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Véronique Gayrard

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AGING IN METROPOLIS DYNAMICS OF THE REM: A PROOF

VÉRONIQUE GAYRARD

ABSTRACT. We consider Metropolis dynamics of the Random Energy Model (REM). We prove that the classical two-time correlation function that allows one to establish aging converges almost surely to the arcsine law distribution function, as predicted in the physics literature, in the optimal domain of the time-scale and temperature parameters where this result can be expected to hold. To do this we link the two-time correlation function to a certain continuous-time clock process which, after proper rescaling, is proven to converge to a stable subordinator almost surely in the random environment and in the fine J_1-topology of Skorohod. This fine topology then enables us to deduce from the arcsine law for stable subordinators the asymptotic behavior of the two-time correlation function that characterizes aging.

1. INTRODUCTION

While there is as yet no established theory for the description of glasses, a consensus exists that this amorphous state of matter is intrinsically dynamical in nature [18]. Measuring suitable two-time correlation functions indeed reveals that glassy dynamics are history dependent and dominated by ever slower transients: they are *aging*. The realization in the late 80's that *mean-field* spin glass dynamics could provide a mathematical formulation for this phenomenon sparked renewed interest in models, such as Derrida's REM and p-spin SK models [15], [16], whose statics had, until then, been the main focus of attention. Despite this, Bouchaud's phenomenological *trap models* first took the center stage as they succeeded in predicting the power-law decay of two-time correlation functions observed experimentally, even though they did so at the cost of an ad hoc construction and drastically simplifying assumptions [9].

It was not until 2003 that a trap model dynamics was shown to result for the microscopic Glauber dynamics of a (random) mean-field spin glass Hamiltonian, namely, the REM endowed with the so-called *Random Hopping* dynamics and observed on time-scales near equilibrium [3, 4, 5]. Quite remarkably, the predicted functional form of two-time correlation functions was recovered. Rapid progress followed over the ensuing decade, beginning with [6]. The optimal domain of temperature and time-scales were this prediction applies was obtained in Ref. [22] (almost surely in the random environment except for times scales near equilibrium where the results hold in probability only) and these results were partially extended to the p-spin SK models [2], [12].

The choice of the Random Hopping dynamics, however, clearly favored the emergence of trap models. Just as in trap model constructions, its trajectories are those of a simple random walk on the underlying graph, and thus, do not depend on the random Hamiltonian. This is in sharp contrast with *Metropolis* dynamics, a choice heralded in the physic's

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literature as *the* natural microscopic Glauber dynamics [26], whose trajectories are biased against increasing the energy. This dependence on the random Hamiltonian makes the analysis of the two-time correlation functions much harder. This problem was first tackled in [24] were a truncated REM is considered, and a natural two-time correlation function is proved to behave as in the Random Hopping dynamics, in the same, optimal range of time-scales and temperatures for which this result holds almost surely in the random environment. In the present paper, we free ourselves of the simplifying truncation assumption and prove that the same result holds true almost surely for the full REM. A partial result was obtained in the recent paper [14] where its is proved that a certain clock process – a key object in the aging mechanism – converges to a stable subordinator in the M_1-topology of Skorohod, in probability with respect to the random environment and in a limited domain of the time-scale and temperature parameters (see the discussion below Theorem (1.4) for details). As noted by the authors of ref. [14], this result and its method of proof did not allow them to deduce aging, namely, convergence of two-time correlation functions.

As explained in detail in the remainder of this introduction, our analysis of two-time correlation functions relies on a scheme that consists in expressing Metropolis dynamics of the REM as an *exploration process* time-changed by a *clock process*, and in studying these two (interrelated) processes. Let us briefly discuss a different approach, initiated in [8, 21] in the simpler context of trap models. In those references, a process defined as "the mean holding time at the currently visited vertex" and known today as the *age process* was introduced in the hope that this process alone would suffice to establish the aging behavior of any two-time correlation functions. However, even within this very simple framework additional results, including explicit knowledge of the clock process, remained necessary to analyse some classical two-time correlation functions (e.g., (1.8) below). A thorough discussion of the age process in the more complex setting of Metropolis dynamics of the REM can be found in the last section of [14]. In order to make sense, the age process must now be defined not as the mean holding time at the currently visited vertex, as in [8], but rather as the mean exit time of the currently visited metastable set containing that vertex, or as some asymptotically equivalent process. This is the idea underlying the generalization of the age process proposed in (8.5) of [14]. However, the authors could not prove that this process converges and mention the missing proof of statement (8.3) of [14] as being one of their main obstacles. Statement (8.3) of [14] is in essence equivalent to Proposition 3.8 of the present paper and, thus, is solved here. The only remaining ingredient needed to prove the desired convergence that we do not provide (nor does [14]) is the exponentiality of metastable exit times. Addressing this question, which can be done using, for example, the techniques of [10, 11], goes beyond the scope of this paper.

1.1. **Main result.** Let us now specify the model. Denote by $\mathcal{V}_n = \{-1,1\}^n$ the n-dimensional discrete cube and by $\mathcal{E}_n$ its edge set. The Hamiltonian (or energy) of the REM is a collection of independent Gaussian random variables, $(\mathcal{H}_n(x), x \in \mathcal{V}_n)$, satisfying

$$\mathbb{E}\mathcal{H}_n(x) = 0, \quad \mathbb{E}\mathcal{H}_n^2(x) = n. \tag{1.1}$$

The sequence $(\mathcal{H}_n(x), x \in \mathcal{V}_n)$, $n > 1$, is defined on a common probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. On $\mathcal{V}_n$, we consider the Markov jump process $(X_n(t), t > 0)$ with rates

$$\lambda_n(x,y) = \frac{1}{n} e^{-\beta[\mathcal{H}_n(y)-\mathcal{H}_n(x)]^+}, \quad \text{if } (x,y) \in \mathcal{E}_n, \tag{1.2}$$

and $\lambda_n(x,y)=0$ else, were $a+=\max\{a,0\}$. This defines the single spin-flip continuous time Metropolis dynamics of the REM at temperature $\beta^{-1}>0$. Note that the rates are reversible with respect to the measure that assigns to $x\in\mathcal{V}_n$ the mass

$$\tau_n(x)\equiv\exp\{-\beta\mathcal{H}_n(x)\}. \tag{1.3}$$

When studying aging the choice of the observation time-scale, c_n, is all-important. Given $0<\varepsilon<1$ and $0<\beta<\infty$, we let $c_n\equiv c_n(\beta,\varepsilon)$ be the two-parameter sequence defined by

$$2^{\varepsilon n}\mathbb{P}(\tau_n(x)\geq c_n)=1. \tag{1.4}$$

Gaussian tails estimates yield the explicit form

$$c_n=\exp\left\{n\beta\beta_c(\varepsilon)-(1/2\alpha(\varepsilon))\left(\log(\beta_c^2(\varepsilon)n/2)+\log 4\pi+o(1)\right)\right\} \tag{1.5}$$

where

$$\beta_c(\varepsilon)=\sqrt{\varepsilon 2\log 2}, \tag{1.6}$$

$$\alpha(\varepsilon)=\beta_c(\varepsilon)/\beta. \tag{1.7}$$

A classical choice of two-time correlation function is the probability $\mathcal{C}_n(t,s)$ to find the process in the same state at the two endpoints of the time interval $[c_nt,c_n(t+s)]$,

$$\mathcal{C}_n(t,s)\equiv\mathcal{P}_{\mu_n}\left(X_n(c_nt)=X_n(c_n(t+s))\right),\quad t,s>0. \tag{1.8}$$

Here $\mathcal{P}_{\mu_n}$ denotes the law of X_n conditional on $\mathcal{F}$ (i.e. for fixed realizations of the random Hamiltonian) when the initial distribution, μ_n, is the uniform measure on $\mathcal{V}_n$.

Theorem 1.1. *For all* $0<\varepsilon<1$ *and all* $\beta>\beta_c(\varepsilon)$*, for all* $t>0$ *and* $s>0$*,* $\mathbb{P}$*-almost surely,*

$$\lim_{n\to\infty}\mathcal{P}_{\mu_n}\left(X_n(c_nt)=X_n(c_n(t+s))\right)=\frac{\sin\alpha(\varepsilon)\pi}{\pi}\int_0^{t/(t+s)}u^{\alpha(\varepsilon)-1}(1-u)^{-\alpha(\varepsilon)}\,du. \tag{1.9}$$

Remark. We in fact prove the more general statement that (1.9) holds along any n-dependent sequences of the form $0<\varepsilon_n\leq 1-c'\beta\sqrt{n^{-1}\log n}+c''n^{-1}\log n$ where $0<c',c''<\infty$ are constants, that satisfy $\lim_{n\to\infty}\varepsilon_n=\varepsilon$, $0<\varepsilon\leq 1$. Relaxation to stationarity is known to occur, to leading order, on time-scales c_n of the form (1.5) with $\varepsilon_n=1$ [20]. At the other extremity, a behavior known as extremal aging is expected to characterize the process on times scales that are sub-exponential in the volume and defined through sequences ε_n that decay to 0 slowly enough [13], [7]. This will be the object of a follow up paper.

As in virtually all papers on aging, the proof of Theorem 1.1 relies on a scheme that seeks to isolate the causes of aging by writing the process of interest, X_n, as an *exploration process* time-changed by (the inverse of) a *clock process*. Aging is then linked to the arcsine law for stable subordinators through the convergence of the suitably rescaled clock process to an α-stable subordinator, $0<\alpha<1$. This, provided that the two-time correlation function at hand can be brought into a suitable function of the clock.

While this scheme offers the methodological underpinnings of the analysis of aging, two distinct ways of implementing it, through *discrete* or *continuous* time objects, respectively, have emerged from the literature (we refer to [24], [25], and [14] for in-depth bibliographies). The first arose from the study of models whose exploration process can be chosen as the simple random walk on the underlying graph. As mentioned earlier, this includes all Random Hopping dynamics and several trap models (e.g. on the complete graph or on $\mathbb{Z}^d$). In physically more realistic dynamics the discrete scheme may quickly become intractable. As shown in Ref. [24] for Metropolis dynamics of a truncated REM, the associated exploration process is itself an aging process that presents the same complexity as the original dynamics. A similar situation arises when considering asymmetric trap models on $\mathbb{Z}^d$. Initiated in that context, the continuous time scheme consists in choosing a (now continuous time) exploration process that mimics the simple random walk.

Since the prescription of the exploration process completely determines the clock process, it is essential to have effective tools to prove that clock processes converge to stable subordinators. Such tools were provided in Ref. [23] and [12] for discrete-time clock processes in the general setting of reversible Markov jumps processes in random environment on sequences of finite graphs and, more recently, for both discrete and continuous-time clock processes of similar Markov jumps processes on infinite graphs [25]. These tools have allowed to both improve all earlier results on the Random Hopping dynamics of mean-field models [22], [12], [13], turning statements previously obtained in law into almost sure statements in the random environment, and to obtain the first aging results for several two-time correlation functions of asymmetric trap model on $\mathbb{Z}^d$ [25].

In Section 1.2 below we fill the gap left by continuous-time clock processes in the case of sequences of finite graphs and, thus, extent the results of Ref. [12] to that setting. This is perhaps no more than an exercise but these results (Theorem 1.2 and Theorem 1.3) are the cornerstone of our approach and, hopefully, of other papers to come. We close this introduction in Section 1.3 by stating a clock process convergence result for Metropolis dynamics of the REM (Theorem 1.4) that is at the heart of the proof of Theorem 1.1.

1.2. **Convergence of continuous-time clock processes.** We now enlarge our focus to the following abstract setting. Let $G_n(\mathcal{V}_n, \mathcal{E}_n)$ be a sequence of loop-free graphs with set of vertices $\mathcal{V}_n$ and set of edges $\mathcal{E}_n$. A *random environment* is a family of possibly dependent positive random variables, $(\tau_n(x), x \in \mathcal{V}_n)$. The sequence $(\tau_n(x), x \in \mathcal{V}_n)$, $n > 1$, is defined on a common probability space denoted by $(\Omega, \mathcal{F}, \mathbb{P})$. On $\mathcal{V}_n$ we consider a Markov jump process, $(X_n(t), t > 0)$, with initial distribution μ_n and jump rates $(\lambda_n(x,y))_{x,y\in\mathcal{V}_n}$ satisfying $\lambda_n(y,x) = 0$ if $(x,y) \notin \mathcal{E}_n$ and

$$\tau_n(x)\lambda_n(x,y) = \tau_n(y)\lambda_n(y,x) \quad \text{if } (x,y) \in \mathcal{E}_n,\ x \neq y. \tag{1.10}$$

Thus X_n is reversible with respect to the (random measure) that assigns to $x \in \mathcal{V}_n$ the mass $\tau_n(x)$. To X_n we associate an *exploration process* Y_n. This is any Markov jump process, $(Y_n(t), t > 0)$, with state space $\mathcal{V}_n$, initial distribution μ_n, and jump rates $(\widetilde{\lambda}_n(x,y))_{x,y\in\mathcal{V}_n}$ chosen such that X_n and Y_n have the same trajectories, that is to say,

$$\frac{\lambda_n(x,y)}{\lambda_n(x)} = \frac{\widetilde{\lambda}_n(x,y)}{\widetilde{\lambda}_n(x)} \quad \forall (x,y) \in \mathcal{E}_n, \tag{1.11}$$

where $\widetilde{\lambda}_n^{-1}(x)$ and $\lambda_n^{-1}(x)$ are, respectively, the mean holding times at x of Y_n and X_n:

$$\widetilde{\lambda}_n(x) \equiv \sum_{y:(x,y)\in\mathcal{E}_n} \widetilde{\lambda}_n(x,y), \tag{1.12}$$

$$\lambda_n(x) \equiv \sum_{y:(x,y)\in\mathcal{E}_n} \lambda_n(x,y). \tag{1.13}$$

Then X_n and Y_n are related to each other through the time change

$$X_n(t) = Y_n(\widetilde{S}_n^{\leftarrow}(t)), \qquad t \geq 0, \tag{1.14}$$

where $\widetilde{S}_n^{\leftarrow}$ denotes the generalized right continuous inverse of $\widetilde{S}_n$, and $\widetilde{S}_n$, the so-called *continuous-time clock process*, is given by

$$\widetilde{S}_n(t) = \int_0^t \lambda_n^{-1}(Y_n(s))\widetilde{\lambda}_n(Y_n(s))ds, \qquad t \geq 0. \tag{1.15}$$

Note that there is considerable freedom in the choice of the exploration process Y_n. We come back to this issue at the end of this subsection and focus, for the time being, on the analysis of the asymptotic behavior of the general clock process (1.15).

For future reference, we denote by $\mathcal{F}^Y$ the σ-algebra generated by the processes Y_n. We write P for the law of the process Y_n conditional on the σ-algebra $\mathcal{F}$, i.e. for fixed realizations of the random environment. Likewise we call $\mathcal{P}$ the law of X_n conditional on $\mathcal{F}$. If the initial distribution, μ_n, has to be specified we write $\mathcal{P}_{\mu_n}$ and P_{μ_n}. Expectation with respect to $\mathbb{P}$, P_{μ_n}, and $\mathcal{P}_{\mu_n}$ are denoted by $\mathbb{E}$, E_{μ_n}, and $\mathcal{E}_{\mu_n}$, respectively.

Our main aim is to obtain simple and robust criteria for the convergence of the (suitably rescaled) clock process (1.15) to a stable subordinator. More precisely, we will ask whether there exist sequences a_n and c_n that make the rescaled clock process

$$S_n(t) = c_n^{-1}\widetilde{S}_n(a_n t)\,, \quad t \geq 0, \tag{1.16}$$

converge weakly, as $n \uparrow \infty$, as a sequence of random elements in Skorokhod's space $D((0,\infty])$, and strive to obtain $\mathbb{P}$-almost sure results in the random environment since such results (also referred to as *quenched*) contain the most useful information from the point of view of physics.

As for discrete-time clock processes [23], [12], the driving force behind our approach is a powerful method developed by Durrett and Resnick [19] to prove functional limit theorems for sums of dependent variables. Clearly this method does not cover the case of our continuous-time clock processes. The simple idea (already present in [25]) is to introduce a suitable "blocking" that turns the rescaled clock process (1.16) into a partial sum process to which Durrett and Resnick method can now be applied. For this we introduce a new scale, θ_n, and set

$$k_n(t) \equiv \lfloor a_n t/\theta_n \rfloor. \tag{1.17}$$

The *blocked clock process*, $S_n^b(t)$, is defined through

$$S_n^b(t) = \sum_{i=1}^{k_n(t)} Z_{n,i} \tag{1.18}$$

where, for each $i \geq 1$,

$$Z_{n,i} \equiv c_n^{-1} \sum_{x\in\mathcal{V}_n} \left(\lambda_n^{-1}(x)\widetilde{\lambda}_n(x)\right)[\ell_n^x(\theta_n i) - \ell_n^x(\theta_n(i-1))], \tag{1.19}$$

and where, for each $x \in \mathcal{V}_n$,

$$\ell_n^x(t) = \int_0^t \mathbb{1}_{\{Y_n(s)=x\}}ds \tag{1.20}$$

is the local time at x. The next theorem gives sufficient conditions for S_n^b to converge. These conditions are expressed in terms of a small number of objects. For each $t > 0$, let

$$\pi_n^{Y,t}(y) = k_n^{-1}(t) \sum_{i=1}^{k_n(t)-1} \mathbb{1}_{\{Y_n(i\theta)=y\}} \tag{1.21}$$

be the empirical measure on $\mathcal{V}_n$ constructed from the sequence $(Y_n(i\theta), i \in \mathbb{N})$. For $y \in \mathcal{V}_n$ and $u > 0$, denote by

$$Q_n^u(y) \equiv P_y(Z_{n,1} > u) \tag{1.22}$$

the tail distribution of the aggregated jumps when X_n (equivalently, Y_n) starts in y. Using these quantities, define the functions

$$\nu_n^{Y,t}(u,\infty) \equiv k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n^{Y,t}(y) Q_n^u(y), \tag{1.23}$$

$$\sigma_n^{Y,t}(u,\infty) \equiv k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n^{Y,t}(y) \left[Q_n^u(y)\right]^2. \tag{1.24}$$

Observe that the sequence of measures $\pi_n^{Y,t}$ as well as the sequence of functions $Q_n^u(y)$, $y \in \mathcal{V}_n$, are random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the random environment. Thus, the functions $\nu_n^{Y,t}$ and $\sigma_n^{Y,t}$ also are random variables on that space.

We now formulate four conditions for the sequence S_n^b to converge to a subordinator. These conditions refer to a given sequence of initial distributions μ_n, given sequences of numbers a_n, c_n, and θ_n as well as a given realization of the random environment.

Condition (A0). For all $u > 0$,

$$\lim_{n\to\infty} P_{\mu_n}(Z_{n,1} > u) = 0. \tag{1.25}$$

Condition (A1). There exists a σ-finite measure ν on $(0,\infty)$ satisfying $\int_0^\infty (x \wedge 1)\nu(dx) < \infty$ and such that for all continuity points x of the distribution function of ν, for all $t > 0$ and all $u > 0$,

$$P_{\mu_n}\left(\left|\nu_n^{Y,t}(u,\infty) - t\nu(u,\infty)\right| < \epsilon\right) = 1 - o(1), \quad \forall \epsilon > 0. \tag{1.26}$$

Condition (A2). For all $u > 0$ and all $t > 0$,

$$P_{\mu_n}\left(\sigma_n^{Y,t}(u,\infty) < \epsilon\right) = 1 - o(1), \quad \forall \epsilon > 0. \tag{1.27}$$

Condition (A3). For all $t > 0$,

$$\lim_{\epsilon\downarrow 0} \limsup_{n\uparrow\infty} k_n(t) \sum_{y \in \mathcal{V}_n} E_{\mu_n}(\pi_n^{Y,t}(y)) E_y(Z_{n,1} \mathbb{1}_{\{Z_{n,1} \le \epsilon\}}) = 0. \tag{1.28}$$

Theorem 1.2. *For all sequences of initial distributions μ_n and all sequences a_n, c_n, and $1 \le \theta_n \ll a_n$ for which Conditions (A0), (A1), (A2), and (A3) are verified, either $\mathbb{P}$-almost surely or in $\mathbb{P}$-probability, the following holds w.r.t. the same convergence mode:*

$$S_n^b \Rightarrow_{J_1} S_\nu, \tag{1.29}$$

where S_ν is the Lévy subordinator with Lévy measure ν and zero drift. Convergence holds weakly on the space $D([0,\infty))$ equipped with the Skorokhod J_1-topology.

Remark. Note that the theorem is stated for the *blocked* process S_n^b rather than the original process S_n of (1.16). This may falsely appear as an undesirable consequence of our techniques. We stress that for applications to correlation functions, one needs statements that are valid in the strong J_1 topology whereas forming blocks is needed in order to make sense of writing J_1 convergence statements in the setting of continuous-time clocks.

As for the discrete-time clocks of Ref. [12], our next step consists in reducing Conditions (A1) and (A2) of Theorem 1.2 to (i) a *mixing condition* for the chain Y_n, and (ii) a *law of large numbers* for the random variables Q_n. Again we formulate three conditions for a given sequence of initial distributions μ_n, given sequences a_n, c_n, and θ_n, and a given realization of the random environment.

Condition (B0). Denote by π_n the invariant measure of Y_n. There exists a sequence $\kappa_n \in \mathbb{N}$ and a positive decreasing sequence ρ_n, satisfying $\rho_n \downarrow 0$ as $n \uparrow \infty$, such that, for all pairs $x, y \in \mathcal{V}_n$, and all $t \geq 0$,

$$|P_x\left(Y_n(t+\kappa_n) = y\right) - \pi_n(y)| \leq \rho_n \pi_n(y). \tag{1.30}$$

Condition (B1). There exists a measure ν as in Condition (A1) such that, for all $t > 0$ and all $u > 0$,

$$\nu_n^t(u, \infty) \equiv k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n(y) Q_n^u(y) \to t\nu(u, \infty), \tag{1.31}$$

Condition (B2). For all $t > 0$ and all $u > 0$,

$$\sigma_n^t(u, \infty) \equiv k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n(y) \left[Q_n^u(y)\right]^2 \to 0. \tag{1.32}$$

Condition (B3). For all $t > 0$,

$$\lim_{\epsilon \downarrow 0} \limsup_{n \uparrow \infty} k_n(t) \sum_{y \in \mathcal{V}_n} \pi_n(y) E_y(Z_{n,1} \mathbb{1}_{\{Z_{n,1} \leq \epsilon\}}) = 0. \tag{1.33}$$

Theorem 1.3. *Assume that for all sequences of initial distributions μ_n and all sequences a_n, c_n, κ_n, and $\kappa_n \leq \theta_n \ll a_n$, Conditions (A0), (B0), (B1), (B2), and (B3) hold $\mathbb{P}$-almost surely, respectively in $\mathbb{P}$-probability. Then, as in (1.29), $S_n^b \Rightarrow_{J_1} S_\nu$, $\mathbb{P}$-almost surely, respectively in $\mathbb{P}$-probability.*

Theorem 1.3 is our key tool for proving convergence of blocked clock processes to subordinators. It is of course essential for the success of our strategy that the convergence criteria we obtained be tractable. Going back to (1.11) we thus now ask, in this light, how best to choose the exploration process Y_n.

