# A Characterization of Displayable Logics Extending Update Logic

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**Abstract**

Correspondence results for substructural logics are proved and a series of correspondence algorithms are introduced for relating analytic inference rules of display calculi and first-order frame conditions. These results and algorithms are obtained thanks to update logic, which is a generalization of the non-associative Lambek calculus. We characterize all the properly displayable logics without (truth) constant extending update logic (and thus the Lambek calculus). Our characterization tells us that a logic without constant extending update logic is properly displayable if, and only if, the class of pointed substructural frames on which the logic is based can be defined by some finite set of specific primitive first-order formulas called prototypic formulas. In that case, we provide algorithms to compute the prototypic formulas defining the class of pointed substructural frames that correspond to the analytic inference rules of the proper display calculus and, vice versa, we also provide algorithms to compute the analytic inference rules of the display calculus that correspond to the prototypic formulas defining the class of substructural frames. Our proofs and algorithms resort to a specific multi-modal tense logic and they use extensively Sahlqvist's as well as Kracht's results and techniques developed for tense logics.

# Contents



## 1 Introduction

Update logic [2] is both a generalization and an extension of the Lambek calculus. The development of update logic is motivated by the intention to capture within the logical framework of substructural logics various logic-based formalisms dealing with common sense reasoning and logical dynamics. This initiative is based on the key observation that an update can be represented abstractly by the ternary relation of the substructural framework: the first argument of the ternary relation represents an initial situation, the second an informative event and the third the resulting situation after the occurrence of the informative event. In [2], a sound and complete display calculus was introduced for update logic and we imported some of the correspondence results obtained for substructural logics to obtain new display calculi for substructural logics, and in particular for (modal) bi-intuitionistic logic.

In general, correspondence theory investigates to what extent specific properties of accessibility relations can be reformulated in terms of the validity of specific (modal or tense) formulas. The following kinds of questions are addressed: when does the truth of a given (modal or tense) formula in a frame corresponds to a first-order property in this frame ? (Sahlqvist correspondence theorem); when does the validity of a (modal or tense) formula on a class of frames corresponds to the fact that this class of frames satisfies a specific first-order property (and vice versa) ? (Sahlqvist and Kracht theorems). We refer the reader to [11] for more details on correspondence theory for modal and tense logic, even if the basics of this theory will be recalled in this report. Given the general and generic nature of update logic, it seems relevant to develop a correspondence theory for logics extending update logic. Naturally, we expect this correspondence theory to address the same questions as the correspondence theory that has been developed for modal and tense logic. Update logic and tense logic both have a relationnal semantics. The only major difference is that update logic deals moreover with (substructural) binary connectives, whereas modal and tense logics only deal with (modal and tense) unary connectives. At the semantic level, this difference is reflected by the addition of a ternary relation to provide a semantics to the (substructural) binary connectives.

In this report, we develop a correspondence theory for logics extending update logic. From a methodological point of view, we resort to a specific multi-modal tense logic which turns out to be as expressive as update logic. This multi-modal tense logic plays the role of a lingua franca and we use the correspondence results already obtained for display calculi of tense logics to obtain our correspondence results for display calculi of substructural logics (more precisely of logics extending update logic). In fact, our proofs and algorithms rely extensively on Sahlqvist's [45] as well as Kracht's [21, 22] results and techniques developed for tense logics. Independently from our work, Palmigiano & Al [20] have recently applied the tools of *unified correspondence* [13] to address the identification of the syntactic shape of *axioms* which can be translated into analytic structural rules of a display calculus, and the definition of an effective procedure for transforming axioms into such rules.

Our main contribution is to provide a characterization of all the properly displayable logics extending update logic (and thus also the Lambek calculus). Our characterization shows that a logic extending update logic is properly displayable if, and only if, it is sound and complete with respect to a class of substructural frames defined by some finite set of prototypic first-order formulas (a subclass of primitive first-order formulas). In that case, we provide algorithms to compute the prototypic first-order formulas defining the class of frames that correspond to the structural rules of the proper display calculus and, vice versa, we also provide algorithms to compute the structural rules of the display calculus that correspond to the prototypic first-order formulas defining the class of substructural frames.

*Weakening:*

$$\frac{U \vdash V}{X, U \vdash V}\, W_A \qquad \frac{U \vdash V}{U \vdash V, X}\, W_K$$

*Contraction:*

$$\frac{X, X, U \vdash V}{X, U \vdash V}\, C_A \qquad \frac{U \vdash V, X, X}{U \vdash V, X}\, C_K$$

*Permutation:*

$$\frac{U, Y, X, V \vdash W}{U, X, Y, V \vdash W}\, P_A \qquad \frac{U \vdash V, Y, X, W}{U \vdash V, X, Y, W}\, P_K$$

Figure 1: Structural Rules of Classical Logic

**Organization of the report.** In Section 2.1, we recall the framework of substructural logics based on the relational semantics. In Section 2.2, we discuss to what extent the ternary relation of substructural logics can be interpreted dynamically as a sort of update. In Section 3, after some preliminary definitions, we motivate and (re)introduce the syntax and semantics of update logic. In Section 4, we provide a display calculus for update logic. Then, in Section 5.1, we define our specific tense logic for which we provide a Hilbert as well as a display calculus. In Section 5.2, we show that our tense logic is as expressive as update logic and we provide translation back and forth between update logic and our tense logic. In Sections 6.2 and 6.2, we recall some results about correspondence theory for tense logic adapted to our substructural framework. In Section 7, we provide correspondence result for inference rules, from analytic inference rules to first-order frames conditions (Section 7.1), and vice versa (Section 7.2). Our main result is Theorem 23, it is stated in Section 7.3. In Section 8, we give some examples of correspondence translations from inference rules to first-order frame conditions (Section 8.1) and vice versa (Section 8.2). Some of these examples of correspondence are already known from the literature, but we will rediscover them by different means.

# 2 Substructural Logics and Updates

Substructural logics are a family of logics lacking some of the structural rules of classical logic. A *structural rule* is a rule of inference which is closed under substitution of formulas [39, Definition 2.23]. In a certain sense, a structural rule allows to manipulate the structure(s) of the sequent/consecution without altering its logical content. The structural rules for classical logic introduced by Gentzen [18] are given in Figure 1. The comma in these sequents has to be interpreted as a conjunction in an antecedent and as a disjunction in a consequent. While *Weakening* $(W_A, W_K)$ and *Contraction* $(C_A, C_K)$ are often dropped as in relevance logic and linear logic, the rule of *Permutation* $(P_A, P_K)$ is often preserved. When some of these rules are dropped, the comma ceases to behave as a conjunction (in the antecedent) or a disjunction (in the consequent). In that case the comma corresponds to other substructural connectives and we often introduce new punctuation marks which do not fulfill all these structural rules to deal with these new substructural connectives.

