ON A DIVISOR PROBLEM RELATED TO THE EPSTEIN
ZETA-FUNCTION, III

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ABSTRACT. In this paper we study the mean square of the error term $\Delta_k^*(Q, x)$ in a divisor problem related to the Epstein zeta-function. An asymptotic formula has been obtained when $k = 2$.

1. Introduction

This is the third part of our series of papers on a divisor problem related to the Epstein zeta-function [10, 11]. First we recall some notation there. Let $\ell \geq 2$, $y := (y_1, \ldots, y_\ell)$ and $A = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \pmod{2}$ for $0 \leq i \leq \ell$. Thus a positive definite quadratic form $Q(y)$ can be written as

$$Q(y) = \frac{1}{2} y^t A y = \sum_{1 \leq i < j \leq \ell} a_{ij} y_i y_j + \frac{1}{2} \sum_{1 \leq i \leq \ell} a_{ii} y_i^2,$$

where $y^t$ is the transpose of $y$. The corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$(1.1) \quad Z_Q(s) := \sum_{y \in \mathbb{Z}^\ell \setminus \{0\}} Q(y)^{-s} = \sum_{n \geq 1} a_n n^{-s} \quad (\Re s > \ell/2),$$

where $a_n$ is the number of the solutions of the equation $Q(y) = n$ with $y \in \mathbb{Z}^\ell$. It is known that $Z_Q(s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ with only a simple pole at $s = \ell/2$, and satisfies a functional equation of Riemann type (cf. [13]). For each integer $k \geq 1$, we define $a_k(n)$ by

$$(1.2) \quad Z_Q(s)^k = \sum_{n \geq 1} a_k(n) n^{-s} \quad (\Re s > \ell/2)$$

and put

$$(1.3) \quad \Delta_k^*(Q, x) := \sum_{n \leq x} a_k(n) - x^{\ell/2} P_k(\log x),$$

where $P_k(\log x) := x^{-\ell/2} \Res_{s=\ell/2} (Z_Q(s)^k x^s s^{-1})$ is a polynomial of $\log x$ of degree $k - 1$. The study on asymptotic behavior of the error term $\Delta_k^*(Q, x)$ has received
much attention [8, 1, 13]. In particular Sankaranarayanan [13] showed that for \( k \geq 2 \) and \( \ell \geq 3 \),
\[
(1.4) \quad \Delta_\ell^*(Q, x) \ll x^{\ell/2-1/k+\varepsilon},
\]
where and throughout this paper \( \varepsilon \) denotes an arbitrarily small positive constant. Recently inspired by Iwaniec’s book [6], Lü [10] marked that (1.4) can been improved for the quadratic forms of level one (see [6, Chapter 11]). These quadratic forms are defined by \( Q(y) = \frac{1}{2} y^T A y \) verifying the following supplementary conditions:
\[
\ell \equiv 0 \pmod{8}, \quad A \text{ is equivalent to } A^{-1}, \quad \det(A) = 1.
\]
Denote by \( Q_\ell \) the set of such quadratic forms. For \( Q \in Q_\ell \), we have [6, (11.32)]
\[
a_n = A_\ell \sigma_{\ell/2-1}(n) + a_f(n, Q) \quad (n \geq 1),
\]
where
\[
A_\ell := \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2) \Gamma(\ell/2)}, \quad \sigma_k(n) = \sum_{d \mid n} d^k,
\]
\( \zeta(s) \) is the Riemann zeta-function, \( \Gamma(s) \) is the Gamma function and \( a_f(n, Q) \) is the nth Fourier coefficient of a cusp form \( f(z, Q) \) of weight \( \ell/2 \) with respect to the full modular group \( \text{SL}(2, \mathbb{Z}) \). Thus
\[
(1.5) \quad Z_Q(s) = A_\ell \zeta(s) \zeta(s-\ell/2+1) + L(s, f) \quad (\Re s > \ell/2),
\]
where \( L(s, f) \) is the Hecke \( L \)-function associated with \( f(z, Q) \). According to Deligne’s well known work [2], we know
\[
(1.6) \quad |a_f(n, Q)| \leq n^{(\ell/2-1)/2} \tau(n) \quad (n \geq 1),
\]
where \( \tau(n) \) is the divisor function. With the help of these properties, Lü [10] (for \( k \geq 4 \)) and Lü, Wu & Zhai [11] (for \( k = 2, 3 \)) obtained
\[
\Delta_\ell^*(Q, x) \ll x^{\ell/2-1+\theta_k+\varepsilon},
\]
where \( \theta_k \) is the exponent in the classical \( k \)-dimension divisor problem
\[
\Delta_k(x) := \sum_{n \leq x} \tau_k(n) - \text{Res}_{s=1}(\zeta(s)^k x^{s-1}) \ll x^{\theta_k+\varepsilon} \quad (x \geq 2).
\]
In particular we can take \( \theta_2 = 131/416 \) [4], \( \theta_3 = 43/96 \) [7] and \( \theta_k = (k-1)/(k+2) \) for \( k \geq 4 \) [15]. Besides, an \( \Omega \)-result has been established in [11]: if \( 8 \nmid \ell \) and \( Q(y) \in Q_\ell \), then we have for \( k = 2, 3 \) that
\[
\Delta_\ell^*(Q, x) = \Omega \left( x^{\ell/2-1+(k-1)/2k}(\log x)^{(k-1)/(2k)}(\log_2 x)^a(\log_3 x)^{-b'} \right),
\]
where \( a = \frac{k+1}{2k}(k/(k+1) - 1) \), \( b' \) is any constant greater than \( \frac{3k-1}{4k} \) and \( \log_r \) denotes the \( r \)-fold iterated logarithm.

