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ON A DIVISOR PROBLEM RELATED TO THE EPSTEIN ZETA-FUNCTION, III

GUANGSHI LÜ, JIE WU & WENGUANG ZHAI

ABSTRACT. In this paper we study the mean square of the error term $\Delta_k^*(Q, x)$ in a divisor problem related to the Epstein zeta-function. An asymptotic formula has been obtained when $k = 2$.

1. INTRODUCTION

This is the third part of our series of papers on a divisor problem related to the Epstein zeta-function [10, 11]. First we recall some notation there. Let $\ell \geq 2$, $\mathbf{y} := (y_1, \dots, y_\ell)$ and $\mathbf{A} = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \pmod{2}$ for $0 \leq i \leq \ell$. Thus a positive definite quadratic form $Q(\mathbf{y})$ can be written as

$$Q(\mathbf{y}) = \frac{1}{2} \mathbf{y}^t \mathbf{A} \mathbf{y} = \sum_{1 \leq i < j \leq \ell} a_{ij} y_i y_j + \frac{1}{2} \sum_{1 \leq i \leq \ell} a_{ii} y_i^2,$$

where $\mathbf{y}^t$ is the transpose of $\mathbf{y}$. The corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$(1.1) \quad Z_Q(s) := \sum_{\mathbf{y} \in \mathbb{Z}^\ell \setminus \{0\}} Q(\mathbf{y})^{-s} = \sum_{n \geq 1} a_n n^{-s} \quad (\Re s > \ell/2),$$

where a_n is the number of the solutions of the equation $Q(\mathbf{y}) = n$ with $\mathbf{y} \in \mathbb{Z}^\ell$. It is known that $Z_Q(s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ with only a simple pole at $s = \ell/2$, and satisfies a functional equation of Riemann type (cf. [13]). For each integer $k \geq 1$, we define $a_k(n)$ by

$$(1.2) \quad Z_Q(s)^k = \sum_{n \geq 1} a_k(n) n^{-s} \quad (\Re s > \ell/2)$$

and put

$$(1.3) \quad \Delta_k^*(Q, x) := \sum_{n \leq x} a_k(n) - x^{\ell/2} P_k(\log x),$$

where $P_k(\log x) := x^{-\ell/2} \operatorname{Res}_{s=\ell/2} (Z_Q(s)^k x^s s^{-1})$ is a polynomial of $\log x$ of degree $k - 1$. The study on asymptotic behavior of the error term $\Delta_k^*(Q, x)$ has received

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much attention [8, 1, 13]. In particular Sankaranarayanan [13] showed that for $k \geq 2$ and $\ell \geq 3$,

$$(1.4) \quad \Delta_k^*(Q, x) \ll x^{\ell/2-1/k+\varepsilon},$$

where and throughout this paper ε denotes an arbitrarily small positive constant. Recently inspired by Iwaniec's book [6], Lü [10] marked that (1.4) can be improved for the quadratic forms of level one (see [6, Chapter 11]). These quadratic forms are defined by $Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^t \mathbf{A} \mathbf{y}$ verifying the following supplementary conditions:

$$\ell \equiv 0 \pmod{8}, \quad \mathbf{A} \text{ is equivalent to } \mathbf{A}^{-1}, \quad \det(\mathbf{A}) = 1.$$

Denote by $\mathcal{Q}_\ell$ the set of such quadratic forms. For $Q \in \mathcal{Q}_\ell$, we have [6, (11.32)]

$$a_n = A_\ell \sigma_{\ell/2-1}(n) + a_f(n, Q) \quad (n \geq 1),$$

where

$$A_\ell := \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}, \quad \sigma_k(n) = \sum_{d|n} d^k,$$

$\zeta(s)$ is the Riemann zeta-function, $\Gamma(s)$ is the Gamma function and $a_f(n, Q)$ is the n th Fourier coefficient of a cusp form $f(z, Q)$ of weight $\ell/2$ with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$. Thus

$$(1.5) \quad Z_Q(s) = A_\ell \zeta(s) \zeta(s - \ell/2 + 1) + L(s, f) \quad (\Re s > \ell/2),$$

where $L(s, f)$ is the Hecke L -function associated with $f(z, Q)$. According to Deligne's well known work [2], we know

$$(1.6) \quad |a_f(n, Q)| \leq n^{(\ell/2-1)/2} \tau(n) \quad (n \geq 1),$$

where $\tau(n)$ is the divisor function. With the help of these properties, Lü [10] (for $k \geq 4$) and Lü, Wu & Zhai [11] (for $k = 2, 3$) obtained

