1-\sigma)/3} \log |\tau| \quad (\tfrac{1}{2} \leq \sigma \leq 1 + \log^{-1} |\tau|, |\tau| \geq 2),$$

$$(2.11) \quad |\zeta(\sigma + i\tau)| \gg \log^{-1}(|\tau| + 3) \quad (\sigma \geq 1 - \sigma_0(\tau), \tau \in \mathbb{R}),$$

where $C > 0$ is an absolute positive constant and

$$(2.12) \quad \sigma_0(t) := \frac{C}{(\log(|t| + 3))^{2/3} (\log \log(|t| + 3))^{1/3}}.$$

In view of (2.10), (2.11) and (1.4), we have

$$\mathcal{F}(s) \ll MT^{\max\{(1-\sigma)(\kappa/3+\delta), 0\}} (\log T)^{|\Re w| + \kappa + A}$$

for $s = \sigma \pm iT$ with $\frac{1}{2} \leq \sigma \leq b$. Thus

$$(2.13) \quad \begin{aligned} |I_1| + |I_6| &\ll \int_{1/2}^b MT^{(1-\sigma)(\kappa/3+\delta)} (\log T)^{|\Re w| + \kappa + A} \frac{x^\sigma}{T} d\sigma \\ &\ll \frac{x}{T} (\log T)^{|\Re w| + \kappa + A} \end{aligned}$$

provided $T \leq x^{1/(\kappa/3+\delta)}$.

D. *Estimations of I_2 and I_5*

For $s = \frac{1}{2} + i\tau \neq \frac{1}{2} + i\gamma$ with $\zeta(\frac{1}{2} + i\gamma) = 0$ and $1/\log x \leq |\tau| \leq T$, the estimates (2.10), (2.11) and (1.4) imply that

$$\mathcal{F}(s) \ll (|\tau| + 1)^{\kappa/6+\delta/2} (\log x)^{|\Re w| + \kappa + A}.$$

This allows us to write

$$(2.14) \quad \begin{aligned} |I_2| + |I_5| &\ll x^{1/2} (\log x)^{|\Re w| + \kappa + A} \int_0^T (\tau + 1)^{-1+\kappa/6+\delta/2} d\tau \\ &\ll x^{1/2} (\log x)^{|\Re w| + \kappa + A} T^{\kappa/6+\delta/2}. \end{aligned}$$

E. *Estimations of the I_ρ*

As in the case C, we have

$$\mathcal{F}(s) \ll M|\gamma|^{(1-\sigma)(\kappa/3+\delta)} (\log |\gamma|)^{|\Re w| + \kappa + A}$$

for $s = \sigma + i\gamma$ with $\frac{1}{2} \leq \sigma \leq \beta < 1 - \sigma_0(\gamma)$. From this and (2.6) we deduce that

$$(2.15) \quad |I_\rho| \ll \int_{1/2}^\beta M|\gamma|^{(1-\sigma)(\kappa/3+\delta)} (\log |\gamma|)^{|\Re w| + \kappa + A} x^{\sigma-1} y d\sigma.$$

Denote by $N(\sigma, T)$ the number of zeros of $\zeta(s)$ in the region $\Re s \geq \sigma$ and $|\Im z| \leq T$. Summing (2.15) over $|\gamma| < T$ and interchanging the summations, we have

$$\sum_{\substack{\beta > \frac{1}{2} \\ |\gamma| < T}} |I_\rho| \ll My (\log x)^{|\Re w| + \kappa + A} \int_{1/2}^{1-\sigma_0(T)} (T^{\kappa/3+\delta}/x)^{1-\sigma} N(\sigma, T) d\sigma.$$

According to [6], it is known that

$$(2.16) \quad N(\sigma, T) \ll T^{(12/5)(1-\sigma)} (\log T)^{44}$$

for $\frac{1}{2} \leq \sigma \leq 1$ and $T \geq 2$. Thus

$$(2.17) \quad \sum_{\substack{\beta > \frac{1}{2} \\ |\gamma| < T}} |I_\rho| \ll My(\log x)^{|\Re w| + \kappa + A + 44} \int_{1/2}^{1 - \sigma_0(T)} (T^{\kappa/3 + \delta + 12/5} / x)^{1 - \sigma} d\sigma \\ \ll y(\log x)^{|\Re w| + \kappa + A + 44} (T^{\kappa/3 + \delta + 12/5} / x)^{\sigma_0(T)}$$

provided $T \leq x^{1/(\kappa/3 + \delta + 12/5)}/2$.