The aim of this paper is to study the mean square of \( \Delta_\ell^*(Q, x) \).

**Theorem 1.** If \( 8 \nmid \ell \), then for any quadratic form \( Q(y) \in Q_\ell \), we have
\[
\int_1^T |\Delta_\ell^*(Q, x)|^2 \, dx = C_T T^{\ell-1/2} + O(T^{\ell-1}(\log T)^3 \log_2 T),
\]
where

\begin{equation}
 g_a(n) := \sum_{d|n} \tau(d) \tau(n/d) / d^a, \quad C_\ell := \frac{3A_\ell^4}{(2\ell - 1)\pi^2} \sum_{n=1}^\infty \frac{g(\ell-3)/2(n)^2}{n^{3/2}}.
\end{equation}

The estimate $O(T^{\ell-1}(\log T)^3 \log_2 T)$ follows from the result of [9] on the mean square of $\Delta_2(x)$.

**Theorem 2.** For $k \geq 2$, $8 \mid \ell$ and $Q(y) \in Q_\ell$, we define

\[ \beta_k := \inf \left\{ b_k : \int_1^T |\Delta_k(x)|^2 \, dx \ll T^{1+2b_k+\varepsilon} \right\}, \]

\[ \beta^*_k := \inf \left\{ b^*_k : \int_1^T |\Delta^*_k(Q, x)|^2 \, dx \ll T^{\ell-1+2b^*_k+\varepsilon} \right\}. \]

Then $\beta^*_k = \beta_k$. Further we have $\beta^*_k \geq (k-1)/2k$ and the equality holds if the Lindelöf hypothesis of $\zeta(s)$ is true.

Ivić [5, ] proved that $\beta_3 = 1/3$, $\beta_4 = 3/8$, $\beta_5 \leq 119/260$, $\beta_6 \leq 1/2$, $\beta_7 \leq 39/70$.

According to Theorem 2, the same estimates for $\beta^*_k$ hold.

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2. **An Expression of $\Delta^*_2(Q, x)$**

In [11], we actually established the formula

\[ \Delta^*_2(Q, x) = A_\ell^2 x^{\ell/2-1} \sum_{d \leq x} \frac{\tau(d)}{d^{\ell/2-1}} \Delta_2 \left( \frac{x}{d} \right) + O(x^{\ell/2-1+\varepsilon}). \]

From it we can deduce $\Omega$-result of $\Delta^*_2(Q, x)$. However, it is not enough to prove Theorem 1. So first we will give a better expression of $\Delta^*_2(Q, x)$.

