A Sharp Estimate for Divisors of Bernoulli Sums

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Abstract

Let $S_n = \varepsilon_1 + \ldots + \varepsilon_n$, where ε_i are i.i.d. Bernoulli r.v.'s. Let $0 \le r_d(n) < 2d$ be the least residue of n mod$(2d)$, $\bar{r}_d(n) = 2d - r_d(n)$ and $\beta(n,d) = \max(\frac{1}{d}, \frac{1}{\sqrt{n}})[e^{-r_d(n)^2/2n} + e^{-\bar{r}_d(n)^2/2n}]$. We show that

$$\sup_{2\le d\le n} \big|\mathbf{P}\big\{d|S_n\big\} - E(n,d)\big| = \mathcal{O}\big(\frac{\log^{5/2} n}{n^{3/2}}\big),$$

where $E(n,d)$ verifies $c_1\beta(n,d) \le E(n,d) \le c_2\beta(n,d)$ and c_1, c_2 are numerical constants.

1 Main result

Let $\{\varepsilon_i,\ i \ge 1\}$ denote a Bernoulli sequence defined on a joint probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbf{P}})$, with partial sums $S_n = \varepsilon_1 + \ldots + \varepsilon_n$. Consider the Theta function

$$\Theta(d,m) = \sum_{\ell\in\mathbf{Z}} e^{im\pi\frac{\ell}{d} - \frac{m\pi^2\ell^2}{2d^2}}.$$

The improvment of the following result, which is Theorem II in [2], is the main purpose of this work.

Proposition 1 *We have the following uniform estimate:*

$$\sup_{2\le d\le n} \Big|\mathbf{P}\big\{d|S_n\big\} - \frac{\Theta(d,n)}{d}\Big| = \mathcal{O}\big(\frac{\log^{5/2} n}{n^{3/2}}\big). \tag{1}$$

This estimate is sharp already when $d < (Bn/\log n)^{1/2}$, otherwise

$$\big|\mathbf{P}\big\{d|S_n\big\} - \frac{1}{d}\big| \le \begin{cases} C\big\{\frac{\log^{5/2} n}{n^{3/2}} + \frac{1}{d}e^{-\frac{n\pi^2}{2d^2}}\big\} & \text{if } d \le n^{1/2}, \\ \frac{C}{n^{1/2}} & \text{if } n^{1/2} \le d \le n. \end{cases} \tag{2}$$

And this is no longer efficient when $d \gg \sqrt{n}$. The purpose of this Note is to remedy this by showing the existence of an extra corrective exponential factor in that case. Introduce a notation. Let $n \ge d \ge 2$ be integers and denote by $r_d(n)$ the least residue of n modulo $2d$: $n \equiv r$ mod$(2d)$ and $0 \le r < 2d$. Let also denote $\bar{r}_d(n) = 2d - r_d(n)$.

Theorem 2 *We have*

$$\sup_{2\le d\le n}\big|\mathbf{P}\{d|S_n\}-E(n,d)\big| = \mathcal{O}\big(\frac{\log^{5/2} n}{n^{3/2}}\big).$$

where $E(n,d)$ *satisfies*

$$\frac{1}{2\sqrt{2\pi}} \le \frac{E(n,d)}{\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)\big[e^{-\frac{r_d(n)^2}{2n}}+e^{-\frac{\bar r_d(n)^2}{2n}}\big]} \le \frac{32}{\sqrt{2\pi}}.$$

This exponential factor is effective when $\min(r_d(n),\bar r_d(n)) \gg \sqrt{n}$. Its importance is easily seen through the following example.