## 2.1 Substructural Logics

Our exposition of substructural logics is based on [39, 40, 14] (see also [33] for a general introduction).[1] The logical framework presented in [39] is more general and studies a wide range of substructural logics: relevant logic, linear logic, Lambek calculus, arrow logic, *etc*. We will only introduce a fragment of this general framework in order to highlight the main new ideas. In particular, we will not consider truth sets and we will assume that our logics do not reject distribution. These other features can be added and our framework can be adapted, following the exposition of Restall [39]. We will moreover assume that we have multiple modalities (one for each agent $j \in \mathbb{G}$).

The semantics of substructural logics is based on the ternary relation of the frame semantics for relevant logic originally introduced by Routley and Meyer [41, 42, 43, 44]. Another semantics proposed independently by Urquhart [46, 47, 48] at about the same time will be discussed at the end of this section.

In the sequel we consider the following set of *logical connectives*:

$$\mathsf{Sub} := \left\{\top, \bot, \Box_j, \Diamond_j^-, \neg, \vee, \wedge, \otimes, \supset, \subset, \Rightarrow \mid j \in \mathbb{G}\right\}$$

We also define the set of connectives $\mathsf{Sub}_{-} := \mathsf{Sub} - \{\Rightarrow\}$ (the connective $\Rightarrow$ corresponds to the intuitionistic implication).

**Definition 1** (Languages $\mathcal{L}(\mathbb{P}, \mathsf{Sub})$ and $\mathcal{L}(\mathbb{P}, \mathsf{Sub}_{-})$)**.** The *language* $\mathcal{L}(\mathbb{P}, \mathsf{Sub})$ is the language associated to $\mathsf{Sub}$, that is, the language built compositionally from the connectives of $\mathsf{Sub}$ and the set of propositional letters $\mathbb{P}$. More formally, it is the set of formulas defined inductively by the following grammar in BNF, where $p$ ranges over $\mathbb{P}$ and $j$ ranges over $\mathbb{G}$:

$$\begin{array}{llll} \mathcal{L}(\mathbb{P}, \mathsf{Sub}): & \varphi & ::= & \top \mid \bot \mid p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \Rightarrow \varphi) \mid \\ & & & \Box_j \varphi \mid \Diamond_j^- \varphi \mid (\varphi \otimes \varphi) \mid (\varphi \subset \varphi) \mid (\varphi \supset \varphi) \end{array}$$

The language $\mathcal{L}(\mathbb{P}, \mathsf{Sub}_{-})$ is the language $\mathcal{L}(\mathbb{P}, \mathsf{Sub})$ without the (intuitionistic) connective $\Rightarrow$. $\Box$

**Definition 2** (Point set, accessibility relation)**.** A *point set* $\mathcal{P} = (P, \sqsubseteq)$ is a non-empty set $P$ together with a partial order $\sqsubseteq$ on $P$. The set $Prop(\mathcal{P})$ of *propositions* on $\mathcal{P}$ is the set of all subsets $X$ of $P$ which are *closed upwards*: that is, if $x \in X$ and $x \sqsubseteq x'$ then $x' \in X$. When $\sqsubseteq$ is the identity relation $=$, we say that $\mathcal{P}$ is *flat*. We abusively write $x \in \mathcal{P}$ for $x \in P$.

- A binary relation $S$ is a *positive two–place accessibility relation* on the point set $\mathcal{P}$ if, and only if, for any $x, y \in \mathcal{P}$ where $xSy$, if $x' \sqsubseteq x$ then there is a $y' \sqsupseteq y$ such that $x'Sy'$. Similarly, if $xSy$ and $y \sqsubseteq y'$ then there is some $x' \sqsubseteq x$ such that $x'Sy'$.

- A binary relation $S$ is a *plump positive two-place accessibility relation* on the point set $\mathcal{P}$ if, and only if, for any $w, v, w', v' \in \mathcal{P}$, where $wSv$, $w' \sqsubseteq w$ and $v \sqsubseteq v'$ it follows that $w'Sv'$.

[1] We very slightly change the definitions of *frames* and *models* as they are defined in [39] (we give the details of these differences in the sequel). The definitions remain equivalent nevertheless.

- A ternary relation $\mathcal{R}$ is a *three–place accessibility relation* on the point set $\mathcal{P}$ if, and only if, whenever $\mathcal{R}xyz$ and $z \sqsubseteq z'$ then there are $y' \sqsupseteq y$ and $x' \sqsupseteq x$ such that $\mathcal{R}x'y'z'$. Similarly, if $x' \sqsubseteq x$ then there are $y' \sqsubseteq y$ and $z' \sqsupseteq z$ such that $\mathcal{R}x'y'z'$, and if $y' \sqsubseteq y$ then there are $x' \sqsubseteq x$ and $z' \sqsupseteq z$, such that $\mathcal{R}x'y'z'$.

- A ternary relation $\mathcal{R}$ is a *plump three-place accessibility relation* on the point set $\mathcal{P}$ if, and only if, for any $w, v, u, w', v', u' \in \mathcal{P}$ such that $\mathcal{R}wvu$, if $w' \sqsubseteq w$, $v' \sqsubseteq v$ and $u \sqsubseteq u'$, then $\mathcal{R}w'v'u'$.

We say that $Q$ is an *accessibility relation* if, and only if, it is either a (positive or negative) two-place accessibility relation or a three-place accessibility relation. $\square$

Note that plump accessibility relations are accessibility relations. The definitions of accessibility relations relate $S, C, \mathcal{R}$ with $\sqsubseteq$. They are set in such a way that condition (Persistence) can be lifted to arbitrary formulas of $\mathcal{L}(\mathbb{P}, \mathsf{Sub})$ and holds not only for the propositional letters of $\mathbb{P}$.

**Definition 3** (substructural model)**.** A *(multi-modal) substructural model* is a tuple $\mathcal{M} = (\mathcal{P}, S_1, \ldots S_m, \mathcal{R}, \mathcal{I})$ where:

- $\mathcal{P} = (P, \sqsubseteq)$ is a point set;

- $S_j \subseteq \mathcal{P} \times \mathcal{P}$ is a (binary) accessibility relation on $\mathcal{P}$, for each $j \in \mathbb{G}$;

- $\mathcal{R} \subseteq \mathcal{P} \times \mathcal{P} \times \mathcal{P}$ is a (ternary) accessibility relation on $\mathcal{P}$;

- $\mathcal{I} : P \to 2^{\mathbb{P}}$ is a function called the *interpretation function* satisfying moreover the condition $\{w \in \mathcal{M} \mid w \in \mathcal{I}(p)\} \in Prop(\mathcal{P})$, which can be reformulated as follows: for all $w, v \in P$ and all $p \in \mathbb{P}$,

$$\text{if } p \in \mathcal{I}(w) \text{ and } w \sqsubseteq v \text{ then } p \in \mathcal{I}(v). \tag{Persistence}$$

We abusively write $w \in \mathcal{M}$ for $w \in \mathcal{P}$ and $(\mathcal{M}, w)$ is called a *pointed substructural model*. The class of all pointed substructural models is denoted $\mathcal{E}_{\sqsubseteq}$. A *(pointed) substructural frame* is a (pointed) substructural model without interpretation function. The class of all pointed substructural frames is denoted $\mathcal{F}_{\sqsubseteq}$. The class of all pointed substructural models (frames) where point sets are flat is denoted $\mathcal{E}$ (resp. $\mathcal{F}$). $\square$