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1+\theta_k+\varepsilon},$$

where θ_k is the exponent in the classical k -dimension divisor problem

$$\Delta_k(x) := \sum_{n \leq x} \tau_k(n) - \mathrm{Res}_{s=1}(\zeta(s)^k x^s s^{-1}) \ll x^{\theta_k+\varepsilon} \quad (x \geq 2).$$

In particular we can take $\theta_2 = 131/416$ [4], $\theta_3 = 43/96$ [7] and $\theta_k = (k-1)/(k+2)$ for $k \geq 4$ [15]. Besides, an Ω -result has been established in [11]: if $8 \mid \ell$ and $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, then we have for $k = 2, 3$ that

$$\Delta_k^*(Q, x) = \Omega\left(x^{\ell/2-1+(k-1)/2k} (\log x)^{(k-1)/(2k)} (\log_2 x)^a (\log_3 x)^{-b'}\right),$$

where $a = \frac{k+1}{2k}(k^{(2k)/(k+1)} - 1)$, b' is any constant greater than $\frac{3k-1}{4k}$ and $\log_r$ denotes the r -fold iterated logarithm.

The aim of this paper is to study the mean square of $\Delta_k^*(Q, x)$.

Theorem 1. *If $8 \mid \ell$, then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we have*

$$\int_1^T |\Delta_2^*(Q, x)|^2 dx = C_\ell T^{\ell-1/2} + O(T^{\ell-1} (\log T)^3 \log_2 T),$$

where

$$(1.7) \quad g_a(n) := \sum_{d|n} \frac{\tau(d)\tau(n/d)}{d^a}, \quad C_\ell := \frac{3A_\ell^4}{(2\ell-1)\pi^2} \sum_{n=1}^{\infty} \frac{g_{(\ell-3)/2}(n)^2}{n^{3/2}}.$$

The estimate $O(T^{\ell-1}(\log T)^3 \log_2 T)$ follows from the result of [9] on the mean square of $\Delta_2(x)$.

Theorem 2. *For $k \geq 2$, $8 \mid \ell$ and $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we define*

$$\beta_k := \inf \left\{ b_k : \int_1^T |\Delta_k(x)|^2 dx \ll T^{1+2b_k+\varepsilon} \right\},$$

$$\beta_k^* := \inf \left\{ b_k^* : \int_1^T |\Delta_k^*(Q, x)|^2 dx \ll T^{\ell-1+2b_k^*+\varepsilon} \right\}.$$

Then $\beta_k^* = \beta_k$. Further we have $\beta_k^* \geq (k-1)/2k$ and the equality holds if the Lindelöf hypothesis of $\zeta(s)$ is true.

Ivić [5,] proved that

$$\beta_3 = 1/3, \quad \beta_4 = 3/8, \quad \beta_5 \leq 119/260, \quad \beta_6 \leq 1/2, \quad \beta_7 \leq 39/70.$$

According to Theorem 2, the same estimates for β_k^* hold.

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2. AN EXPRESSION OF $\Delta_2^*(Q, x)$

In [11], we actually established the formula

$$\Delta_2^*(Q, x) = A_\ell^2 x^{\ell/2-1} \sum_{d \leq x} \frac{\tau(d)}{d^{\ell/2-1}} \Delta_2\left(\frac{x}{d}\right) + O(x^{\ell/2-1+\varepsilon}).$$

From it we can deduce Ω -result of $\Delta_2^*(Q, x)$. However, it is not enough to prove Theorem 1. So first we will give a better expression of $\Delta_2^*(Q, x)$.

Lemma 2.1. *If $8 \mid \ell$, then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we have*

$$\Delta_2^*(Q, x) = A_\ell^2 x^{\ell/2-1} \sum_{d \leq x} \frac{\tau(d)}{d^{\ell/2-1}} \left(\Delta_2\left(\frac{x}{d}\right) - \frac{1}{4} \right) \\ - 2A_\ell x^{\ell/2-1} \sum_{d \leq x} \frac{b(d)}{d^{\ell/2-1}} \psi\left(\frac{x}{d}\right) + O(x^{\ell/2-5/4}),$$

where $\psi(t) := \{t\} - \frac{1}{2}$ and $\{t\}$ denotes the fractional part of t .