Let $0<c<1$ and let $1\le \varphi_1(n)\le c\varphi_2(n)$ be non-decreasing. Suppose d is such that $2d\ge \sqrt{n}\varphi_2(n)$ with $r_d(n)$ large so that $\sqrt{n}\varphi_1(n)\le r_d(n)\le c\sqrt{n}\varphi_2(n)$. Then $\bar r_d(n)\ge (1-c)\sqrt{n}\varphi_2(n)$ and so

$$E(n,d)\le \frac{32}{\sqrt{2\pi n}}\big[e^{-\frac{\varphi_1^2(n)}{2}}+e^{-\frac{(1-c)^2\varphi_2^2(n)}{2}}\big].$$

Let $0<A_1\le A_2$. By taking $\varphi_i(n)=\sqrt{2A_i\log n}$, $i=1,2$, we get

$$E(n,d)\le C\max\big(n^{-1/2-A_1},n^{-1/2-(1-c)^2A_2}\big)\ll n^{-1/2}.$$

Thus we get a much better upper bound than in (2). The proof uses estimates for Theta functions, which are provided in the next Section.

2 Theta Function Estimates

Let $E(n,d):=\frac{\Theta(d,n)}{d}$. By the Poisson summation formula

$$\sum_{\ell\in\mathbf{Z}} e^{-(\ell+\delta)^2\pi x^{-1}} = x^{1/2}\sum_{\ell\in\mathbf{Z}} e^{2i\pi\ell\delta-\ell^2\pi x},$$

where x is any real and $0\le\delta\le 1$, we get with the choices $x=\pi n/(2d^2)$, $\delta=n/(2d)$

$$E(n,d)=\sqrt{\frac{2}{\pi n}}\sum_{h\in\mathbf{Z}} e^{-2(\{\frac{n}{2d}\}+h)^2\frac{d^2}{n}}. \tag{3}$$

Let $a>0$, $0\le\mu\le 1$ and write $\bar\mu:=1-\mu$. We begin with elementary estimates of

$$S(\mu,a):=\sum_{h\in\mathbf{Z}} e^{-a(\mu+h)^2} = e^{-a\mu^2}+e^{-a\bar\mu^2}+\sum_{h=1}^{\infty} e^{-a(\mu+h)^2}+\sum_{h=1}^{\infty} e^{-a(h+\bar\mu)^2}.$$

Lemma 3 *Define for* $0\le\mu\le 1$ *and* $a>0$, $\varphi(\mu,a)=\frac{1}{\sqrt{2a}+2a\mu}$. *Then*

$$(\varphi(\mu,a)-1)e^{-a\mu^2}\le \sum_{h=1}^{\infty} e^{-a(\mu+h)^2}\le 2\varphi(\mu,a)e^{-a\mu^2}.$$

Proof. Consider Mill's ratio $R(x) = e^{x^2/2}\int_x^\infty e^{-t^2/2}\,dt$. Then ([1] section 2.26)

$$\frac{1}{1+x} \le \frac{2}{\sqrt{x^2+4}+x} \le R(x) \le \frac{2}{\sqrt{x^2+8/\pi}+x} \le \frac{2}{1+x}, \qquad x \ge 0.$$

First

$$\int_0^\infty e^{-a(\mu+x)^2}dx - e^{-a\mu^2} \le \sum_{h=1}^\infty e^{-a(\mu+h)^2} \le \int_0^\infty e^{-a(\mu+x)^2}dx.$$

But

$$\int_0^\infty e^{-a(\mu+x)^2}dx = \frac{e^{-\mu^2 a}}{\sqrt{2a}}R(\mu\sqrt{2a}) \quad \text{and} \quad \frac{1}{1+\mu\sqrt{2a}} \le R(\mu\sqrt{2a}) \le \frac{2}{1+\mu\sqrt{2a}}.$$

Thus

$$\frac{1}{\sqrt{2a}+2a\mu}e^{-a\mu^2} \le \int_0^\infty e^{-a(\mu+x)^2}dx \le \frac{2}{\sqrt{2a}+2a\mu}e^{-a\mu^2}.$$

Hence

$$(\varphi(\mu,a)-1)e^{-a\mu^2} \le \sum_{h=1}^\infty e^{-a(\mu+h)^2} \le 2\varphi(\mu,a)e^{-a\mu^2},$$

as claimed. ∎

Corollary 4 *Put $\psi(\mu,a) := \big(1+\varphi(\mu,a)\big)e^{-a\mu^2}$. Then for every $0 \le \mu \le 1$ and $a > 0$*

$$\frac{1}{2} \le \frac{S(\mu,a)}{\psi(\mu,a)+\psi(\bar{\mu},a)} \le 2.$$

Proof. At first by the previous Lemma

$$A := e^{-a\mu^2} + \sum_{h=1}^\infty e^{-a(\mu+h)^2} \le e^{-a\mu^2}\big(1+2\varphi(\mu,a)\big).$$