**Definition 4** (Evaluation relation)**.** We define the *evaluation relation* $\Vdash \subseteq \mathcal{E}_{\sqsubseteq} \times \mathcal{L}(\mathbb{P}, \mathsf{Sub})$ as follows. Let $\mathcal{M}$ be a substructural model, $w \in \mathcal{M}$ and $\varphi, \psi \in \mathcal{L}(\mathbb{P}, \mathsf{Sub})$. The truth conditions for the atomic facts and the connectives of $\mathsf{Sub}$ are defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash \top & & \text{always;} \\
\mathcal{M}, w \Vdash \bot & & \text{never;} \\
\mathcal{M}, w \Vdash p & \text{iff} & p \in \mathcal{I}(w); \\
\mathcal{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi; \\
\mathcal{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ or } \mathcal{M}, w \Vdash \psi; \\
\mathcal{M}, w \Vdash \Box_j \varphi & \text{iff} & \text{for all } v \in \mathcal{P}, \text{ such that } wS_j v, \mathcal{M}, v \Vdash \varphi; \\
\mathcal{M}, w \Vdash \Diamond_j^- \varphi & \text{iff} & \text{there is } v \in \mathcal{P} \text{ such that } vS_j w \text{ and } \mathcal{M}, v \Vdash \varphi; \\
\mathcal{M}, w \Vdash \varphi \otimes \psi & \text{iff} & \text{there are } v, u \in \mathcal{P} \text{ such that } \mathcal{R}vuw, \\
 & & \mathcal{M}, v \Vdash \varphi \text{ and } \mathcal{M}, u \Vdash \psi; \\
\mathcal{M}, w \Vdash \varphi \supset \psi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \mathcal{R}wvu, \\
 & & \text{if } \mathcal{M}, v \Vdash \varphi \text{ then } \mathcal{M}, u \Vdash \psi; \\
\mathcal{M}, w \Vdash \psi \subset \varphi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \mathcal{R}vwu, \\
 & & \text{if } \mathcal{M}, v \Vdash \varphi \text{ then } \mathcal{M}, u \Vdash \psi; \\
\mathcal{M}, w \Vdash \varphi \Rightarrow \psi & \text{iff} & \text{for all } v \in P, \text{ if } w \sqsubseteq v \text{ then not } \mathcal{M}, v \Vdash \varphi \text{ or } \mathcal{M}, v \Vdash \psi.
\end{array}
$$

We extend these definitions to the class of pointed substructural frames. We define the *evaluation relation* $\Vdash \subseteq \mathcal{F}_{\sqsubseteq} \times \mathcal{L}(\mathbb{P}, \mathsf{Sub})$ as follows. Let $(F, w)$ be a pointed frame and let $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Sub})$. Then, we have that

$$
\begin{array}{lll}
F, w \Vdash \varphi & \text{iff} & \text{for all interpretation functions } \mathcal{I} \text{ such that } (F, \mathcal{I}) \text{ satisfies Persistence,} \\
 & & (F, \mathcal{I}), w \Vdash \varphi
\end{array}
$$

□

A substructural model stripped out from its interpretation function corresponds to a *frame* as defined in [39, Definition 11.8] and without *truth sets*. In [39], a *model* is a frame together with an evaluation relation.

**Urquhart's semantics.** The Urquhart's semantics for relevance logic was developed independently from the Routley–Meyer's semantics in the early 1970's. An *operational frame* is a set of points $\mathcal{P}$ together with a function which gives us a new point from a pair of points:

$$\sqcup : \mathcal{P} \times \mathcal{P} \to \mathcal{P}. \tag{1}$$

An *operational model* is then an operational frame together with a relation $\Vdash$ which indicates what formulas are true at what points. The truth conditions for the implication $\supset$ are defined as follows:

$$w \Vdash \varphi \supset \psi \text{ iff for each } v, \text{ if } v \Vdash \varphi \text{ then } w \sqcup v \Vdash \psi \tag{2}$$

As one can easily notice, an operational frame is a Routley-Meyer frame where $\mathcal{R}wvu$ holds if and only if $w \sqcup v = u$. Hence, the ternary relation $\mathcal{R}$ of the Routley–Meyer semantics is a generalization of the function $\sqcup$ of the Urquhart's semantics. Because it is a *relation*, it allows moreover to apply $w$ to $v$ and yield either a set of outcomes or no outcome at all.

## 2.2 Updates as Ternary Relations

The ternary relation of the Routley and Meyer semantics was introduced originally for technical reasons: any 2-ary ($n$-ary) connective of a logical language can be given a semantics by resorting to a 3-ary (resp. $n+1$-ary) relation on worlds. In fact, this may be the most general and abstract way of providing a semantics for two-ary conditionals of the form $\varphi \supset \psi$. Subsequently, a number of philosophical interpretations of this ternary relation have been proposed and we will briefly recall some of them at the end of this section (see [9, 40, 29] for more details). However, one has to admit that providing a non-circular and conceptually grounded interpretation of this relation remains problematic [9]. In this article we propose a new *dynamic* interpretation.

Our proposal is based on the key observation that an update can be represented abstractly as a ternary relation: the first argument of the ternary relation represents the initial situation/state, the second the event that occurs in this initial situation (the informative input) and the third the resulting situation/state after the occurrence of the event. With this interpretation in mind, $\mathcal{R}xyz$ reads as 'the occurrence of event $y$ in world $x$ results in the world $z$' and the corresponding conditional $\chi \supset \varphi$ reads as 'the occurrence in the current world of an event satisfying property $\chi$ results in a world satisfying $\varphi$'.

This interpretation is coherent with a number of interpretations of the ternary relation proposed in substructural logic. In substructural logics, points are sometimes also called worlds, states, situations, set-ups, and as explained by Restall:

> "We have a class of points (over which $w$ and $v$ vary), and a function $\sqcup$ which gives us new points from old. The point $w \sqcup v$ is supposed, on Urquhart's interpretation, to be the body of information given by combining $w$ with $v$." [40, p. 363]

and also, keeping in mind the truth conditions for the connective $\supset$ of Expression (2):

> "To be committed to $A \supset B$ is to be committed to $B$ whenever we gain the information that $A$. To put it another way, a body of information warrants $A \supset B$ if and only if whenever you *update* that information with new information which warrants $A$, the resulting (perhaps new) body of information warrants $B$." (emphasis added) [40, p. 362]

Moreover, as explained by Restall, this substructural "update" can be nonmonotonic and may correspond to some sort of *revision*:

> "[C]ombination is sometimes nonmonotonic in a natural sense. Sometimes when a body of information is combined with another body of information, some of the original body of information might be lost. This is simplest to see in the case motivating the failure of $A \vdash B \supset A$. A body of information might tell us that $A$. However, when we combine it with something which tells us $B$, the resulting body of information might no longer warrant $A$ (as $A$ might with $B$). Combination might not simply result in the addition of information. It may well warrant its *revision*." (emphasis added) [40, p. 363]

Our dynamic interpretation of the ternary relation is consistent with the above considerations: sometimes updating beliefs amounts to *revise* beliefs.