A tentative answer to this question is to mimic the exploration process of the Random Hopping dynamics, which means choose Y_n such that its invariant measure, π_n, is "close" to the uniform measure and its mixing time, κ_n, is short compared to that of the process X_n. The following class of jump rates, inspired from an ingenious choice made in Ref. [14], is intended to favor the emergence of these properties. Given a fresh sequence $\eta_n \geq 0$, set

$$\widetilde{\lambda}_n(x, y) = \max(\eta_n, \tau_n(x))\lambda_n(x, y). \tag{1.34}$$

One easily checks that (1.11) is verified, that Y_n is reversible with respect to the measure

$$\pi_n(x) = \frac{\min\bigl(\eta_n, \tau_n(x)\bigr)}{\sum_{x \in \mathcal{V}_n} \min\bigl(\eta_n, \tau_n(x)\bigr)} \mathbb{1}_{\{\eta_n > 0\}} + |\mathcal{V}_n|^{-1} \mathbb{1}_{\{\eta_n = 0\}}, \quad x \in \mathcal{V}_n, \tag{1.35}$$

and that the clock (1.15) becomes

$$\widetilde{S}_n(t) = \int_0^t \max\bigl(\eta_n, \tau_n(Y_n(s))\bigr) ds. \tag{1.36}$$

We will see in Section 3.1 that in Metropolis dynamics of REM the parameter η_n has a capping effect on the mixing time of the exploration process, namely, κ_n in (1.30) can be made as small as needed by taking η_n large enough, while on the other hand π_n can be kept as close as desired to the uniform measure (obtained when choosing $\eta_n = 0$ in (1.35)) by keeping η_n small enough. This still gives us plenty of freedom to choose η_n.

Let us finally stress that the sole convergence statement (1.29) does not suffice to deduce aging, namely, the specific power law decay of the two-time correlation function of (1.9). One still has to show that the correlation function can be reduced, asymptotically, to the arcsine law for stable subordinators, and this typically requires extra information on the behavior of the exploration process within the blocks of S_n^b and in between given blocks.

1.3. **Application to Metropolis dynamics of the REM.** From that point onwards we focus on Metropolis dynamics of the REM (see (1.1)-(1.2)) started in the uniform measure on $\mathcal{V}_n$. Applying the abstract results of Section 1.2 enables us to prove $\mathbb{P}$-almost sure convergence of the blocked clock process $S_n^b(t)$, defined in (1.18), when the continuous-time clock process $\widetilde{S}_n(t)$, given by (1.15), is chosen as in (1.36).

To sate this result we must specify several quantities: the parameter η_n, the time-scales, a_n and c_n, and the block length, θ_n, entering the definitions of $\widetilde{S}_n(t)$ and $S_n^b(t)$. We begin by defining a sequence, $r_n^\star$, that is ubiquitous throughout the rest of the paper: given $\beta > 0$ and a constant $c_\star > 1 + \log 4$, we let $r_n^\star \equiv r_n(\beta, c_\star)$ be the solution of

$$n^{c_\star}\mathbb{P}(\tau_n(x) \geq r_n^\star) = 1. \tag{1.37}$$

In explicit form

$$r_n^\star = \exp\left\{\beta\sqrt{2c_\star n \log n}\left(1 - \tfrac{\log\log n}{8c_\star \log n}(1 + o(1))\right)\right\}. \tag{1.38}$$

We now take $\eta_n \equiv (r_n^\star)^{-1}$ in (1.34) which, combined with (1.2), yields

$$\widetilde{\lambda}_n(x, y) = \frac{1}{nr_n^\star}\frac{\min(\tau_n(y), \tau_n(x))}{\min\left(\frac{1}{r_n^\star}, \tau_n(x)\right)}, \quad \text{if } (x, y) \in \mathcal{E}_n, \tag{1.39}$$

and $\widetilde{\lambda}_n(x, y) = 0$ else. The observation time-scale, c_n, is chosen as in (1.4). It is naturally the same as in the Random Hopping dynamics. On the contrary, the definition of the auxiliary time-scale, a_n, contrasts sharply with the simple choice $a_n = 2^{\varepsilon n}$ made in the Random Hopping dynamics. We here must take

$$a_n = 2^{\varepsilon n}/b_n \tag{1.40}$$

where the sequence b_n is defined as follows. Recalling (1.6) and (1.7), define

$$F_{\beta,\varepsilon,n}(x) \equiv x^{\alpha_n(\varepsilon) - \frac{\log x}{2n\beta^2}}\left(1 - \tfrac{\log x}{n\beta\beta_c(\varepsilon)}\right)^{-1}, \quad x > 0, \tag{1.41}$$

where $\alpha_n(\varepsilon) \equiv (n\beta^2)^{-1}\log c_n$, that is, in view of (1.5), $\alpha_n(\varepsilon) = \alpha(\varepsilon)(1 - o(1))$. Further introduce the random set

$$T_n \equiv \left\{x \in \mathcal{V}_n \mid \tau_n(x) \geq c_n(n^2\theta_n)^{-1}\right\}. \tag{1.42}$$

Then, for ℓ_n^x as in (1.20), we set

$$b_n \equiv (\theta_n\pi_n(T_n))^{-1}\sum_{x\in T_n} E_{\pi_n}\left[F_{\beta,\varepsilon,n,}(\ell_n^x(\theta_n))\right]. \tag{1.43}$$

This somewhat daunting definition is discussed below. One of the strengths of the method, however, is that it does not require a deep understanding of b_n whose fine properties ultimately do not matter.

It now only remains to choose the block length θ_n. (The notation $x_n \ll y_n$ means that the sequences $x_n > 0$ and $y_n > 0$ satisfy $x_n/y_n \to 0$ as $n \to \infty$.)

Theorem 1.4. *Given* $0 < \varepsilon < 1$ *let* θ_n *be any sequence such that*

$$\tfrac{4}{1-\alpha(\varepsilon)} \log r_n^\star < \log \theta_n \ll n \tag{1.44}$$

and let c_n *and* a_n *be as in (1.4) and (1.40)-(1.43), respectively. Then, for all* $0 < \varepsilon < 1$ *and all* $\beta > \beta_c(\varepsilon)$*,* $\mathbb{P}$*-almost surely,*

$$S_n^b \Rightarrow_{J_1} V_{\alpha(\varepsilon)} \tag{1.45}$$

where $V_{\alpha(\varepsilon)}$ *is a stable subordinator with zero drift and Lévy measure* ν *defined through*

$$\nu(u, \infty) = u^{-\alpha(\varepsilon)}, \quad u > 0, \tag{1.46}$$

and where $\Rightarrow_{J_1}$ *denotes weak convergence in the space* $D([0,\infty))$ *of càdlàg functions equipped with the Skorokhod* J_1*-topology.*

We again emphasize (see the remark below Theorem (1.2)) that the J_1 convergence statement of Theorem 1.4 is a necessary ingredient of the proof of the convergence of the correlation function. Of course, Theorem 1.4 immediately implies that the original (non blocked) clock process (1.16) converges to the same limit in the M_1 topology of Skorokhod, but this strictly weaker result does not allow to retrieve information on the correlation function. Such a result was proved in Ref. [14] (for the clock obtained by taking $\eta_n = 1$ in (1.36)) albeit only in $\mathbb{P}$-probability and in the restricted domain of parameters $\beta > \beta_c(\varepsilon)$ and $1/2 < \varepsilon < 1$. When $1/2 < \varepsilon < 1$ the graph structure of the set T_n reduces, as shown in [24] (see lemma 2.1), to a collection a collection of *isolated* vertices, namely, no element of T_n has a neighbor in T_n. This feature of the REM's random landscape leads to drastic simplifications. In particular, it has the remarkable implication that given T_n, the law of the exploration process Y_n becomes independent of the random environment in T_n, as can easily be seen from (1.39).

Let us now examine the sequence b_n introduced in (1.40) and defined in (1.43). We do not have much intuition to offer for this complicated definition except that it emerges in a straightforward way from the verification of Condition (B1) of Theorem 1.3. One sees that b_n is a priori random in the random environment and depends on a sequence, θ_n, that can itself be chosen within the two widely different bounds of (1.44). The next proposition provides deterministic upper and lower bounds on b_n that are not affected by the choice of θ_n and are valid $\mathbb{P}$-almost surely.

Proposition 1.5. *Given* $0 < \varepsilon < 1$*, let* c_n *and* θ_n *be as in Theorem 1.4. Then, there exists a subset* $\Omega' \subseteq \Omega$ *with* $\mathbb{P}(\Omega') = 1$ *such that on* Ω'*, for all but a finite number of indices* n

$$\left(n^{c_-}(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\right)^{-1} \le b_n \le n^{c_+}(r_n^\star)^{1+\alpha_n(\varepsilon)} \tag{1.47}$$

where $0 < c_-, c_+ \le \infty$ *are numerical constants. Thus* $\lim_{n\to\infty} n^{-1}\log a_n = \varepsilon$ $\mathbb{P}$*-a.s..*

Remark. The form of (1.43) naturally prompts us to ask whether b_n converges as n diverges and, if so, whether the limit remains random or not. We have not been able to answer these questions. Indeed, the randomness of b_n enters mainly through the local times which depend on the fine details of the random environment locally, in some vicinity of the set T_n, and are delicate to control. However, as already mentioned, a strong side of the method is that no knowledge of the fine asymptotic properties of b_n is needed. Deterministic bounds suffice.

Remark. The precision of Theorem 1.1 does depend on the precision of the bounds on b_n through the domain of validity of the parameters ε and β (bad bounds would have affected this domain) but not the nature of the aging result itself: b_n is an auxiliary time-scale whose properties have no impact on aging.

Remark. The definition (1.40)-(1.43) of a_n and that of the sequence R_N in (2.10) of Ref. [14] bear a distinct resemblance. Our control of a_n through Proposition 1.5, which is sharp up to error terms of order $e^{\pm cst\sqrt{n\log n}}$,must be compared to the bounds on R_N of Lemma 4.4 of [14] that differ by multiplicative error terms of order $e^{\pm\epsilon n}$, $\epsilon > 0$.

Remark. One may wonder whether the lower bound of (1.44) can be improved. The main obstacle to doing so is the lower bound on mean hitting times of Lemma 3.4. In particular, trying to improve the bound (3.3) on the spectral gap by choosing η_n larger, say as large as 1 as in Ref. [14], can at best improve the constant $\frac{4}{1-\alpha(\varepsilon)}$ in front of $\log r^\star_n$ in (1.44).

The rest of the paper is organized as follows. Section 2 is concerned with the properties of the REM's landscape: several level sets that play an important role in our analysis are introduced and their properties collected. Section 3 gathers all needed results on the exploration process Y_n. The proof of Theorem 1.4 can then begin. Section 4, 5, and 6 are devoted, respectively, to the verification of Condition (B1), (B2), and (B3) of Theorem 1.3. The proof of Proposition 1.5 is given at the end of Subsection 4.2. The proof of Theorem 1.4 is completed in Section 7. Also in Section 7, the link between the blocked clock process of (1.45) and the two-time correlation function (1.8) is made, and the proof of Theorem 1.1 is concluded. An appendix (Section 8) contains the proof of the results of Section 1.2.

2. Level sets of the REM's landscape: the Top and other sets

Given $V \subseteq \mathcal{V}_n$ we denote by $G \equiv G(V)$ the undirected graph which has vertex set V and edge set $E(G(V)) \subseteq \mathcal{E}_n$ consisting of pairs of vertices $\{x, y\}$ in V with $\mathrm{dist}(x, y) = 1$, where $\mathrm{dist}(x, x') \equiv \frac{1}{2}\sum_{i=1}^n |x_i - x'_i|$ is the graph distance on $\mathcal{V}_n$. When $\mathrm{dist}(x, y) = 1$ we simply write $x \sim y$. We now introduce several sets that play key roles in our analysis: they are level sets of the form

$$V_n(\rho) = \{x \in \mathcal{V}_n \mid \tau_n(x) \geq r_n(\rho)\} \tag{2.1}$$

where, for different values of $\rho > 0$, the threshold level $r_n(\rho)$ is the sequence defined by

$$2^{\rho n}\mathbb{P}(\tau_n(x) \geq r_n(\rho)) = 1. \tag{2.2}$$

• ***The sets $V^\star_n$ and $\overline{V}^\star_n$ (of local valleys and hills).*** Set set $V^\star_n \equiv V_n(\rho^\star_n)$ where

$$\rho^\star_n \equiv \frac{c_\star \log n}{n \log 2} \tag{2.3}$$

for $c_\star$ as in (1.37). $V^\star_n$ can uniquely be decomposed into a collection of subsets

$$V^\star_n = \cup_{l=1}^{L^\star} C^\star_{n,l}, \quad C^\star_{n,l} \cap C^\star_{n,k} \ \forall l \neq k, \quad L^\star \equiv L_n(\rho^\star_n), \tag{2.4}$$

such that each graph $G(C^\star_{n,l})$ is connected but any two distinct graphs $G(C^\star_{n,l})$ and $G(C^\star_{n,k})$ are disconnected. With a little abuse of terminology we call the sets $C^\star_{n,l}$ the connected components of the graph $G(V^\star_n)$. From now on we write $r^\star_n \equiv r_n(\rho^\star_n)$. Let

$$\overline{V}^\star_n \equiv \overline{V}_n(\rho^\star_n) = \left\{x \in \mathcal{V}_n \mid \tau_n^{-1}(x) \geq r^\star_n\right\} \tag{2.5}$$

be the set obtained from $V_n(\rho_n^\star)$ by substituting $-\mathcal{H}_n(x)$ for $\mathcal{H}_n(x)$ in (1.3). Since $\mathcal{H}_n(x)$ is symmetrical $\overline{V}_n^\star$ has the same random graph properties as $V_n^\star$. Note that the form of the rates (1.39) depend on the set $\overline{V}_n^\star$, namely,

$$\widetilde{\lambda}_n(x,y) = \begin{cases} \frac{1}{n} e^{-\beta \max(\mathcal{H}_n(y),\mathcal{H}_n(x))}, & \text{if } x \notin \overline{V}_n^\star, \\ \frac{1}{n r_n^\star} e^{-\beta[\mathcal{H}_n(y)-\mathcal{H}_n(x)]^+}, & \text{if } x \in \overline{V}_n^\star. \end{cases} \tag{2.6}$$

Key properties of these rates are gathered at the end on this section.

As shown later in Lemma 2.1, the sets $V_n^\star$ and $\overline{V}_n^\star$ contain only “small” connected components and their complement, $\mathcal{V}_n \setminus (V_n^\star \cup \overline{V}_n^\star)$, forms a totally connected “giant” component (see [20]). We may thus think of the connected components of $V_n^\star$ and $\overline{V}_n^\star$ as containing, respectively, the local ”valleys” and ”hills” of the random energy landscape, $\mathcal{H}_n$, whereas in their complement, or “horizon level”, $\mathcal{H}_n$ has only small fluctuations.

• ***Immersions in $V_n^\star$.*** Given any subset $A \subset V_n^\star$ we call *immersion of A in $V_n^\star$* and denote by $A^\star$ the set

$$A^\star \equiv \cup_{l=1}^{L^\star} A_{n,l}^\star, \quad A_{n,l}^\star = \begin{cases} C_{n,l}^\star, & \text{if } C_{n,l}^\star \cap A \neq \emptyset, \\ \emptyset, & \text{else.} \end{cases} \tag{2.7}$$

Thus the sets $A_{n,l}^\star$ are the valleys $C_{n,l}^\star$ that contain at least one element of A. Clearly, $\overline{V}_n^\star \cap V_n^\star = \emptyset$. Hence by (2.6), immersed sets have the property that

$$\widetilde{\lambda}_n(x,y) \le n^{-1} r_n^\star \text{ for all } x \sim y \text{ such that } x \in A^\star, y \notin A^\star \text{ or } y \in A^\star, x \notin A^\star. \tag{2.8}$$

• ***The top, T_n, and the associated sets $T_n^\star$, T_n° and $I_n^\star$.*** Given a sequence $\delta_n \downarrow 0$ as $n \uparrow \infty$, set $\varepsilon_n \equiv \varepsilon - \delta_n$ and let the *top* be the set

$$T_n \equiv V_n(\varepsilon_n) \tag{2.9}$$

obtained by taking $\rho = \varepsilon_n$ in (2.1). (δ_n will later be chosen so that the definitions (2.9) and (1.42) coincide.) Clearly, T_n contains the top of the order statistics of $-\mathcal{H}_n$ (that is, the deepest valleys of the random landscape). Since $\rho_n^\star \ll \varepsilon_n$, $T_n \subset V_n^\star$, so that T_n can be immersed in $V_n^\star$. According to (2.7) we write

$$T_n^\star \equiv \cup_{l=1}^{L^\star} T_{n,l}^\star. \tag{2.10}$$

To each $x \in T_n$ corresponds a unique index $1 \le l \equiv l(x) \le L^\star$ such that $x \in T_{n,l(x)}^\star$. Of course a given valley $T_{n,l}^\star$ may contain several vertices of T_n. A set that is of special importance in the sequel is the subset T_n° of vertices of T_n that are alone in their valley,

$$T_n^\circ \equiv \left\{ x \in T_n \mid T_{n,l(x)}^\star \cap T_n = \{x\} \right\}. \tag{2.11}$$

Finally, define

$$I_n^\star \equiv \{x \in \mathcal{V}_n \mid \tau_n(x) \ge r_n(\varepsilon_n), \forall_{y\sim x} (r_n^\star)^{-1} < \tau_n(y) < r_n^\star\} \subseteq T_n^\circ. \tag{2.12}$$

The content of the next three lemmata is taken from [24]: the first one gives estimates on the size of various sets, the second one expresses the function $r_n(\rho)$ defined through (2.2) and the last one states needed bounds, in particular, on the maximal jump rate.

Lemma 2.1. *There exists $\Omega^\star \subset \Omega$ with $\mathbb{P}(\Omega^\star) = 1$ such that on $\Omega^\star$, for all but a finite number of indices n,*

$$1 \le |C_{n,l}^\star| \le \{\rho_n^\star[1 - 2c_\star^{-1}(1 + \mathcal{O}(\log n/n))]\}^{-1}, \quad 1 \le l \le L^\star. \tag{2.13}$$

Furthermore,

$$
\begin{aligned}
|V_n^\star| &= 2^n n^{-c_\star}(1+o(n^{-c_\star})) \textit{ and } |\overline{V}_n^\star| = 2^n n^{-c_\star}(1+o(n^{-c_\star})), &(2.14)\\
|T_n| &= 2^{n(1-\varepsilon_n)}(1+\mathcal{O}(n2^{-n\varepsilon_n/2})), &(2.15)\\
|T_n^\circ| &= 2^{n(1-\varepsilon_n)}(1+\mathcal{O}(n2^{-n\varepsilon_n/2})), &(2.16)\\
|T_n\setminus T_n^\circ| &\le n^4 2^{n(1-2\varepsilon_n)}(1+o(1)), &(2.17)\\
|I_n^\star| &= 2^{n(1-\varepsilon_n)}(1-2n^{-c_\star+1}(1+o(1)), &(2.18)\\
|T_n^\circ\setminus I_n^\star| &= 2n^{-c_\star+1}2^{n(1-\varepsilon_n)}(1+o(1)). &(2.19)
\end{aligned}
$$

Proof of Lemma 2.1. Recall that $c_\star > 2$. Eq. (2.13) is (2.9) of Lemma 2.2 of Ref. [24]. The estimate (2.14) on $|V_n^\star|$ is (2.11) of Ref. [24]. and the estimate on $|\overline{V}_n^\star|$ follows by symmetry of $\mathcal{H}_n$. Eq. (2.15) and (2.18) are proved, respectively, as (2.11) of and (2.10) of Ref. [24]. The proof of (2.17) is a simple adaptation of the proof of lemma 7.1 of Ref. [24]. Clearly, (2.16) follows from (2.15) and (2.17), and (2.19) follows from (2.16) and (2.18). □

Lemma 2.2 (Lemma 2.3 of [24])**.** *For all $\rho > 0$, possibly depending on n, and such that $\rho n \uparrow \infty$ as $n \uparrow \infty$,*

$$r_n(\rho) = \exp\left\{n\beta\beta_c(\rho) - (\beta/2\beta_c(\rho))\left[\log(\beta_c^2(\rho)n/2) + \log 4\pi\right] + o(\beta/\beta_c(\rho))\right\}. \quad (2.20)$$

Lemma 2.3 (Lemma 2.4 of [24])**.** *There exists a subset $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that on Ω_0, for all but a finite number of indices n the following holds:*

$$
\begin{aligned}
e^{-\beta\min\{\max(H_n(y),H_n(x))\,|\,(x,y)\in\mathcal{E}_n\}} &\le e^{\beta n\sqrt{\log 2}(1+2\log n/n\log 2)} \equiv \nu_n, &(2.21)\\
e^{-\beta\min\{H_n(x)\,|\,x\in\mathcal{V}_n\}} &\le e^{\beta n\sqrt{2\log 2}(1+2\log n/n)}. &(2.22)
\end{aligned}
$$

To close this section let us collect some elementary but key properties of the rates. First note that by (2.6) and (1.12), denoting by $\partial A = \{x \in \mathcal{V}_n \mid \mathrm{dist}(x, A) = 1\}$ the outer boundary of $A \subset \mathcal{V}_n$, we have that for all $x \in \partial V_n^\star$,

$$\widetilde{\lambda}_n(x) = \sum_{y\in(V_n^\star)^c}\widetilde{\lambda}_n(x,y) + \left((nr_n^\star)^{-1}\mathbb{1}_{\{x\in\overline{V}_n^\star\}} + \tau_n(x)n^{-1}\mathbb{1}_{\{x\in(\overline{V}_n^\star)^c\}}\right)|\partial x\cap V_n^\star|. \quad (2.23)$$

Hence, given $V_n^\star$, the mean holding time at $x \in (V_n^\star)^c$ does not depend on the variables $\{\tau_n(y), y \in V_n^\star\}$ but only depends on the variables $\{\tau_n(y), y \in (V_n^\star)^c\}$. Next, introduce the set

$$\mathcal{M}_n \equiv \{x \in \mathcal{V}_n \mid \tau_n(x) > \tau_n(y) \text{ for all } y \sim x\} \quad (2.24)$$

of local minima of $\mathcal{H}_n$ and observe that by (2.6), for all $x \in \mathcal{M}_n \cap V_n^\star$ and all $y \sim x$,

$$\widetilde{\lambda}_n(x,y) = n^{-1}\tau_n(y) \;\text{ and }\; \widetilde{\lambda}_n(y,x) = \begin{cases} n^{-1}\tau_n(y), & \text{if } y \notin \overline{V}_n^\star,\\ n^{-1}, & \text{if } y \in \overline{V}_n^\star,\end{cases} \quad (2.25)$$

Hence, given $\mathcal{M}_n \cap V_n^\star$, the generator of the process Y_n does not depend on the variables $\{\tau_n(x), x \in \mathcal{M}_n \cap V_n^\star\}$. (One in fact may show that on a set of full measure, for all large enough n, it does not depend on the variables $\{\tau_n(x), x \in \mathcal{M}_n\}$.) Since

$$T_n^\circ \subseteq \{x \in \mathcal{V}_n \mid \tau_n(x) \ge r_n(\varepsilon_n), \forall_{y\sim x}\tau_n(y) < r_n(\varepsilon_n)\} \subseteq \mathcal{M}_n, \quad (2.26)$$

the generator of the process Y_n does not depend on the variables $\{\tau_n(x), x \in T_n^\circ\}$.