Proof. From (1.5) we have

$$Z_Q(s)^2 = A_\ell^2 \zeta(s)^2 \zeta(s - \ell/2 + 1)^2 + A_\ell \zeta(s) \zeta(s - \ell/2 + 1) L(s, f) + L(s, f)^2.$$

Here the last two terms do not appear when $\ell = 8, 16$ since there are no cusp forms of weights 4 and 8 with respect to $\mathrm{SL}(2, \mathbb{Z})$. Thus we can write

$$(2.1) \quad \sum_{n \leq x} a_2(n) = A_\ell^2 \sum_{d \leq x} \tau(d) \sum_{m \leq x/d} \tau(m) m^{\ell/2-1} \\ + 2A_\ell \sum_{d \leq x} b(d) \sum_{m \leq x/d} m^{\ell/2-1} + \sum_{d \leq x} c(d),$$

where $b(n)$ and $c(n)$ are defined by

$$\zeta(s)L(s, f) = \sum_{n=1}^{\infty} b(n)n^{-s} \quad \text{and} \quad L(s, f)^2 = \sum_{n=1}^{\infty} c(n)n^{-s}$$

for $\Re s > \ell/2$, respectively. By using Deligne's bound (1.6), it is easy to see that

$$(2.2) \quad |b(n)| \leq n^{(\ell/2-1)/2} \tau_3(n) \quad \text{and} \quad |c(n)| \leq n^{(\ell/2-1)/2} \tau_4(n).$$

Thus

$$(2.3) \quad \sum_{n \leq x} (|b(n)| + |c(n)|) \ll x^{\ell/4+1/2} (\log x)^3.$$

By partial summation we have

$$\sum_{m \leq x} \tau(m) m^{\ell/2-1} = \frac{2}{\ell} x^{\ell/2} \left(\log x - \frac{2}{\ell} + 2\gamma \right) + x^{\ell/2-1} \Delta_2(x) \\ - (\ell/2 - 1) \int_1^x \Delta_2(t) t^{\ell/2-2} dt.$$

By using Voronoï's well known formula [16]:

$$\int_1^t \Delta_2(u) du = \frac{t}{4} + O(t^{3/4}),$$

a simple partial summation leads to

$$(\ell/2 - 1) \int_1^x \Delta_2(t) t^{\ell/2-2} dt = \frac{1}{4} x^{\ell/2-1} + O(x^{\ell/2-5/4}).$$

Combining these, we find that

$$(2.4) \quad \sum_{m \leq x} \tau(m) m^{\ell/2-1} = \frac{2}{\ell} x^{\ell/2} \left(\log x - \frac{2}{\ell} + 2\gamma \right) \\ + x^{\ell/2-1} \left(\Delta_2(x) - \frac{1}{4} \right) + O(x^{\ell/2-5/4}).$$

Similarly (even easier)

$$(2.5) \quad \sum_{m \leq x} m^{\ell/2-1} = \frac{2}{\ell} x^{\ell/2} - x^{\ell/2-1} \psi(x) + O(x^{\ell/2-2}).$$

Now the required result follows from (2.1), (2.3), (2.4) and (2.5). $\square$

3. PROOF OF THEOREM 1

3.1. Beginning of the Proof. Let

$$\tilde{\Delta}_2^*(Q, x) := \frac{\Delta_2^*(Q, x)}{A_\ell^2 x^{\ell/2-1}} \quad \text{and} \quad \tilde{C}_\ell := \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{g_{(\ell-3)/2}(n)^2}{n^{3/2}}.$$

Clearly it is sufficient to prove that

$$(3.1) \quad \int_1^T |\tilde{\Delta}_2^*(Q, x)|^2 dx = \frac{\tilde{C}_\ell}{6} T^{3/2} + O(T(\log T)^3 \log_2 T).$$