The dynamic reading of the ternary relation and its corresponding conditional is very much in line with the so-called "Ramsey Test" of conditional logic. The Ramsey test can be viewed as the very first modern contribution to the logical study of conditionals and much of the contemporary work on conditional logic can be traced back to the famous footnote of Ramsey [37].[2] Roughly, it consists in defining a counterfactual conditional in terms of belief revision: an agent currently believes that $\varphi$ would be true if $\psi$ were true (*i.e.* $\psi \supset \varphi$) if and only if he should believe $\varphi$ after *learning* $\psi$. A first attempt to provide truth conditions for conditionals, based on Ramsey's ideas, was proposed by Stalnaker. He defined his semantics by means of selection functions over possible worlds $f : W \times 2^W \to W$. As one can easily notice, Stalnaker's selection functions could also be considered from a formal point of view as a special kind of ternary relation, since a relation $\mathcal{R}_f \subseteq W \times 2^W \times W$ can be canonically associated to each selection function $f$.[3] So, the dynamic reading of the ternary semantics is consistent with the dynamic reading of conditionals proposed by Ramsey.

This dynamic reading was not really considered or investigated by substructural logicians when they connected the substructural ternary semantics with conditional logic [9]. On the other hand, the dynamic reading of inferences has been stressed to a large extent by van Benthem [50, 51] and also by Baltag & Smets [4, 5, 6]. Our *dynamic* interpretation of the ternary semantics of substructural logics is consistent with the interpretations proposed by substructural logicians. In fact, our point of view is also very much in line with the claim of Gärdenfors and Makinson [17, 27] that non-monotonic reasoning and belief revision are "two sides of the same coin": as a matter of fact, non-monotonic reasoning is a reasoning style and belief revision is a sort of update. The formal connection in this case also relies on a similar idea based on the Ramsey test.

To summarize our discussion, our dynamic interpretation of the ternary relation of substructural logic is intuitive and consistent, in the sense that the intuitions underlying this dynamic interpretation are coherent with those underlying the ternary semantics of substructural logics, as witnessed by our quotes and citations from the substructural literature.

**Other interpretations of the ternary relation.** One interpretation, due to Barwise [8] and developed by Restall [38], takes worlds to be 'sites' or 'channels', a site being possibly a channel and a channel being possibly a site. If $x, y$ and $z$ are sites, $\mathcal{R}xyz$ reads as '$x$ is a channel between $y$ and $z$'. Hence, if $\varphi \supset \psi$ is true at channel $x$, it means that all sites $y$ and $z$ connected by channel $x$ are such that if $\varphi$ is information available in $y$, then $\psi$ is information available in $z$. Another similar interpretation due to Mares [28] adapts Israel and Perry's theory of information [34] to

[2]Here is Ramsey's footnote: "If two people are arguing 'If $p$, then $q$?' and are both in doubt as to $p$, they are adding $p$ hypothetically to their stock of knowledge and arguing on that basis about $q$; so that in a sense 'If $p$, $q$' and 'If $p$, $\neg q$' are contradictories. We can say that they are fixing their degree of belief in $q$ given $p$. If $p$ turns out false, these degrees of belief are rendered void. If either party believes not $p$ for certain, the question ceases to mean anything to him except as a question about what follows from certain laws or hypotheses."[37, 154–155]

[3]Note that Burgess [12] already proposed a ternary semantics for conditionals, but his truth conditions and his interpretation of the ternary relation were quite different from ours.

the relational semantics. In this interpretation, worlds are situations in the sense of Barwise and Perry's situation semantics [7] and pieces of information – called *infons* – can carry information about other infons: an infon might carry the information that a red light on a mobile phone carries the information that the battery of the mobile phone is low. In this interpretation, the ternary relation $\mathcal{R}$ represents the informational links in situations: if there is an informational link in situation $x$ that says that an infon $\sigma$ carries the information that the infon $\pi$ also holds, then if $\mathcal{R}xyz$ holds and $y$ contains the infon $\sigma$, then $z$ contains the infon $\pi$. Other interpretations of the ternary relation have been proposed by Beall & Al. [9], with a particular focus on their relation to conditionality. For more information on this topic the reader is invited to consult [30] which covers the material briefly reviewed in this paragraph.

# 3 Update Logic

In this section, we define update logic. After introducing some mathematical definitions in Section 3.1, we motivate in Section 3.2 the introduction of three triples of logical connectives. These connectives generalize the triple $(\otimes, \supset, \subset)$ of substructural logics and will be given a semantics based on cyclic permutations in Section 3.3.

## 3.1 Preliminary Definitions

The general definitions of this section will be used in the rest of the article.

**Definition 5** (Logic)**.** A *logic* is a triple $\mathsf{L} := (\mathcal{L}(\mathbb{P}, \mathsf{F}), E, \models)$ where

- $\mathcal{L}(\mathbb{P}, \mathsf{F})$ is a *logical language* defined as a set of well-formed expressions built from a set of *logical (and structural) connectives* $\mathsf{F}$ and a set of *propositional letters* $\mathbb{P}$;
- $E$ is a *class of pointed models or frames*;
- $\models$ is a *satisfaction relation* which relates in a compositional manner elements of $\mathcal{L}(\mathbb{P}, \mathsf{F})$ to models of $E$ by means of so-called *truth conditions*. $\square$

Note that the above semantically–based definition of a logic is also used by French et Al. [16].

**Example 1.** The triples $(\mathcal{L}(\mathbb{P}, \mathsf{Sub}), \mathcal{E}, \Vdash)$ and $(\mathcal{L}(\mathbb{P}, \mathsf{Sub}_{-}), \mathcal{E}, \Vdash)$ are logics. We list in Figure 2 logics that we deem to be 'classical'. $\square$

**Definition 6** (Expressiveness)**.** Let two logics $\mathsf{L} = (\mathcal{L}, E, \models)$ and $\mathsf{L}' = (\mathcal{L}', E, \models')$ be given (interpreted over the same class of models $E$). Let $\varphi \in \mathcal{L}$ and $\varphi' \in \mathcal{L}'$. We say that $\varphi$ is *as expressive as* $\varphi'$ when $\{\mathcal{M} \in E \mid \mathcal{M} \models \varphi\} = \{\mathcal{M} \in E \mid \mathcal{M} \models \varphi'\}$. We say that $\mathsf{L}$ *has at least the same expressive power as* $\mathsf{L}'$, denoted $\mathsf{L} \geq \mathsf{L}'$, when for all $\varphi' \in \mathcal{L}'$, there is $\varphi \in \mathcal{L}$ such that $\varphi$ is as expressive as $\varphi'$. When $\mathsf{L}$ has at least the same expressive power as $\mathsf{L}'$ and vice versa, we say that $\mathsf{L}$ and $\mathsf{L}'$ *have the same expressive power* and we write it $\mathsf{L} \equiv \mathsf{L}'$. Otherwise, $\mathsf{L}$ is *strictly more expressive than* $\mathsf{L}'$ and we write it $\mathsf{L} > \mathsf{L}'$. $\square$