3. PROPERTIES OF THE EXPLORATION PROCESS Y_n

In this Section we establish the properties of the exploration process needed in the rest of the paper. By (1.35) with $\eta_n \equiv (r^\star_n)^{-1}$ and (2.5), the invariant measure π_n of Y_n can be written as

$$\pi_n(x) = \left(\mathbb{1}_{\{x \notin \overline{V}^\star_n\}} + r^\star_n \tau_n(x) \mathbb{1}_{\{x \in \overline{V}^\star_n\}}\right) Z^{-1}_{\beta,n}, \quad x \in \mathcal{V}_n \tag{3.1}$$

where $Z_{\beta,n} \equiv |\mathcal{V}_n \setminus \overline{V}^\star_n| + \sum_{x \in \overline{V}^\star_n} r^\star_n \tau_n(x)$.

Lemma 3.1. *On $\Omega^\star$, for all but a finite number of indices n, for all subset $A \subseteq \mathcal{V}_n$ such that $A \cap \overline{V}^\star_n = \emptyset$*

$$\pi_n(A) = |A| 2^{-n}(1 + o(1)) \tag{3.2}$$

whereas for arbitrary A, (3.2) holds with equality replaced by "less than or equal". In both cases $o(1)$ is independent of A.

Proof. Since $\{x \in \overline{V}^\star_n\} = \{r^\star_n \tau_n(x) \leq 1\}$, $|\mathcal{V}_n \setminus \overline{V}^\star_n| \leq Z_{\beta,n} \leq |\mathcal{V}_n \setminus \overline{V}^\star_n| + |\overline{V}^\star_n| \leq 2^n$. Eq. (2.14) of Lemma 2.1 then yields $2^n(1 - n^{-c_\star}(1 + o(n^{-c_\star})) \leq Z_{\beta,n} \leq 2^n$. The claim of the lemma directly follows. □

3.1. **Spectral gap and mixing condition.** Denote by $\widetilde{L}_n$ the Markov generator matrix of Y_n (that is, the matrix with off-diagonal entries $\widetilde{\lambda}_n(x, y)$ and diagonal entries $-\widetilde{\lambda}_n(x)$), and by $0 = \vartheta_{n,0} < \vartheta_{n,1} \leq \cdots \leq \vartheta_{n,2^n-1}$ the eigenvalues of $-\widetilde{L}_n$.

Proposition 3.2. *If $c_\star > 1 + \log 4$ then for all $\beta > 0$, there exists a subset $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that, on Ω_1, for all but a finite number of indices n,*

$$1/\vartheta_{n,1} \leq \tfrac{5}{2} n^2 r^\star_n (1 + o(1)) \equiv \tilde{\kappa}_n \tag{3.3}$$

As a direct consequence on Proposition 3.2, Condition (B0) of Theorem 1.3 is satisfied $\mathbb{P}$-almost surely with e.g.

$$\kappa_n \equiv \lfloor n^4 r^\star_n \rfloor. \tag{3.4}$$

Proposition 3.3. *On Ω_1, for all but a finite number of indices n, for all pairs $x, y \in \mathcal{V}_n$ and all $t \geq 0$,*

$$|P_x(Y_n(t + \kappa_n) = y) - \pi_n(y)| \leq \rho_n \pi_n(y), \tag{3.5}$$

where κ_n is given by (3.4) and $\rho_n < e^{-n}$.

Proof of Proposition 3.2. This is a simple adaptation of the proof of [20] (or [14], albeit with other constants). □

Proof of Proposition 3.3. Using (3.1), the bound $Z_{\beta,n} \leq 2^n$ and (2.22) of Lemma 2.3 to bound $\sup_{z \in \mathcal{V}_n} \pi^{-1}_n(z)$ from above, the claim of Proposition 3.3 readily follows from the bound (1.10) of Proposition 3 of Ref. [17] and Proposition 3.2, choosing κ_n as in (3.4). □

3.2. **Hitting time for the stationary chain.** Drawing heavily on Aldous and Brown's work [1], this section collects results on hitting times for the process Y_n at stationarity. Let

$$H(A) = \inf\{t \geq 0 \mid Y_n(t) \in A\} \tag{3.6}$$

be the hitting time of $A \subseteq \mathcal{V}_n$. We begin with bounds on the mean value of $H(A)$.

Lemma 3.4. *On Ω_1, for all but a finite number of indices n, for all $A \subseteq \mathcal{V}_n$,*

$$\frac{(1 - n\pi_n(A))^2}{r^\star_n n \pi_n(A)(1 - \pi_n(A))} \leq \frac{E_{\pi_n} H(A)}{1 - \pi_n(A)} \leq \frac{\tilde{\kappa}_n}{\pi_n(A)}. \tag{3.7}$$

The next lemma gives bounds on the density function $h_{n,A}(t)$, $t>0$, of $H(A)$ when Y_n starts in its invariant measure, π_n.

Lemma 3.5. *On Ω_1, for all but a finite number of indices n, for all $A\subseteq\mathcal{V}_n$ and all $t>0$,*

$$\frac{1}{E_{\pi_n}H(A)}\left(1-\frac{\tilde{\kappa}_n}{E_{\pi_n}H(A)}\right)^2\left(1-\frac{t}{E_{\pi_n}H(A)}\right)\leq h_{n,A}(t)\leq\frac{1}{E_{\pi_n}H(A)}\left(1+\frac{\tilde{\kappa}_n}{2t}\right).$$

The bounds of Lemma 3.5 imply that $h_{n,A}(t)\approx\frac{1}{E_{\pi_n}H(A)}$ when $\tilde{\kappa}_n\ll t\ll E_{\pi_n}H(A)$. Complementing this, Lemma 3.6 is well suited to dealing with "small" values of t.

Lemma 3.6. *On $\Omega^\star$, for all but a finite number of indices n, for all $A\subseteq\mathcal{V}_n$ and all $t>0$,*

$$P_{\pi_n}(H(A)>t)\geq(1-n\pi_n(A))\exp\left(-t\frac{r_n^\star n\pi_n(A)}{1-n\pi_n(A)}\right). \tag{3.8}$$

In particular, for any A and any sequence t_n such that $t_nr_n^\star n\pi_n(A)\to 0$ as $n\to\infty$,

$$P_{\pi_n}(H(A)\leq t_n)<t_nr_n^\star n\pi_n(A)\left(1+t_nr_n^\star n\pi_n(A)\right) \tag{3.9}$$

for all large enough n.

The next Corollary is stated for later convenience.

Corollary 3.7. *Under the assumptions of Lemma 3.6 the following holds: For all $0<\varepsilon<1$, for any sequence t_n such that $t_nr_n^\star n2^{-n\varepsilon_n}\to 0$ as $n\to\infty$*

$$P_{\pi_n}(H(T_n\setminus T_n^\circ)\leq t_n)\leq t_nr_n^\star n^5 2^{-2n\varepsilon_n}(1+o(1)), \tag{3.10}$$

$$P_{\pi_n}(H(T_n^\circ)\leq t_n)\leq t_nr_n^\star n2^{-n\varepsilon_n}(1+o(1)). \tag{3.11}$$

We now prove these results, beginning with Lemma 3.6.

Proof of Lemma 3.6. Write $A=B\cup B^c$ where $B=A\cap V_n^\star$ and $B^c=A\setminus B$. Let $B^\star$ be the immersion of B in $V_n^\star$ (see (2.7)). Since $A\subseteq B^\star\cup B^c$, $H(A)\geq H(B^\star\cup B^c)$, and

$$P_{\pi_n}(H(A)>t)\geq P_{\pi_n}(H(B^\star\cup B^c)>t). \tag{3.12}$$

To bound the right-hand side of (3.12), we use a well know lower bound on hitting times for stationary reversible chains taken from Ref. [1] (combine Theorem 3 and Lemma 2 therein) that states that for all $C\subseteq\mathcal{V}_n$ and all $t>0$,

$$P_{\pi_n}(H(C)>t)\geq(1-\pi_n(C))\exp\left(-t\frac{q_n(C,C^c)}{1-\pi_n(C)}\right) \tag{3.13}$$

where, for for any two sets C and $\widetilde{C}$ such that $C\cap\widetilde{C}=\emptyset$,

$$q_n(C,\widetilde{C})\equiv\sum_{x\in C}\sum_{y\in\widetilde{C}}\pi_n(x)\widetilde{\lambda}_n(x,y). \tag{3.14}$$

Let us thus evaluate (3.14) with $C=B^\star\cup B^c$. Clearly $q_n(B^\star\cup B^c,(B^\star\cup B^c)^c)\leq q_n(B^\star,(B^\star\cup B^c)^c)+q_n(B^c,(B^\star\cup B^c)^c)$. Clearly also, by (2.6), $\widetilde{\lambda}_n(x,y)\leq n^{-1}r_n^\star$ for any $x\in B^c$ and any $y\sim x$. Thus $q_n(B^c,(B^\star\cup B^c)^c)\leq r_n^\star\pi_n(B^c)$. Next, by (2.8), $q_n(B^\star,(B^\star\cup B^c)^c)\leq r_n^\star\pi_n(B^\star)$. Thus

$$q_n(B^\star\cup B^c,(B^\star\cup B^c)^c)\leq r_n^\star[\pi_n(B^\star)+\pi_n(B^c)]. \tag{3.15}$$

Denoting by $C^\star_{n,l(x)}$ the (unique) component of $B^\star$ (see (2.7)) that contains x, we have $|B^\star| \leq |\cup_{x\in B} C^\star_{n,l(x)}| \leq |B| \max_{x\in B} |C^\star_{n,l(x)}|$ where by (2.13), on $\Omega^\star$, $|C^\star_{n,l(x)}| \ll n$. By this and (3.1) we get $\pi_n(B^\star) = Z^{-1}_{\beta,n}|B^\star| \leq nZ^{-1}_{\beta,n}|B| = n\pi_n(B)$. Therefore,

$$\pi_n(B^\star \cup B^c) \leq \pi_n(B^\star) + \pi_n(B^c) \leq n\pi_n(B) + \pi_n(B^c) \leq n\pi_n(B\cup B^c) = n\pi_n(A). \quad (3.16)$$

Using (3.16) in the right-and side of (3.15) and plugging the result in (3.13) finally yields (3.8). □

Proof of Corollary 3.7. This follows from (3.2) of Lemma 3.1, (2.16), and (2.17). □

Proof of Lemma 3.5. Proceed as in Lemma 13 of Ref. [1] and use Proposition 3.2. □

Proof of Lemma 3.4. The rightmost inequality is that of Lemma 2 of Ref. [1] combined with Proposition 3.2. Lemma 2 of Ref. [1] also states that for $C \subseteq \mathcal{V}_n$ and $q_n(C, C^c)$ defined as in (3.14),

$$\frac{E_{\pi_n} H(C)}{1-\pi_n(C)} \geq \frac{1-\pi_n(C)}{q_n(C, C^c)}. \quad (3.17)$$

Given $A \subseteq \mathcal{V}_n$ let $B^\star$ and B^c be defined as in the first line of the proof of Lemma 3.6. Since $H(A) \geq H(B^\star \cup B^c)$, $E_{\pi_n} H(A) \geq E_{\pi_n} H(B^\star \cup B^c)$. Using (3.17) with $C = B^\star \cup B^c$, (3.7) follows from (3.15) and the bound on $\pi_n(B^\star \cup B^c)$ of (3.16). □

3.3. **On hitting the top starting in the top.** Let T°_n and $I^\star_n$ be as in (2.11) and (2.12).

Proposition 3.8. *Given $\epsilon > 0$ there exists a subset $\Omega^\circ \subset \Omega$ with $\mathbb{P}(\Omega^\circ) = 1$ such that on Ω°, for all but a finite number of indices n, for all $s > 0$*

$$|T^\circ_n|^{-1} \sum_{x\in T^\circ_n} P_x\left(H(T^\circ_n \setminus x) \leq s\right) \leq s n^{c_\star+3} r^\star_n \pi_n(T^\circ_n). \quad (3.18)$$

The next proposition is a variant of Proposition 3.8 that we state for later convenience.

Proposition 3.9. *Under the assumptions and with the notation of Proposition 3.8, on Ω°, for all but a finite number of indices n, for all $s > 0$*

$$|T^\circ_n \setminus I^\star_n|^{-1} \sum_{x\in T^\circ_n \setminus I^\star_n} P_x\left(H(I^\star_n) \leq s\right) \leq s n^2 r^\star_n \pi_n(I^\star_n)(1 + o(1)). \quad (3.19)$$

Proof of Proposition 3.8. A key ingredient of the proof is an explicit expression of the density function $h^x_{n,A}(t)$, $t \geq 0$, of the hitting time $H(A)$ when Y_n starts in $x \in A^c \equiv \mathcal{V}_n \backslash A$ that we take from [27] (see Section 6.2, p. 83). Consider the matrix $\widetilde{P}_n = (\widetilde{p}_n(x, y))$ defined by $\widetilde{P}_n = I + \nu_n^{-1}\widetilde{L}_n$ where I denotes the identity matrix, $\widetilde{L}_n$ the Markov generator matrix of Y_n and ν_n is defined in (2.22). By Lemma 2.3, on Ω_0,

$$0 < \max_{(x,y)\in\mathcal{E}_n} \widetilde{\lambda}_n(x, y) \leq n^{-1}\nu_n < \infty \quad (3.20)$$

for all large enough n, hence $\widetilde{P}_n$ is a well defined the stochastic matrix (namely, its entries obey $0 \leq \widetilde{p}_n(x, y) \leq 1$ and $\sum_{y\in\mathcal{V}_n} \widetilde{p}_n(x, y) = 1$). Denote by $Q_n = (q_n(x, y))$ the matrix with entries $q_n(x, y) : A^c \times A^c \to \mathbb{R}$ given by $q_n(x, y) = \widetilde{p}_n(x, y)$. This is the sub-matrix of $\widetilde{P}_n$ on $A^c \times A^c$. Thus Q_n is sub-stochastic. Similarly, denote by $R_n = (r_n(x, y))$ the sub-matrix of $\widetilde{P}_n$ on $A^c \times A$. Let 1_A be the vector of 1's on A and let δ_x be the vector on A^c taking value 1 at x and zero else. Then, for all $x \in A^c$,

$$h^x_{n,A}(t) = \nu_n \sum_{k=0}^{\infty} \frac{(\nu_n t)^k}{k!} e^{-\nu_n t}\left(\delta_x, Q^k_n R_n 1_A\right), \quad t \geq 0, \quad (3.21)$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^{|A^c|}$. Consequently, for all $s>0$,

$$P_x\left(H(A)\leq s\right)=\int_0^s \nu_n\sum_{k=0}^{\infty}\frac{(\nu_n t)^k}{k!}e^{-\nu_n t}\left(\delta_x, Q_n^k R_n 1_A\right)dt. \tag{3.22}$$

For later reference we also denote by $(h^x_{n,y,A}(t))_{y\in A}$ the vector whose components are, for each $y\in A$, the joint density that A is reached at time t, and that arrival to that set occurs in state y, namely, $h^x_{n,y,A}(t)$ is defined as in (3.21) substituting δ_y for 1_A therein; as a result $h^x_{n,A}(t)=\sum_{y\in A}h^x_{n,y,A}(t)$.

Returning to (3.18), a first order Tchebychev inequality yields, for all $\epsilon>0$

$$\mathbb{P}\left[\sum_{x\in T_n^\circ}P_x\left(H(T_n^\circ\setminus x)\leq s\right)\geq\epsilon\right]\leq\epsilon^{-1}\mathbb{E}\left[\sum_{x\in T_n^\circ}P_x\left(H\left(T_n^\star\setminus T^\star_{n,l(x)}\right)\leq s\right)\right] \tag{3.23}$$

where $T_n^\star\equiv\cup_{l=1}^{L^\star}T^\star_{n,l}$ is defined in (2.10) and $l(x)$ is as in (2.11). Calling $\mathcal{W}_n$ the expectation in the right-hand side of (3.23) we have, by (3.22) with $A=T_n^\star\setminus T^\star_{n,l(x)}$,

$$\mathcal{W}_n=\sum_{x\in\mathcal{V}_n}\int_0^s dt\sum_{k=1}^{\infty}\frac{(\nu_n t)^k}{k!}e^{-\nu_n t}\mathcal{W}_{n,k}(x) \tag{3.24}$$

where

$$\mathcal{W}_{n,k}(x)\equiv\mathbb{E}\left[\mathbb{1}_{\{x\in T_n^\circ\}}\nu_n\left(\delta_x, Q_n^k R_n 1_{T_n^\star\setminus T^\star_{n,l(x)}}\right)\right]. \tag{3.25}$$

Note that the term $k=0$ is zero. For $k\geq 1$ the matrix term in (3.25) reads,

$$\mathbb{1}_{\{x\in T_n^\circ\}}\nu_n\left(\delta_x, Q_n^k R_n 1_{T_n^\star\setminus T^\star_{n,l(x)}}\right)=\mathbb{1}_{\{x\in T_n^\circ\}}\sum_{y\in(T_n^\star\setminus T^\star_{n,l(x)})^c}q_n^{(k)}(x,y)\sum_{z\in T_n^\star\setminus T^\star_{n,l(x)}}\nu_n r_n(y,z) \tag{3.26}$$

where $q_n^{(k)}(x,y)$ denotes the entries of Q_n^k. By (2.8), for all $y\in(T_n^\star\setminus T^\star_{n,l(x)})^c$,

$$\sum_{z\in T_n^\star\setminus T^\star_{n,l(x)}}\nu_n r_n(y,z)=\sum_{z\in T_n^\star\setminus T^\star_{n,l(x)}}\widetilde{\lambda}_n(y,z)\leq n^{-1}r_n^\star\sum_{z\in T_n^\star\setminus T^\star_{n,l(x)}}\mathbb{1}_{\{z\sim y\}}. \tag{3.27}$$

Therefore, inserting (3.27) in (3.26), (3.25) yields

$$\mathcal{W}_{n,k}(x)\leq\frac{r_n^\star}{n}\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{x\in T_n^\circ\}}\sum_{y\in(T_n^\star\setminus T^\star_{n,l(x)})^c}q_n^{(k)}(x,y)\sum_{z\in T_n^\star\setminus T^\star_{n,l(x)}}\mathbb{1}_{\{z\sim y\}}\,\middle|\,V_n^\star\right]\right] \tag{3.28}$$

where $\mathbb{E}[\cdot\,|\,V_n^\star]$ denotes the conditional expectation given a realization of the set $V_n^\star$, namely, setting $C^\star_{n,0}\equiv\mathcal{V}_n\setminus V_n^\star$ and using (2.4) to write $\mathcal{V}_n=\cup_{0\leq l\leq L^\star}C^\star_{n,l}$, expectation with respect to the measure

$$\mathbb{P}(\cdot\,|\,V_n^\star)=\frac{\mathbb{P}(\cdot\cap\{\forall_{1\leq l\leq L^\star}\forall_{x\in C^\star_{n,l}}\tau_n(x)\geq r_n^\star\}\cap\{\forall_{x\in C^\star_{n,0}}\tau_n(x)<r_n^\star\})}{\mathbb{P}(\{\forall_{1\leq l\leq L^\star}\forall_{x\in C^\star_{n,l}}\tau_n(x)\geq r_n^\star\}\cap\{\forall_{x\in C^\star_{n,0}}\tau_n(x)<r_n^\star\})}. \tag{3.29}$$

Observe now that, conditionally on $V_n^\star$, the entries of the matrix Q_n are functions of the variables $\{\tau_n(y),y\in(T_n^\star\setminus T^\star_{n,l(x)})^c\}$ only: for off-diagonal entries this is an immediate consequence of (2.6), and for diagonal entries this claim follows from (1.12) and (2.6) if $x\notin\partial V_n^\star$ and from (2.23) and (2.6) if $x\in\partial V_n^\star$. To build on this property let us rewrite the sums in (3.28) in such a way that the variables $\{\tau_n(y),y\in T_n^\star\setminus T^\star_{n,l(x)}\}$ no longer appear in the summations sets but only in the summands. For this note that the sum over $y\in(T_n^\star\setminus T^\star_{n,l(x)})^c$ in (3.28) can be restricted to the sum over $y\in\partial V_n^\star\subseteq C^\star_{n,0}$ and use the definitions (2.10) and (2.11) of $T_n^\star$ and T_n° to write that for all $x\in T_n^\circ$

$$\begin{aligned}
&\sum_{y\in(T_n^\star\setminus T^\star_{n,l(x)})^c} q_n^{(k)}(x,y) \sum_{z\in T_n^\star\setminus T^\star_{n,l(x)}} \mathbb{1}_{\{z\sim y\}}\\
&=\sum_{x_1\in\mathcal{V}_n}\cdots\sum_{x_{k-1}\in\mathcal{V}_n}\sum_{y\in\partial V_n^\star}\sum_{z\in\partial y}\sum^{\star}_{0\le l_1\le L^\star}\cdots\sum^{\star}_{0\le l_{k-1}\le L^\star}\sum^{\star}_{1\le l\neq l(x)\le L^\star} \qquad (3.30)\\
&\quad q_n(x,x_1)\dots q_n(x_{k-1},y)\prod_{i=1}^{k-1}\mathbb{1}_{\{\forall_{x_i'\in C^\star_{n,l_i}\setminus\{x\}}\tau_n(x_i')<r_n(\varepsilon_n)\}}\mathbb{1}_{\{\exists_{z'\in C^\star_{n,l}}\tau_n(z')\ge r_n(\varepsilon_n)\}}
\end{aligned}$$

where the starred sums are defined as

$$\sum^{\star}_{0\le l_i\le L^\star}\cdot\equiv\sum_{l_i=0}^{L^\star}\cdot\mathbb{1}_{\{C^\star_{n,l_i}\cap x_i\neq\emptyset\}}\quad\text{and}\quad\sum^{\star}_{1\le l\neq l(x)\le L^\star}\cdot\equiv\sum_{l=1}^{L^\star}\cdot\mathbb{1}_{\{l\neq l(x)\}}\mathbb{1}_{\{C^\star_{n,l}\cap z\neq\emptyset\}}. \tag{3.31}$$

