AN INVERSE SOURCE PROBLEM FOR THE DIFFUSION EQUATION WITH FINAL OBSERVATION

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ABSTRACT. We investigate the inverse problem involving recovery of source temperature from the information of final temperature profile. We prove that we can uniquely recover the source of a n-dimensional heat equation from the measurement of the temperature at fixed time provided that the source is known in an arbitrary subdomain. The algorithm is based on the Carleman estimate. By using a Bukhgeim-Klibanov method, as a first step, we determine the source term by two measurements. A compacity and analyticity arguments procedure help to reduce the number of measurements.
Keywords: Inverse source problem, parabolic equation, Carleman estimates, Final overdetermination.

1. INTRODUCTION AND MAIN RESULT

Let us consider the diffusion equation which is a partial differential equation which describes density fluctuations in a material undergoing diffusion occupying an open and bounded domain of $\mathbb{R}^n$ with $\mathcal{C}^\infty$ boundary $\Gamma = \partial\Omega$. Given $T > 0$, we consider the following boundary value problem for the reaction-diffusion with homogeneous Dirichet boundary condition

$$\begin{aligned} &y'(x,t) - \mathrm{div}(D(x)\nabla y(x,t)) = F(x,t) && (x,t) \in Q = \Omega \times (0,T), \\ &y(x,t) = 0 && (x,t) \in \Sigma = \Gamma \times (0,T), \\ &y(x,0) = 0 && x \in \Omega, \end{aligned} \tag{1.1}$$

where prime stands for the time derivative. Throughout this paper, t and $x = (x_1, \dots, x_n)$ denote the time variable and the spatial variable respectively, and y denotes the temperature, is a scalar function, $F \in L^2(Q)$ is the heat source. We will assume that the diffusion matrix $D(x) = (d_{ij})_{1\leqslant i,j\leqslant n}$, where the coefficients $d_{ij}(x)$ are smooth on $\overline{\Omega}$ for $1 \leqslant i, j \leqslant n$, is positive-definite and satisfy for some positive constant $C > 0$

$$C^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} d_{ij}\xi_i\xi_j \leqslant C|\xi|^2, \quad d_{ij}(x) = d_{ji}(x), \quad x \in \overline{\Omega},\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n. \tag{1.2}$$

It is well known that system (1.1) possesses an unique solution y such that

$$y \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)). \tag{1.3}$$

1.1. **Inverse Source Problem.** We assume that the heat source is given by

$$F(x,t) = q(x)k(x,t), \tag{1.4}$$

Let $0 < \theta < T$ and let $\omega \Subset \Omega$ be an arbitrary fixed subdomain. Moreover let us assume that $k = k(x,t)$ is known. Then determine $q = q(x)$, $x \in \Omega$, from only the overdetermining data

$$y(x,\theta), \quad x \in \Omega$$

provided that $q(x) = 0$ in ω.

In our inverse problem, we are required to determine the x-dependent function $q(x)$, provided that the (x,t)-dependent factor $k(x,t)$ is known. As one simple example of such a source term, we can take

$k(x,t) = e^{-\sigma t}$ with a constant $\sigma > 0$ which is x-independent. This model describes a heat source by the decay of a radioactive isotope and then $q = q(x)$ corresponds to the spatial density of the isotope. We note that $\theta > 0$, and we do not assume any initial condition at $t = 0$, but we alternatively assume to be able to measure $y(\cdot, \theta)$ in Ω at any fixed moment $\theta > 0$. In the two dimensional case, such an observation can be realized for example by means of the thermography at θ, while it is difficult to determine $y(\cdot, 0)$ by observations when the heat process is already going on. Therefore our formulation of the inverse problem is meaningful from the practical point of view.

In the area of mathematical and physical inverse problems, the inverse source identification problems is one of the most addressed. Indeed, the notion of source term in the form $q(x)k(x,t)$ where $k(x,t)$ is assumed to be known can be associated easily to the notion of reaction term in a linear reaction diffusion equation. This type of equation appears in several fields of applications such as hydrology [3] or in heat transfer [4], in population genetics [1], chemistry [6], biology and also in spacial ecology to modelise the dynamic of a population [30]. It is well known that the solution $y(x,t)$ and the behavior of $y(x,t)$ depend mainly on the linear part of the reaction term: in population dynamic, the extinction of the population, its persistence and its evolution is conditioned by this term [11]. Therefore, it is crucial to know or determine this coefficient. But, in many physical applications, this term is often unknown or partially known and this coefficient cannot be directly measured since it is the resultant of mixed effects of several factors. Thus, this coefficient is generally measured through the solution $y(x,t)$, and in most of the cases $y(x,t)$ is not available simultaneously for all x and all time t.