**Example 2.** It holds that $(\mathcal{L}(\mathbb{P}, \mathsf{Sub}), \mathcal{E}_{\sqsubseteq}, \Vdash) > (\mathcal{L}(\mathbb{P}, \mathsf{Sub}_{-}), \mathcal{E}_{\sqsubseteq}, \Vdash)$. $\square$

| Models $E$ | | | Connectives $\mathsf{F}$ | Logic $(\mathcal{L}(\mathbb{P}, \mathsf{F}), E, \Vdash)$ |
|---|---|---|---|---|
| $\sqsubseteq$ | $S$ | $\mathcal{R}$ | | |
| $=$ | | | $\neg\ \wedge$ | Propositional Logic |
| $=$ | $\bullet$ | | $\neg\ \wedge\ \Box_j$ | Modal Logic |
| $=$ | | $\bullet$ | $\otimes\ \supset\ \subset$ | Lambek Calculus |
| $=$ | $\bullet$ | $\bullet$ | $\neg\ \wedge\ \Box_j\ \otimes\ \supset\ \subset$ | Modal Lambek Calculus |

Figure 2: 'Classical' Logics

### 3.2 Talking about Ternary Relations

If we want to reason about updates, we must be able to express properties of updates. In other words, we need a language for talking about updates. Since we represent them by ternary relations, it seems natural to require that our language be able to express properties that relate what is true at each point of the ternary relations, that is, what is true at: 1. the initial situation (expressed by a formula $\varphi$), 2. the event occurring in this situation (expressed by a second formula $\chi$), 3. the resulting situation after the event has occurred (expressed by a third formula $\psi$):

$$1, \varphi \longrightarrow 2, \chi \longrightarrow 3, \psi$$

This leads us to the following general question: assume that we stand in one of these three time points $x$ (be it 1, 2 or 3), what kind of property can we express and infer about the other time points $y$ and $z$? Here is a non-exhaustive list of the possible and most natural expressions that we would want to state:

(a) For all $y$, if $y$ satisfies $\varphi$ then for all $z$, $z$ satisfies $\psi$: "$x \models \forall y \forall z(\varphi(y) \to \psi(z))$".

For example, in the initial state 1, is it the case that *any* event satisfying $\chi$ will *always* lead to a state 3 satisfying $\psi$? Or, in state 3, is it the case that before the occurrence of *any* event satisfying $\chi$, $\varphi$ held in *all* initial states 1?

(b) There exist $y$ and $z$ such that $y$ satisfies $\varphi$ and $z$ satisfies $\psi$: "$x \models \exists y \exists z(\varphi(y) \wedge \psi(z))$".

For example, in state 1, is it the case that there exists an event satisfying $\chi$ that may lead to a state where $\psi$ holds ? Or, in state 3, is it possible that our current state might have been the result of an event satisfying $\chi$ in an initial state where $\varphi$ held?

(c) For all $y$ satisfying $\varphi$, there exists $z$ satisfying $\psi$: "$x \models \forall y \exists z(\varphi(y) \to \psi(z))$".

For example, in state 1, is it the case that any events satisfying $\chi$ may lead possibly to a state where $\psi$ holds ? Or, in state 3, is it the case that an event satisfying $\chi$ might have occurred so that any former situation before this event satisfied $\varphi$?

This list of expressions is obviously non-exhaustive. Providing formal tools that answer these kinds of questions leads to applications in artificial intelligence and theoretical computer science, and as it turns out, some of these questions have already been addressed in dynamic epistemic logic and other logical formalisms (see the companion article [3, Sect. 7.2] for more details and examples). Typically, most of the works about conditionals and belief dynamics deal with the first kind of statements (a) or (b). In fact, the conditionals $\supset$ and $\subset$ of substructural and relevance logics of the previous section are of the form (a), whereas the substructural connective $\otimes$ is of the form (b). The language that we will define will only deal with the first two kinds of expressions (a) and (b) (Section 3.3). This language is intended to capture the various conditionals and belief change operators which have been introduced in the philosophical and artificial intelligence literature. As shown in [3], it captures very well the operators of Dynamic Epistemic Logic.

### 3.3 Syntax and Semantics of Update Logic

We define formally formulas, structures and then consecutions (sometimes called sequents in the literature). This is an incremental definition and each of these objects is defined on the basis of the previous one. Moreover, in the sequel, we will view sets of formulas, sets of structures and sets of consecutions as logical languages.

**Notation 1.** *In the rest of this article, we will use the following* logical connectives $\mathsf{Form}$ *and* structural connectives $\mathsf{Struc}$*:*

$$\mathsf{Form} := \left\{\Box_j, \Diamond_j^-, \Diamond_j, \Box_j^-, \neg, \vee, \wedge, \rightarrow, \otimes_i, \supset_i, \subset_i, \oplus_i, \prec_i, \succ_i \mid i \in \{1,2,3\}, j \in \mathbb{G}\right\}$$
$$\mathsf{Struc} := \{*, \bullet_j, \ ,_i \mid i \in \{0,1,2,3\}, j \in \mathbb{G}\}$$

*The connectives* $\vee, \wedge, \rightarrow, \otimes_i, \supset_i, \subset_i, \oplus_i, \prec_i, \succ_i, \ ,_0, \ ,_i$ *(where $i$ ranges over $\{1,2,3\}$) are binary connectives and* $\Box_j, \Diamond_j^-, \Diamond_j, \Box_j^-, \neg, *, \bullet_j$ *are unary connectives (where $j$ ranges over $\mathbb{G}$). The structural connective* $,_0$ *will often simply be denoted* $,$ .

**Definition 7** (Formula, structure and consecution)**.**

- Let $\mathsf{F} \subseteq \mathsf{Form}$ be a non-empty set of logical connectives. The *language associated to* $\mathsf{F}$, denoted $\mathcal{L}(\mathbb{P}, \mathsf{F})$, is the language built compositionally from the connectives of $\mathsf{F}$ and the set of propositional letters $\mathbb{P}$. Elements of the language $\mathcal{L}(\mathbb{P}, \mathsf{F})$ are called $\mathcal{L}(\mathbb{P}, \mathsf{F})$–*formulas* and are generally denoted $\varphi, \chi, \psi, \ldots$

- Let $\mathsf{S} \subseteq \mathsf{Struc}$ and $\mathsf{F} \subseteq \mathsf{Form}$ be non-empty sets of structural connectives and logical connectives. The *set of structures associated to* $\mathsf{F}$ *and* $\mathsf{S}$, denoted $\mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{S})$, is the language built compositionally from the structural connectives of $\mathsf{S}$ and the set $\mathcal{L}(\mathbb{P}, \mathsf{F})$. Elements of the language $\mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{S})$ are called $\mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{S})$–*structures* and are generally denoted $X, Y, Z, \ldots$

  The *structural connectives associated to* $\mathsf{F}$, denoted $\mathsf{Struc}(\mathsf{F})$, is the set of structural connectives $\{*, \ ,_0\}$ together with $\{\ ,_1\ , \ ,_2\ , \ ,_3\}$ if $\mathsf{F} \cap \{\otimes_i, \supset_i, \subset_i, \oplus_i, \prec_i, \succ_i \mid i \in \{1,2,3\}\} \neq \emptyset$ and with $\{\bullet_j\}$ if $\mathsf{F} \cap \left\{\Box_j, \Diamond_j^-, \Diamond_j, \Box_j^- \mid j \in \mathbb{G}\right\} \neq \emptyset$. We denote by $\mathcal{S}(\mathbb{P}, \mathsf{F})$ the set of all $\mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{Struc}(\mathsf{F}))$–structures.