Notice that each of the starred sums over l_i has only one term given by the index, l_i, of the set $C^\star_{n,l_i}$ that contains x_i. Similarly, the starred sum over l has at most one term. Since $\mathbb{1}_{\{\forall_{z'\in C^\star_{n,\ell}\setminus\{x\}}\tau_n(z')<r_n(\varepsilon_n)\}}\mathbb{1}_{\{\exists_{z'\in C^\star_{n,\ell}}\tau_n(z')\ge r_n(\varepsilon_n)\}}=0$ for all $\ell\neq l(x)$, the starred sum over l in (3.30) can be restricted to $1\le l\neq l(x), l\neq l_1,\dots,l\neq l_{k-1}\le L^\star$. We may now multiply (3.30) by $\mathbb{1}_{\{x\in T_n^\circ\}}$ and take the conditional expectation. The variables $\{\tau_n(z'), z'\in C^\star_{n,l}\}$ being independent of the variables $\{\tau_n(x'), x'\in\cup_{0\le\ell\neq l\le L^\star}C^\star_{n,\ell}\}$, they can be integrated out first, yielding, for all $y\in\partial V_n^\star$

$$\begin{aligned}
&\sum_{z\in\partial y}\sum^{\star}_{1\le l\neq l(x),l\neq l_1,\dots,l\neq l_{k-1}\le L^\star}\mathbb{P}\left[\exists_{z'\in C^\star_{n,l}}\tau_n(z')\ge r_n(\varepsilon_n)\,\middle|\,V_n^\star\right] &(3.32)\\
&\le n\max_{1\le l\le L^\star}|C^\star_{n,l}|2^{-(\varepsilon_n-\rho_n^\star)n} &(3.33)\\
&\le n^2 2^{-(\varepsilon_n-\rho_n^\star)n}, &(3.34)
\end{aligned}$$

where we used in (3.33) that the starred sum over l contains at most one term while the sum over z contains at most n terms. Eq. (3.34) then follows from (2.13) and so, is valid on $\Omega^\star$ for all large enough n. This bound is uniform in $y\in\partial V_n^\star$. Therefore, using (3.34) in (3.30) and re-summing, (3.28) becomes

$$\begin{aligned}
\mathcal{W}_{n,k}(x)&\le\frac{r_n^\star}{n}n^2 2^{-(\varepsilon_n-\rho_n^\star)n}\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{x\in T_n^\circ\}}\sum_{y\in\partial V_n^\star}q_n^{(k)}(x,y)\,\middle|\,V_n^\star\right]\right] &(3.35)\\
&\le\frac{r_n^\star}{n}n^2 2^{-(\varepsilon_n-\rho_n^\star)n}\mathbb{P}(x\in T_n^\circ) &(3.36)
\end{aligned}$$

where we used in (3.36) that since Q_n is sub-stochastic, $\sum_{y\in\partial V_n^\star}q_n^{(k)}(x,y)\le 1$ for all x. Now, by (2.11) and (2.2), $\mathbb{P}(x\in T_n^\circ)\le\mathbb{P}(\tau_n(x)\ge r_n(\varepsilon_n))=2^{-\varepsilon_n n}$. Thus

$$\mathcal{W}_{n,k}(x)\le r_n^\star n 2^{-2\varepsilon_n n}2^{\rho_n^\star n}=n^{c_\star+1}r_n^\star 2^{-2\varepsilon_n n}. \tag{3.37}$$

The last equality is (2.3). Using this bound in (3.24) finally yields that on $\Omega^\star$, for all large enough n,

$$\mathcal{W}_n=\sum_{x\in\mathcal{V}_n}\int_0^{\theta_n}dt\sum_{k=1}^{\infty}\frac{(\nu_n t)^k}{k!}e^{-\nu_n t}S_{n,k}(x)\le\theta_n n^{c_\star+1}r_n^\star 2^n 2^{-2\varepsilon_n n}. \tag{3.38}$$

It only remains to observe that by (2.16) and (3.2) of Lemma 3.1, on $\Omega^\star$, $\pi_n(T_n^\circ) = 2^{-n\varepsilon_n}(1+o(1))$ for all but a finite number of indices n. Hence

$$\mathbb{P}\left[|T_n^\circ|^{-1}\sum_{x\in T_n^\circ} P_x\left(H(T_n^\star \setminus T_{n,l(x)}^\star)\leq s\right)\geq \epsilon\right] \leq \epsilon^{-1} s n^{c_\star+1} r_n^\star \pi_n(T_n^\circ)(1+o(1)).$$

Choosing $\epsilon = n^2 n^{c_\star+1} r_n^\star \pi_n(T_n^\circ)$, the claim of the proposition follows from Borel-Cantelli Lemma. □

Proof of Proposition 3.9. This is a rerun of the proof of Proposition 3.8. □

3.4. **Rough bounds on local times.**

Lemma 3.10. *For all* $0\leq\alpha\leq 1$, *all* $x\in\mathcal{V}_n$, *and all* $s>0$,

$$E_x\left[\ell_n^x(s)\right]^\alpha \geq (\widetilde{\lambda}_n^{-1}(x))^\alpha\Gamma(1+\alpha)[1-c_1\exp(-c_2 s\widetilde{\lambda}_n(x))] + s^\alpha \exp(-s\widetilde{\lambda}_n(x)) \quad (3.39)$$

where $0<c_1,c_2<\infty$ *are constants, and if moreover* $sr_n^\star n\pi_n(x)\to 0$ *as* $n\to\infty$,

$$E_x\left[\ell_n^x(s)\right]^\alpha \leq \kappa_n^\alpha + \mathbb{1}_{\{s>\kappa_n\}} s^\alpha (s-\kappa_n) r_n^\star n\pi_n(x)(1+o(1)). \quad (3.40)$$

Proof of Lemma 3.10. The lower bound follows from the trite observation that $\ell_n^x(s)$ is at least as large as the minimum between the first jump of Y_n and s, that is,

$$\ell_n^x(s) \geq \widetilde{\lambda}_n^{-1}(x) e_1 \mathbb{1}_{s>\widetilde{\lambda}_n^{-1}(x)e_1} + s\mathbb{1}_{s\leq\widetilde{\lambda}_n^{-1}(x)e_1}, \quad (3.41)$$

where e_1 is an exponential random variable of mean one. Thus

$$E_x\left[\ell_n^x(s)\right]^\alpha \geq E_x\left[\widetilde{\lambda}_n^{-1}(x)e_1\mathbb{1}_{s>\widetilde{\lambda}_n^{-1}(x)e_1}\right]^\alpha + s^\alpha E_x\left[\mathbb{1}_{s\leq\widetilde{\lambda}_n^{-1}(x)e_1}\right]^\alpha. \quad (3.42)$$

Eq. (3.39) now readily follows. To get an upper bound write $E_x\left[\ell_n^x(s)\right]^\alpha \leq \kappa_n^\alpha$ if $s\leq\kappa_n$. Otherwise write

$$\begin{aligned} E_x\left[\ell_n^x(s)\right]^\alpha &\leq E_x\left[\kappa_n + \textstyle\int_{\kappa_n}^s \mathbb{1}_{\{Y_n(s)=x\}}ds\right]^\alpha & (3.43)\\ &\leq (1+\rho_n)E_{\pi_n}\left[\kappa_n + \textstyle\int_0^{s-\kappa_n}\mathbb{1}_{\{Y_n(s)=x\}}ds\right]^\alpha & (3.44)\end{aligned}$$

where the last line follows from Proposition 3.3 and the Markov property. Next,

$$\begin{aligned} E_{\pi_n}\left[\kappa_n + \textstyle\int_0^{s-\kappa_n}\mathbb{1}_{\{Y_n(s)=x\}}ds\right]^\alpha &\leq E_{\pi_n}\left(\kappa_n^\alpha\mathbb{1}_{\{H(x)>s-\kappa_n\}} + s^\alpha\mathbb{1}_{\{H(x)\leq s-\kappa_n\}}\right)\\ &\leq \kappa_n^\alpha + s^\alpha P_{\pi_n}(H(x)\leq s-\kappa_n). \end{aligned} \quad (3.45)$$

Eq. (3.40) now follows from (3.9) of Lemma 3.6. □

4. Verification of Condition (B1)

In this section we prove a strong law of large number for the function $\nu_n^t(u,\infty)$ defined in (1.31). Recall that for $r_n^\star$ defined in (1.37), we take $\eta_n\equiv(r_n^\star)^{-1}$ in (1.34), (1.35), and (1.36). Then by (1.18)-(1.19), (1.22), and (1.34),

$$\nu_n^t(u,\infty) = k_n(t)P_{\pi_n}\left(\int_0^{\theta_n}\max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s))\right)ds > u\right) \quad (4.1)$$

where π_n is the invariant measure (1.35) of Y_n, θ_n is the block length of the blocked clock process (1.18), $k_n(t)=\lfloor a_n t/\theta_n\rfloor$, and, given $0<\varepsilon<1$, c_n and a_n are defined in (1.4) and (1.40)-(1.43), respectively. By Theorem 1.3, θ_n and a_n must obey

$$\lfloor n^4 r_n^\star(1+o(1))\rfloor \equiv \kappa_n \leq \theta_n \ll a_n, \quad (4.2)$$

where the left-most equality is (3.4). Further recall from Section 2 that for $\rho_n^\star$ as in (2.3),

$$\rho_n^\star \ll \varepsilon_n \equiv \varepsilon - \delta_n. \tag{4.3}$$

(Recall that $0 < x_n \ll y_n$ means that $x_n/y_n \to 0$ as $n \to \infty$.) From now on we take δ_n such that $2^{n\delta_n} = (n^2\theta_n)^{\alpha(\varepsilon)}$, i.e.

$$\delta_n \equiv \frac{1}{n\beta}\sqrt{\frac{2\varepsilon}{\log 2}}\log\left(n^2\theta_n\right). \tag{4.4}$$

Thus, given $0 < \varepsilon < 1$ and $\beta > 0$, all sequences except θ_n are determined.

Proposition 4.1. *Given $0 < \varepsilon < 1$ and $\beta > 0$ let the sequences c_n and a_n be defined as in (1.4) and (1.40)-(1.43), respectively, and let θ_n be such that*

$$(r_n^\star)^4 \ll \theta_n^{1-\alpha(\varepsilon)}, \tag{4.5}$$

$$n^{-1}\log\theta_n \ll 1. \tag{4.6}$$

Then, for all $0 < \varepsilon < 1$ and $\beta > 0$, $\mathbb{P}$-almost surely,

$$\lim_{n\to\infty} \nu_n^t(u,\infty) = tu^{\alpha(\varepsilon)}, \quad \forall\, t > 0, u > 0. \tag{4.7}$$

Remark. Eq. (4.6) implies that $\delta_n \ll 1$ and that $\theta_n \ll c_n$ for all $\varepsilon > 0$. In view of (1.38), (3.3), (4.4) and (3.4), (4.6) also implies that

$$c_0 n^{c_1}\tilde{\kappa}_n^{c_2}\kappa_n^{c_3}(r_n^\star)^{c_4}\theta_n^{c_5} \ll 2^{\varepsilon n} \quad \text{and} \quad c_0 n^{c_1}\tilde{\kappa}_n^{c_2}\kappa_n^{c_3}(r_n^\star)^{c_4}\theta_n^{c_5} \ll 2^{\varepsilon_n n} \tag{4.8}$$

for all $\varepsilon > 0$ and any choice of constants $0 \le c_i < \infty$.

Remark. In order to guarantee strict equivalence of the definitions (1.42) and (2.9) of the set T_n when δ_n is given by (4.4), we should replace the term $c_n(n\theta_n)^{-1}$ in (1.42) by

$$c_n \exp\left\{-\log(n^2\theta_n)\left[1 + (1+o(1))(2n\beta\beta_c(\varepsilon))^{-1}\log(n^2\theta_n)\right]\right\} \tag{4.9}$$

(use Lemma 2.2). We didn't state this precise formula to keep the presentation simple.

The rest of the section is organized as follows. In Section 4.1 we show that $\nu_n^t(u,\infty)$ can be reduced to the quantity $\nu_n^{\circ,t}(u,\infty)$ defined in (4.32). In Section 4.2 we prove upper et lower bounds on a sequence, b_n°, defined as b_n with T_n° substituted for T_n, and show that b_n and b_n° behave in the same way to leading order. In Section 4.3 we show that $\nu_n^{\circ,t}(u,\infty)$ concentrates around its mean value when choosing $a_n = 2^{\varepsilon n}/b_n^\circ$. The proof of Proposition 4.1 is finally completed in Section 4.2.

4.1. **Preparations.** To begin with, we bring the function $\nu_n^t(u,\infty)$ given in (4.1) into a form amenable to treatment. Let T_n be as in (2.9). For all $0 < \varepsilon < 1$ and δ_n as in (4.4),

$$0 \le \int_0^{\theta_n} \max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s))\right)\mathbb{1}_{\{Y_n(s)\notin T_n\}}ds \le \theta_n\frac{r_n(\varepsilon_n)}{r_n(\varepsilon)} \le n^{-2} \tag{4.10}$$

as follows from (2.20). Hence visits of Y_n outside the set T_n only yield a negligible contribution to the event in (4.1), implying that

$$\check{\nu}_n^t(u,\infty) \le \nu_n^t(u,\infty) \le \check{\nu}_n^t\left(u - n^{-2},\infty\right) \tag{4.11}$$

where

$$\check{\nu}_n^t(u,\infty) \equiv k_n(t)P_{\pi_n}\left(\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\}}ds > u\right). \tag{4.12}$$

Our next step consists in reducing visits to T_n in $\check{\nu}_n^t(u,\infty)$ to visits to the subset T_n° defined in (2.11). Set

$$\bar{\nu}_n^t(u,\infty) \equiv k_n(t)P_{\pi_n}\left(\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}}ds > u\right). \tag{4.13}$$

Lemma 4.2. *Assume that (4.6) holds. Then on $\Omega^\star$, for all but a finite number of indices n,*

$$|\check{\nu}_n^t(u,\infty) - \bar{\nu}_n^t(u,\infty)| \le 2k_n(t)\theta_n r_n^\star n^5 2^{-2n\varepsilon_n}(1+o(1)). \tag{4.14}$$

Proof of Lemma 4.2. Decomposing the event appearing in the probability in (4.12) according to whether $\{H(T_n\setminus T_n^\circ)\le\theta_n\}$ or $\{H(T_n\setminus T_n^\circ)>\theta_n\}$, (4.14) follows from (3.10) of Corollary 3.7 applied with $t_n=\theta_n$, which is licit by virtue of (4.6) (see also (4.8)). □

We next decompose (4.13) according to the hitting time, $H(T_n^\circ)$, and hitting place, $Y_n(H(T_n^\circ))$, of the set T_n°. The density of the joint distribution of $H(T_n^\circ)$ and $Y_n(H(T_n^\circ))$ is a $|T_n^\circ|$-dimensional vector, $(h_{n,x})_{x\in T_n^\circ}$, whose components are, for each $x\in T_n^\circ$, the joint density that T_n° is reached at time v, and that arrival to that set occurs in state x,

$$P_{\pi_n}(H(T_n^\circ)\le s, Y_n(H(T_n^\circ))=x) = \int_0^s h_{n,x}(v)dv. \tag{4.15}$$

For this vector of densities we have

$$\sum_{x\in T_n^\circ}\int_0^\infty h_{n,x}(v)dv = 1, \tag{4.16}$$

and, denoting by h_{n,T_n° the density of $H(T_n^\circ)$,

$$h_{n,T_n^\circ} = \sum_{x\in T_n^\circ} h_{n,x}. \tag{4.17}$$

In the notation of Section 3.3 (see the paragraph below (3.22)) $h_{n,x}=\sum_{y\in\mathcal{V}_n}\pi_n(y)h^y_{n,x,T_n^\circ}$ where, for $y\in T_n^\circ$, $h^y_{n,x,T_n^\circ}=\delta_y$. From this and the strong Markov property it follows that

$$\bar{\nu}_n^t(u,\infty) = k_n(t)\sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}(v)P_x\left(\int_0^{\theta_n-v} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}}ds > u\right)dv. \tag{4.18}$$

Denote by $\overline{Q}_n^{u,v}(x)$ the probability appearing in (4.18). Notice that Y_n starts in $x\in T_n^\circ$ and further decompose this probability according to whether $\{H(T_n^\circ\setminus x)\le\theta_n-v\}$ or $\{H(T_n^\circ\setminus x)>\theta_n-v\}$, that is, write $\overline{Q}_n^{u,v}(x)\equiv\widetilde{Q}_n^{u,v}(x)+\widehat{Q}_n^{u,v}(x)$,

$$\widetilde{Q}_n^{u,v}(x) = P_x\left(\int_0^{\theta_n-v} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}}ds > u, H(T_n^\circ\setminus x)\le\theta_n-v\right), \tag{4.19}$$

$$\widehat{Q}_n^{u,v}(x) = P_x\left(\int_0^{\theta_n-v} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}}ds > u, H(T_n^\circ\setminus x)>\theta_n-v\right), \tag{4.20}$$

and split (4.18) accordingly. Clearly, for all $v>0$

$$\widetilde{Q}_n^{u,v}(x)\le P_x\left(H(T_n^\circ\setminus x)\le\theta_n\right). \tag{4.21}$$

This and the bound $\int_0^{\theta_n} h_{n,x}(v)dv \leq P_{\pi_n}(H(x) \leq \theta_n)$ (that follows from (4.15)), yield

$$k_n(t) \sum_{x \in I_n^\circ} \int_0^{\theta_n} h_{n,x}(v) \widetilde{Q}_n^{u,v}(x) dv \tag{4.22}$$

$$\leq k_n(t) \sum_{x \in T_n^\circ} P_{\pi_n}(H(T_n^\circ) \leq \theta_n, Y_n(H(T_n^\circ)) = x) P_x \left(H(T_n^\circ \setminus x) \leq \theta_n\right) \tag{4.23}$$

$$\leq \widetilde{\nu}_n^t \tag{4.24}$$

where

$$\widetilde{\nu}_n^t \equiv k_n(t) \sum_{x \in T_n^\circ} P_{\pi_n}(H(x) \leq \theta_n) P_x \left(H(T_n^\circ \setminus x) \leq \theta_n\right). \tag{4.25}$$

Lemma 4.3. *Assume that (4.6) holds. Then on $\Omega^\star$, for all but a finite number of indices n,*

$$\widetilde{\nu}_n^t \leq k_n(t) n^{c_\star + 4} \left(\theta_n \pi_n(T_n^\circ) r_n^\star\right)^2 (1 + o(1)). \tag{4.26}$$

Proof of Lemma 4.3. By (3.2), (2.16), (4.3) and (4.4), on $\Omega^\star$, for all large enough n, $\theta_n n \pi_n(T_n^\circ) r_n^\star = n^{1+2\alpha(\varepsilon)} r_n^\star \theta_n^{1+\alpha(\varepsilon)} 2^{-n\varepsilon}(1 + o(1))$, wich decays to zero as n diverges by (4.6) (see also (4.8)). We may thus use (3.9) of Lemma 3.6 to bound the term $P_{\pi_n}(H(x) \leq \theta_n)$ in (4.25), and by this and (3.2) we get that on $\Omega^\star$, for all large enough n,

$$\widetilde{\nu}_n^t \leq k_n(t) \theta_n n \pi_n(T_n^\circ) r_n^\star (1 + o(1)) |T_n^\circ|^{-1} \textstyle\sum_{x \in T_n^\circ} P_x \left(H(T_n^\circ \setminus x) \leq \theta_n\right). \tag{4.27}$$

The lemma now follows from Proposition 3.8. $\square$

Consider now the contribution to (4.18) coming from (4.20). By definition,

$$\widehat{Q}_n^{u,v}(x) = P_x \left(c_n^{-1} \tau_n(x) \ell_n^x(\theta_n - v) > u, H(T_n^\circ \setminus x) > \theta_n - v\right). \tag{4.28}$$

Thus

$$\widehat{\nu}_n^t(u, \infty) \tag{4.29}$$

$$\equiv k_n(t) \sum_{x \in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) \widehat{Q}_n^{u,v}(x) dv \tag{4.30}$$

$$= k_n(t) \sum_{x \in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) P_x \left(c_n^{-1} \tau_n(x) \ell_n^x(\theta_n - v) > u, H(T_n^\circ \setminus x) > \theta_n - v\right) dv. \tag{4.31}$$

Setting

$$\nu_n^{\circ,t}(u, \infty) \equiv k_n(t) \sum_{x \in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) P_x \left(c_n^{-1} \tau_n(x) \ell_n^x(\theta_n - v) > u\right) dv, \tag{4.32}$$

we have

$$\nu_n^{\circ,t}(u, \infty) - w_n^t(u, \infty) \leq \widehat{\nu}_n^t(u, \infty) \leq \nu_n^{\circ,t}(u, \infty) \tag{4.33}$$

where

$$w_n^t(u, \infty) \equiv k_n(t) \sum_{x \in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) P_x \left(c_n^{-1} \tau_n(x) \ell_n^x(\theta_n - v) > u, H(T_n^\circ \setminus x) \leq \theta_n - v\right) dv$$

$$\leq k_n(t) \sum_{x \in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) P_x \left(H(T_n^\circ \setminus x) \leq \theta_n - v\right) dv \leq \widetilde{\nu}_n^t. \tag{4.34}$$

Inserting our bounds in (4.18), we finally get that for all $u > 0$

$$\left|\nu_n^{\circ,t}(u,\infty) - \bar{\nu}_n^t(u,\infty)\right| \leq \widetilde{\nu}_n^t. \tag{4.35}$$

Our aim now is to prove almost sure convergence of $\nu_n^{\circ,t}(u,\infty)$. To do so we will need certain properties a sequence, b_n°, associated to the sequence b_n, that we now define.