Inserting (2.8), (2.9), (2.13), (2.14) and (2.17) into (2.1), we find that

$$\sum_{x < n \leq x+y} f(n) = y(\log x)^{\kappa-1} \left\{ \sum_{\ell=0}^N \frac{\lambda_\ell(\kappa, w)}{(\log x)^\ell} + O(E_N^*(x, y)) \right\} + \mathcal{R}_T(x, y),$$

where

$$\mathcal{R}_T(x, y) := y(\log x)^{|\Re w| + \kappa + A + 44} \left(\frac{T^{\kappa/3 + \delta + 12/5}}{x} \right)^{\sigma_0(T)} \\ + \frac{x^{1+\varepsilon}}{T} + x^{1/2} (\log x)^{|\Re w| + \kappa + A} T^{\kappa/6 + \delta/2}.$$

Taking

$$T = x^{1/(\kappa/3 + \delta + 12/5) - 10\varepsilon},$$

we obtain the required result.

3. PROOF OF THEOREM 1.2

Firstly we establish the following lemma with the help of Theorem 1.1.

Lemma 3.1. *For any $\varepsilon > 0$, we have*

$$\sum_{x < n \leq x+y} \frac{1}{\tau(dn)} = \frac{hy}{\sqrt{(\pi \log x)}} \left\{ g(d) + O_\varepsilon \left(\frac{(3/4)^{\omega(d)}}{\log x} \right) \right\}$$

uniformly for $d \geq 1$, $x \geq 2$ and $x^{47/77+\varepsilon} \leq y \leq x$, where $\omega(n)$ is the number of distinct prime factors of n and

$$h := \prod_p \sqrt{p(p-1)} \log(1 - 1/p)^{-1}, \\ g(d) := \prod_{p^\nu \parallel d} \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j + \nu + 1} \right) \left(\sum_{j=0}^{\infty} \frac{p^{-j}}{j + 1} \right)^{-1}.$$

Proof. As usual, we denote by $v_p(n)$ the p -adic valuation of n . By using the formula

$$\tau(dn) = \prod_p (v_p(n) + v_p(d) + 1),$$

we write, for $\Re s > 1$,

$$\begin{aligned}\mathcal{F}_d(s) &:= \sum_{n=1}^{\infty} \tau(dn)^{-1} n^{-s} \\ &= \prod_p \sum_{j=0}^{\infty} \frac{p^{-js}}{j + v_p(d) + 1} \\ &= \frac{\zeta(s)^{1/2}}{\zeta(2s)^{1/24}} \mathcal{G}_d(s; 1/2, 1/24),\end{aligned}$$

where

$$\begin{aligned}\mathcal{G}_d(s; 1/2, 1/24) &:= \prod_p \sum_{j=0}^{\infty} \frac{p^{-js}}{j+1} \left(1 - \frac{1}{p^s}\right)^{1/2} \left(1 - \frac{1}{p^{2s}}\right)^{-1/24} \\ &\quad \times \prod_{p^\nu \parallel d} \left(\sum_{j=0}^{\infty} \frac{p^{-js}}{j+\nu+1}\right) \left(\sum_{j=0}^{\infty} \frac{p^{-js}}{j+1}\right)^{-1}\end{aligned}$$

is a Dirichlet series that converges absolutely for $\Re s > \frac{1}{3}$. For $\Re s \geq \frac{1}{2}$, we easily see that

$$\left| \sum_{j=0}^{\infty} \frac{p^{-js}}{j+1} \right| = \left| \frac{\log(1 - p^{-s})}{p^{-s}} \right| \geq \frac{\log(1 + p^{-\sigma})}{p^{-\sigma}} \geq \frac{1}{1 + p^{-1/2}}.$$

This implies

$$|\mathcal{G}_d(s; 1/2, 1/24)| \ll \prod_{p^\nu \parallel d} \left\{ \frac{1}{1+\nu} + O\left(\frac{1}{\sqrt{p}}\right) \right\} \leq C \left(\frac{3}{4}\right)^{\omega(d)}$$

for $\Re s \geq \frac{1}{2}$, where $C > 0$ is an absolute constant.