On the other hand, the classical way to recover the source term in a parabolic problem such as (1.1) is to involve additional information. Different methods and different hypothesis on the additional informations have produced several interesting works. Around the topic of recovering a source term or the linear part of the reaction term in a reaction-diffusion equation, a lot of authors have used additional information coming from a finite number of boundary measurements [16, 29, 32, 31]. Thanks to the Carleman inequalities, the method introduced by Bukgheim and Klibanov [8] has permitted to establish important results involving stability inequality linking the coefficient to be recovered with the observations [2, 14, 21, 39]. These stability inequalities can play an important role in view to implement numerical simulations. The observations involved in these papers correspond to local observation on $\omega \times (0,T)$ where ω is a subdomain of Ω plus the observation of the solution on all the domain Ω at one time $\theta > 0$. Recently [13, 36], a new approach involving only pointwise observation of the solution $y(x,t)$ has been applied to obtain uniqueness results in several inverse parabolic problems including the recovering of the source terms in the one dimensional case. Another important inverse parabolic problems are associated to the integral overdetermination in which the additive observation can take the form:

$$\int_\Omega y(x,t)u(x)dx = \varphi(t), \text{ for } t \in (0,T),$$

where u and φ are the known functions, or the form

$$\int_{(0,T)} y(x,t)v(t)dt = \psi(x), \text{ for } x \in \Omega,$$

where v and ψ are the known functions. We find these additive informations in [37] associated to parabolic equations parametrized by a diffusion coefficient, and in the monograph [33], in which the authors have established uniqueness results for special classes of coefficients. They deal with an equivalent formulation of their inverse problem and they use a Fredholm approach. In [25, 27] the linear parabolic case in addressed and in [5], the authors are interested by the case of Neumann boundary conditions and they study a certain class of non linear reaction term.

The particular case of the final overdetermination in which the additive observation is the value of the solution at one fixed time θ : $y(x, \theta)$ has been addressed by Prilepko and al in [33] using the same method described above and the same special classes of coefficients.

The works of Isakov [22, 23] and Prilepko and Solov'ev [34] involve point fixed theorem. In his paper [22], Isakov carried out a stability result requiring additional boundary observations with some sign conditions. In [9], Bushuyev considers the case of special non linear source term and in [12], the authors deal with parabolic equations parametrized by a diffusion coefficient and establish a local stability inequality except for a countable set of parameters.
In [15], the authors used the optimal control framework to solve an optimization problem with final overdetermination to recover a potential in a nonlinear parabolic problem in the one dimensional case and in [17], Hasanov used a weak solution approach to minimize a coast functional.
On the other hand, there are several papers addressing the inverse problem of the determination of zeroth order coefficient in parabolic equation. Isakov [24] proved the uniqueness where the boundary observation $\partial_\nu y|_{\Gamma_1 \times (0,T)}$ with some $\Gamma_1 \subset \partial\Omega$, is adopted in place of $y|_{\omega \times (0,T)}$. Imanuvilov and Yamamoto [21] established the Lipschitz stability in determining a coefficient $q \in L^2(\Omega)$, and Imanuvilov and Yamamoto [20] proved the uniqueness within $q \in H^{-m}(\Omega)$; a Sobolev space of negative order $-m$. In the case of x-independent $k = k(t)$ and $\theta = 0$, we refer to Cannon [10] and Yamamoto [40], [41]. Cannon [10] proved the conditional stability in the special case of $k \equiv 1$, and [40] - [41] established the conditional stability of the logarithmic type by means of boundary observation data.

This above list of papers and methods is non exhaustive, but summarizing the existing results associated to each kind of additive observations, it seems that the most interesting subsidiary information in view of practical applications is the final overdetermination. Indeed, the final overdetermination $y(x, \theta)$ corresponds to a finite number of direct measurement of the solution. In other words, the choice of the final overdetermination corresponds to a realistic and rather simple observation which can be measured by an appropriate sensor.
Furthermore, we stress out that a mathematical stability inequality linking the coefficient to be reconstructed with finite number of physical observations (in this paper physical observation corresponds to the final overdetermination $y(x, \theta)$) is not proposed in most of the previous results excepted for the results using Carleman inequalities. But, in all the existing results, the stability inequalities obtained via Carleman estimates require additional observations, e.g. local measurement in space and time of the solution of the problem.
We propose here, a new result coupling stability inequality and final overdetermination observation at one time θ in a multidimensional case.