4.2. **Properties of the sequences b_n and b_n°.** For $F_{\beta,\varepsilon,n}(x)$ as in (1.41) define

$$b_n^\circ \equiv (\theta_n \pi_n(T_n^\circ))^{-1} \sum_{x\in T_n^\circ} \int_0^{\theta_n} h_{n,x}(v) E_x[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n - v))]dv. \tag{4.36}$$

Thus b_n° is nothing but b_n (see (1.43)) with T_n° substituted for T_n. The next lemma collects properties of the sequences b_n and b_n° needed in the verification of both Condition (B1) and (B2). Set $\mathcal{I}_n(a,b) = (\theta_n\pi_n(T_n^\circ))^{-1}\sum_{x\in T_n^\circ}\mathcal{J}_n^x(a,b)$,

$$\mathcal{J}_n^x(a,b) = \int_a^b h_{n,x}(v) E_x[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n - v))]dv, \tag{4.37}$$

and given $0 < \zeta_n < \theta_n$ split b_n° into $b_n^\circ = \mathcal{I}_n(0,\kappa_n) + \mathcal{I}_n(\kappa_n, \theta_n - \zeta_n) + \mathcal{I}_n(\theta_n - \zeta_n, \theta_n)$.

Lemma 4.4. *Assume that (4.5) and (4.6) hold. Let $\zeta_n > 0$ be a sequence satisfying*

$$n^{-1}|\log\zeta_n| \ll 1, \ \text{ and } \ \tilde{\kappa}_n (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \zeta_n^{\alpha_n(\varepsilon)+o(1)} \downarrow 0 \ \text{ as } \ n\uparrow\infty. \tag{4.38}$$

Then, on $\Omega_1\cap\Omega^\circ\cap\Omega^\star$, for all but a finite number of indices n,

$$\frac{\mathcal{I}_n(0,\kappa_n)}{\mathcal{I}_n(\kappa_n,\theta_n-\zeta_n)} \leq \theta_n^{-1}\tilde{\kappa}_n\kappa_n^{1+\alpha_n(\varepsilon)}(nr_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}, \tag{4.39}$$

$$0 \leq (b_n - b_n^\circ)/b_n^\circ \leq n(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\kappa_n^{1+\alpha_n(\varepsilon)}2^{-n\varepsilon_n}, \tag{4.40}$$

and the right-hand sides of (4.39) and (4.40) decay to zero as n diverges. Furthermore

$$\kappa_n^{-1}(r_n^\star)^{-\{\alpha_n(\varepsilon)+o(1)\}} \leq b_n^\circ \leq (1+o(1))nr_n^\star\kappa_n^{\alpha_n(\varepsilon)}. \tag{4.41}$$

Proof of Lemma 4.4. We first prove a lower bound on $\mathcal{I}_n(\kappa_n,\theta_n-\zeta_n)$. For this write

$$\mathcal{J}_n^x(\kappa_n,\theta_n-\zeta_n) \geq \mathcal{J}_{n,1}^x \equiv \int_{\kappa_n}^{\theta_n-\zeta_n} h_{n,x}(v)E_x[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n-v))\mathbb{1}_{\{\zeta_n<\ell_n^x(\theta_n-v)\leq\theta_n\}}]dv.$$

Since $F_{\beta,\varepsilon,n}(x) = (1+o(1))x^{\alpha_n(\varepsilon)+o(1)}$ for all $\zeta_n < x \leq \theta_n$,

$$\begin{aligned}\mathcal{J}_{n,1}^x &\geq (1+o(1))\int_{\kappa_n}^{\theta_n-\zeta_n} h_{n,x}(v)E_x[\ell_n^x(\theta_n-v)]^{\alpha_n(\varepsilon)+o(1)}(1-\mathbb{1}_{\{\ell_n^x(\theta_n-v)<\zeta_n\}})dv \\ &\equiv \mathcal{J}_{n,3}^x - \mathcal{J}_{n,4}^x\end{aligned} \tag{4.42}$$

where we used the left-most inequality in (4.72) to relax the constraint $\ell_n^x(\theta_n - v) \leq \theta_n$. Let us bound $\mathcal{J}_{n,3}^x$ for $x\in I_n^\star$. Note that by (2.12) and (2.6)

$$(r_n^\star)^{-1} \leq \widetilde{\lambda}_n(x) \leq r_n^\star, \quad \forall x\in I_n^\star. \tag{4.43}$$

Thus, setting $\zeta_n' \equiv nr_n^\star$, it follows from (3.39) of Lemma 3.10 that for all $x\in I_n^\star$,

$$\mathcal{J}_{n,3}^x \geq c_3(\widetilde{\lambda}_n^{-1}(x))^{\alpha_n(\varepsilon)+o(1)}\int_{\kappa_n}^{\theta_n-\zeta_n'} h_{n,x}(v)dv \tag{4.44}$$

for some numerical constant $0 < c_3 < \infty$. Summing over x, wet get

$$\sum_{x\in T_n^\circ}\mathcal{J}_{n,3}^x \geq \sum_{x\in I_n^\star}\mathcal{J}_{n,3}^x \geq c_3(r_n^\star)^{-\{\alpha_n(\varepsilon)+o(1)\}}\sum_{x\in I_n^\star}\int_{\kappa_n}^{\theta_n-\zeta_n'} h_{n,x}(v)dv \tag{4.45}$$

where the last sum in the right-hand side of (4.45) is equal to

$$P_{\pi_n}(\kappa_n < H(I_n^\star) < \theta_n - \zeta_n', H(I_n^\star) < H(T_n^\circ \setminus I_n^\star)). \tag{4.46}$$

Decomposing this probability into

$$p_1 - p_2 \equiv P_{\pi_n}(\kappa_n < H(I_n^\star) < \theta_n - \zeta_n') - P_{\pi_n}(\kappa_n < H(I_n^\star) < \theta_n - \zeta_n', H(I_n^\star) > H(T_n^\circ \setminus I_n^\star))$$

we have, by Lemma 3.5 and (3.7), whenever $\theta_n r_n^\star n \pi_n(I_n^\star) \to 0$,

$$p_1 \geq \tilde\kappa_n^{-1}\theta_n\pi_n(I_n^\star)(1-\theta_n^{-1}\zeta_n')(1+o(1)) = \tilde\kappa_n^{-1}\theta_n\pi_n(I_n^\star)(1+o(1)) \tag{4.47}$$

where the last equality follows from (4.5). To get an upper bound on p_2, write

$$p_2 \leq P_{\pi_n}(H(T_n^\circ \setminus I_n^\star) < \kappa_n) + P_{\pi_n}(H(T_n^\circ \setminus I_n^\star) < H(I_n^\star) < \theta_n) \equiv p_3 + p_4. \tag{4.48}$$

By (3.9), $p_3 \leq \kappa_n r_n^\star n \pi_n(T_n^\circ \setminus I_n^\star)(1+o(1))$, whereas proceeding as in (4.22)-(4.25),

$$p_4 \leq \sum_{x \in T_n^\circ \setminus I_n^\star} P_{\pi_n}(H(x) \leq \theta_n) P_x\left(H(I_n^\star) \leq \theta_n\right) \tag{4.49}$$

$$= n^3(\theta_n r_n^\star)^2 \pi_n(T_n^\circ \setminus I_n^\star)\pi_n(I_n^\star)(1+o(1)) \tag{4.50}$$

where the last equality follows from (3.9) and (3.19). By (2.18), (2.19), and (3.2), on $\Omega^\star$ and for large enough n, $\pi_n(I_n^\star) = 2^{-n\varepsilon_n}(1 - n^{-c_\star}(1+o(1)))$ and $\pi_n(T_n^\circ \setminus I_n^\star) = n^{-c_\star+1}2^{-n\varepsilon_n}(1+o(1))$ (thus in particular, $\pi_n(I_n^\star)/\pi_n(T_n^\circ) = 1+o(1)$). In view of this, (4.5), and (4.6), one checks that $\theta_n r_n^\star n \pi_n(I_n^\star) \to 0$ (as requested above (4.47)) and that $p_2 = o(p_1)$. Thus $p_1 - p_2 = p_1(1+o(1))$ and by this, (4.47), and (4.45),

$$(\theta_n \pi_n(T_n^\circ))^{-1} \sum_{x \in T_n^\circ} \mathcal{J}_{n,3}^x \geq \tilde\kappa_n^{-1}(r_n^\star)^{-\{\alpha_n(\varepsilon)+o(1)\}}(1+o(1)). \tag{4.51}$$

Turning to $\mathcal{J}_{n,4}^x$ we have

$$\sum_{x \in T_n^\circ} \mathcal{J}_{n,4}^x \leq (1+o(1))\zeta_n^{\alpha_n(\varepsilon)+o(1)} \sum_{x \in T_n^\circ} \int_{\kappa_n}^{\theta_n - \zeta_n} h_{n,x}(v)dv, \tag{4.52}$$

where the last sum is equal to $P_{\pi_n}(\kappa_n < H(T_n^\circ) < \theta_n - \zeta_n)$. Since by Lemma 3.5 and (3.7), $P_{\pi_n}(\kappa_n < H(T_n^\circ) < \theta_n - \zeta_n) \leq (1+o(1)) r_n^\star n \theta_n \pi_n(T_n^\circ)$, we get

$$(\theta_n \pi_n(T_n^\circ))^{-1} \sum_{x \in T_n^\circ} \mathcal{J}_{n,4}^x \leq (1+o(1)) n r_n^\star \zeta_n^{\alpha_n(\varepsilon)+o(1)}. \tag{4.53}$$

At this point we may observe that the right-most condition in (4.38) is tailored to guarantee that $\sum_{x \in T_n^\circ} \mathcal{J}_{n,3}^x \gg \sum_{x \in T_n^\circ} \mathcal{J}_{n,4}^x$. Hence, collecting our bounds,

$$\mathcal{I}_n(\kappa_n, \theta_n - \zeta_n) = \frac{1+o(1)}{\theta_n \pi_n(T_n^\circ)} \sum_{x \in T_n^\circ} \int_{\kappa_n}^{\theta_n - \zeta_n} h_{n,x}(v) E_x[\ell_n^x(\theta_n - v)]^{\alpha_n(\varepsilon)+o(1)} \tag{4.54}$$

$$\geq \tilde\kappa_n^{-1}(r_n^\star)^{-\{\alpha_n(\varepsilon)+o(1)\}}. \tag{4.55}$$

We now prove an upper bound on $\mathcal{I}_n(0, \kappa_n)$. Using that $F_{\beta,\varepsilon,n}(x) \leq (1+o(1))x^{\alpha_n(\varepsilon)}$ for all $0 < x \leq \theta_n$, (3.40) of Lemma 3.10 (which by (4.6) and Lemma 3.1 is licit) gives

$$\mathcal{J}_n^x(0, \kappa_n) \leq (1+o(1))\kappa_n^{\alpha_n(\varepsilon)} \int_0^{\kappa_n} h_{n,x}(v)dv. \tag{4.56}$$

Summing over $x \in T_n^\circ$ and using (3.11) and (4.6) to bound the resulting probability,

$$\mathcal{I}_n(0, \kappa_n) \leq (1+o(1)) n r_n^\star \theta_n^{-1} \kappa_n^{1+\alpha_n(\varepsilon)}. \tag{4.57}$$

One proves in the same way that

$$\mathcal{I}_n(0,\theta_n) \leq (1+o(1))nr_n^\star\kappa_n^{\alpha_n(\varepsilon)}\left[1+\theta_n^{1+\alpha_n(\varepsilon)}r_n^\star n\kappa_n^{-\alpha_n(\varepsilon)}2^{-n}\right], \tag{4.58}$$

where by (4.6) the term in square brackets (that comes from (3.40)) is equal to $1+o(1)$.

Combining (4.57) and (4.55) proves (4.39). Since $\mathcal{I}_n(\kappa_n,\theta_n-\zeta_n) \leq b_n^\circ = \mathcal{I}_n(0,\theta_n)$, (4.55) and (4.58) yield, respectively, the lower and upper bounds of (4.41). It remains to prove (4.40). By definition (see (1.43), (4.36), and the second remark below (4.7) on the definition of T_n)

$$|T_n|b_n - |T_n^\circ|b_n^\circ = 2^n\theta_n^{-1}\sum_{x\in T_n\setminus T_n^\circ} E_{\pi_n}\left[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n))\right]. \tag{4.59}$$

Conditioning on the time of the first visit to x, and proceeding as in (4.57)-(4.58) to bound the expectation starting in x, $E_{\pi_n}\left[F_{\beta,\varepsilon,n}(\ell_n^x(\theta_n))\right] \leq (1+o(1))P_{\pi_n}(H(x)\leq\theta_n)\kappa_n^{\alpha_n(\varepsilon)}$. From this and (3.9), $|T_n|b_n - |T_n^\circ|b_n^\circ \leq (1+o(1))r_n^\star n2^n\pi_n(T_n\setminus T_n^\circ)\kappa_n^{\alpha_n(\varepsilon)}$. Now by (2.15)-(2.17), $|T_n| = |T_n^\circ|(1+o(1))$ and $|T_n\setminus T_n^\circ| = |T_n^\circ|n^4 2^{-n\varepsilon_n}(1+o(1))$. Hence $b_n - b_n^\circ \leq (1+o(1))n^5 r_n^\star\kappa_n^{\alpha_n(\varepsilon)}2^{-n\varepsilon_n}$. Combining this and (4.55) yields (4.40). The proof of Lemma 4.4 is now complete. □

Proof of Proposition 1.5. This is a straightforward consequence of (4.40), (4.41), the assumptions of (1.44), and (1.38). □

4.3. **Concentration of $\nu_n^{\circ,t}(u,\infty)$.** Let us now focus on the term $\nu_n^{\circ,t}(u,\infty)$ of (4.32). Recall the definitions of $k_n(t)$ and b_n° from (1.17) and (4.36), respectively.

Proposition 4.5. *Choose $a_n = 2^{\varepsilon n}/b_n^\circ$ in $k_n(t)$ and assume that (4.6) holds. Let $\mathbb{P}^\circ$ denote the law of the collection $\{\tau_n(x), x\in T_n^\circ\}$ conditional on T_n°,*

$$\mathbb{P}^\circ(\cap_{x\in T_n^\circ}\{\tau_n(x)\in\cdot\}) = \mathbb{P}(\cap_{x\in T_n^\circ}\{\tau_n(x)\in\cdot\}\mid T_n^\circ). \tag{4.60}$$

Then, for any sequence $u_n>0$ such that $0<u-u_n<n^{-1}$ and all $u>0$ and $t>0$,

$$\mathbb{P}^\circ\left(\left|\nu_n^{\circ,t}(u_n,\infty)-\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)\right| > n\sqrt{t\Xi_n\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)}\right) \leq n^{-2}(1+o(1)) \tag{4.61}$$

where $\Xi_n \equiv (2^{\varepsilon n}/b_n^\circ)nr_n^\star 2^{-n}$ and

$$\lim_{n\to\infty}\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty) = tu^{\alpha(\varepsilon)}. \tag{4.62}$$

Proof of Proposition 4.5. We assume throughout that $\omega\in\Omega^\star$. A key ingredient of the proof is the observation (see (2.25)-(2.26)) that the generator $\widetilde{L}_n$ of Y_n does not depend on the variables $\{\tau_n(x), x\in T_n^\circ\}$. Furthermore, one easily checks that $\mathbb{P}^\circ$ in (4.60) is the product measure

$$\mathbb{P}^\circ(\cap_{x\in T_n^\circ}\{\tau_n(x)\in\cdot\}) = \prod_{x\in T_n^\circ}\frac{\mathbb{P}(\tau_n(x)\in\cdot,\tau_n(x)\geq r_n(\varepsilon_n))}{\mathbb{P}(\tau_n(x)\geq r_n(\varepsilon_n))}. \tag{4.63}$$

Consequently, for fixed T_n°, the collection $\{X_n(x), x\in T_n^\circ\}$,

$$X_n(x) \equiv \int_0^{\theta_n} h_{n,x}(v)P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v)>u_n\right)dv, \tag{4.64}$$

viewed as a collection of r.v.'s on the sub-sigma field $\mathcal{F}_n^\circ = \sigma(\{\tau_n(x), x\in T_n^\circ\})$, forms a collection of independent random variables under $\mathbb{P}^\circ$ (that of course still depend on the variables $\tau_n(x)$ in $(T_n^\circ)^c$). The proof now hinges on a simple mean and variance argument. We deal with the variance first. By (4.32) and (4.64),

$$\mathbb{E}^\circ \nu_n^{\circ,t}(u_n,\infty) = k_n(t) \sum_{x\in T_n^\circ} \mathbb{E}^\circ X_n(x), \tag{4.65}$$

and by independence

$$\mathbb{E}^\circ(\nu_n^{\circ,t}(u_n,\infty) - \mathbb{E}^\circ \nu_n^{\circ,t}(u_n,\infty))^2 \le k_n^2(t) \sum_{x\in T_n^\circ} \mathbb{E}^\circ (X_n(x))^2. \tag{4.66}$$

Note that since

$$X_n(x) \le \int_0^{\theta_n} h_{n,x}(v)dv \le P_{\pi_n}(H(x)\le \theta_n) \le \theta_n r_n^\star n 2^{-n}(1+o(1)), \tag{4.67}$$

(the last inequality is (3.9) combined with (3.2)) then

$$k_n^2(t) \sum_{x\in T_n^\circ} \mathbb{E}^\circ (X_n(x))^2 \le t(2^{\varepsilon n}/b_n^\circ) r_n^\star n 2^{-n}(1+o(1))\mathbb{E}^\circ \nu_n^{\circ,t}(u_n,\infty), \tag{4.68}$$

where we used that for $a_n = 2^{\varepsilon n}/b_n^\circ$, $\theta_n k_n(t) = \theta_n\lfloor t(2^{\varepsilon n}/b_n^\circ)/\theta_n\rfloor = t(2^{\varepsilon n}/b_n^\circ)(1+o(1))$. Inserting (4.68) in (4.66), a second order Tchebychev inequality then yields (4.61).

To estimate $\mathbb{E}^\circ \nu_n^{\circ,t}(u_n,\infty)$ in (4.65) we first use Fubini to write,

$$\mathbb{E}^\circ X_n(x) = \int_0^{\theta_n} h_{n,x}(v) E_x \mathbb{P}^\circ\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u_n\right) dv. \tag{4.69}$$

Denoting by $\mathbb{P}^x$ the law of the single variable $\tau_n(x)$,

$$\mathbb{P}^\circ\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u_n\right) = \frac{\mathbb{P}^x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u_n, \tau_n(x)\ge r_n(\varepsilon_n)\right)}{\mathbb{P}^x(\tau_n(x)\ge r_n(\varepsilon_n))} \tag{4.70}$$

$$= \frac{\mathbb{P}^x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v) > u_n\right)}{\mathbb{P}^x(\tau_n(x)\ge r_n(\varepsilon_n))} \tag{4.71}$$

where (4.71) follows from the definition of c_n (see (1.4)), the a priory bound

$$\ell_n^x(\theta_n - v) \le \theta_n - v \ll c_n, \quad 0\le v\le \theta_n, \tag{4.72}$$

and the fact that δ_n in (4.4) in chosen in such a way that $\theta_n r_n(\varepsilon_n) r_n^{-1}(\varepsilon) \le n^{-2}\downarrow 0$ as $n\uparrow\infty$ (see the last inequality in (4.10)). Using classical estimates on the asymptotics of gaussian integrals, Lemma 2.2, and again the definition of c_n, simple calculations yield that for all $0<u<\infty$ and $0\le v<\theta_n$, (4.71) is equal to

$$(1+o(1))F_{\beta,\varepsilon,n}\left(\tfrac{\ell_n^x(\theta_n - v)}{u_n}\right)\frac{\mathbb{P}\left(\tau_n(x) > c_n\right)}{\mathbb{P}(\tau_n(x)\ge r_n(\varepsilon_n))} \tag{4.73}$$

where $F_{\beta,\varepsilon,n}(x)$ is defined in (1.41). Furthermore, by (1.4), $2^{\varepsilon n}\mathbb{P}(\tau_n(x)\ge c_n) = 1$ whereas by (2.2), (2.16), and (3.2), $\mathbb{P}(\tau_n(x)\ge r_n(\varepsilon_n)) = \pi_n(T_n^\circ)(1+o(1))$. In view of this and (4.36) we get, combining (4.73), (4.69), (4.65), and using the a priori bound (4.72) that

$$\mathbb{E}^\circ \nu_n^{\circ,t}(u_n,\infty) = (1+o(1))k_n(t)\theta_n(b_n^\circ/2^{\varepsilon n})\frac{I_{(0,\theta_n)}(u_n)}{I_{(0,\theta_n)}(1)} \tag{4.74}$$

where for $w>0$

$$I_{(a,b)}(w) \;=\; \sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}(v) E_x\left[F_{\beta,\varepsilon,n}\left(\tfrac{\ell_n^x(\theta_n - v)}{w}\right)\right]\mathbb{1}_{\{a\le \ell_n^x(\theta_n - v)<b\}}dv. \tag{4.75}$$

To evaluate the ratio in (4.74) set $0 < \zeta_n \equiv e^{-n^{9/10}} \downarrow 0$ and split the integral in $I_{(0,\theta_n)}(u_n)$ into $I_{(0,\theta_n)}(u_n) \equiv I_{(0,\zeta_n)}(u_n) + I_{(\zeta_n,\theta_n)}(u_n)$. Note that $n^{-1}|\log\zeta_n| = n^{-1/10}$, $n^{-1}(\log\zeta_n)^2 = n^{4/5}$, while for all $u > 0$, $n^{-1}\log u_n \downarrow 0$, $n^{-1}(\log u_n)^2 \downarrow 0$ as $n \uparrow \infty$. Using that $F_{\beta,\varepsilon,n}(x)$ is increasing on the domain $(0, \zeta_n/u_n)$

$$I_{(0,\zeta_n)}(u_n) \leq F_{\beta,\varepsilon,n}\left(\tfrac{\zeta_n}{u_n}\right) P_{\pi_n}(H(T_n^\circ) < \theta_n) \tag{4.76}$$

where $F_{\beta,\varepsilon,n}\left(\frac{\zeta_n}{u_n}\right) = e^{o(1)\log u_n} F_{\beta,\varepsilon,n}(\zeta_n) F_{\beta,\varepsilon,n}(u_n^{-1})$ and $F_{\beta,\varepsilon,n}(\zeta_n) \leq e^{-\alpha_n(\varepsilon)n^{9/10} - n^{4/5}/2\beta^2}$. By this, (3.9), the lower bound (4.41) on b_n°, and our assumptions on u_n,

$$\frac{I_{(0,\zeta_n)}(u_n)}{I_{(0,\theta_n)}(1)} = e^{o(1)\log u_n} F_{\beta,\varepsilon,n}(u_n^{-1}) F_{\beta,\varepsilon,n}(\zeta_n) n\kappa_n (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \to 0 \tag{4.77}$$

as $n \to \infty$. Next, since $n^{-1}\log l \downarrow 0$ as $n \uparrow \infty$ for all $\zeta_n \leq l \leq \theta_n$ we have, using (4.72),

$$\frac{I_{(\zeta_n,\theta_n)}(u_n)}{I_{(0,\theta_n)}(1)} = e^{o(1)\log u_n} F_{\beta,\varepsilon,n}(u_n^{-1}) \left[1 - \tfrac{I_{(0,\zeta_n)}(u_n)}{I_{(0,\theta_n)}(1)}\right] \to u^{-\alpha(\varepsilon)} \tag{4.78}$$

as $n \to \infty$ for all $u > 0$. Inserting (4.77) and (4.78) in (4.74), choosing $a_n = 2^{\varepsilon n}/b_n^\circ$, and passing to the limit $n \to \infty$ finally gives (4.62). The proof of the lemma is done. □