According to Lemma 2.1, we can write

$$\tilde{\Delta}_2^*(Q, x) = U(x) - V(x) + O(x^{\ell/2-5/4}),$$

where

$$U(x) := \sum_{d \leq x} \frac{\tau(d)}{d^{\ell/2-1}} \left(\Delta_2\left(\frac{x}{d}\right) - \frac{1}{4} \right), \quad V(x) := \frac{2}{A_\ell} \sum_{d \leq x} \frac{b(d)}{d^{\ell/2-1}} \psi\left(\frac{x}{d}\right).$$

Next we shall prove

$$(3.2) \quad \int_1^T U^2(x) dx = \frac{\tilde{C}_\ell}{6} T^{3/2} + O(T(\log T)^3 \log_2 T),$$

$$(3.3) \quad \int_1^T U(x)V(x) dx \ll T(\log T)^2,$$

which imply (3.1).

3.2. Preparation. In this subsection, we shall prove some preliminary estimates, which are useful later.

Lemma 3.1. *Let $a > 0, b > 1, \ell > a + b$ and $A \geq 1$. We have*

$$(3.4) \quad \sum_{\substack{d_1, d_2 \leq T \\ m_1, m_2 \leq M \\ d_1 m_2 = d_2 m_1}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1 d_2)^{\ell/2-a/2} (m_1 m_2)^{b/2}} = \sum_{n=1}^{\infty} \frac{g_{(\ell-a-b)/2}(n)^2}{n^b} + O_A\left(\frac{(\log T)^3}{T^{b-1}}\right),$$

$$(3.5) \quad \sum_{\substack{d_1, d_2 \leq T \\ m_1, m_2 \leq M \\ d_1 m_2 \neq d_2 m_1}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1 d_2)^{\ell/2-3/4} (m_1 m_2)^{3/4}} \frac{1}{|\sqrt{m_1/d_1} - \sqrt{m_2/d_2}|} \ll_A (\log T)^3 \log_2 T,$$

$$(3.6) \quad \sum_{\substack{d_1, d_2 \leq T \\ m_1, m_2 \leq M}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1 d_2)^{\ell/2-3/4} (m_1 m_2)^{3/4}} \frac{1}{\sqrt{m_1/d_1} + \sqrt{m_2/d_2}} \ll_A (\log T)^3 \log_2 T.$$

uniformly for $1 \leq T \leq M \leq T^A$, where $g_r(n)$ is defined as in (1.7).

Proof. First we write

$$\begin{aligned}
 (3.7) \quad S_1(T, M) &= \sum_{n \leq TM} \frac{1}{n^b} \left(\sum_{\substack{d \leq T; m \leq M \\ dm=n}} \frac{\tau(d)\tau(m)}{d^{(\ell-a-b)/2}} \right)^2 \\
 &= \sum_{n=1}^{\infty} \frac{g_{(\ell-a-b)/2}(n)^2}{n^b} + O\left(\sum_{n>T} \frac{g_{(\ell-a-b)/2}(n)^2}{n^b} \right).
 \end{aligned}$$

It is easy to see that $g_r(n)$ is multiplicative, $g_r(p) = 2 + 2/p^r$ and $g_r(p^\nu) \ll_r (\nu + 1)$ for all p and $\nu \geq 1$. Applying Theorem 2.1 of [14] with $x = y$ and $\kappa = 4$ to $g_r(n)^2$ leads to the following inequality

$$\sum_{n \leq x} g_r(n)^2 \ll_r x(\log x)^3 \quad (r > 0, x \geq 2).$$

From it and (3.7), we can easily deduce (3.4).

Similarly we can write

$$S_2(T, M) \leq \sum_{\substack{n, n' \leq TM \\ n \neq n'}} \frac{g_{\ell/2-2}(n)g_{\ell/2-2}(n')}{(nn')^{3/4}} \frac{1}{|\sqrt{n} - \sqrt{n'}|} \ll_A (\log T)^3 \log_2 T.$$

In the last step we have used the bound of Lau & Tsang [9].