**Lemma 2.1.** If $8 \mid \ell$, then for any quadratic form $Q(y) \in Q_\ell$, we have

\[ \Delta^*_2(Q, x) = A_\ell^2 x^{\ell/2-1} \sum_{d \leq x} \frac{\tau(d)}{d^{\ell/2-1}} \left( \Delta_2 \left( \frac{x}{d} \right) - \frac{1}{4} \right) \]

\[ - 2A_\ell x^{\ell/2-1} \sum_{d \leq x} \frac{b(d)}{d^{\ell/2-1}} \psi \left( \frac{x}{d} \right) + O(x^{\ell/2-5/4}), \]

where $\psi(t) := \{t\} - \frac{1}{2}$ and $\{t\}$ denotes the fractional part of $t$.

**Proof.** From (1.5) we have

\[ Z_Q(s)^2 = A_\ell^2 \zeta(s)^2 \zeta(s - \ell/2 + 1)^2 + A_\ell \zeta(s) \zeta(s - \ell/2 + 1) L(s, f) + L(s, f)^2. \]
Here the last two terms do not appear when \( \ell = 8, 16 \) since there are no cusp forms of weights 4 and 8 with respect to \( \text{SL}(2, \mathbb{Z}) \). Thus we can write

\[
\sum_{n \leq x} a_2(n) = A^2_\ell \sum_{d \leq x} \tau(d) \sum_{m \leq x/d} \tau(m)m^{\ell/2-1} \\
+ 2A_\ell \sum_{d \leq x} b(d) \sum_{m \leq x/d} m^{\ell/2-1} + \sum_{d \leq x} c(d),
\]

where \( b(n) \) and \( c(n) \) are defined by

\[
\zeta(s)L(s, f) = \sum_{n=1}^{\infty} b(n)n^{-s} \quad \text{and} \quad L(s, f)^2 = \sum_{n=1}^{\infty} c(n)n^{-s}
\]

for \( \Re s > \ell/2 \), respectively. By using Deligne’s bound (1.6), it is easy to see that

\[
|b(n)| \leq n^{(\ell/2-1)/2}\tau_3(n) \quad \text{and} \quad |c(n)| \leq n^{(\ell/2-1)/2}\tau_4(n).
\]

Thus

\[
\sum_{n \leq x} (|b(n)| + |c(n)|) \ll x^{\ell/4+1/2}(\log x)^3.
\]

By partial summation we have

\[
\sum_{m \leq x} \tau(m)m^{\ell/2-1} = \frac{2}{\ell}x^{\ell/2} \left( \log x - \frac{2}{\ell} + 2\gamma \right) + x^{\ell/2-1}\Delta_2(x) \\
- \left( \frac{\ell}{2} - 1 \right) \int_1^x \Delta_2(t)t^{\ell/2-2}dt.
\]

By using Voronoï’s well known formula [16]:

\[
\int_1^t \Delta_2(u)du = \frac{t}{4} + O(t^{3/4}),
\]

a simple partial summation leads to

\[
\left( \frac{\ell}{2} - 1 \right) \int_1^x \Delta_2(t)t^{\ell/2-2}dt = \frac{1}{4}x^{\ell/2-1} + O(x^{\ell/2-5/4}).
\]

Combining these, we find that

\[
\sum_{m \leq x} \tau(m)m^{\ell/2-1} = \frac{2}{\ell}x^{\ell/2} \left( \log x - \frac{2}{\ell} + 2\gamma \right) \\
+ x^{\ell/2-1}\left( \Delta_2(x) - \frac{1}{4} \right) + O(x^{\ell/2-5/4}).
\]

Similarly (even easier)

\[
\sum_{m \leq x} m^{\ell/2-1} = \frac{2}{\ell}x^{\ell/2} - x^{\ell/2-1}\psi(x) + O(x^{\ell/2-2}).
\]

Now the required result follows from (2.1), (2.3), (2.4) and (2.5). \( \square \)
3. Proof of Theorem 1

3.1. Beginning of the Proof. Let

$$\tilde{\Delta}_2^*(Q, x) := \frac{\Delta_2^*(Q, x)}{A_2^{d/2-1}}$$

and

$$\tilde{C}_t := \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{g(t-3/2)(n)^2}{n^{3/2}}.$$