4.4. **Proof of Proposition 4.1.** By (4.6), (4.3)-(4.4), and the bound $\kappa_n \leq \theta_n$, (4.40) implies that on $\Omega_1 \cap \Omega^\circ \cap \Omega^\star$, for large enough n, $b_n = b_n^\circ(1 + o(1))$. The assumption that $a_n = 2^{\varepsilon n}/b_n$ in (4.1) can thus be replaced by $a_n = 2^{\varepsilon n}/b_n^\circ$. Consider now (4.61) and note that by (4.41), (3.4), (1.38), and (4.6) (see also (4.8)), for all $0 < \varepsilon < 1$,

$$(2^{\varepsilon n}/b_n^\circ) r_n^\star n^3 2^{-n} \leq \kappa_n (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} n^3 2^{n\varepsilon} 2^{-n} \to 0 \tag{4.79}$$

as $n \to \infty$. Thus, by Proposition 4.5 and Borel-Cantelli Lemma we get that for all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} \nu_n^{\circ,t}(u,\infty) = tu^{\alpha(\varepsilon)} \quad \mathbb{P} - \text{almost surely.} \tag{4.80}$$

In the same way we get that for all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} \nu_n^{\circ,t}(u,\infty) = tu^{\alpha(\varepsilon)} \quad \mathbb{P} - \text{almost surely.} \tag{4.81}$$

Next, by Lemma 4.2, Lemma 4.3, and (4.35) we have that on $\Omega^\star$, for all but a finite number of indices n,

$$\left|\breve{\nu}_n^t(u,\infty) - \nu_n^{\circ,t}(u,\infty)\right| \tag{4.82}$$
$$\leq t(b_n^\circ)^{-1}[2r_n^\star n^5 \theta_n 2^{-n\varepsilon+2\delta_n n} + n^{c_\star+4} 2^{n\varepsilon} (\theta_n \pi_n(T_n^\circ) r_n^\star)^2](1 + o(1)) \tag{4.83}$$
$$\leq 2tn^{c_\star+4(1+\alpha_n(\varepsilon))} (r_n^\star)^{\alpha_n(\varepsilon)+2+o(1)} \kappa_n \theta_n^{2+2\alpha(\varepsilon)} 2^{-n\varepsilon}(1 + o(1)) \tag{4.84}$$

where the last inequality follows from (4.41), (2.16), (4.3), and (4.4). Since $\kappa_n \leq \theta_n$, (4.6) (see also (4.8)) implies that (4.84) decays to zero as $n \to \infty$. From this and (4.80) we get that for all $u > 0$ and all $t > 0$, $\lim_{n\to\infty} \breve{\nu}_n^t(u,\infty) = tu^{\alpha(\varepsilon)}$ $\mathbb{P}$-almost surely. One proves in the same way that for all $u > 0$ and all $t > 0$, $\lim_{n\to\infty} \breve{\nu}_n^t(u - n^{-2}, \infty) = tu^{\alpha(\varepsilon)}$ $\mathbb{P}$-almost surely. Therefore, by (4.11), for all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} \nu_n^t(u,\infty) = tu^{\alpha(\varepsilon)} \quad \mathbb{P} - \text{almost surely.} \tag{4.85}$$

Since ν_n^t is increasing both in t and u and since its limit continuous in those two variables, (4.85) implies that $\mathbb{P}$-almost surely,

$$\lim_{n\to\infty} \nu_n^t(u,\infty) = tu^{\alpha(\varepsilon)}, \quad \forall\, u > 0, t > 0. \tag{4.86}$$

The proof of Proposition 4.1 is done.

5. VERIFICATION OF CONDITION (B2)

By (1.18)-(1.19), (1.22), and (1.34), Condition (B2) in (1.32) states that

$$\sigma_n^t(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[P_y \left(\int_0^{\theta_n} \max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s)) \right) ds > u \right) \right]^2$$

decays to zero as n diverges. We prove in this section that this holds true $\mathbb{P}$-almost surely.

Proposition 5.1. *Under the assumptions of Proposition 4.1, for all* $0<\varepsilon<1$ *and* $\beta>0$, $\mathbb{P}$-*almost surely,*

$$\lim_{n\to\infty} \sigma_n^t(u,\infty) = 0, \quad \forall\, t>0, u>0. \tag{5.1}$$

As in the proof of Proposition 4.1 we first bring $\sigma_n^t(u,\infty)$ into a suitable form. Proceeding as in (4.11)-(4.12), we first write

$$\check{\sigma}_n^t(u,\infty) \le \sigma_n^t(u,\infty) \le \check{\sigma}_n^t(u-n^{-2},\infty) \tag{5.2}$$

where

$$\check{\sigma}_n^t(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[P_y \left(\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s)) \mathbb{1}_{\{Y_n(s)\in T_n\}} ds > u \right) \right]^2, \tag{5.3}$$

and next reduce visits to T_n in (5.3) to visits to visits to T_n°, just as in Lemma 4.2. Set

$$\bar{\sigma}_n^t(u,\infty) \equiv k_n(t) \sum_{y\in\mathcal{V}_n} \pi_n(y) \left[P_y \left(\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s)) \mathbb{1}_{\{Y_n(s)\in T_n^\circ\}} ds > u \right) \right]^2. \tag{5.4}$$

Lemma 5.2. *Assume that (4.6) holds. Then on* $\Omega^\star$, *for all but a finite number of indices* n,

$$|\check{\sigma}_n^t(u,\infty) - \bar{\sigma}_n^t(u,\infty)| \le 6k_n(t)\theta_n n^5 r_n^\star 2^{-2n\varepsilon_n}(1+o(1)). \tag{5.5}$$

Proof of lemma 5.2. As in the Proof of Lemma 4.2 we decompose the event appearing in the probability in (5.3) according to whether $\{H(T_n\setminus T_n^\circ)\le\theta_n\}$ or not, that is, setting

$$q_1(y) = P_y\left(\textstyle\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\}}ds > u, H(T_n\setminus T_n^\circ)\le\theta_n\right), \tag{5.6}$$

$$q_2(y) = P_y\left(\textstyle\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\}}ds > u, H(T_n\setminus T_n^\circ)>\theta_n\right), \tag{5.7}$$

we write $\check{\sigma}_n^t(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)[q_1(y)+q_2(y)]^2$. In the same way write $\bar{\sigma}_n^t(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)[\bar{q}_1(y)+\bar{q}_2(y)]^2$ where $\bar{q}_1(y)$ and $\bar{q}_2(y)$ are defined as in (5.6) and (5.7), respectively, substituting T_n° for T_n. Note that

$$[x_1+x_2]^2 \le 3x_1 + x_2^2, \quad 0\le x_1,x_2\le 1. \tag{5.8}$$

Applying (5.8) to the terms $[q_1(y)+q_2(y)]^2$ and $[\bar{q}_1(y)+\bar{q}_2(y)]^2$, and observing that $q_2^2=\bar{q}_2^2$, we get

$$\begin{aligned} |\check{\sigma}_n^t(u,\infty) - \bar{\sigma}_n^t(u,\infty)| &\le 3k_n(t)\sum_{y\in\mathcal{V}_n}\pi_n(y)(q_1(y)+\bar{q}_1(y)) && (5.9)\\ &\le 6k_n(t)P_{\pi_n}(H(T_n\setminus T_n^\circ)\le\theta_n). && (5.10)\end{aligned}$$

The Lemma now follows from (3.10) of Corollary 3.7. □

We continue our parallel with the proof of Proposition 4.1 and decompose (5.4) according to the hitting time and hitting place of the set T_n°. We slightly abuse the notation of Section 3 (see the paragraph below (3.22)) and denote by $h_{n,x}^y$ (instead of $h_{n,x,T_n^\circ}^y$) the joint density that T_n° is reached at time t, and that arrival to that set occurs in state x, given that the process starts in y. As already observed (see the paragraph below (4.17)), $h_{n,x} = \sum_{y\in\mathcal{V}_n} \pi_n(y)h_{n,x}^y$. Proceeding as in (4.18)-(4.20) we then get

$$\bar{\sigma}_n^t(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n} \pi_n(y)\left[\overline{R}_n^u(y)\right]^2 \tag{5.11}$$

where, using (4.19) and (4.20),

$$\overline{R}_n^u(y) \equiv \sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}^y(v)\left(\widetilde{Q}_n^{u,v}(x)+\widehat{Q}_n^{u,v}(x)\right)dv \equiv \widetilde{R}_n^u(y)+\widehat{R}_n^u(y). \tag{5.12}$$

By analogy with (4.30) we also set

$$\widehat{\sigma}_n^t(u,\infty) \equiv k_n(t)\sum_{y\in\mathcal{V}_n} \pi_n(y)\left[\widehat{R}_n^u(y)\right]^2. \tag{5.13}$$

The next lemma plays the role of Lemma 4.3.

Lemma 5.3. *Assume that (4.6) holds. Then $\Omega^\star$, for all but a finite number of indices n,*

$$0 \le \bar{\sigma}_n^t(u,\infty) - \widehat{\sigma}_n^t(u,\infty) \le 3k_n(t)n^{c_\star+4}\left(\theta_n\pi_n(T_n^\circ)r_n^\star\right)^2(1+o(1)). \tag{5.14}$$

Proof of Lemma 5.3. As in the proof of Lemma 5.2, the proof of Lemma 5.3 relies on the observation that since $0 \le \widetilde{R}_n^u(y), \widehat{R}_n^u(y) \le 1$ in (5.12) for all $y \in \mathcal{V}_n$, then by (5.8),

$$0 < \bar{\sigma}_n^t(u,\infty) - \widehat{\sigma}_n^t(u,\infty) \le 3k_n(t)\sum_{y\in\mathcal{V}_n} \pi_n(y)\widetilde{R}_n^u(y) \tag{5.15}$$

$$= 3k_n(t)\sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}(v)\widetilde{Q}_n^{u,v}(x)dv \le 3\widetilde{\nu}_n^t. \tag{5.16}$$

The equality in (5.16) follows from the identity $h_{n,x}(v) = \sum_{y\in\mathcal{V}_n} \pi_n(y)h_{n,x}^y(v)$, and the final inequality is (4.24). The claim of the lemma now follows from Lemma 4.3. $\square$

We now need an upper bound on $\widehat{\sigma}_n^t(u,\infty)$. For this we proceed as in (4.31)-(4.33) and write that $0 \le \widehat{\sigma}_n^t(u,\infty) \le \sigma_n^{\circ,t}(u,\infty)$ where, by analogy with (4.33),

$$\sigma_n^{\circ,t}(u,\infty) = k_n(t)\sum_{y\in\mathcal{V}_n} \pi_n(y)\left[\sum_{x\in T_n^\circ}\int_0^{\theta_n} h_{n,x}^y(v)P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v)>u\right)dv\right]^2$$

Again, the quantity in between the square brackets is in $[0,1]$. Thus, splitting the integral into the sum of the integrals over $[0,\kappa_n]$ and $[\kappa_n,\theta_n]$, we get, using (5.8) and reasoning as in (5.15)-(5.16),

$$\sigma_n^{\circ,t}(u,\infty) \le 3\bar{\eta}_n^{\circ,t}(u,\infty) + \eta_n^{\circ,t}(u,\infty) \tag{5.17}$$

where

$$\bar{\eta}_n^{\circ,t}(u,\infty) \equiv k_n(t)\sum_{x\in T_n^\circ}\int_0^{\kappa_n} h_{n,x}(v)P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v)>u\right)dv, \tag{5.18}$$

$$\eta_n^{\circ,t}(u,\infty) \equiv k_n(t)\sum_{y\in\mathcal{V}_n} \pi_n(y)\left[\sum_{x\in T_n^\circ}\int_{\kappa_n}^{\theta_n} h_{n,x}^y(v)P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v)>u\right)dv\right]^2 \tag{5.19}$$

The next two propositions bound (5.18) and (5.19) in terms of the quantities $\nu_n^{\circ,t}(u_n,\infty)$ and $\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)$ defined in (4.32) and (4.65), respectively.

Proposition 5.4. *Choose $a_n = 2^{\varepsilon n}/b_n^\circ$ in (1.17). Then, for any sequence $u_n > 0$ such that $0 < u - u_n < n^{-1}$ and all $u > 0$,*

$$\mathbb{P}\left(\bar{\eta}_n^{\circ,t}(u_n,\infty) \geq t\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)n^2\theta_n^{-1}\tilde{\kappa}_n\kappa_n^{1+\alpha_n(\varepsilon)}(nr_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\right) \leq n^{-2}. \tag{5.20}$$

Proposition 5.5. *On $\Omega^\star \cap \Omega_1$, for all but a finite number of indices n and all $u > 0$,*

$$\eta_n^{\circ,t}(u,\infty) \leq \nu_n^{\circ,t}(u,\infty)\theta_n r_n^\star n 2^{-n\varepsilon_n}(1+o(1)). \tag{5.21}$$

Proof of Proposition 5.4. As in the proof of Proposition 4.5 denote by $\mathbb{P}^\circ$ the law of the collection $\{\tau_n(x), x \in T_n^\circ\}$ conditional on T_n°. By a first order Tchebychev inequality,

$$\mathbb{P}\left(\bar{\eta}_n^{\circ,t}(u_n,\infty) \geq \epsilon\right) \leq \epsilon^{-1}\mathbb{E}\left[\mathbb{E}^\circ\bar{\eta}_n^{\circ,t}(u_n,\infty)\right]. \tag{5.22}$$

Note that $\mathbb{E}^\circ\bar{\eta}_n^{\circ,t}(u,\infty)$ only differs from the term $\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)$ of (4.65) in that the integral in (5.18) is over $[0,\kappa_n]$ instead of $[0,\theta_n]$. Taking $a_n = 2^{\varepsilon n}/b_n^\circ$, a simple adaptation of the proof of(4.62) (see (4.69)-(4.78)) yields

$$\mathbb{E}^\circ\bar{\eta}_n^{\circ,t}(u_n,\infty) = t(1+o(1))\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty)\frac{\mathcal{I}_n(0,\kappa_n)}{\mathcal{I}_n(0,\theta_n)} \tag{5.23}$$

where $\mathcal{I}_n(a,b)$ is defined above (4.37). Eq. (4.39) of Lemma 4.4 was designed precisely to control the ratio in (5.23). Namely, on $\Omega^\circ \cap \Omega^\star$, for all but a finite number of indices n,

$$\frac{\mathcal{I}_n(0,\kappa_n)}{\mathcal{I}_n(0,\theta_n)} \leq \frac{\mathcal{I}_n(0,\kappa_n)}{\mathcal{I}_n(\kappa_n,\theta_n-\zeta_n)} \leq \theta_n^{-1}\tilde{\kappa}_n\kappa_n^{1+\alpha_n(\varepsilon)}(nr_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}. \tag{5.24}$$

The combination of (5.22), (5.23), and (5.24) gives (5.20). The proof is complete. □

Proof of Proposition 5.5. To prove (5.21) first observe that

$$\begin{aligned}\sum_{x\in T_n^\circ}\int_{\kappa_n}^{\theta_n} h_{n,x}^y(v)P_x\left(c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v) > u\right)dv &\leq P_y(\kappa_n < H(T_n^\circ) \leq \theta_n)\\ &\leq (1+o(1))P_{\pi_n}(H(T_n^\circ)\leq\theta_n)\end{aligned}$$

where the last line follows from Proposition 3.3 and the Markov property, and is valid on Ω_1, for all but a finite number of indices n. Applying this bound to one of the two square brackets in (5.19) and using (4.32) to bound the remaining term, we get, under the same assumptions as above, that

$$\eta_n^{\circ,t}(u,\infty) \leq (1+o(1))\nu_n^{\circ,t}(u,\infty)P_{\pi_n}(H(T_n^\circ)\leq\theta_n). \tag{5.25}$$

Using Corollary (3.11) to bound the last probability yields the claim of the proposition. □

We are now ready to complete the

Proof of Proposition 5.1. Recall from the proof of Proposition 4.1 that on $\Omega_1 \cap \Omega^\circ \cap \Omega^\star$ $a_n = 2^{\varepsilon n}/b_n = 2^{\varepsilon n}/b_n^\circ(1+o(1))$ for large enough n and consider (5.20). By (4.5), $n^2\theta_n^{-1}\tilde{\kappa}_n\kappa_n^{1+\alpha_n(\varepsilon)}(nr_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \downarrow 0$ as $n \uparrow \infty$ and by (4.62), for all $u > 0$ and $t > 0$ $\lim_{n\to\infty}\mathbb{E}^\circ\nu_n^{\circ,t}(u_n,\infty) = tu^{\alpha(\varepsilon)}$. Thus, by Proposition 5.4 and Borel-Cantelli Lemma we get that for all $u > 0$ and $t > 0$,

$$\lim_{n\to\infty}\bar{\eta}_n^{\circ,t}(u,\infty) = 0 \quad \mathbb{P}-\text{almost surely}. \tag{5.26}$$

Turning to (5.21) and invoking (4.6) (see also (4.8)), it follows from Proposition 5.4 that for all $0 < \varepsilon < 1$ and for all $u > 0$ and $t > 0$,

$$\lim_{n\to\infty} \eta_n^{\circ,t}(u,\infty) = 0 \quad \mathbb{P}-\text{almost surely}. \tag{5.27}$$

Hence by (5.17), for all $u > 0$ and $t > 0$,

$$\lim_{n\to\infty} \sigma_n^{\circ,t}(u,\infty) = 0 \quad \mathbb{P}-\text{almost surely}. \tag{5.28}$$

From there on the proof is a rerun of the proof of Proposition 4.1 with Lemma 5.2 and Lemma 5.3 playing the role of Lemma 4.2 and Lemma 4.3, respectively. We omit the details. □

6. VERIFICATION OF CONDITION (B3)

By (1.18)-(1.20), (1.22), and (1.34), Condition (B3) in (1.33) will be verified if we can establish that:

Proposition 6.1. *Under the assumptions of Proposition 4.1, for all $0 < \varepsilon < 1$ and all $\beta > \beta_c(\varepsilon)$, $\mathbb{P}$-almost surely,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\uparrow\infty} k_n(t) E_{\pi_n} \int_0^{\theta_n} \mathcal{M}_n(Y_n(s)) \mathbb{1}_{\{\int_0^{\theta_n} \mathcal{M}_n(Y_n(s))ds \le \epsilon\}} = 0, \quad \forall t > 0. \tag{6.1}$$

where $\mathcal{M}_n(Y_n(s)) = \max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s))\right)$.