The estimate (3.6) is an immediate consequence of (3.4) with $a = b = 2$ and (3.5) if noting that

$$\frac{1}{\sqrt{m_1/d_1} + \sqrt{m_2/d_2}} \ll \begin{cases} \left(\frac{d_1 d_2}{m_1 m_2} \right)^{1/4} & \text{if } m_1/d_1 = m_2/d_2 \\ \frac{1}{|\sqrt{m_1/d_1} - \sqrt{m_2/d_2}|} & \text{if } m_1/d_1 \neq m_2/d_2. \end{cases}$$

□

3.3. Proof of (3.2). According to Meurman [12], we have

$$(3.8) \quad \Delta_2(x) - \frac{1}{4} = \frac{x^{1/4}}{\sqrt{2}\pi} \sum_{m \leq M} \frac{\tau(m)}{m^{3/4}} \cos \left(4\pi\sqrt{xm} - \frac{\pi}{4} \right) + E(x)$$

for all $M > x > 1$, where

$$(3.9) \quad E(x) \ll \begin{cases} x^{-1/4} & \text{if } \|x\| \geq x^{5/2} M^{-1/2}, \\ x^\varepsilon & \text{if } \|x\| \leq x^{5/2} M^{-1/2}. \end{cases}$$

Thus we can write, with the choice of $M = T^{10} > x$,

$$(3.10) \quad U(x) = A(x) + B(x),$$

where

$$\begin{aligned}
 A(x) &:= \frac{x^{1/4}}{\sqrt{2}\pi} \sum_{d \leq x} \frac{\tau(d)}{d^{\ell/2-3/4}} \sum_{m \leq M} \frac{d(m)}{m^{3/4}} \cos \left(4\pi\sqrt{\frac{m}{d}}x - \frac{\pi}{4} \right), \\
 B(x) &:= \sum_{d \leq x} \frac{\tau(d)}{d^{\ell/2-1}} E\left(\frac{x}{d}\right).
 \end{aligned}$$

In view of the identity $2 \cos u \cos v = \cos(u - v) + \cos(u + v)$, we easily see that

$$A(x)^2 = A_1(x) + A_2(x) + A_3(x),$$

where

$$\begin{aligned} A_1(x) &:= \frac{x^{1/2}}{4\pi^2} \sum_{\substack{d_1, d_2 \leq x \\ m_1, m_2 \leq M \\ m_1 d_2 = m_2 d_1}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1 d_2)^{\ell/2-3/4}(m_1 m_2)^{3/4}}, \\ A_2(x) &:= \frac{x^{1/2}}{4\pi^2} \sum_{\substack{d_1, d_2 \leq x \\ m_1, m_2 \leq M \\ m_1 d_2 \neq m_2 d_1}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1 d_2)^{\ell/2-3/4}(m_1 m_2)^{3/4}} \cos \left(4\pi \left(\sqrt{\frac{m_1}{d_1}} - \sqrt{\frac{m_2}{d_2}} \right) \sqrt{x} \right), \\ A_3(x) &:= \frac{x^{1/2}}{4\pi^2} \sum_{\substack{d_1, d_2 \leq x \\ m_1, m_2 \leq M}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1 d_2)^{\ell/2-3/4}(m_1 m_2)^{3/4}} \cos \left(4\pi \left(\sqrt{\frac{m_1}{d_1}} + \sqrt{\frac{m_2}{d_2}} \right) \sqrt{x} \right). \end{aligned}$$

By using (3.4) we have

$$\begin{aligned} \int_1^T A_1(x) dx &= \frac{1}{4\pi^2} \sum_{\substack{d_1, d_2 \leq T \\ m_1, m_2 \leq M \\ m_1 d_2 = m_2 d_1}} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1 d_2)^{\ell/2-3/4}(m_1 m_2)^{3/4}} \int_{\max\{d_1, d_2\}}^T x^{1/2} dx \\ &= \frac{\tilde{C}_\ell}{6} T^{3/2} + O(T(\log T)^3). \end{aligned}$$

With the help of the first derivative test and (3.5), we get

$$\begin{aligned} \int_1^T A_2(x) dx &\leq \sum_{1 \leq k \leq 2 \log T} \left| \int_{T/2^k}^{T/2^{k-1}} A_2(x) dx \right| \\ &\ll \sum_{1 \leq k \leq 2 \log T} (T/2^k) S_2(T/2^{k-1}, M) \\ &\ll T(\log T)^3 \log_2 T. \end{aligned}$$