Clearly it is sufficient to prove that

$$\int_1^T |\tilde{\Delta}_2^*(Q, x)|^2 \, dx = \frac{\tilde{C}_t}{6} T^{3/2} + O(T(\log T)^3 \log_2 T). \tag{3.1}$$

According to Lemma 2.1, we can write

$$\tilde{\Delta}_2^*(Q, x) = U(x) - V(x) + O(x^{\ell/2-5/4}),$$

where

$$U(x) := \sum_{d \leq x} \tau(d) \left( \Delta_2 \left( \frac{x}{d} \right) - \frac{1}{4} \right), \quad V(x) := \frac{2}{A_\ell} \sum_{d \leq x} \frac{b(d)}{d^{\ell/2-1}} \psi \left( \frac{x}{d} \right).$$

Next we shall prove

$$\int_1^T U^2(x) \, dx = \frac{\tilde{C}_t}{6} T^{3/2} + O(T(\log T)^3 \log_2 T), \tag{3.2}$$

$$\int_1^T U(x) V(x) \, dx \ll T(\log T)^2, \tag{3.3}$$

which imply (3.1).

3.2. Preparation. In this subsection, we shall prove some preliminary estimates, which are useful later.

Lemma 3.1. Let $a > 0$, $b > 1$, $\ell > a + b$ and $A \geq 1$. We have

$$\sum_{\substack{d_1, d_2 \leq T \\text{prime} \\text{to each other}}} \frac{\tau(d_1) \tau(d_2) \tau(m_1) \tau(m_2)}{(d_1 d_2)^{\ell/2-a/2}(m_1 m_2)^{b/2}} = \sum_{n=1}^{\infty} \frac{g(t-3/2)(n)^2}{n^b} + O_A \left( \frac{(\log T)^3}{T^{b-1}} \right), \tag{3.4}$$

$$\sum_{d_1, d_2 \leq T \\text{prime} \\text{to each other}} \frac{\tau(d_1) \tau(d_2) \tau(m_1) \tau(m_2)}{(d_1 d_2)^{\ell/2-3/4}(m_1 m_2)^{3/4}} \frac{1}{|\sqrt{m_1/d_1} - \sqrt{m_2/d_2}|} \ll_A (\log T)^3 \log_2 T, \tag{3.5}$$

$$\sum_{d_1, d_2 \leq T \\text{prime} \\text{to each other}} \frac{\tau(d_1) \tau(d_2) \tau(m_1) \tau(m_2)}{(d_1 d_2)^{\ell/2-3/4}(m_1 m_2)^{3/4}} \frac{1}{\sqrt{m_1/d_1} + \sqrt{m_2/d_2}} \ll_A (\log T)^3 \log_2 T. \tag{3.6}$$

uniformly for $1 \leq T \leq M \leq T^A$, where $g_\ell(n)$ is defined as in (1.7).
Proof. First we write
\[
S_1(T, M) = \sum_{n \leq TM} \frac{1}{n^b} \left( \sum_{d \leq T; m \leq M} \frac{\tau(d) \tau(m)}{d(t-a-b)/2} \right)^2
\]
(3.7)
\[
= \sum_{n=1}^{\infty} \frac{g(t-a-b)/2(n)}{n^b} + O\left( \sum_{n>T} \frac{g(t-a-b)/2(n)^2}{n^b} \right).
\]

It is easy to see that \(g_r(n)\) is multiplicative, \(g_r(p) = 2 + 2/p\) and \(g_r(p^\nu) \ll (\nu + 1)\) for all \(p\) and \(\nu \geq 1\). Applying Theorem 2.1 of [14] with \(x = y\) and \(\kappa = 4\) to \(g_r(n)^2\) leads to the following inequality
\[
\sum_{n \leq x} g_r(n)^2 \ll x(\log x)^3 \quad (r > 0, \ x \geq 2).
\]
From it and (3.7), we can easily deduce (3.4).

Similarly we can write
\[
S_2(T, M) \leq \sum_{n, n' \leq TM \atop n \neq n'} \frac{\tau_2(n) \tau_2(n')}{(nn')^{3/4}} \frac{1}{|\sqrt{n} - \sqrt{n'}|} \ll_A (\log T)^3 \log_2 T.
\]
In the last step we have used the bound of Lau & Tsang [9].