The Lemma below is central to the proof. Recall that $\alpha_n(\varepsilon) \equiv (n\beta^2)^{-1}\log c_n$, that is, in view of (1.5),

$$\alpha_n(\varepsilon) = \alpha(\varepsilon) - (2n\beta\beta_c(\varepsilon))^{-1}[\log(\beta_c^2(\varepsilon)n/2) + \log 4\pi + o(1)]. \tag{6.2}$$

Lemma 6.2. *There are constants $K, K' < \infty$ such that for $\alpha_n(\varepsilon)$ as in (6.2) and any sequence $\epsilon_n > 0$ such that $i\alpha_c^{-1}(\varepsilon) - 1 - \frac{\log \epsilon_n}{n\beta\beta_c(\varepsilon)} > 0$ where $i = 1$ in (6.3) and $i = 2$ in (6.4), we have, for all large enough n,*

$$\mathbb{E} 2^{\varepsilon n} c_n^{-1}\tau_n(x)\mathbb{1}_{\{c_n^{-1}\tau_n(x)\le\epsilon_n\}} \le K \frac{\epsilon_n^{1-\alpha_n(\varepsilon)-\frac{\log\epsilon_n}{2n\beta^2}}}{\alpha_c^{-1}(\varepsilon) - 1 - \frac{\log\epsilon_n}{n\beta\beta_c(\varepsilon)}}, \tag{6.3}$$

$$\mathbb{E}\left(2^{\varepsilon n} c_n^{-1}\tau_n(x)\mathbb{1}_{\{c_n^{-1}\tau_n(x)\le\epsilon_n\}}\right)^2 \le K' \frac{\epsilon_n^{2-\alpha_n(\varepsilon)-\frac{\log\epsilon_n}{2n\beta^2}}}{2\alpha_c^{-1}(\varepsilon) - 1 - \frac{\log\epsilon_n}{n\beta\beta_c(\varepsilon)}}. \tag{6.4}$$

Proof of Lemma 6.2. Using standard estimates on the asymptotics of Gaussian integrals the claimed result follows from straightforward computations. □

Proof of Proposition 6.1. We assume throughout that $\omega \in \Omega_1 \cap \Omega^\circ \cap \Omega^\star$ and that n is as large as desired. Note that $\mathcal{M}_n(Y_n(s)) \le (c_n r_n^\star)^{-1} + c_n^{-1}\tau_n(Y_n(s))$ and that the contribution to (6.1) coming from the term $(c_n r_n^\star)^{-1}$ if or order $o(1)$. Indeed by (1.17), (1.40), the lower bound on b_n obtained by combining (4.41) and (4.40), the expression (1.5) of c_n, the expression (3.4) of κ_n, and the fact, that follows from (1.6), that $2^n = e^{n\beta_c^2(\varepsilon)/2}$,

$$k_n(t)\theta_n(c_n r_n^\star)^{-1} \le 2tn^4(r_n^\star)^{\alpha_n(\varepsilon)+o(1)} e^{n\beta_c^2(\varepsilon)/2} e^{-n\beta\beta_c(\varepsilon)(1+o(1))} \tag{6.5}$$

and so, for all $0 < \varepsilon < 1$ and $\beta > \beta_c(\varepsilon)$, by virtue of (4.6) (see also (4.8))

$$k_n(t)\theta_n(c_n r_n^\star)^{-1} \le 2tn^4(r_n^\star)^{\alpha_n(\varepsilon)+o(1)} e^{-n\beta_c^2(\varepsilon)(1+o(1))/2} \to 0$$

as $n \to \infty$. To prove Proposition 6.1 it thus suffices to establish that $\mathbb{P}$-almost surely,

$$\lim_{\epsilon\downarrow 0}\limsup_{n\uparrow\infty} k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds\leq\epsilon\}} = 0, \quad \forall t>0. \tag{6.6}$$

For T_n as in (2.9) with δ_n given by (4.4), set

$$\mathcal{S}^{(1)}_{n,\epsilon}(t) \equiv k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\}}\mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds\leq\epsilon\}}ds,$$
$$\mathcal{S}^{(2)}_{n,\epsilon}(t) \equiv k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\notin T_n\}}\mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds\leq\epsilon\}}ds.$$

To bound $\mathcal{S}^{(2)}_{n,\epsilon}(t)$ simply note that, using Lemma 3.1,

$$\begin{aligned}\mathcal{S}^{(2)}_{n,\epsilon}(t) &\leq k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{\tau_n(Y_n(s))\leq r_n(\varepsilon_n)\}}ds\\ &\leq k_n(t)\theta_n 2^{-n}(1+o(1))\sum_{x\in\mathcal{V}_n} c_n^{-1}\tau_n(x)\mathbb{1}_{\{\tau_n(x)\leq r_n(\varepsilon_n)\}}.\end{aligned}$$

Take $\epsilon_n = c_n^{-1}r_n(\varepsilon_n)$ and note that by (2.20), the definition of c_n, and (4.6),

$$-\left(n\beta\beta_c(\varepsilon)\right)^{-1}\log\epsilon_n = o(1) \;\text{ and }\; \left(n^{2(1+c_\star 6^2/\alpha(\varepsilon))}\theta_n\right)^{-1} \leq \epsilon_n \leq (n^2\theta_n)^{-1}. \tag{6.7}$$

Thus, by Lemma 6.2 and a first order Tchebychev inequality, for all large enough n,

$$\mathbb{P}\left(\mathcal{S}^{(2)}_{n,\epsilon}(t) \geq n^2 t b_n^{-1}(c_n^{-1}r_n(\varepsilon_n))^{1-\alpha(\varepsilon)+o(1)}\right) \leq n^{-2}K'' \tag{6.8}$$

for some constant $K''>0$. Using the upper bound on ϵ_n of (6.7) and the lower bound on b_n of Lemma 4.4 obtained by combining (4.41) and (4.40),

$$n^2 b_n^{-1}(c_n^{-1}r_n(\varepsilon_n))^{1-\alpha(\varepsilon)+o(1)} \leq n^2\kappa_n(r_n^\star)^{\alpha_n(\varepsilon)+o(1)}\left(n^2\theta_n\right)^{-1+\alpha(\varepsilon)+o(1)} \to 0 \tag{6.9}$$

as $n\to\infty$ by (4.5). Hence by (6.8), (6.9), and Borel-Cantelli Lemma, for all $\epsilon>0$,

$$\lim_{n\to\infty}\mathcal{S}^{(2)}_{n,\epsilon}(t) = 0, \quad \mathbb{P}-\text{almost surely.} \tag{6.10}$$

To deal with $\mathcal{S}^{(1)}_{n,\epsilon}(t)$ we further decompose it into $\mathcal{S}^{(1)}_{n,\epsilon}(t) = \mathcal{S}^{(3)}_{n,\epsilon}(t) + \mathcal{S}^{(4)}_{n,\epsilon}(t)$, where

$$\mathcal{S}^{(3)}_{n,\epsilon}(t) \equiv k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n^\circ\}}\mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds\leq\epsilon\}}ds,$$
$$\mathcal{S}^{(4)}_{n,\epsilon}(t) \equiv k_n(t)E_{\pi_n}\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in T_n\setminus T_n^\circ\}}\mathbb{1}_{\{\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))ds\leq\epsilon\}}ds.$$

Since $\mathcal{S}^{(4)}_{n,\epsilon}(t)$ is non zero only if the event $\{H(T_n\setminus T_n^\circ)\leq\theta_n\}$ occurs,

$$\mathcal{S}^{(4)}_{n,\epsilon}(t) \leq \epsilon k_n(t)E_{\pi_n}\mathbb{1}_{\{H(T_n\setminus T_n^\circ)\leq\theta_n\}}. \tag{6.11}$$

Using assertion (ii) of Corollary 3.7 with $t_n=\theta_n$ as in the proof of Lemma 4.2, we get, assuming (4.6), that on $\Omega^\star$, for all but a finite number of indices n,

$$\mathcal{S}^{(4)}_{n,\epsilon}(t) \leq \epsilon k_n(t)\theta_n r_n^\star n 2^{-2n\varepsilon_n}(1+o(1)),$$

Proceeding as in (6.9) to bound b_n, (4.6) (see also (4.8)) guarantees that for all $\epsilon>0$

$$\lim_{n\to\infty}\mathcal{S}^{(4)}_{n,\epsilon}(t) = 0, \quad \mathbb{P}-\text{almost surely.} \tag{6.12}$$

Using next that $\int_0^{\theta_n} c_n^{-1}\tau_n(Y_n(s))\mathbb{1}_{\{Y_n(s)\in A\}} = \sum_{x\in A} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n)$ for any $A \subseteq \mathcal{V}_n$,

$$\mathcal{S}^{(3)}_{n,\epsilon}(t) \le \mathcal{S}^{(5)}_{n,\epsilon}(t) \equiv k_n(t)E_{\pi_n}\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n)\mathbb{1}_{\{\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n)\le\epsilon\}}.$$

With the notation of (4.15)-(4.17),

$$\mathcal{S}^{(5)}_{n,\epsilon}(t) = k_n(t)\sum_{y\in T_n^\circ}\int_0^{\theta_n} dv h_{n,y}(v)E_y\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v)\mathbb{1}_{\{\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v)\le\epsilon\}}.$$

We further split the sum over x above into $x = y$ and $x \neq y$. The latter contribution is

$$\mathcal{S}^{(6)}_{n,\epsilon}(t) \equiv k_n(t)\sum_{y\in T_n^\circ}\int_0^{\theta_n} dv h_{n,y}(v)E_y\sum_{x\in T_n^\circ\setminus y} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v)\mathbb{1}_{\{\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v)\le\epsilon\}}.$$

Observing that

$$E_y\sum_{x\in T_n^\circ\setminus y} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n - v)\mathbb{1}_{\{\sum_{x\in T_n^\circ} c_n^{-1}\tau_n(x)\ell_n^x(\theta_n-v)\le\epsilon\}} \le \epsilon P_y(H(T_n^\circ\setminus y)\le\theta_n)$$

yields the bound $\mathcal{S}^{(6)}_{n,\epsilon}(t) \le \epsilon\widetilde{\nu}_n^t$, where $\widetilde{\nu}_n^t$ is defined in (4.25). Thus by Lemma 4.3, reasoning as in the paragraph below (4.84), we get that for all $\epsilon > 0$

$$\lim_{n\to\infty}\mathcal{S}^{(6)}_{n,\epsilon}(t) = 0, \quad \mathbb{P}-\text{almost surely.} \tag{6.13}$$

It remains to bound $\mathcal{S}^{(5)}_{n,\epsilon}(t) - \mathcal{S}^{(6)}_{n,\epsilon}(t)$. For this we write $\mathcal{S}^{(5)}_{n,\epsilon}(t) - \mathcal{S}^{(6)}_{n,\epsilon}(t) \le \mathcal{S}^{(7)}_{n,\epsilon}(t)$ where

$$\mathcal{S}^{(7)}_{n,\epsilon}(t) \equiv k_n(t)\sum_{y\in T_n^\circ}\int_0^{\theta_n} dv h_{n,y}(v)E_y c_n^{-1}\tau_n(y)\ell_n^y(\theta_n - v)\mathbb{1}_{\{c_n^{-1}\tau_n(y)\ell_n^x(\theta_n-v)\le\epsilon\}}.$$

Let us now establish that for b_n° as in (4.36), $\mathcal{S}^{(7)}_{n,\epsilon}(t)$ obeys the following

Lemma 6.3. *Let the sequences a_n, c_n, θ_n be as in Proposition 6.1. Then, under the assumptions and with the notation of Proposition 4.5,*

$$\mathbb{P}^\circ\left(\left|\mathcal{S}^{(7)}_{n,\epsilon}(t) - \mathbb{E}^\circ\mathcal{S}^{(7)}_{n,\epsilon}(t)\right| > t\epsilon^{1/2}n2^{-n(1-\varepsilon)/2}\right) \le n^{-2}(1+o(1)) \tag{6.14}$$

for all $\epsilon > 0$, and

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\mathbb{E}^\circ\mathcal{S}^{(7)}_{n,\epsilon}(t) = 0. \tag{6.15}$$

Proof of lemma 6.3. The proof closely follows that of Proposition 4.5. We only point out the main differences. The random variables (4.64) are now replaced by

$$X_n(y) \equiv \int_0^{\theta_n} dv h_{n,y}(v)E_y c_n^{-1}\tau_n(y)\ell_n^y(\theta_n - v)\mathbb{1}_{\{c_n^{-1}\tau_n(y)\ell_n^y(\theta_n-v)\le\epsilon\}} \tag{6.16}$$

and

$$\mathbb{E}^\circ\mathcal{S}^{(7)}_{n,\epsilon}(t) = k_n(t)\textstyle\sum_{y\in T_n^\circ}\mathbb{E}^\circ X_n(y).$$

Proceeding as in (4.70)-(4.72) to deal with the conditional expectation and using that $\mathbb{P}(\tau_n(x) \ge r_n(\varepsilon_n)) = \pi_n(T_n^\circ)(1+o(1))$ (see the paragraph below (4.73)), we get

$$\mathbb{E}^\circ\mathcal{S}^{(7)}_{n,\epsilon}(t) = \frac{k_n(t)(1+o(1))}{\pi_n(T_n^\circ)}\sum_{y\in T_n^\circ}\int_0^{\theta_n} dv h_{n,y}(v)E_y\ell_n^y(\theta_n - v)\mathbb{E}^y c_n^{-1}\tau_n(y)\mathbb{1}_{\{c_n^{-1}\tau_n(y)\le\epsilon_n\}}$$

where $\mathbb{P}^y$ denotes the law of $\tau_n(y)$ and where $\epsilon_n \equiv \epsilon_n(y) = \epsilon/\ell_n^y(\theta_n - v)$. Using (6.3) if $\ell_n^y(\theta_n - v) > \epsilon e^{-n\beta\beta_c(\varepsilon)(\alpha_c^{-1}(\varepsilon)-1)}$ and using that if $\ell_n^y(\theta_n - v) \leq \epsilon e^{-n\beta\beta_c(\varepsilon)(\alpha_c^{-1}(\varepsilon)-1)}$ then

$$E_y \ell_n^y(\theta_n - v)\mathbb{E}^y c_n^{-1}\tau_n(y)\mathbb{1}_{\{c_n^{-1}\tau_n(y)\leq\epsilon_n\}} \leq \epsilon e^{-n\beta\beta_c(\varepsilon)(\alpha_c^{-1}(\varepsilon)-1)} c_n^{-1} e^{n\beta^2/2},$$

we readily see that

$$\begin{aligned}\mathbb{E}^\circ \mathcal{S}_{n,\epsilon}^{(7)}(t) \leq& \, C_1 t \frac{\epsilon^{1-\alpha_n(\varepsilon)-\frac{\log\epsilon}{2n\beta^2}}}{b_n^\circ \theta_n \pi_n(T_n^\circ)} \sum_{y\in T_n^\circ} \int_0^{\theta_n} dv h_{n,y}(v) E_y \widetilde{F}_{\beta,\varepsilon,\epsilon,n}(\ell_n^y(\theta_n - v)) \\ &+ C_2 \epsilon n^{\alpha_n(\varepsilon)/2} e^{-n\beta^2/2} k_n(t) (\pi_n(T_n^\circ))^{-1} P_{\pi_n}(H(T_n^\circ) \leq \theta_n)\end{aligned} \tag{6.17}$$

where here and below $C_i > 0$, $i = 1, 2, \dots$ are constants, and for $F_{\beta,\varepsilon,n}$ as in (1.41),

$$\widetilde{F}_{\beta,\varepsilon,\epsilon,n}(z) = F_{\beta,\varepsilon,n}(z) \frac{z^{\frac{\log\epsilon}{n\beta^2}}\left(1 - \frac{\log z}{n\beta\beta_c(\varepsilon)}\right)}{\alpha_c^{-1}(\varepsilon) - 1 - \frac{\log\epsilon}{n\beta\beta_c(\varepsilon)} + \frac{\log z}{n\beta\beta_c(\varepsilon)}} \mathbb{1}_{\left\{z > \epsilon e^{-n\beta\beta_c(\varepsilon)(\alpha_c^{-1}(\varepsilon)-1)}\right\}}. \tag{6.18}$$

By the leftmost inequality of (4.72) and (4.6), $\widetilde{F}_{\beta,\varepsilon,\epsilon,n}(z) \leq C_3 F_{\beta,\varepsilon,n}(z)$. Thus, by (4.36), the first summand in (6.17) is bounded above by

$$C_4 t \epsilon^{1-\alpha_n(\varepsilon)-\frac{\log\epsilon}{2n\beta^2}}. \tag{6.19}$$

Using (3.11) and proceeding as in (6.5) to bound $k_n(t)$, the second summand is bounded above by

$$C_5 t e^{-n(\beta^2-\beta_c^2(\varepsilon))/2} \kappa_n n^{\alpha_n(\varepsilon)/2+1} (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)} \to 0 \tag{6.20}$$

as $n \to \infty$ by virtue of (3.4), (1.38), and the assumption that $\beta > \beta_c(\varepsilon)$ where $0 < \varepsilon < 1$. Note in particular that $\lim_{n\to\infty} \alpha_n(\varepsilon) = \alpha(\varepsilon) < 1$. Hence, inserting (6.19) and (6.20) in (6.17) and passing to the limit

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \mathbb{E}^\circ \mathcal{S}_{n,\epsilon}^{(7)}(t) = 0, \quad \forall t > 0.$$

This proves (6.15). Turning to the variance we have, as in (4.66), by independence, that

$$\mathbb{V}^\circ(\mathcal{S}_{n,\epsilon}^{(7)}(t)) \equiv \mathbb{E}^\circ(\mathcal{S}_{n,\epsilon}^{(7)}(t) - \mathbb{E}^\circ \mathcal{S}_{n,\epsilon}^{(7)}(t))^2 \leq k_n^2(t) \sum_{y\in T_n^\circ} \mathbb{E}^\circ(X_n(y))^2.$$

Proceeding as in the proof of (6.17) but using (6.4) and the line below (6.18), we get that

$$\begin{aligned}\mathbb{V}^\circ(\mathcal{S}_{n,\epsilon}^{(7)}(t)) \leq& \, C_6 t^2 \frac{\epsilon^{2-\alpha_n(\varepsilon)-\frac{\log\epsilon}{2n\beta^2}}}{(b_n^\circ\theta_n)^2 \pi_n(T_n^\circ)} \sum_{y\in T_n^\circ} \left(\int_0^{\theta_n} dv h_{n,y}(v) E_y F_{\beta,\varepsilon,\epsilon,n}(\ell_n^y(\theta_n - v))\right)^2 \\ &+ C_7 \epsilon n^{\alpha_n(\varepsilon)/2} e^{-n\beta\beta_c(\varepsilon)} \frac{k_n^2(t)\theta}{\pi_n(T_n^\circ)} \sum_{y\in T_n^\circ} \left(\int_0^{\theta_n} dv h_{n,y}(v)\right)^2.\end{aligned}$$

From the bound $\int_0^{\theta_n} dv h_{n,y}(v) E_y F_{\beta,\varepsilon,\epsilon,n}(\ell_n^y(\theta_n - v)) \leq (1 + o(1)) \int_0^{\theta_n} dv h_{n,y}(v) \theta_n^{\alpha_n(\varepsilon)} \leq (1 + o(1))\theta_n^{\alpha_n(\varepsilon)} P_{\pi_n}(H(y) \leq \theta_n)$ and (3.9), (4.41), we get that on $\Omega^\star$, for all but a finite number of indices n, the first summand is bounded above by

$$C_8 t^2 \epsilon^{2-\alpha_n(\varepsilon)-\frac{\log\epsilon}{2n\beta^2}} \left(n\kappa_n \theta_n^{\alpha_n(\varepsilon)} (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\right)^2 2^{-n}.$$

Using the bound $\sum_{y\in T_n^\circ}\left(\int_0^{\theta_n} dv h_{n,y}(v)\right)^2 \leq \sup_{y\in T_n^\circ} P_{\pi_n}(H(y) \leq \theta_n) P_{\pi_n}(H(T_n^\circ) \leq \theta_n)$, and proceeding as in (6.20), the second summand is bounded above by

$$C_9 t^2 \epsilon n^{\alpha_n(\varepsilon)/2} \left(n^2 \kappa_n (r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\right)^2 \theta_n e^{-n\beta_c(\varepsilon)(\beta-\beta_c(\varepsilon))} 2^{-n}.$$

Since by assumption $\beta > \beta_c(\varepsilon)$ and $0 < \varepsilon < 1$, (4.6) (see also (4.8)) enables us to conclude that on $\Omega^\star$, for all large enough n,

$$\mathbb{V}^\circ(\mathcal{S}^{(7)}_{n,\epsilon}(t)) \leq C_{10} t^2 \epsilon 2^{-n(1-\varepsilon)}.$$

This yields (6.14) and concludes the proof of the Lemma. □

Arguing as in the proof of Proposition 4.1 that $b_n = b_n^\circ(1+o(1))$ on $\Omega_1 \cap \Omega^\circ \cap \Omega^\star$ for all large enough n, it follows from Lemma 6.3 and Borel-Cantelli Lemma that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\left(\mathcal{S}^{(5)}_{n,\epsilon}(t) - \mathcal{S}^{(6)}_{n,\epsilon}(t)\right) = 0, \quad \mathbb{P}-\text{almost surely.} \tag{6.21}$$

Collecting (6.10), (6.12), (6.13) and (6.21) yields (6.6). The proof of Proposition 6.1 is complete. □

7. Proof of Theorem 1.1 and Theorem 1.4

Proof of Theorem 1.4. By Proposition (3.3), Proposition (4.1), Proposition (5.1) and Proposition (6.1), under the assumptions of Proposition (4.1) and Proposition (6.1), Conditions (B0), (B1), (B2), and (B3) of Theorem 1.3 are satisfied $\mathbb{P}$-a.s.. It remains to check Condition (A0), i.e. to prove that $\mathbb{P}$-a.s., for all $u > 0$,

$$\lim_{n\to\infty} P_{\mu_n}(Z_{n,1} > u) = 0 \tag{7.1}$$

where $Z_{n,1} = \int_0^{\theta_n} \max\left((c_n r_n^\star)^{-1}, c_n^{-1}\tau_n(Y_n(s))\right) ds$ and μ_n is the uniform measure on $\mathcal{V}_n$. By (3.2) and Lemma 3.1

$$\begin{aligned} P_{\mu_n}(Z_{n,1} > u) &\leq (1+o(1))P_{\pi_n}(Z_{n,1} > u) + \textstyle\sum_{x\in\overline{V}_n^\star} \mu_n(x)P_x(Z_{n,1} > u) \\ &\leq (1+o(1))P_{\pi_n}(Z_{n,1} > u) + n^{-c_\star}(1+o(1)) \end{aligned}$$

where the last line is (2.14). Thus (7.1) is an immediate consequence of Proposition (4.1). One readily checks that the assumptions on a_n, c_n, and θ_n of the theorem imply that the conditions (4.5) and (4.6) of Proposition (4.1) are verified. The proof of 1.4 is done. □

Proof of Theorem 1.1. Reasoning as in the proof of Theorem 1.4, we may assume that the process starts in its invariant measure π_n. The main idea behind the proof is now classical. Suppose that

$$\mathcal{P}_{\pi_n}\left(A_n(t,s)\right) = P_{\pi_n}\left(\{\mathcal{R}_n \cap (t,t+s) = \emptyset\}\right) + o(1) \tag{7.2}$$

where $A_n(t,s) \equiv \{X(c_n t) = X(c_n(t+s))\}$ and where $\mathcal{R}_n$ denotes the range of the rescaled blocked clock process $S_n^b(t)$. Then Theorem 1.1 is a direct consequence of Theorem 1.4 and the arcsine law for stable subordinators. We refer to Ref. [23] for a detailed proof (see the proof of Theorem 1.6 therein) and again stress that the J_1 topology in (1.45) is necessary for this statement to hold.

We now focus on establishing (7.2). For $k \geq 1$ and $Z_{n,i}$ as in (1.19) set

$$\mathcal{B}_k = \left\{\textstyle\sum_{i=1}^k Z_{n,i} < t, \sum_{i=1}^{k+1} Z_{n,i} > t+s\right\}. \tag{7.3}$$

Then by (1.18), $\{\mathcal{R}_n \cap (t,t+s) = \emptyset\} = \{\cup_{k\geq 1}\mathcal{B}_k\}$. Furthermore, for any $T > 0$,

$$P_{\pi_n}\left(\cup_{k>k_n(T)}\mathcal{B}_k\right) \leq P_{\pi_n}\left(S_n^b(T) < t\right) \underset{n\to\infty}{\longrightarrow} P\left(V_{\alpha(\varepsilon)}(T) < t\right) \leq \delta \tag{7.4}$$

where convergence is almost sure in the random environment, as follows from Theorem 1.4, and where δ can be made as small as desired by taking T large enough. Therefore

$$0 \leq P_{\pi_n}\left(\{\mathcal{R}_n \cap (t,t+s) = \emptyset\}\right) - P_{\pi_n}\left(\cup_{1\leq k\leq k_n(T)}\mathcal{B}_k\right) \leq \delta. \tag{7.5}$$

Note that the event $\mathcal{B}_k$ is non empty if and only if the increment $Z_{n,k+1}$ straddles over the interval $(t, t+s)$. To show that (7.2) holds it now suffices to prove the following two facts:
Fact 1. $\mathbb{P}$-a.s,

$$P_{\pi_n}\left(A_n(t,s)\cap\{\cup_{1\le k\le k_n(T)}\mathcal{B}_k\}\right)\ge P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}\mathcal{B}_k\right)+o(1). \tag{7.6}$$

Fact 2. $\mathbb{P}$-a.s,

$$P_{\pi_n}\left(A_n(t,s)\cap(\cap_{1\le k\le k_n(T)}\mathcal{B}_k^c)\right)\to 0,\quad n\to\infty. \tag{7.7}$$

Combining (7.5), (7.6) and (7.7) then establishes that

$$|P_{\pi_n}(A_n(t,s))-P_{\pi_n}(\{\mathcal{R}_n\cap(t,t+s)=\emptyset\})|\le\delta+o(1) \tag{7.8}$$

which is tantamount to (7.2). The proofs of Facts 1 and 2 follow a now classical pattern (see e.g. Ref. [24], [12]) which mostly uses information already obtained in the course of the verification of Conditions (B1)-(B3).