Consequently, $\mathcal{F}_d(s)$ is a Dirichlet series of type $\mathcal{P}(\frac{1}{2}, \frac{1}{24}, \frac{1}{2}, 0, 0, \frac{1}{2}, C(\frac{3}{4})^{\omega(d)})$. Applying Theorem 1.1 with $N = 0$ and noticing that $\lambda_0(\frac{1}{2}) = hg(d)/\Gamma(\frac{1}{2}) = hg(d)/\sqrt{\pi}$, we get that

$$\sum_{x < n \leq x+y} \frac{1}{\tau(dn)} = \frac{hy}{\sqrt{(\pi \log x)}} \left\{ g(d) + O_\varepsilon \left(\frac{g(d)y}{x \log x} + \frac{(3/4)^{\omega(d)}}{\log x} \right) \right\}$$

uniformly for $d \geq 1$, $x \geq 2$ and $x^{47/77+\varepsilon} \leq y \leq x$. This implies the required result since $g(d) \ll (3/4)^{\omega(d)}$ and $y \leq x$. $\square$

We are now ready to prove Theorem 1.2.

In view of the symmetry of the divisors of n about $\sqrt{n}$, we have

$$\begin{aligned}F_n(t) &= \text{Prob}(D_n \geq 1 - t) \\ &= 1 - \text{Prob}(D_n < 1 - t) \\ &= 1 - F_n(1 - t) + O(\tau(n)^{-1}).\end{aligned}$$

Summing over $x < n \leq x + y$ and treating the O -term by Lemma 3.1 with $d = 1$, we find that

$$S(x, y; t) + S(x, y; 1 - t) = 1 + O\left(\frac{1}{\sqrt{\log x}}\right) \quad (0 \leq t \leq 1),$$

where

$$S(x, y; t) := \frac{1}{y} \sum_{x < n \leq x+y} F_n(t).$$

On the other hand, we have the identity

$$\frac{2}{\pi} \arcsin \sqrt{t} + \frac{2}{\pi} \arcsin \sqrt{1-t} = 1 \quad (0 \leq t \leq 1).$$

Therefore it is sufficient to prove (1.11) for $0 \leq t \leq \frac{1}{2}$.

For $0 \leq t \leq \frac{1}{2}$, we can write

$$\begin{aligned} (3.1) \quad S(x, y; t) &= \frac{1}{y} \sum_{x < n \leq x+y} \frac{1}{\tau(n)} \sum_{d|n, d \leq n^t} 1 \\ &= S_1(x, y; t) - S_2(x, y; t), \end{aligned}$$

where

$$\begin{aligned} S_1(x, y; t) &:= \frac{1}{y} \sum_{x < n \leq x+y} \frac{1}{\tau(n)} \sum_{d|n, d \leq (x+y)^t} 1, \\ S_2(x, y; t) &:= \frac{1}{y} \sum_{x < n \leq x+y} \frac{1}{\tau(n)} \sum_{d|n, n^t < d \leq (x+y)^t} 1. \end{aligned}$$

Firstly we evaluate $S_1(x, y; t)$. Changing the order of summations, we have

$$S_1(x, y; t) = \frac{1}{y} \sum_{d \leq (x+y)^t} \sum_{x/d < m \leq (x+y)/d} \frac{1}{\tau(dm)}.$$

For $d \leq (x+y)^t \leq (2x)^{1/2}$ and $y \geq x^{62/77+\varepsilon}$, it is easy to verify that

$$(y/d) \geq (x/d)^{47/77+\varepsilon}.$$

Thus we can apply Lemma 3.1 with $(x/d, y/d)$ in place of (x, y) to write

$$S_1(x, y; t) = \frac{h}{\sqrt{\pi}} \sum_{d \leq (x+y)^t} \frac{1}{d\sqrt{\log(x/d)}} \left\{ g(d) + O_\varepsilon\left(\frac{(3/4)^{\omega(d)}}{\log x}\right) \right\}$$