The estimate (3.6) is an immediate consequence of (3.4) with \(a = b = 2\) and (3.5) if noting that
\[
1 \geq \sqrt{m_1/d_1} + \sqrt{m_2/d_2} \ll \begin{cases} 
\left( \frac{d_1d_2}{m_1m_2} \right)^{1/4} & \text{if } m_1/d_1 = m_2/d_2 \\
\frac{1}{|\sqrt{m_1/d_1} - \sqrt{m_2/d_2}|} & \text{if } m_1/d_1 \neq m_2/d_2.
\end{cases}
\]

3.3. Proof of (3.2). According to Meurman [12], we have
\[
\Delta_2(x) - \frac{1}{4} = \frac{x^{1/4}}{\sqrt{2\pi}} \sum_{m \leq M} \frac{\tau(m)}{m^{3/4}} \cos \left( 4\pi \sqrt{mx} - \frac{\pi}{4} \right) + E(x)
\]
(3.8)
for all \(M > x > 1\), where
\[
E(x) \ll \begin{cases} 
x^{-1/4} & \text{if } \|x\| \geq x^{5/2}M^{-1/2}, \\
x^\varepsilon & \text{if } \|x\| \leq x^{5/2}M^{-1/2}.
\end{cases}
\]
(3.9)
Thus we can write, with the choice of \(M = T^{10} > x\),
\[
U(x) = A(x) + B(x),
\]
(3.10)
where
\[
A(x) := \frac{x^{1/4}}{\sqrt{2\pi}} \sum_{d \leq x} \frac{\tau(d)}{d^{1/2-3/4}} \sum_{m \leq M} \frac{d(m)}{m^{3/4}} \cos \left( 4\pi \sqrt{m/d}x - \frac{\pi}{4} \right),
\]
\[
B(x) := \sum_{d \leq x} \frac{\tau(d)}{d^{1/2-1}} E\left( \frac{x}{d} \right).
\]
In view of the identity $2 \cos u \cos v = \cos(u - v) + \cos(u + v)$, we easily see that

$$A(x)^2 = A_1(x) + A_2(x) + A_3(x),$$

where

$$A_1(x) := \frac{x^{1/2}}{4\pi^2} \sum_{d_1,d_2 \leq x \atop m_1,m_2 \leq M \atop m_1d_2 = m_2d_1} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{1/2 - 3/4}(m_1m_2)^{3/4}},$$

$$A_2(x) := \frac{x^{1/2}}{4\pi^2} \sum_{d_1,d_2 \leq x \atop m_1,m_2 \leq M \atop m_1d_2 \neq m_2d_1} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{1/2 - 3/4}(m_1m_2)^{3/4}} \cos \left( 4\pi \left( \sqrt{x} - \sqrt{\frac{m_1}{d_1}} + \sqrt{\frac{m_2}{d_2}} \right) \right),$$

$$A_3(x) := \frac{x^{1/2}}{4\pi^2} \sum_{d_1,d_2 \leq x \atop m_1,m_2 \leq M} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{1/2 - 3/4}(m_1m_2)^{3/4}} \cos \left( 4\pi \left( \sqrt{x} - \sqrt{\frac{m_1}{d_1}} + \sqrt{\frac{m_2}{d_2}} \right) \right).$$

By using (3.4) we have

$$\int_1^T A_1(x) \, dx = \frac{1}{4\pi^2} \sum_{d_1,d_2 \leq T \atop m_1,m_2 \leq M \atop m_1d_2 = m_2d_1} \frac{\tau(d_1)\tau(d_2)\tau(m_1)\tau(m_2)}{(d_1d_2)^{1/2 - 3/4}(m_1m_2)^{3/4}} \int_{\max\{d_1,d_2\}}^T x^{1/2} \, dx$$

$$= \frac{\tilde{C}_r}{6}T^{3/2} + O(T(\log T)^3).$$

With the help of the first derivative test and (3.5), we get

$$\int_1^T A_2(x) \, dx \ll \sum_{1 \leq k \leq 2\log T} \left| \int_{T/2^k}^{T/2^{k-1}} A_2(x) \, dx \right|$$

$$\ll \sum_{1 \leq k \leq 2\log T} (T/2^k)S_2(T/2^{k-1}, M)$$

$$\ll T(\log T)^3 \log_2 T.$$

Similarly we have

$$\int_1^T A_3(x) \, dx \ll T.$$

Combining these estimates, we find that

$$\int_1^T A(x)^2 \, dx = \frac{\tilde{C}_r}{6}T^{3/2} + O(T(\log T)^3 \log_2 T).$$