1.2. **Notations and statement of main results.** We define the following spaces:

$$H^{1,2}(Q) = H^1(0, T; L^2(\Omega)) \cap L^2(0, T, H^2(\Omega))$$

and

$$\mathcal{C}^{1,0}(\overline{Q}) = \mathcal{C}^1([0, T]; \mathcal{C}(\overline{\Omega})).$$

Let $\theta \in (0, T)$. We assume that the known part of the source term satisfy the following properties:

$$k \in \mathcal{C}^{1,0}(\overline{Q}), \quad \|k\|_{\mathcal{C}^{1,0}(\overline{Q})} \leqslant M, \quad |k(x, \theta)| \geqslant r_0 > 0, \quad x \in \overline{\Omega} \tag{1.5}$$

with some constants $M > 0$ and $r_0 > 0$. Let y the solution of (1.1). Then by the regularity of the parabolic system (see Evans [18] Thm.5, p.360) we have

$$y, y' \in H^{1,2}(Q)$$

moreover there exists $C > 0$ such that

$$\|y\|_{L^2(Q)} + \|\nabla y\|_{L^2(Q)} + \|y'\|_{L^2(Q)} \leqslant C \|q\|_{L^2(\Omega)} \tag{1.6}$$

and

$$\|y(\cdot,t)\|_{H^2(\Omega)} \leqslant C\,\|q\|_{L^2(\Omega)} \tag{1.7}$$

where the constant C depends only on Ω, T, $\|D\|_\infty$ and M.

We are ready to state the main result.

Theorem 1.1. *Let $\omega \Subset \Omega$ be an arbitrary open subdomain of Ω. Let $k \in \mathcal{C}^{1,0}(\overline{Q})$ satisfies (1.5) with $\theta \in (0,T)$. If $y' \in H^{1,2}(Q)$ satisfies (1.1), then there exists a constant $C = C(\omega, M, \theta) > 0$ such that*

$$\|q\|_{L^2(\Omega)} \leqslant C\|y(\cdot,\theta)\|_{H^2(\Omega)} \tag{1.8}$$

for all $q \in L^2(\Omega)$ satisfying $q = 0$ in ω.

As for the corresponding inverse problem in the case of $\theta = 0$, the uniqueness and the stability by $y_{|\omega_T}$ with an arbitrary ω are very difficult and open problems. As for related inverse problems, see Bukhgeim [7], Isakov [23], Lavrent'ev, Romanov and Shishat·skiĭ [28], Romanov [35].

In Section 2, we recall the key Carleman estimates from [21] and the Sections 3 and 4 are devoted to the proof of the theorem 1.1.

2. Carleman estimate

Our proof is the application of a Carleman estimate, which originates from the paper by Bukhgeim and Klibanov [8], and in [26] the uniqueness is proved in the inverse problem.

We set $\omega_T = \omega \times (0,T)$. We assume that the matrix D satisfies (1.2). In this section, we will detail a Carleman estimate for the problem:

$$\begin{aligned} &y'(x,t) - \operatorname{div}(D(x)\nabla y)(x,t) = F(x,t) && (x,t) \in Q = \Omega \times (0,T), \\ &y(x,t) = 0 && (x,t) \in \Sigma = \Gamma \times (0,T), \end{aligned} \tag{2.1}$$

which is the analogue to the Carleman estimate by Imanuvilov [19]. First we will define appropriate weight functions.

Lemma 2.1 (see [21])**.** *Let $\omega \Subset \Omega$ be an arbitrary open subdomain, for every open set $\omega_0 \Subset \omega$, there exists a function $\widetilde{\beta}$ such that: $\widetilde{\beta} \in \mathcal{C}^2(\overline{\Omega})$,*

$$\widetilde{\beta} = 0, \quad \partial_\nu \widetilde{\beta} < 0 \qquad \text{on } \partial\Omega,$$
$$|\nabla \widetilde{\beta}| > 0 \qquad \text{in } \overline{\Omega}\backslash\omega_0,$$

We take $K > 0$, such that

$$K \geqslant 5 \max_{\overline{\Omega}} \widetilde{\beta}$$

and set

$$\beta = \widetilde{\beta} + K, \quad \widehat{\beta} = \frac{5}{4} \max_{\overline{\Omega}} \beta.$$

Then we introduce the weight functions:

$$\varphi(x,t) = \frac{e^{\lambda\beta(x)}}{\ell(t)}, \quad \alpha(x,t) = \frac{e^{\lambda\beta(x)} - e^{\lambda\widehat{\beta}}}{\ell(t)}, \qquad \ell(t) = t(T-t)$$

where $\lambda > 0$ is a parameter.