- Let $\mathsf{S} \subseteq \mathsf{Struc}$ and $\mathsf{F} \subseteq \mathsf{Form}$ be non-empty sets of structural connectives and logical connectives. A $\mathcal{S}(\mathbb{P}, F, S)$–*consecution* is an expression of the form $X \vdash Y$, $X \vdash$ or $\vdash Y$, where $X, Y \in \mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{S})$. The $\mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{S})$–structure $X$ is called the *antecedent* and the $\mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{S})$–structure $Y$ is called the *consequent*. We denote by $\mathcal{C}(\mathbb{P}, \mathsf{F})$ the set of all $\mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{Struc}(\mathsf{F}))$–consecutions.

To avoid any ambiguity, every occurence of any binary connective is surrounded by brackets. $\square$

**Example 3.** If $\mathsf{F} = \{\neg, \wedge, \otimes_3, \supset_1, \subset_2\}$, then the language $\mathcal{L}(\mathbb{P}, \mathsf{F})$ is defined by the following grammar in BNF, where $p$ ranges over $\mathbb{P}$:

$$\mathcal{L}(\mathbb{P}, \mathsf{F}): \quad \varphi \ ::= \ p \ | \ \neg\varphi \ | \ (\varphi \wedge \varphi) \ | \ (\varphi \otimes_3 \varphi) \ | \ (\varphi \supset_1 \varphi) \ | \ (\varphi \subset_2 \varphi)$$

Then, we have that $\mathsf{Struc}(\mathsf{F}) = \{*, \ ,_0 \ , \ ,_i \ | \ i \in \{1, 2, 3\}\}$. So, the language $\mathcal{S}(\mathbb{P}, \mathsf{F}) := \mathcal{S}(\mathbb{P}, \mathsf{F}, \mathsf{Struc}(\mathsf{F}))$ is defined by the following grammar in BNF, where $\varphi$ ranges over $\mathcal{L}(\mathbb{P}, \mathsf{F})$ and $i$ ranges over $\{1, 2, 3\}$:

$$\mathcal{S}(\mathbb{P}, \mathsf{F}): \ X \ ::= \ \varphi \ | \ *X \ | \ (X \ ,_0 X) \ | \ (X \ ,_i X)$$

$\square$

**Notation 2.** *To save parenthesis, we use the following ranking of binding strength: $\otimes_i, \supset_i, \subset_i, \wedge, \vee, \rightarrow$ (where $i$ ranges over $\{1, 2, 3\}$). For example, $\square_1\neg p \wedge q \rightarrow \neg r \otimes_3 s$ stands for $((\square_1(\neg p)) \wedge q) \rightarrow ((\neg r) \otimes_3 s)$ (additional brackets have been added for the unary connectives $\square_1$ and $\neg$, even if they are not needed and will not appear in any formula anyway). For every binary connective $\star$, we use the following notation: $X_1 \star \ldots \star X_n := ((\ldots(X_1 \star \ldots \star X_{n-2}) \star X_{n-1}) \star X_n)$. For example, $\varphi_1 \vee \ldots \vee \varphi_n := ((\ldots(\varphi_1 \vee \ldots \vee \varphi_{n-2}) \vee \varphi_{n-1}) \vee \varphi_n)$ and $X_1, \ \ldots, X_n := ((\ldots(X_1, \ \ldots, X_{n-2}), X_{n-1}), X_n)$. Moreover, if $\Gamma := \{\varphi_1, \ldots, \varphi_n\}$ is a finite set of formulas and $\star$ is a binary connective over formulas, we use the following notation: $\star\Gamma := \varphi_1 \star \ldots \star \varphi_n$. For example, $\bigvee\Gamma := \varphi_1 \vee \ldots \vee \varphi_n$ and $\bigwedge\Gamma := \varphi_1 \wedge \ldots \wedge \varphi_n$.*

In the sequel, we assume that the point sets of all substructural models and frames are flat, *i.e.* $\sqsubseteq$ is the equality relation $=$. So, the class of pointed substructural models and frames that we consider are $\mathcal{E}$ and $\mathcal{F}$ respectively. This entails that we will not consider the connective $\Rightarrow$ of substructural logics.

**Definition 8** (Update logic)**.** Let $E$ be an arbitrary set of three elements. For each $i \in \{1, 2, 3\}$, we define the cyclical permutations $\sigma_i : E^3 \mapsto E^3$ as follows: for all $x, y, z \in E$,

$$\sigma_1(x, y, z) = (x, y, z) \qquad \sigma_2(x, y, z) = (z, x, y) \qquad \sigma_3(x, y, z) = (y, z, x).$$

- We define the *evaluation relation* $\Vdash \ \subseteq \mathcal{E} \times \mathcal{L}(\mathbb{P}, \mathsf{Form})$ inductively as follows. Let $(\mathcal{M}, w) \in \mathcal{E}$ be a pointed substructural model and let $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form})$. The truth conditions for the connectives $\square_j, \Diamond_j^-, \wedge, \vee$ are defined like in Definition 4. The truth condition for the Boolean negation is defined as follows:

$$\mathcal{M}, w \Vdash \neg\varphi \quad \text{iff} \quad \text{it is not the case that } \mathcal{M}, w \Vdash \varphi.$$

The truth conditions for the connectives $\otimes_{\text{i}}, \supset_{\text{i}}, \subset_{\text{i}}$ are defined as follows: for all $i \in \{1,2,3\}$, we have that

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash \varphi \otimes_{\text{i}} \psi & \text{iff} & \text{there are } v, u \in \mathcal{P} \text{ such that } \sigma_i(w,v,u) \in \mathcal{R}, \\
& & \mathcal{M}, v \Vdash \varphi \text{ and } \mathcal{M}, u \Vdash \psi; \\
\mathcal{M}, w \Vdash \varphi \supset_{\text{i}} \psi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \sigma_i(w,v,u) \in \mathcal{R}, \\
& & \text{if } \mathcal{M}, v \Vdash \varphi \text{ then } \mathcal{M}, u \Vdash \psi; \\
\mathcal{M}, w \Vdash \varphi \subset_{\text{i}} \psi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \sigma_i(w,v,u) \in \mathcal{R}, \\
& & \text{if } \mathcal{M}, u \Vdash \psi \text{ then } \mathcal{M}, v \Vdash \varphi.
\end{array}
$$