Similarly we have

$$\int_1^T A_3(x) dx \ll T.$$

Combining these estimates, we find that

$$(3.11) \quad \int_1^T A(x)^2 dx = \frac{\tilde{C}_\ell}{6} T^{3/2} + O(T(\log T)^3 \log_2 T).$$

By Cauchy's inequality, it follows

$$B(x)^2 \leq \sum_{d \leq x} \frac{\tau(d)^2}{d^2} \sum_{d \leq x} \frac{1}{d^{\ell-4}} E\left(\frac{x}{d}\right)^2 \ll \sum_{d \leq x} \frac{1}{d^{\ell-4}} E\left(\frac{x}{d}\right)^2,$$

which combining (3.9) allows us to deduce that

$$\begin{aligned}
 \int_1^T B(x)^2 dx &\ll \sum_{d \leq T} \frac{1}{d^{\ell-5}} \left(\int_1^{T/d} t^\varepsilon dt + \int_1^{T/d} t^{-1/2} dt \right) \\
 &\ll \sum_{d \leq T} \frac{1}{d^{\ell-5}} \left\{ \left(\frac{T}{d} \right)^{7/2+\varepsilon} \frac{1}{M^{1/2}} + \left(\frac{T}{d} \right)^{1/2} \right\} \\
 &\ll T^{1/2}.
 \end{aligned}
 \tag{3.12}$$

From (3.11) and (3.12), we get, via Cauchy's inequality, that

$$\int_1^T A(x)B(x) dx \ll T.
 \tag{3.13}$$

Now the asymptotic formula (3.2) follows from (3.10), (3.11), (3.12) and (3.13).

3.4. Proof of (3.3). By using Theorem 4.5 in Graham and Kolesnik [3]

$$\Delta_2(u) = -2 \sum_{m \leq \sqrt{u}} \psi(u/m) + O(1)$$

and (2.2), we have

$$\int_1^T U(x)V(x) dx \ll \sum_{d \leq T} \frac{\tau(d)}{d^{\ell/2-1}} \sum_{m \leq (T/d)^{1/2}} \sum_{n \leq T} \frac{\tau_3(n)}{n^{\ell/4-1/2}} |I(d, m, n)| + T,
 \tag{3.14}$$

where

$$I(d, m, n) := \int_{\max\{dm^2, n\}}^T \psi\left(\frac{x}{dm}\right) \psi\left(\frac{x}{n}\right) dx.$$

For $\psi(u)$, it is well-known that the finite Fourier expansion

$$\psi(u) = - \sum_{1 \leq h \leq H} \frac{\sin(2\pi hu)}{\pi h} + O\left(\min\left\{1, \frac{1}{H\|u\|}\right\}\right)$$

holds for any $H \geq 2$. It is easily seen that for any $r > 0$

$$\begin{aligned}
 \int_{\max\{dm^2, n\}}^T \min\left\{1, \frac{1}{H\|x/r\|}\right\} dx &= r \int_{m^2}^{T/r} \min\left\{1, \frac{1}{H\|t\|}\right\} dt \\
 &\ll T \int_0^{1/2} \min\left\{1, \frac{1}{Ht}\right\} dt \\
 &\ll TH^{-1} \log H.
 \end{aligned}$$

From these we deduce

$$I(d, m, n) \ll \sum_{h_1, h_2 \leq H} \frac{|I(h_1, h_2)|}{h_1 h_2} + \frac{T(\log H)^2}{H}
 \tag{3.15}$$

where

$$I(h_1, h_2) := \int_{\max\{dm^2, n\}}^T \sin\left(\frac{2\pi h_1 x}{dm}\right) \sin\left(\frac{2\pi h_2 x}{n}\right) dx \\ \ll \begin{cases} 1/|h_1/dm - h_2/n| & \text{if } h_1 n \neq h_2 dm, \\ T & \text{if } h_1 n = h_2 dm. \end{cases}$$

Here we have used the identity $2 \sin u \sin v = \cos(u - v) - \cos(u + v)$ and the first derivative test when $h_1 n \neq h_2 dm$.