Now we are ready to recall the first key Carleman estimate.

Lemma 2.2. *Under the above assumptions, there exists $\lambda_0 > 0$ so that for any $\lambda > \lambda_0$, there exists a constant $s_0(\lambda) > 0$ satisfying the following property: there exists a constant $C > 0$, independent of s, such that*

$$s^3 \int_Q e^{2s\alpha} \ell(t)^{-3} |y|^2 dxdt + s \int_Q e^{2s\alpha} \ell(t)^{-1} |\nabla y|^2 dxdt \leqslant Cs^3 \int_{\omega_T} e^{2s\alpha} \ell(t)^{-3} |y|^2 dxdt + C \int_Q e^{2s\alpha} |F|^2 dxdt \tag{2.2}$$

for all $y \in H^{1,2}(Q)$ solves (2.1) and all $s \geqslant s_0$.

For the application to the inverse problem, we further need to estimate the term y'.

Lemma 2.3. *Under the above assumptions, there exists $\lambda_0 > 0$ so that for any $\lambda > \lambda_0$, there exists a constant $s_0(\lambda) > 0$ satisfying the following property: there exists a constant $C > 0$, independent of s, such that*

$$\frac{1}{s} \int_Q e^{2s\alpha} \ell(t) |y'|^2 dxdt \leqslant Cs^3 \int_{\omega_T} e^{2s\alpha} \ell(t)^{-3} |y|^2 dxdt + C \int_Q e^{2s\alpha} |F|^2 dxdt \tag{2.3}$$

for all $y \in H^{1,2}(Q)$ solves (2.1) and all $s \geqslant s_0$.

3. Local estimates for parabolic equation

For further reference we now establish the following technical results, which are a kind of weighted regularity local estimates for parabolic equations. We consider the following parabolic system:

$$y' - \mathrm{div}(D(x)\nabla y) = F, \quad x \in \Omega,\ t \in (0, T), \tag{3.1}$$

we assume that $F \in L^2(Q)$ and

$$F(x,t) = 0, \quad x \in \omega,\ t \in (0,T). \tag{3.2}$$

Lemma 3.1. *Let $\gamma \in [0, +\infty)$ and $\omega' \Subset \omega \subset \Omega$. Then there exists a constant $C = C(\gamma, \omega', \omega) > 0$ such that the parabolic estimate*

$$\| e^{s\alpha} \ell^{-\gamma} \nabla y \|_{L^2(\omega'_T)} \leqslant Cs \| e^{s\alpha} \ell^{-(\gamma+1)} y \|_{L^2(\omega_T)}, \tag{3.3}$$

hold for any $y \in H^{1,2}(Q)$ satisfies (3.1)-(3.2) and $s \geqslant 1$.

Proof. Let $\chi \in \mathcal{C}_0^\infty(\Omega)$ be supported in ω with $\chi(x) = 1$ for all $x \in \omega'$ and $\mathrm{Supp}(\chi) \subset \omega$.
Multiplying F by $e^{2s\alpha} \ell^{-2\gamma} \chi y$ and integrating over $Q = \Omega \times (0,T)$ we get that

$$0 = \int_Q e^{2s\alpha} \ell^{-2\gamma} \chi(x) F(x,t) y(x,t) dxdt = U + W, \tag{3.4}$$

with

$$U = \frac{1}{2} \int_{\Omega \times (0,T)} e^{2s\alpha} \ell^{-2\gamma} \chi(x) \frac{d}{dt} (|y(x,t)|^2) dxdt,$$

$$W = - \int_{\Omega \times (0,T)} e^{2s\alpha} \ell^{-2\gamma} \chi(x) \mathrm{div}(D(x)\nabla y) y(x,t) dxdt.$$