The truth conditions for the connectives $\oplus_{\text{i}}, \prec_{\text{i}}, \succ_{\text{i}}$ are defined as follows: for all $i \in \{1,2,3\}$, we have that

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash \varphi \oplus_{\text{i}} \psi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \sigma_i(w,v,u) \in \mathcal{R}, \\
& & \mathcal{M}, v \Vdash \varphi \text{ or } \mathcal{M}, u \Vdash \psi; \\
\mathcal{M}, w \Vdash \varphi \prec_{\text{i}} \psi & \text{iff} & \text{there are } v, u \in \mathcal{P} \text{ such that } \sigma_i(w,v,u) \in \mathcal{R}, \\
& & \mathcal{M}, u \Vdash \varphi \text{ and not } \mathcal{M}, v \Vdash \psi; \\
\mathcal{M}, w \Vdash \varphi \succ_{\text{i}} \psi & \text{iff} & \text{there are } v, u \in \mathcal{P} \text{ such that } \sigma_i(w,v,u) \in \mathcal{R}, \\
& & \mathcal{M}, v \Vdash \psi \text{ and not } \mathcal{M}, u \Vdash \varphi.
\end{array}
$$

The truth conditions for the connectives $\Diamond_j, \Box_j^-$ are defined as follows:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash \Diamond_j \varphi & \text{iff} & \text{there is } v \in \mathcal{P} \text{ such that } wRv \text{ and } \mathcal{M}, v \Vdash \varphi; \\
\mathcal{M}, w \Vdash \Box_j^- \varphi & \text{iff} & \text{for all } v \in \mathcal{P} \text{ such that } vSw, \mathcal{M}, v \Vdash \varphi.
\end{array}
$$

- We extend the scope of the evaluation relation $\Vdash$ simultaneously in two different ways in order to also relate points to $\mathcal{S}(\mathbb{P}, \mathsf{Form})$–structures. The *antecedent evaluation relation* $\Vdash^{A} \subseteq \mathcal{E} \times \mathcal{S}(\mathbb{P}, \mathsf{Form})$ is defined inductively as follows: for all $i \in \{1,2,3\}$,

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash^{A} \varphi & \text{iff} & \mathcal{M}, w \Vdash \varphi; \\
\mathcal{M}, w \Vdash^{A} * X & \text{iff} & \text{it is not the case that } \mathcal{M}, w \Vdash^{K} X; \\
\mathcal{M}, w \Vdash^{A} \bullet_{\text{j}} X & \text{iff} & \text{there is } v \in \mathcal{M} \text{ such that } vS_j w \\
& & \text{and it holds that } \mathcal{M}, v \Vdash^{A} X; \\
\mathcal{M}, w \Vdash^{A} X, Y & \text{iff} & \mathcal{M}, w \Vdash^{A} X \text{ and } \mathcal{M}, w \Vdash^{A} Y; \\
\mathcal{M}, w \Vdash^{A} X \,;_{\text{i}}\, Y & \text{iff} & \text{there are } v, u \in \mathcal{M} \text{ such that } \sigma_i(w,v,u) \in S, \\
& & \mathcal{M}, v \Vdash^{A} X \text{ and } \mathcal{M}, u \Vdash^{A} Y.
\end{array}
$$

The *consequent evaluation relation* $\Vdash^{K} \subseteq \mathcal{E} \times \mathcal{S}(\mathbb{P}, \mathsf{Form})$ is defined inductively as follows: for all $i \in \{1,2,3\}$,

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash^{K} \varphi & \text{iff} & \mathcal{M}, w \Vdash \varphi; \\
\mathcal{M}, w \Vdash^{K} * X & \text{iff} & \text{it is not the case that } \mathcal{M}, w \Vdash^{A} X; \\
\mathcal{M}, w \Vdash^{K} \bullet_{\text{j}} X & \text{iff} & \text{for all } v \in \mathcal{M} \text{ such that } wS_j v, \\
& & \text{it holds that } \mathcal{M}, v \Vdash^{K} X; \\
\mathcal{M}, w \Vdash^{K} X, Y & \text{iff} & \mathcal{M}, w \Vdash^{K} X \text{ or } \mathcal{M}, w \Vdash^{K} Y; \\
\mathcal{M}, w \Vdash^{K} X \,;_{\text{i}}\, Y & \text{iff} & \text{for all } v, u \in \mathcal{M} \text{ such that } \sigma_i(w,v,u) \in \mathcal{R}, \\
& & \mathcal{M}, v \Vdash^{K} X \text{ or } \mathcal{M}, u \Vdash^{K} Y.
\end{array}
$$

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash \chi \otimes_1 \psi & \text{iff} & \text{there are } v, u \in \mathcal{P} \text{ such that } \mathcal{R}wvu, \\
& & \mathcal{M}, v \Vdash \chi \text{ and } \mathcal{M}, u \Vdash \psi; \\
\mathcal{M}, w \Vdash \chi \supset_1 \psi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \mathcal{R}wvu, \\
& & \text{if } \mathcal{M}, v \Vdash \chi \text{ then } \mathcal{M}, u \Vdash \psi; \\
\mathcal{M}, w \Vdash \chi \subset_1 \psi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \mathcal{R}wvu, \\
& & \text{if } \mathcal{M}, u \Vdash \psi \text{ then } \mathcal{M}, v \Vdash \chi; \\
\mathcal{M}, w \Vdash \psi \otimes_2 \varphi & \text{iff} & \text{there are } v, u \in \mathcal{P} \text{ such that } \mathcal{R}uwv, \\
& & \mathcal{M}, u \Vdash \varphi \text{ and } \mathcal{M}, v \Vdash \psi; \\
\mathcal{M}, w \Vdash \psi \supset_2 \varphi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \mathcal{R}uwv, \\
& & \text{if } \mathcal{M}, v \Vdash \psi \text{ then } \mathcal{M}, u \Vdash \varphi; \\
\mathcal{M}, w \Vdash \psi \subset_2 \varphi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \mathcal{R}uwv, \\
& & \text{if } \mathcal{M}, u \Vdash \varphi \text{ then } \mathcal{M}, v \Vdash \psi; \\
\mathcal{M}, w \Vdash \varphi \otimes_3 \chi & \text{iff} & \text{there are } v, u \in \mathcal{P} \text{ such that } \mathcal{R}vuw, \\
& & \mathcal{M}, v \Vdash \varphi \text{ and } \mathcal{M}, u \Vdash \chi; \\
\mathcal{M}, w \Vdash \varphi \supset_3 \chi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \mathcal{R}vuw, \\
& & \text{if } \mathcal{M}, v \Vdash \varphi \text{ then } \mathcal{M}, u \Vdash \chi; \\
\mathcal{M}, w \Vdash \varphi \subset_3 \chi & \text{iff} & \text{for all } v, u \in \mathcal{P} \text{ such that } \mathcal{R}vuw, \\
& & \text{if } \mathcal{M}, u \Vdash \chi \text{ then } \mathcal{M}, v \Vdash \varphi.
\end{array}
$$