Next

$$A \ge e^{-a\mu^2} + \frac{1}{2}\sum_{h=1}^\infty e^{-a(\mu+h)^2} \ge e^{-a\mu^2} + \frac{1}{2}\big(\varphi(\mu,a)-1\big)e^{-a\mu^2} = \frac{1}{2}\big(1+\varphi(\mu,a)\big)e^{-a\mu^2}.$$

Thereby $1/2 \le \frac{A}{\psi(\mu,a)} \le 2$. Operating similarly with $\bar{A} = e^{-a\bar{\mu}^2} + \sum_{h=1}^\infty e^{-a(\bar{\mu}+h)^2}$ leads to

$$\frac{1}{2} \le \frac{S(\mu,a)}{\psi(\mu,a)+\psi(\bar{\mu},a)} \le 2.$$

∎

Notice that $\varphi(0,a) = 1/\sqrt{2a}$ and

$$\frac{1}{2}\big(1+\frac{1}{\sqrt{2a}}\big) \le S(0,a) = 1 + 2\sum_{h=1}^\infty e^{-ah^2} \le 4\big(1+\frac{1}{\sqrt{2a}}\big). \qquad (4)$$

We now need an extra Lemma.

Lemma 5 *Let $n = 2dK + r$ with $1 \leq r \leq 2d$. Then*

$$\frac{1}{2}\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}} \leq \frac{\psi(\frac{r}{2d},\frac{2d^2}{n})}{\sqrt{n}} \leq 2\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}}.$$

Proof. We have

$$\psi(\frac{r}{2d},\frac{2d^2}{n}) = \Big(1+\frac{\sqrt{n}}{2d}\,\frac{1}{1+\frac{r}{\sqrt{n}}}\Big)e^{-\frac{r^2}{2n}}.$$

We consider three cases.

Case a. $2d \leq \sqrt{n}$. Then $\frac{r}{\sqrt{n}} < \frac{2d}{\sqrt{n}} \leq 1$, and so $\frac{\sqrt{n}}{4d}e^{-\frac{r^2}{2n}} \leq \psi(\frac{r}{2d},\frac{2d^2}{n}) \leq \frac{\sqrt{n}}{d}e^{-\frac{r^2}{2n}}$, which implies

$$\frac{1}{2}\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}} = \frac{e^{-\frac{r^2}{2n}}}{4d} \leq \frac{\psi(\frac{r}{2d},\frac{2d^2}{n})}{\sqrt{n}} \leq \frac{e^{-\frac{r^2}{2n}}}{d} = 2\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}}.$$

Case b. $2d \geq \sqrt{n}$ and $r \leq \sqrt{n}$. Here we have $e^{-\frac{r^2}{2n}} \leq \psi(\frac{r}{2d},\frac{2d^2}{n}) \leq 2e^{-\frac{r^2}{2n}}$, which implies

$$\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}} = \frac{e^{-\frac{r^2}{2n}}}{\sqrt{n}} \leq \frac{\psi(\frac{r}{2d},\frac{2d^2}{n})}{\sqrt{n}} \leq \frac{2e^{-\frac{r^2}{2n}}}{\sqrt{n}} = 2\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}}.$$

Case c. $2d \geq \sqrt{n}$ and $r \geq \sqrt{n}$. The exponential factor $e^{-\frac{r^2}{2n}}$ may this time play a role (if $r \gg \sqrt{n}$), and we have $e^{-\frac{r^2}{2n}} \leq \psi(\frac{r}{2d},\frac{2d^2}{n}) \leq \frac{3}{2}e^{-\frac{r^2}{2n}}$ which implies

$$\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}} = \frac{e^{-\frac{r^2}{2n}}}{\sqrt{n}} \leq \frac{\psi(\frac{r}{2d},\frac{2d^2}{n})}{\sqrt{n}} \leq \frac{3e^{-\frac{r^2}{2n}}}{2\sqrt{n}} = \frac{3}{2}\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}}.$$

Summarizing cases a) to c), we have that

$$\frac{1}{2}\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}} \leq \frac{\psi(\frac{r}{2d},\frac{2d^2}{n})}{\sqrt{n}} \leq 2\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big)e^{-\frac{r^2}{2n}}.$$

■

3 Proof

A first case is simple.