We treat each of the two terms appearing in the rhs of (3.4) separately. Integrating by parts in the first one we get

$$|U| = \left| \int_Q e^{2s\alpha} \ell^{-2\gamma} \chi(x) (s\alpha_t - \gamma \ell^{-1}(t)(T - 2t)) \, |y|^2 \, dxdt \right| \leqslant Cs \| e^{s\alpha} \ell^{-(\gamma+1)} y \|^2_{L^2(\omega_T)}, \tag{3.5}$$

since $|\alpha_t| \leqslant C\ell^{-2}$.
Next we obtain in the same way that

$$W = \int_Q e^{2s\alpha}\ell^{-2\gamma}\chi^\tau \nabla y \cdot D(x) \cdot \nabla y dxdt + \int_Q e^{2s\alpha}\ell^{-2\gamma\tau}(\nabla\chi + 2s\chi\nabla\alpha) \cdot D(x) \cdot \nabla y(x,t) dxdt. \tag{3.6}$$

Further, since

$$\begin{aligned} 2\int_Q e^{2s\alpha}\ell^{-2\gamma\tau}\nabla\chi \cdot D(x) \cdot \nabla y y dxdt &= \int_Q e^{2s\alpha}\ell^{-2\gamma\tau}\nabla\chi \cdot D(x) \cdot \nabla(|y|^2) dxdt \\ &= -\int_Q e^{2s\alpha}\ell^{-2\gamma}(\mathrm{div}(D(x)\nabla\chi) + 2s\nabla\chi \cdot \nabla\alpha)\,|y|^2\,dxdt, \end{aligned} \tag{3.7}$$

we get that

$$\left|\int_Q e^{2s\alpha}\ell^{-2\gamma\tau}\nabla\chi \cdot D(x) \cdot \nabla y(x,t) y(x,t) dxdt\right| \leqslant Cs\|e^{s\alpha}\ell^{-(\gamma+1/2)}y\|^2_{L^2(\omega_T)}. \tag{3.8}$$

Moreover, we have

$$\begin{aligned} \left|\int_Q e^{2s\alpha}\ell^{-2\gamma}\chi^\tau\nabla\alpha \cdot D(x) \cdot \nabla y y dxdt\right| &\leqslant s^{-1}\|e^{s\alpha}\ell^{-\gamma}\chi^{1/2}\nabla y\|^2_{L^2(Q)} + s\|e^{s\alpha}\ell^{-\gamma}\chi^{1/2}\nabla\alpha y\|^2_{L^2(Q)} \\ &\leqslant s^{-1}\|e^{s\alpha}\ell^{-\gamma}\chi^{1/2}\nabla y\|^2_{L^2(Q)} + s\|e^{s\alpha}\ell^{-(\gamma+1)}y\|^2_{L^2(\omega_T)}. \end{aligned}$$

as we have $|\nabla\alpha| \leqslant C\ell^{-1}$. Putting this together with (3.5)-(3.6) and recalling from (1.3) that

$$\int_Q e^{2s\alpha}\ell^{-2\gamma}\chi\nabla y \cdot D(x) \cdot \nabla y dxdt \geqslant C^{-1}\|e^{s\alpha}\ell^{-\gamma}\nabla y\|^2_{L^2(\omega'_T)}$$

we end up getting (3.3). □

Lemma 3.2. *Let $\gamma \in [0,+\infty)$ and $\omega' \Subset \omega \subset \Omega$. Then there exists a constant $C = C(\gamma,\omega',\omega) > 0$ such that the parabolic estimate*

$$\|e^{s\alpha}\ell^{-\gamma}y'\|_{L^2(\omega'_T)} \leqslant Cs^2\|e^{s\alpha}\ell^{-(\gamma+2)}y\|_{L^2(\omega_T)}. \tag{3.9}$$

holds for any $y \in H^{1,2}(Q)$ satisfying (3.1)-(3.2) and $s \geqslant s_0$.

Proof. Choose ω'' such that $\omega' \Subset \omega'' \Subset \omega$. Let $\chi \in \mathcal{C}_0^\infty(\Omega)$ such that $\chi = 1$ in ω' and $\mathrm{Supp}(\chi) \subset \omega''$. Let $y_* = \chi y$. From the very definition of F we have

$$y'_* - \mathrm{div}(D(x)\nabla y_*) = \chi F + \mathcal{Q}_1 y \equiv \mathcal{Q}_1 y \quad \text{in } Q,$$

where $\mathcal{Q}_1$ is a first order differential operator supported in ω''. Let $\tilde{y} = \ell^{-(\gamma+1/2)}y_*$. Then $\tilde{y}$ satisfy

$$\tilde{y}' - \mathrm{div}(D(x)\nabla\tilde{y}) = \ell^{-(\gamma+1/2)}\mathcal{Q}_1 y - (\gamma + \frac{1}{2})(T-2t)\ell^{-(\gamma+3/2)}y_* \equiv G$$