uniformly for $0 \leq t \leq \frac{1}{2}$, $x \geq 2$ and $x \geq y \geq x^{62/77+\varepsilon}$. Bounding $(3/4)^{\omega(d)}$ by 1, the contribution of the error term to S_1 is $\ll 1/\sqrt{\log x}$. According to [10, Chapter II.6], we have

$$\frac{h}{\sqrt{\pi}} \sum_{d \leq x^t} \frac{g(d)}{d\sqrt{\log(x/d)}} = \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right),$$

which implies that

$$\frac{h}{\sqrt{\pi}} \sum_{d \leq (x+y)^t} \frac{g(d)}{d\sqrt{\log(x/d)}} = \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right),$$

since

$$\sum_{x^t < d \leq (x+y)^t} \frac{g(d)}{d\sqrt{\log(x/d)}} \ll \frac{1}{\sqrt{\log x}} \sum_{x^t < d \leq (x+y)^t} \frac{1}{d} \ll \frac{1}{\sqrt{\log x}}.$$

Combining these estimates, we obtain

$$(3.2) \quad S_1(x, y; t) = \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right)$$

uniformly for $0 \leq t \leq \frac{1}{2}$, $x \geq 2$ and $x \geq y \geq x^{62/77+\varepsilon}$.

Next, a similar treatment leads to

$$(3.3) \quad \begin{aligned} S_2(x, y; t) &\leq \frac{1}{y} \sum_{x^t < d \leq (x+y)^t} \sum_{x/d < m \leq (x+y)/d} \frac{1}{\tau(m)} \\ &\ll \frac{1}{\sqrt{\log x}} \sum_{x^t < d \leq (x+y)^t} \frac{1}{d} \\ &\ll \frac{1}{\sqrt{\log x}}. \end{aligned}$$

Inserting (3.2) and (3.3) into (3.1), we find that

$$S(x, y; t) = \frac{2}{\pi} \arcsin \sqrt{t} + O_\varepsilon\left(\frac{1}{\sqrt{\log x}}\right)$$

uniformly for $0 \leq t \leq \frac{1}{2}$, $x \geq 2$ and $x \geq y \geq x^{62/77+\varepsilon}$.

Finally we prove that (1.10) follows from (1.11) with $y = x$. Since $0 \leq F_n(t) \leq 1$, we have

$$\begin{aligned} \sum_{n \leq x} F_n(t) &= \sum_{\sqrt{x} < n \leq x} F_n(t) + O(\sqrt{x}) \\ &= \sum_{0 \leq k \leq (\log x)/(2 \log 2)} \sum_{x/2^{k+1} < n \leq x/2^k} F_n(t) + O(\sqrt{x}). \end{aligned}$$

Applying (1.11) with $y = x$ to the inner sum, we deduce that

$$\begin{aligned} \sum_{n \leq x} F_n(t) &= \sum_{k=0}^{[(\log x)/(2 \log 2)]} \left\{ \frac{x}{2^{k+1}} \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{x/2^{k+1}}{\sqrt{\log(x/2^{k+1})}}\right) \right\} + O(\sqrt{x}) \\ &= x \frac{2}{\pi} \arcsin \sqrt{t} + O\left(\frac{x}{\sqrt{\log x}}\right), \end{aligned}$$

since

$$2^{[(\log x)/(2 \log 2)]+1} \asymp \sqrt{x} \quad \text{and} \quad \sum_{k=0}^{[(\log x)/(2 \log 2)]} \frac{1}{2^{k+1}} = 1 + O\left(\frac{1}{\sqrt{x}}\right).$$

This completes the proof of Theorem 1.2.

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DEPARTMENT OF MATHEMATICS, SHANGHAI JIAO TONG UNIVERSITY, CHINA

E-mail address: `zcui@sjtu.edu.cn`

JIE WU, SCHOOL OF MATHEMATICS, SHANDONG UNIVERSITY, JINAN, SHANDONG 250100, CHINA. CNRS, INSTITUT ÉLIE CARTAN DE LORRAINE, UMR 7502, 54506 VANDŒUVRE-LÈS-NANCY, FRANCE

Current address: Université de Lorraine, Institut Élie Cartan de Lorraine, UMR 7502, 54506 Vandœuvre-lès-Nancy, France

E-mail address: `jie.wu@univ-lorraine.fr`