Proof of Fact 1 . Fix $0<T<\infty$ and assume that the assumptions of Proposition (4.1) are satisfied. Let $H_k(A)=\inf\{t\ge\theta_n k \mid Y_n(t)\in A\}$ be the first hitting time of $A\subseteq\mathcal{V}_n$ after time $\theta_n k$. Note first that $\mathcal{B}_k=\mathcal{B}_k\cap\{Z_{n,k+1}>s\}$ so that, by (4.10),

$$P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}(\mathcal{B}_k\cap\{H_k(T_n)>\theta_n\})\right)=0 \tag{7.9}$$

for all large enough n. Note next that reasoning as in (6.11)-(6.12), on $\Omega^\circ\cap\Omega^\star$,

$$P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}(\mathcal{B}_k\cap\{H_k(T_n\setminus T_n^\circ)\le\theta_n\})\right)\le k_n(T)P_{\pi_n}\left(H_k(T_n\setminus T_n^\circ)\le\theta_n\right)\to 0$$

as $n\to\infty$ by virtue of (4.6). Hence on $\Omega^\circ\cap\Omega^\star$, for all large enough n,

$$\begin{aligned}&P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}\mathcal{B}_k\right)\\=&P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}(\mathcal{B}_k\cap\{H_k(T_n^\circ)\le\theta_n\}\cap\{H_k(T_n\setminus T_n^\circ)>\theta_n)\}\right)+o(1).\end{aligned}$$

This means that for $\mathcal{B}_k$ to be non-empty asymptotically, the increment $Z_{n,k+1}$ must be produced by visits of Y_n to T_n°, and T_n° only. Let us now prove that all these visits, if there are several of them, must be to a single vertex. For this it suffices to show that as $n\to\infty$,

$$p_n\equiv P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}(\mathcal{B}_k\cap\{H_k(T_n^\circ)\le\theta_n\}\cap\mathcal{C}_n(Y_n(H_k(T_n^\circ))))\right)\to 0,$$

where

$$\mathcal{C}_n(Y_n(H_k(T_n^\circ)))\equiv\left\{\inf\{t>H_k(T_n^\circ)\mid Y_n(t)\in T_n^\circ\setminus Y_n(H_k(T_n^\circ))\}\le\theta_n\right\}.$$

Now,

$$\begin{aligned}p_n&=P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}\cup_{x\in T_n^\circ}(\mathcal{B}_k\cap\{H_k(T_n^\circ)\le\theta_n,Y_n(H_k(T_n^\circ))=x\}\cap\mathcal{C}_n(x)\right)\\&\le\widetilde{\nu}_n^T\end{aligned}$$

where $\widetilde{\nu}_n^T$ is defined in (4.25) and bounded in Lemma 4.3. Reasoning as in the paragraph below (4.84) then yields that under the assumptions (4.5) and (4.6), on $\Omega^\circ\cap\Omega^\star$, $\lim_{n\to\infty}\widetilde{\nu}_n^T=0$. Thus, the increment $Z_{n,k+1}$ in $\mathcal{B}_k$ cannot be produced by visits of Y_n to two or more distinct vertices of T_n°. Setting

$$\mathcal{W}_n=\bigcup_{1\le k\le k_n(T)}\bigcup_{x\in T_n^\circ}(\mathcal{B}_k\cap\{H_k(T_n^\circ)\le\theta_n,Y_n(H_k(T_n^\circ))=x,H_k(T_n\setminus x)>\theta_n\})$$

and combining our results, we get that for all large enough n, $A_n(t,s)\supseteq\mathcal{W}_n$ so that $P_{\pi_n}(A_n(t,s)\cap\mathcal{W}_n)\ge P_{\pi_n}(\mathcal{W}_n)$, whereas $|P_{\pi_n}(\mathcal{W}_n)-P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}\mathcal{B}_k\right)|=o(1)$ on $\Omega^\circ\cap\Omega^\star$. Eq. (7.6) of Fact 1 is now proved. □

Proof of Fact 2. In view of the information gathered in the proof of Fact 1, Fact 2 will be established if we can prove that no two distinct clock increments $Z_{n,k+1}$ and $Z_{n,k'+1}$ can be produced by visits to the same vertex T_n°, asymptotically. More precisely, as $n \to \infty$,

$$\bar{p}_n \equiv P_{\pi_n}\left(\cup_{1\le k\le k_n(T)}\left(\{H_k(T_n^\circ)\le\theta_n\}\cap\mathcal{D}_{n,k}(Y_n(H_k(T_n^\circ)))\right)\right)\to 0, \tag{7.10}$$

where

$$\mathcal{D}_n(Y_n(H_k(T_n^\circ)))\equiv\left\{\inf\{t>(k+1)\theta_n \mid Y_n(t)=Y_n(H_k(T_n^\circ))\}\le\theta_n k_n(T)\right\}.$$

To prove this, observe that the event in (7.10) can be written as

$$\cup_{x\in T_n^\circ}\cup_{y\in T_n^\circ}\left(\{H_k(T_n^\circ)\le\theta_n, Y_n(H_k(T_n^\circ))=x\}\cap\{Y_n(\theta_n(k+1))=y\}\cap\mathcal{D}_{n,k}(x)\right)$$

Thus, by the Markov property we have, using the notation of (4.15)-(4.17) and the bound $P_y\left(H(x)\le\theta_n(k_n(T)-(k+1))\right)\le P_y\left(H(x)\le\theta_n k_n(T)\right)$,

$$\bar{p}_n\le\sum_{1\le k\le k_n(T)}\sum_{x\in T_n^\circ}\sum_{y\in T_n^\circ}\int_0^{\theta_n}dv h_{n,x}(v)P_x\left(Y_n(\theta_n-v)=y\right)P_y\left(H(x)\le\theta_n k_n(T)\right).$$

To proceed, we split the domain of integration into $[0,\theta_n-\kappa_n)\cup[\theta_n-\kappa_n,\theta_n]$. Using that by Proposition 3.3, on Ω_1, for all n large enough, $P_x\left(Y_n(\theta_n-v)=y\right)=\pi_n(y)(1+o(1))$ for all $v\in[0,\theta_n-\kappa_n)$, the contribution coming from this domain is at most

$$\begin{aligned}&(1+o(1))\sum_{1\le k\le k_n(T)}\sum_{x\in T_n^\circ}\int_0^{\theta_n}dv h_{n,x}(v)\sum_{y\in T_n^\circ}\pi_n(y)P_y\left(H(x)\le\theta_n k_n(T)\right)\\&\le(1+o(1))k_n(T)P_{\pi_n}(H(T_n^\circ)\le\theta_n)\sup_{y\in T_n^\circ}P_{\pi_n}(H(y)\le\theta_n k_n(T))\\&\le(1+o(1))\left(\theta_n k_n(T)r_n^\star n2^{-n}\right)^2 2^n\pi_n(T_n^\circ)\end{aligned} \tag{7.11}$$

where we used (3.11) with $t_n=\theta_n$ (which is licit as we many times saw) and (3.9) with $t_n=\theta_n k_n(T)$, which is licit provided that $\theta_n k_n(T)r_n^\star n2^{-n}\to 0$ as $n\to\infty$, and this is guaranteed by our assumptions on a_n. Indeed, proceeding as in the proof of Proposition 4.1 (see (4.79) and the paragraph above) we get that on $\Omega^\circ\cap\Omega^\star\cap\Omega_1$, for large enough n,

$$\theta_n k_n(T)r_n^\star n2^{-n}\le\kappa_n(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}n2^{-(1-\varepsilon)n}\to 0 \tag{7.12}$$

as $n\to\infty$ for all $0<\varepsilon<1$. Since furthermore $2^n\pi_n(T_n^\circ)=(1+o(1))2^{(1-\varepsilon)n}\left(n^2\theta_n\right)^{\alpha(\varepsilon)}$ by (4.4), (3.2), and (2.16), and we get that on $\Omega^\circ\cap\Omega^\star\cap\Omega_1$, (7.11) is bounded above by

$$(1+o(1))\left(\kappa_n(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}n\right)^2\left(n^2\theta_n\right)^{\alpha(\varepsilon)}2^{-(1-\varepsilon)n}, \tag{7.13}$$

and by (4.6) this decays to zero as $n\to\infty$ for all $0<\varepsilon<1$.

Consider next the domain $[\theta_n-\kappa_n,\theta_n]$ and note that since

$$\sum_{y\in T_n^\circ}P_x\left(Y_n(\theta_n-v)=y\right)P_y\left(H(x)\le\theta_n k_n(T)\right)\le 1 \tag{7.14}$$

the corresponding contribution is bounded above by $k_n(T)P_{\pi_n}\left(\theta_n-\kappa_n\le H(T_n^\circ)\le\theta_n\right)$. By the upper bound of (3.5) and the lower bound of (3.4), on $\Omega^\star$, for all but a finite number of indices n, this is in turn bounded above by

$$n^{1+2\alpha_n(\varepsilon)}\theta_n^{-(1-\alpha(\varepsilon))}\kappa_n^2(r_n^\star)^{1+\alpha_n(\varepsilon)+o(1)}\to 0 \tag{7.15}$$

as $n \to \infty$, where we again used that $2^{n\delta_n} = (n^2\theta_n)^{\alpha(\varepsilon)}$ by (4.4) whereas $0 < \alpha(\varepsilon) < 1$ by assumption; the final convergence then follows from (4.5). Combining the conclusions of (7.12) and (7.15) we get that on $\Omega^\circ \cap \Omega^\star \cap \Omega_1$,

$$\lim_{n\to\infty} \bar{p}_n = 0. \tag{7.16}$$

Now this implies that if $\mathcal{B}_k$ and $\mathcal{B}'_k$, $1 \le k \neq k' \le k_n(T)$, are two non-empty events then, on $\Omega^\circ \cap \Omega^\star \cap \Omega_1$, the increments $Z_{n,k+1}$ and $Z_{n,k'+1}$ are produced by visits to two distinct elements of T_n° with probability $1 - o(1)$. This readily implies (7.7) and concludes the proof of Fact 2. □

The proof of Theorem 1.1 is now complete. □

8. Appendix: Proof of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. The proof closely follows that of Theorem 1.2 of Ref. [12]. Throughout we fix a realization $\omega \in \Omega$ of the random environment but do not make this explicit in the notation. We set

$$\widehat{S}_n^b(t) \equiv S_n^b(t) - Z_{n,1}. \tag{8.1}$$

Condition (A0) ensures that $S_n^b - \widehat{S}_n^b$ converges to zero, uniformly. Thus we must show that under Conditions (A1), (A2), and (A3), $\widehat{S}_n^b \Rightarrow_{J_1} S_\nu$. For this we rely on Theorem 1.1 of Ref. [12]. Namely, we want to show that Conditions (A1), (A2), and (A3) imply the conditions of Theorem 1.1 of Ref. [12]. To this end let $\{\mathcal{F}_{n,i}, n \ge 1, i \ge 0\}$ be the array of sub-sigma fields of $\mathcal{F}^Y$ defined (with obvious notation) through $\mathcal{F}_{n,i} = \sigma(Y_n(s), s \le \theta_n i)$, for $i \ge 0$. Note that for each n and $i \ge 1$, $Z_{n,i}$ is $\mathcal{F}_{n,i}$ measurable and $\mathcal{F}_{n,i-1} \subset \mathcal{F}_{n,i}$. Next observe that by the Markov property and the fact that, for all $i \ge 1$ and $y \in \mathcal{V}_n$, $\mathcal{P}_y(Z_{n,i} > u) = \mathcal{P}_y(Z_{n,1} > u)$,

$$\mathcal{P}_{\mu_n}\left(Z_{n,i} > u \,\middle|\, \mathcal{F}_{n,i-1}\right) = \sum_{y\in\mathcal{V}_n} \mathbb{1}_{\{Y_n((i-1)\theta)=y\}} \mathcal{P}_y(Z_{n,1} > u). \tag{8.2}$$

In view of this, (1.21), (1.22), and (1.23)

$$\textstyle\sum_{i=2}^{k_n(t)} \mathcal{P}_{\mu_n}\left(Z_{n,i} > u \mid \mathcal{F}_{n,i-1}\right) = \nu_n^{Y,t}(u,\infty), \tag{8.3}$$

and in view of (1.24)

$$\textstyle\sum_{i=2}^{k_n(t)} \left[\mathcal{P}_{\mu_n}\left(Z_{n,i} > u \mid \mathcal{F}_{n,i-1}\right)\right]^2 = \sigma_n^{Y,t}(u,\infty). \tag{8.4}$$

From (8.3) and (8.4) it follows that Conditions (A1) and (A2) of Theorem 1.2 are exactly the conditions of Theorem 1.1 of Ref. [12]. Similarly Condition (A3) is condition (1.9). Therefore the conditions of Theorem 1.1 of Ref. [12] are verified, and so $\widehat{S}_n^b \Rightarrow_{J_1} S_\nu$ in $D([0,\infty))$ where S_ν is a subordinator with Lévy measure ν and zero drift. □

The proof of Theorem 1.3 centers of the

Proposition 8.1. *Assume that Condition (B1) is satisfied. Then, choosing $\theta_n \ge \kappa_n$, the following holds for all initial distributions μ_n: for all $t > 0$, all $u > 0$, and all $\epsilon > 0$,*

$$P_{\mu_n}\left(\left|\nu_n^{Y,t}(u,\infty) - \nu_n^t(u,\infty)\right| \ge \epsilon\right) \le 5\epsilon^{-2}\left[\rho_n\left(\nu_n^t(u,\infty)\right)^2 + \sigma_n^t(u,\infty)\right], \tag{8.5}$$

and

$$P_{\mu_n}\left(\sigma_n^{Y,t}(u,\infty) \ge \epsilon\right) \le \epsilon^{-1}(1+\rho_n)\sigma_n^t(u,\infty). \tag{8.6}$$

Proof of Proposition 8.1. We assume throughout that $\theta_n \geq \kappa_n$. To prove (8.6), simply note that by a first order Tchebychev inequality

$$\begin{aligned} P_{\mu_n}\left(\sigma_n^{Y,t}(u,\infty) \geq \epsilon\right) &\leq \epsilon^{-1} k_n(t) \textstyle\sum_{y\in\mathcal{V}_n} E_{\mu_n}(\pi_n^{Y,t}(y))\left[Q_n^u(y)\right]^2 && (8.7)\\ &\leq \epsilon^{-1}(1+\rho_n)\sigma_n^t(u,\infty), && (8.8)\end{aligned}$$

where we used in the last line that by (1.30),

$$|E_{\mu_n}(\pi_n^{Y,t}(y)) - \pi_n(y)| \leq \rho_n \pi_n(y). \tag{8.9}$$

Turning to (8.5), a second order Chebychev inequality yields

$$\begin{aligned} &P_{\mu_n}\left(\left|\nu_n^{Y,t}(u,\infty) - \nu_n^t(u,\infty)\right| \geq \epsilon\right)\\ &\leq \epsilon^{-2} \textstyle\sum_{x\in\mathcal{V}_n}\sum_{y\in\mathcal{V}_n} Q_n^u(x)Q_n^u(y) \sum_{i=1}^{k_n(t)-1}\sum_{j=1}^{k_n(t)-1} \Delta_{ij}(x,y) \qquad (8.10)\end{aligned}$$

where

$$\begin{aligned} \Delta_{ij}(x,y) \equiv{}& P_{\mu_n}\left(Y_n(i\theta_n)=x, Y_n(j\theta_n)=y\right) + \pi_n(x)\pi_n(y)\\ &- \pi_n(y)P_{\mu_n}\left(Y_n(i\theta_n)=x\right) - \pi_n(x)P_{\mu_n}\left(Y_n(j\theta_n)=y\right). \end{aligned} \tag{8.11}$$

Using again (1.31) yields

$$|\Delta_{ij}(x,y)| \leq \begin{cases} \rho_n(4+\rho_n)\pi_n(x)\pi_n(y), & \text{if } i\neq j,\\ (1+\rho_n)\pi_n(x) + (1+2\rho_n)\pi_n^2(x), & \text{if } i=j \text{ and } x=y,\\ 0 & \text{else.}\end{cases} \tag{8.12}$$

Thus (8.10) is bounded above by $\epsilon^{-2}\rho_n(4+\rho_n)\left(\nu_n^t(u,\infty)\right)^2 + \epsilon^{-2}(2+3\rho_n)\sigma_n^t(u,\infty)$. Since by assumption $\rho_n \downarrow 0$ as $n\uparrow\infty$, Proposition 8.1 is proven. □

Proof of Theorem 1.3. Condition (B2) combined with the conclusions of Proposition 8.1 implies both conditions (A1) and (A2), and Condition (B3) combined with (8.9) implies Condition (A3). □

REFERENCES

[1] D. J. Aldous and M. Brown. Inequalities for rare events in time-reversible Markov chains. I. In *Stochastic inequalities (Seattle, WA, 1991)*, volume 22 of *IMS Lecture Notes Monogr. Ser.*, pages 1–16. Inst. Math. Statist., Hayward, CA, 1992.

[2] G. Ben Arous, A. Bovier, and J. Černý. Universality of the REM for dynamics of mean-field spin glasses. *Commun. Math. Phys.*, 282(3):663–695, 2008.

[3] G. Ben Arous, A. Bovier, and V. Gayrard. Aging in the random energy model. *Phys. Rev. Lett.*, 88(8):087201, 2002.

[4] G. Ben Arous, A. Bovier, and V. Gayrard. Glauber dynamics of the random energy model. I. Metastable motion on the extreme states. *Commun. Math. Phys.*, 235(3):379–425, 2003.

[5] G. Ben Arous, A. Bovier, and V. Gayrard. Glauber dynamics of the random energy model. II. Aging below the critical temperature. *Commun. Math. Phys.*, 236(1):1–54, 2003.

[6] G. Ben Arous and J. Černý. The arcsine law as a universal aging scheme for trap models. *Comm. Pure Appl. Math.*, 61(3):289–329, 2008.

[7] G. Ben Arous and O. Gün. Universality and extremal aging for dynamics of spin glasses on subexponential time scales. *Comm. Pure Appl. Math.*, 65(1):77–127, 2012.

[8] S. C. Bezerra, L. R. G. Fontes, R. J. Gava, V. Gayrard, and P. Mathieu. Scaling limits and aging for asymmetric trap models on the complete graph and K processes. *ALEA Lat. Am. J. Probab. Math. Stat.*, 9(2):303–321, 2012.

[9] J.-P. Bouchaud and D. S. Dean. Aging on Parisi's tree. *J. Phys I(France)*, 5:265, 1995.

[10] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in stochastic dynamics of disordered mean-field models. *Probab. Theory Related Fields*, 119(1):99–161, 2001.

[11] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability and low lying spectra in reversible Markov chains. *Comm. Math. Phys.*, 228(2):219–255, 2002.

[12] A. Bovier and V. Gayrard. Convergence of clock processes in random environments and ageing in the p-spin SK model. *Ann. Probab.*, 41(2):817–847, 2013.
[13] A. Bovier, V. Gayrard, and A. Švejda. Convergence to extremal processes in random environments and extremal ageing in SK models. *Probab. Theory Related Fields*, 157(1-2):251–283, 2013.
[14] J. Černý and T. Wassmer. Aging of the metropolis dynamics on the random energy model. *Probability Theory and Related Fields*, pages 1–51, 2015.
[15] B. Derrida. Random-energy model: limit of a family of disordered models. *Phys. Rev. Lett.*, 45(2):79–82, 1980.
[16] B. Derrida. A generalization of the random energy model which includes correlations between energies. *J. Physique Lett.*, 46:401–407, 1985.
[17] P. Diaconis and D. Stroock. Geometric bounds for eigenvalues of Markov chains. *Ann. Appl. Probab.*, 1(1):36–61, 1991.
[18] B. Duplantier, T. C. Halsey, and V. Rivasseau, editors. *"Glasses and Grains"*, volume 61 of *Progress in Mathematical Physics*. Birkhäuser/Springer Basel AG, Basel, 2011. Papers from the 13th Poincaré Seminar held in Paris, November 21, 2009.
[19] R. Durrett and S. I. Resnick. Functional limit theorems for dependent variables. *Ann. Probab.*, 6(5):829–846, 1978.
[20] L. R. G. Fontes, M. Isopi, Y. Kohayakawa, and P. Picco. The spectral gap of the REM under Metropolis dynamics. *Ann. Appl. Probab.*, 8(3):917–943, 1998.
[21] L. R. G. Fontes and P. Mathieu. On the dynamics of trap models in $\mathbb{Z}^d$. *Proc. Lond. Math. Soc. (3)*, 108(6):1562–1592, 2014.
[22] V. Gayrard. Aging in reversible dynamics of disordered systems. II. Emergence of the arcsine law in the random hopping time dynamics of the REM. preprint, 2010. arXiv:1008.3849.
[23] V. Gayrard. Convergence of clock process in random environments and aging in Bouchaud's asymmetric trap model on the complete graph. *Electron. J. Probab.*, 17:no. 58, 33, 2012.
[24] V. Gayrard. Convergence of clock processes and aging in Metropolis dynamics of a truncated REM. *Annales Henri Poincaré*, 17(3):537–614, 2015.
[25] V. Gayrard and A. Švejda. Convergence of clock processes on infinite graphs and aging in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 11:no. 2, 781–822, 2015.
[26] I. Junier and J. Kurchan. Microscopic realizations of the trap model. *Journal of Physics A: Mathematical and General*, 37(13):3945, 2004.
[27] J. Keilson. *Markov chain models—rarity and exponentiality*, volume 28 of *Applied Mathematical Sciences*. Springer-Verlag, New York-Berlin, 1979.

V. GAYRARD, AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE
E-mail address: veronique.gayrard@math.cnrs.fr