Next, applying Lemma 2.3 with y replaced by $\tilde{y}$, we get that

$$\begin{aligned} \frac{1}{s}\int_Q e^{2s\alpha}\ell(t)|\tilde{y}'|^2 dxdt &\leqslant Cs^3\int_{\omega_T} e^{2s\alpha}\ell(t)^{-3}|\tilde{y}|^2 dxdt + C\int_Q e^{2s\alpha}|G|^2 dxdt \\ &\leqslant Cs^3\int_{\omega_T} e^{2s\alpha}\ell(t)^{-2(\gamma+2)}|y|^2 dxdt + C\int_Q e^{2s\alpha}|G|^2 dxdt. \end{aligned} \tag{3.10}$$

Furthermore

$$\int_Q e^{2s\alpha}|G|^2 dxdt \leqslant C\int_{\omega''_T} e^{2s\alpha}\ell^{-(2\gamma+1)}|\nabla y|^2 dxdt + C\int_{\omega''_T} e^{2s\alpha}\ell^{-(2\gamma+3)}|y|^2 dxdt. \tag{3.11}$$

Applying Lemma 3.1, with ω' replaced by ω'' and γ replaced by $\gamma+1/2$, to estimate the second integral

$$\int_{\omega''_T} e^{2s\alpha}\ell^{-(2\gamma+1)}|\nabla y|^2dxdt \leqslant Cs^2\int_{\omega_T} e^{2s\alpha}\ell^{-(2\gamma+3)}|y|^2dxdt \tag{3.12}$$

So, inserting (3.12) in (3.11), we obtain

$$\int_Q e^{2s\alpha}|G|^2dxdt \leqslant Cs^2\int_{\omega''_T} e^{2s\alpha}\ell^{-(2\gamma+3)}|y|^2dxdt \tag{3.13}$$

Moreover, since

$$\int_{\omega'_T} e^{2s\alpha}\ell^{-2\gamma}|y'|^2dxdt \leqslant \int_{\omega'_T} e^{2s\alpha}\ell|\tilde{y}'|^2dxdt + C\int_{\omega'_T} e^{2s\alpha}\ell^{-2(\gamma+1)}|y|^2dxdt \tag{3.14}$$

we deduce that

$$\int_{\omega'_T} e^{2s\alpha}\ell^{-2\gamma}|y'|^2dxdt \leqslant Cs^4\int_{\omega_T} e^{2s\alpha}\ell(t)^{-2(\gamma+2)}|y|^2dxdt. \tag{3.15}$$

The proof is complete. □

4. PROOF OF MAIN THEOREM

Now we can prove Theorem 1.1 by an argument similar to Imanuvilov and Yamamoto [21]. In a first step, we establish a new stability inequality (4.9) involving weaker norm than one obtained in [21].
For this, we set $Au(x) = \mathrm{div}(D(x)\nabla u(x))$ when $u \in \mathcal{C}^2(\overline{\Omega})$, and $u|_{\partial\Omega} = 0$.
Since we can change scales of t, without loss of generality, we may assume that $\theta = \frac{T}{2}$. By (1.1), the function $z = y' \in H^{1,2}(Q)$ satisfies

$$z' = Az + q(x)k'(x,t) \quad \text{in } Q \tag{4.1}$$

and

$$z\,(x,\theta) = Ay_\theta + q(x)k\,(x,\theta) \quad \text{in } \Omega, \tag{4.2}$$

where we set $y_\theta(x) = y\,(x,\theta)$, $x \in \Omega$. Therefore by Lemmas 2.2-2.3, using (1.5), we obtain

$$\begin{aligned} s^3\int_Q e^{2s\alpha}\ell(t)^{-3}|z|^2dxdt + s\int_Q e^{2s\alpha}\ell(t)^{-1}|\nabla z|^2dxdt + \frac{1}{s}\int_Q e^{2s\alpha}\ell(t)|z'|^2dxdt \\ \leqslant Cs^3\int_{\omega_T} e^{2s\alpha}\ell(t)^{-3}|z|^2dxdt + C\int_Q e^{2s\alpha}|q(x)|^2dxdt \end{aligned} \tag{4.3}$$