Case I. $2d|n$. We have $E(n,d) = \sqrt{\frac{2}{\pi n}S(0,\frac{2d^2}{n})}$. But by (4)

$$\frac{1}{2}\max\big(1,\frac{\sqrt{n}}{2d}\big) \leq \frac{1}{2}\big(1+\frac{\sqrt{n}}{2d}\big) \leq S(0,\frac{2d^2}{n}) \leq 4\big(1+\frac{\sqrt{n}}{2d}\big) \leq 8\max\big(1,\frac{\sqrt{n}}{2d}\big).$$

Hence

$$\frac{1}{\sqrt{2\pi}}\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big) \leq E(n,d) \leq \frac{16}{\sqrt{2\pi}}\max\big(\frac{1}{2d},\frac{1}{\sqrt{n}}\big). \tag{5}$$

Case II. Now if $2d \not| n$, write $n = 2dK + \rho$ with $0 < \rho < 2d$. In our setting $a = \frac{2d^2}{n}$, $\mu = \{\frac{n}{2d}\} = \frac{\rho}{2d}$ and by (3), $E(n,d) = \sqrt{2/\pi n}\, S(\{\frac{n}{2d}\}, \frac{2d^2}{n})$. Applying Lemma 5 with $r = \rho$ gives

$$\frac{1}{2}\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)e^{-\frac{\rho^2}{2n}} \le \frac{\psi(\frac{\rho}{2d}, \frac{2d^2}{n})}{\sqrt{n}} \le 2\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)e^{-\frac{\rho^2}{2n}}. \tag{6}$$

As to $\psi(\bar{\mu}, \frac{2d^2}{n})$, we have $\bar{\mu} = \frac{2d-\rho}{2d} := \frac{\bar{\rho}}{2d}$ and $0 < \bar{\rho} < 2d$. Applying Lemma 5 with $r = \bar{\rho}$ gives

$$\frac{1}{2}\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)e^{-\frac{\bar{\rho}^2}{2n}} \le \frac{\psi(\frac{\bar{\rho}}{2d}, \frac{2d^2}{n})}{\sqrt{n}} \le 2\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)e^{-\frac{\bar{\rho}^2}{2n}}. \tag{7}$$

Consequently, by Corollary 4

$$\frac{1}{2\sqrt{2\pi}} \le \frac{E(n,d)}{\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)\big[e^{-\frac{\rho^2}{2n}} + e^{-\frac{\bar{\rho}^2}{2n}}\big]} \le \frac{8}{\sqrt{2\pi}}. \tag{8}$$

When $\rho = 0$, it follows from estimate (5) that

$$\begin{aligned}\frac{\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)}{\sqrt{2\pi}}\big[\frac{1 + e^{-\frac{\bar{\rho}^2}{2n}}}{2}\big] \quad &\le \quad \frac{\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)}{\sqrt{2\pi}} \le E(n,d) \le \frac{16\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)}{\sqrt{2\pi}} \\ &\le \quad \frac{32\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)}{\sqrt{2\pi}}\big[\frac{1 + e^{-\frac{\bar{\rho}^2}{2n}}}{2}\big].\end{aligned}$$

Finally in either case

$$\frac{1}{2\sqrt{2\pi}} \le \frac{E(n,d)}{\max\big(\frac{1}{2d}, \frac{1}{\sqrt{n}}\big)\big[e^{-\frac{\rho^2}{2n}} + e^{-\frac{\bar{\rho}^2}{2n}}\big]} \le \frac{32}{\sqrt{2\pi}}. \tag{9}$$

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References

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