By Cauchy’s inequality, it follows

$$B(x)^2 \leq \sum_{d \leq x} \frac{\tau(d)^2}{d^2} \sum_{d \leq x} \frac{1}{d^{k-4}}E \left( \frac{x}{d} \right)^2 \ll \sum_{d \leq x} \frac{1}{d^{k-4}}E \left( \frac{x}{d} \right)^2,$$
which combining (3.9) allows us to deduce that

\[
\int_1^T B(x)^2 \, dx \ll \sum_{d \leq T} \frac{1}{d^{t-5}} \left( \int_1^{T/d} \frac{t^\epsilon}{\|t\|^{t^{5/2} M^{-1/2}}} \, dt + \int_1^{T/d} \frac{t^{-1/2}}{\|t\|^{t^{5/2} M^{-1/2}}} \, dt \right)
\]

(3.12)

\[
\ll \sum_{d \leq T} \frac{1}{d^{t-5}} \left\{ \left( \frac{T}{d} \right)^{7/2+\epsilon} + \frac{1}{M^{1/2}} \left( \frac{T}{d} \right)^{1/2} \right\}
\]

\[
\ll T^{1/2}.
\]

From (3.11) and (3.12), we get, via Cauchy’s inequality, that

(3.13)

\[
\int_1^T A(x)B(x) \, dx \ll T.
\]

Now the asymptotic formula (3.2) follows from (3.10), (3.11), (3.12) and (3.13).

3.4. **Proof of** (3.3). By using Theorem 4.5 in Graham and Kolesnik [3]

\[
\Delta_2(u) = -2 \sum_{m \leq \sqrt{u}} \psi(u/m) + O(1)
\]

and (2.2), we have

(3.14)

\[
\int_1^T U(x)V(x) \, dx \ll \sum_{d \leq T} \frac{\tau(d)}{d^{t^{3/2}-1}} \sum_{m \leq (T/d)^{1/2}} \sum_{n \leq T} \frac{\tau_3(n)}{n^{t^{3/4}-1/2}} |I(d, m, n)| + T,
\]

where

\[
I(d, m, n) := \int_1^T \psi \left( \frac{x}{dm} \right) \psi \left( \frac{x}{n} \right) \, dx.
\]

For \( \psi(u) \), it is well-known that the finite Fourier expansion

\[
\psi(u) = -\sum_{1 \leq h \leq H} \frac{\sin(2\pi hu)}{\pi h} + O\left( \min \left\{ 1, \frac{1}{H\|u\|} \right\} \right)
\]

holds for any \( H \geq 2 \). It is easily seen that for any \( r > 0 \)

\[
\int_{\max\{dm^2, n\}}^T \min \left\{ 1, \frac{1}{H\|x/r\|} \right\} \, dx = r \int_{m^2}^{T/r} \min \left\{ 1, \frac{1}{H\|t\|} \right\} \, dt
\]

\[
\ll T \int_0^{1/2} \min \left\{ 1, \frac{1}{Ht} \right\} \, dt
\]

\[
\ll T H^{-1} \log H.
\]

From these we deduce

(3.15)

\[
I(d, m, n) \ll \sum_{h_1, h_2 \leq H} \frac{|I(h_1, h_2)|}{h_1 h_2} + T (\log H)^2
\]

\[
\frac{H}{H}.
\]
where
\[ I(h_1, h_2) := \int_{\max\{dm^2, n\}}^{T} \sin \left( \frac{2\pi h_1 x}{d} \right) \sin \left( \frac{2\pi h_2 x}{n} \right) \, dx \]
\[ \ll \begin{cases} 1/|h_1/dm - h_2/n| & \text{if } h_1n \neq h_2dm, \\ T & \text{if } h_1n = h_2dm. \end{cases} \]

Here we have used the identity \(2 \sin u \sin v = \cos(u - v) - \cos(u + v)\) and the first derivative test when \(h_1n \neq h_2dm\).