for all large $s > 0$.
By $e^{2s\alpha(x,0)} = 0$, the Cauchy-Schwarz inequality and (4.3), we have

$$\begin{aligned} \int_\Omega s\,|z\,(x,\theta)|^2\, e^{2s\alpha(x,T/2)}dxdt &= \int_0^\theta \frac{\partial}{\partial t}\left(\int_\Omega s|z(x,t)|^2e^{2s\alpha}dx\right)dt \\ &= \int_\Omega\int_0^\theta 2s^2(\alpha' e^{2s\alpha})z^2dxdt + \int_\Omega\int_0^\theta 2\left(\frac{1}{\sqrt{s}}\sqrt{\ell(t)}z'\right)\left(s\sqrt{s}z\frac{1}{\sqrt{\ell(t)}}\right)e^{2s\alpha}dxdt \\ &\leqslant C\int_\Omega\int_0^\theta s^2\ell(t)^{-2}e^{2s\alpha}z^2dxdt + C\int_\Omega\int_0^\theta\left(\frac{1}{s}\ell(t)|z'|^2 + s^3z^2\ell(t)^{-1}\right)e^{2s\alpha}dxdt \\ &\leqslant Cs^3\int_{\omega_T} e^{2s\alpha}\ell(t)^{-3}z^2dxdt + C\int_Q e^{2s\alpha}|q(x)|^2dxdt \end{aligned} \tag{4.4}$$

for all large $s > 0$.
Hence (4.2) and (4.4) yield

$$s\int_{\Omega}|q(x)|^{2}\left|k\left(x,\theta\right)\right|^{2}e^{2s\alpha(x,\theta)}dx\leqslant Cs\int_{\Omega}|Ay_{\theta}|^{2}e^{2s\alpha(x,\theta)}dx + Cs^{3}\int_{\omega_{T}}e^{2s\alpha}\ell(t)^{-3}|z|^{2}dxdt+C\int_{Q}e^{2s\alpha}|q(x)|^{2}dxdt \tag{4.5}$$

for all large $s > 0$. Since $\alpha(x,t)\leqslant\alpha\left(x,\theta\right)$ for all $(x,t)\in Q$, we have

$$\int_{Q}e^{2s\alpha}|q(x)|^{2}dx\leqslant\int_{\Omega}|q(x)|^{2}\left(\int_{0}^{T}e^{2s\alpha(x,\theta)}dt\right)dx=T\int_{\Omega}|q(x)|^{2}e^{2s\alpha(x,\theta)}dx. \tag{4.6}$$

Hence, by the last condition in (4.6), we can absorb the last term at the right hand side of (4.5) into the left hand side if $s > 0$ is sufficiently large, so that

$$s\int_{\Omega}|q(x)|^{2}e^{2s\alpha(x,\theta)}dx\leqslant Cs\int_{\Omega}|Ay_{\theta}|^{2}e^{2s\alpha(x,\theta)}dx+Cs^{3}\int_{\omega_{T}}e^{2s\alpha}\ell(t)^{-3}|y'|^{2}dxdt \tag{4.7}$$

for all large $s > 0$. By Lemma 3.2, we get

$$s\int_{\Omega}|q(x)|^{2}e^{2s\alpha(x,\theta)}dx\leqslant Cs\int_{\Omega}|Ay_{\theta}|^{2}e^{2s\alpha(x,\theta)}dx+Cs^{7}\int_{\omega_{T}}e^{2s\alpha}\ell(t)^{-7}|y|^{2}dxdt \tag{4.8}$$

Since

$$\sup_{x\in\Omega}e^{2s\alpha(x,\theta)},\qquad \sup_{(x,t)\in Q}\ell(t)^{-7}e^{2s\alpha(x,t)}<\infty,$$

fixing $s > 0$ sufficiently large, we obtain the following key result:

Proposition 4.1. *Let $\omega\Subset\Omega$ be an arbitrary open subdomain of Ω. Let $k\in\mathcal{C}^{1,0}(\overline{Q})$ satisfies (1.5) with $\theta\in(0,T)$. If $y\in H^{1,2}(Q)$ satisfies (1.1), then there exists a constant $C=C(\omega,M,\theta)>0$ such that*

$$\|q\|_{L^{2}(\Omega)}\leqslant C(\|y(\cdot,\theta)\|_{H^{2}(\Omega)}+\|y\|_{L^{2}(\omega_{T})}) \tag{4.9}$$

for all $q\in L^{2}(\Omega)$ satisfying $q=0$ in ω.