Inserting (3.15) into (3.14), we get

$$\int_1^T U(x)V(x) dx \ll TS_4(T, H) + S_5(T, H) + T + \frac{T^{3/2}(\log H)^2}{H},$$

where

$$S_4(T, H) := \sum_{d \leq T} \frac{\tau(d)}{d^{\ell/2-1}} \sum_{m \leq (T/d)^{1/2}} \sum_{n \leq T} \frac{\tau_3(n)}{n^{\ell/4-1/2}} \sum_{\substack{h_1, h_2 \leq H \\ h_1 n = h_2 dm}} \frac{1}{h_1 h_2} \\ \leq \sum_{r \leq HT} \frac{1}{r^2} \sum_{n|r} \frac{\tau_3(n)}{n^{\ell/4-3/2}} \sum_{h_2 dm=r} \frac{\tau(d)m}{d^{\ell/2-2}} \ll \sum_{r \leq HT} \frac{1}{r} \ll \log(HT)$$

and

$$S_5(T, H) := \sum_{d \leq T} \frac{\tau(d)}{d^{\ell/2-1}} \sum_{m \leq (T/d)^{1/2}} \sum_{n \leq T} \frac{\tau_3(n)}{n^{\ell/4-1/2}} \sum_{\substack{h_1, h_2 \leq H \\ h_1 n \neq h_2 dm}} \frac{dmn}{h_1 h_2 |h_1 n - h_2 dm|} \\ = \sum_{\substack{r_1, r_2 \leq HT \\ r_1 \neq r_2}} \frac{1}{r_1 r_2 |r_1 - r_2|} \sum_{\substack{h_2 \leq H, d \leq T, m \leq (T/d)^{1/2} \\ h_2 dm = r_1}} \frac{\tau(d)(dm)^2}{d^{\ell/2-1}} \sum_{\substack{n \leq T, h_1 \leq H \\ h_1 n = r_2}} \frac{\tau_3(n)}{n^{\ell/4-5/2}} \\ \leq T \sum_{\substack{r_1, r_2 \leq HT \\ r_1 \neq r_2}} \frac{1}{r_1 r_2 |r_1 - r_2|} \sum_{h_2 dm=r_1} \frac{\tau(d)}{d^{\ell/2-2}} \sum_{h_1 n=r_2} \frac{\tau_3(n)}{n^{\ell/4-5/2}} \\ \ll T \sum_{|r| \leq HT} \frac{1}{|r|} \sum_{r_2 \leq HT} \frac{1}{r_2} \\ \ll T(\log HT)^2.$$

This proves (3.3) with the choice of $H = T$.

4. PROOF OF THEOREM 2

For each $r \geq 2$, let δ_r and δ_r^* denote the infimum of $\sigma > 0$ such that

$$\int_{-\infty}^{\infty} \frac{|\zeta(\sigma + it)|^r}{|\sigma + it|^2} dt \ll 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{|Z_Q(\sigma + it)|^r}{|\sigma + it|^2} dt \ll 1,$$

respectively. According to [5, Lemma 13.1], we have

$$(4.1) \quad \beta_k = \delta_{2k}.$$

On the other hand, following the proof of this lemma word by word by replacing $\zeta(s)$ by $Z_Q(s)$ and $\Delta_k(x)$ by $\Delta_k^*(Q, x)$ respectively, we can prove

$$(4.2) \quad \beta_k^* + \ell/2 - 1 = \delta_{2k}^*.$$

Finally it is easy to see that

$$|\zeta(s - \ell/2 + 1)| \ll |Z_Q(s)| \ll |\zeta(s - \ell/2 + 1)|$$

for $\ell/2 - 1 \leq \sigma \leq \ell/2$. Thus

$$(4.3) \quad \delta_r^* = \ell/2 - 1 + \delta_r.$$

Now Theorem 2 follows from (4.1), (4.2) and (4.3) by noting that the Lindelöf hypothesis implies $\delta_r = 1/2 - 1/r$ for any $r \geq 2$.

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INSTITUT ELIE CARTAN NANCY, CNRS, UNIVERSITÉ HENRI POINCARÉ (NANCY 1), INRIA,
BOULEVARD DES AIGUILLETES, B.P. 239, 54506 VANDŒUVRE-LÈS-NANCY, FRANCE
E-mail address: wujie@iecn.u-nancy.fr

DEPARTMENT OF MATHEMATICS, CHINA UNIVERSITY OF MINING AND TECHNOLOGY, BEI-
JING 100083, CHINA
E-mail address: zhaiwg@hotmail.com