Inserting (3.15) into (3.14), we get
\[ \int_{1}^{T} U(x)V(x) \, dx \ll TS_4(T, H) + S_5(T, H) + T + \frac{T^{3/2}(\log H)^2}{H}, \]
where
\[
S_4(T, H) := \sum_{d \leq T} \frac{\tau(d)}{d^{k/2-1}} \sum_{m \leq (T/d)^{1/2}} \sum_{n \leq T} \frac{\tau_3(n)}{n^{\ell/4-1/2}} \sum_{\substack{h_1, h_2 \leq H \\ h_1n \neq h_2dm}} \frac{1}{h_1h_2} \]
\[ \leq \sum_{r \leq HT} \frac{1}{r^2} \sum_{n|\tau} \frac{\tau_3(n)}{n^{\ell/4-3/2}} \sum_{h_2dm=r} \frac{\tau(d)m}{d^{q/2-2}} \ll \sum_{r \leq HT} \frac{1}{r} \ll \log(HT) \]
and
\[
S_5(T, H) := \sum_{d \leq T} \frac{\tau(d)}{d^{k/2-1}} \sum_{m \leq (T/d)^{1/2}} \sum_{n \leq T} \frac{\tau_3(n)}{n^{\ell/4-1/2}} \sum_{\substack{h_1, h_2 \leq H \\ h_1n \neq h_2dm}} \frac{dmn}{h_1h_2|h_1n - h_2dm|} \]
\[ = \sum_{\substack{r_1, r_2 \leq HT \\ r_1 \neq r_2}} \frac{1}{r_1r_2|r_1 - r_2|} \sum_{\substack{h_2 \leq H, d \leq T, m \leq (T/d)^{1/2} \\ h_2dm=r}} \frac{\tau(d)(dm)^2}{d^{q/2-1}} \sum_{n \leq T, h_1 \leq H} \frac{\tau_3(n)}{n^{\ell/4-5/2}} \]
\[ \ll T \sum_{\substack{r_1, r_2 \leq HT \\ r_1 \neq r_2}} \frac{1}{r_1r_2|r_1 - r_2|} \sum_{h_2dm=r_1} \frac{\tau(d)}{d^{q/2-2}} \sum_{h_1n=r_2} \frac{\tau_3(n)}{n^{\ell/4-5/2}} \]
\[ \ll T \sum_{|r| \leq HT} \frac{1}{|r|} \sum_{r_2 \leq HT} \frac{1}{r_2} \]
\[ \ll T(\log HT)^2. \]

This proves (3.3) with the choice of \(H = T\).

### 4. Proof of Theorem 2

For each \(r \geq 2\), let \(\delta_r\) and \(\delta_r^*\) denote the infimum of \(\sigma > 0\) such that
\[ \int_{-\infty}^{\infty} \frac{|\zeta(\sigma + it)|^r}{|\sigma + it|^2} \, dt \ll 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{|Z_Q(\sigma + it)|^r}{|\sigma + it|^r} \, dt \ll 1, \]
respectively. According to [5, Lemma 13.1], we have
\[
\beta_k = \delta_{2k}. \tag{4.1} \]
On the other hand, following the proof of this lemma word by word by replacing $\zeta(s)$ by $Z_Q(s)$ and $\Delta_k(x)$ by $\Delta_k^*(Q,x)$ respectively, we can prove

$$\beta_k^* + \ell/2 - 1 = \delta_{2^k}.$$  \hspace{1cm} (4.2)

Finally it is easy to see that

$$|\zeta(s - \ell/2 + 1)| \ll |Z_Q(s)| \ll |\zeta(s - \ell/2 + 1)|$$

for $\ell/2 - 1 \leq \sigma \leq \ell/2$. Thus

$$\delta_r^* = \ell/2 - 1 + \delta_r.$$  \hspace{1cm} (4.3)

Now Theorem 2 follows from (4.1), (4.2) and (4.3) by noting that the Lindelöf hypothesis implies $\delta_r = 1/2 - 1/r$ for any $r \geq 2$.

References


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