We stress out that in the previous proposition, contrary to the classical results, the local observation in ω_T is done with a norm in $L^2(\omega_T)$.
In a second step, we recall the following lemma Inspired by [38] :

Lemma 4.2. *Let X, Y , Z be three Banach spaces, let $\mathcal{A}:X\to Y$ be a bounded injective linear operator with domain $\mathscr{D}(\mathcal{A})$, and let $\mathcal{K}:X\to Z$ be a compact linear operator. Assume that there exists $C>0$ such that*

$$\|f\|_{X}\leqslant C_{1}\left\|\mathcal{A}f\right\|_{Y}+\left\|\mathcal{K}f\right\|_{Z},\quad\forall f\in\mathscr{D}(\mathcal{A}). \tag{4.10}$$

Then there exists $C>0$ such that

$$\|f\|_{X}\leqslant C\left\|\mathcal{A}f\right\|_{Y},\quad\forall f\in\mathscr{D}(\mathcal{A}). \tag{4.11}$$

Proof. Given $\mathcal{A}$ bounded and injective we argue by contradiction by assuming the opposite to (4.11). Then there exists a sequence $(f_n)_n$ in X such that $\|f_n\|_X=1$ for all n and $\mathcal{A}f_n\to 0$ in Y as n go to infinity. Since $\mathcal{K}:X\to Z$ is compact, there is a subsequence, still denoted by f_n, such that $(\mathcal{K}f_n)_n$ converges in Z. Therefore this is a Cauchy sequence in Z, hence, by applying (4.10) to f_n-f_m, we get that $\|f_n-f_m\|_X\to 0$, as $n,m\to\infty$. As a consequence $(f_n)_n$ is a Cauchy sequence in X so $f_n\to f$ as $n\to\infty$ for some $f\in X$. Since $\|f_n\|_X=1$ for all n, then $\|f\|_X=1$. Moreover we have $\mathcal{A}f_n\to\mathcal{A}f=0$ as $n\to\infty$, which is a contradiction to the fact that $\mathcal{A}$ is injective. ☐

We end the proof by defining

$$X = \left\{ q \in L^2(\Omega),\ q(x) = 0 \text{ in } \omega \right\}, \qquad Y = H^2(\Omega), \qquad Z = L^2(\omega_T),$$

and

$$\begin{array}{rccc} \mathcal{A}: & X & \longrightarrow & Y \\ & q & \longmapsto & \mathcal{A}q = y(\cdot,\theta) \end{array} ; \qquad \begin{array}{rccc} \mathcal{K}: & X & \longrightarrow & Z \\ & q & \longmapsto & \mathcal{K}q = y_{|\omega_T} \end{array}$$

where y denotes the unique solution to (1.1). Then it suffices to prove the two following lemma:

Lemma 4.3. *The operator $\mathcal{A}$ is bounded and injective.*

Proof. First the boundedness of $\mathcal{A}$ follows readily from (1.7). Second y being solution to the system formed by the two first equations in (1.1), we deduce from the identities $y(\cdot,\theta) = 0$ and $q = 0$ on ω that $y'(\cdot,\theta) = 0$ on ω. Arguing in the same way we get that the successive derivatives of y wrt t vanish on $\omega \times \{\theta\}$. Since y is solution to some initial value problem with time independent coefficients, it is time analytic so we necessarily have $y = 0$ on $\omega \times (0,\theta)$. Therefore $q = 0$ on Ω by (4.9), and the proof is complete. □

Lemma 4.4. *$\mathcal{K}$ is a compact operator from X into Z.*

Proof. The operator $\mathcal{K}$ being bounded from X to $H^1(\omega_T)$ as we have

$$\|y\|_{L^2(\omega_T)} + \|\nabla y\|_{L^2(\omega_T)} + \|y'\|_{L^2(\omega_T)} \leqslant \|q\|_{L^2(\Omega)},$$

from (1.6) so the result follows readily from the compactness of the injection $H^1(\omega_T) \hookrightarrow Z = L^2(\omega_T)$. □

Finally, putting Lemmas 4.1, 4.2, 4.3 and 4.4 together, we end up getting that

$$\|q\|_{L^2(\Omega)} \leqslant C \|\mathcal{A}q\|_Y = C \|y(\cdot,\theta)\|_{H^2(\Omega)},$$

which gives Theorem 1.1.

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