Figure 3: Spelling out the Truth Conditions

- We extend the scope of the relation $\Vdash$ to also relate points to $\mathcal{S}(\mathbb{P}, \mathsf{Form})$–consecutions. Depending on the form of the $\mathcal{S}(\mathbb{P}, \mathsf{Form})$–consecution, that is, whether it is of the form $X \vdash Y$, $\vdash Y$ or $X \vdash$, we have:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash X \vdash Y & \text{iff} & \text{if } \mathcal{M}, w \Vdash^{A} X, \text{ then } \mathcal{M}, w \Vdash^{K} Y; \\
\mathcal{M}, w \Vdash \ \vdash Y & \text{iff} & \mathcal{M}, w \Vdash^{K} Y; \\
\mathcal{M}, w \Vdash X \vdash & \text{iff} & \text{it is not the case that } \mathcal{M}, w \Vdash^{A} X.
\end{array}
$$

So, for all $\mathsf{F} \subseteq \mathsf{Form}$, the triples $(\mathcal{L}(\mathbb{P}, \mathsf{F}), \mathcal{E}, \Vdash)$, $(\mathcal{S}(\mathbb{P}, \mathsf{F}), \mathcal{E}, \Vdash^{A})$, $(\mathcal{S}(\mathbb{P}, \mathsf{F}), \mathcal{E}, \Vdash^{K})$ and $(\mathcal{C}(\mathbb{P}, \mathsf{F}), \mathcal{E}, \Vdash)$ are logics (as defined in Definition 5). The triple $(\mathcal{L}(\mathbb{P}, \mathsf{Form}), \mathcal{F}, \Vdash)$ is also a logic, called *update logic*. $\square$

Spelling out the truth conditions for the connectives $\otimes_i, \supset_i$ and $\subset_i$ for $i \in \{1, 2, 3\}$, we obtain the expressions of Figure 3. The indices 1, 2 and 3 of our connectives indicate when formulas are evaluated. The connectives $\supset_1, \subset_1$ and $\otimes_1$ express properties of updates *before* the event, the connectives $\subset_2, \supset_2$ and $\otimes_2$ properties *during* the event and the connectives $\supset_3, \subset_3$ and $\otimes_3$ properties *after* the event. Typically, the formula $\varphi$ deals with the initial situation, the formula $\chi$ deals with the event and the formula $\psi$ deals with the final situation. The direction of the arrow ($\subset$ or $\supset$) indicates the conditional direction in which the formula should be read. For example, the formula $\psi \supset_2 \varphi$ tells us that it should be evaluated during an event (2) and reads as "*if* the final situation will satisfy $\psi$ *then* the initial situation must necessarily satisfy

$\varphi$", whereas $\psi \subset_2 \varphi$ reads as "*if* the initial situation satisfies $\varphi$ *then* the final situation will necessarily satisfy $\psi$". The formula $\chi \supset_1 \psi$ reads as "$\psi$ will hold after the occurrence of any events satisfying $\chi$" and the formula $\varphi \subset_3 \chi$ reads as "$\varphi$ held before the occurrence of any events satisfying $\chi$". The connectives $\otimes_1, \otimes_2, \otimes_3$ are of the form (b) and the connectives $\supset_1, \subset_1, \supset_2, \subset_2$ , $\supset_3, \subset_3$ are of the form (a) (see page 10). Note that the classical substructural connectives $\otimes, \supset$ and $\subset$ of the previous section correspond to our connectives $\otimes_3, \supset_1$ and $\subset_2$. So, our language $\mathcal{L}(\mathbb{P}, \mathsf{Form})$ extends the language $\mathcal{L}(\mathbb{P}, \mathsf{Sub}_-)$ of substructural logics presented in Section 2.1 and the logic $(\mathcal{L}(\mathbb{P}, \mathsf{Form}), \mathcal{E}, \Vdash)$ is therefore at least as expressive as $(\mathcal{L}(\mathbb{P}, \mathsf{Sub}_-), \mathcal{E}, \Vdash)$. In fact, $(\mathcal{L}(\mathbb{P}, \mathsf{Form}), \mathcal{E}, \Vdash)$ is strictly more expressive than $(\mathcal{L}(\mathbb{P}, \mathsf{Sub}_-), \mathcal{E}, \Vdash)$, as proved in [2].

# 4 Display Calculus for Update Logic

Extending Gentzen's original sequent calculi with modalities has turned out over the years to be difficult. Many of the interesting theoretical properties of sequent calculi are lost when one adds modalities (see for example [35, Chapter 1] for more details). A number of methods have been proposed to overcome these difficulties: display calculi, labelled sequents, tree hypersequents (see [36] for an accessible introduction to these different sorts of calculi). In this section, we provide a display calculus for our update logic. This display calculus is a generalization of the display calculus for modal logic introduced by Wansing [52] and the sequent calculus will be a generalization of the non-associative Lambek calculus NL [25, 26].

## 4.1 Preliminary Definitions

The general definitions of this section will be used in the rest of the article. Our definition of a proof system and of an inference rule is taken from [31].

**Definition 9** (Proof system and sequent calculus)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. A *proof system* $\mathcal{P}$ for $\mathcal{L}$ is a set of elements of $\mathcal{L}$ called *axioms* and a set of *inference rules*. Most often, one can effectively decide whether a given element of $\mathcal{L}$ is an axiom. To be more precise, an *inference rule* $R$ in $\mathcal{L}$ is a relation among elements of $\mathcal{L}$ such that there is a unique $l \in \mathbb{N}^*$ such that, for all $\varphi, \varphi_1, \ldots, \varphi_l \in \mathcal{L}$, one can effectively decide whether $(\varphi_1, \ldots, \varphi_l, \varphi) \in R$. The elements $\varphi_1, \ldots, \varphi_l$ are called the *premises* and $\varphi$ is called the *conclusion* and we say that $\varphi$ is a *direct consequence of* $\varphi_1, \ldots, \varphi_l$ by virtue of $R$. Let $\Gamma \subseteq \mathcal{L}$ and let $\varphi \in \mathcal{L}$. We say that $\varphi$ is *provable* (from $\Gamma$) in $\mathcal{P}$ or a *theorem* of $\mathcal{P}$, denoted $\vdash_{\mathcal{P}} \varphi$ (resp. $\Gamma \vdash_{\mathcal{P}} \varphi$), when there is a *proof* of $\varphi$ (from $\Gamma$) in $\mathcal{P}$, that is, a finite sequence of formulas ending in $\varphi$ such that each of these formulas is:

1. either an instance of an axiom of $\mathcal{P}$ (or a formula of $\Gamma$);
2. or the direct consequence of preceding formulas by virtue of an inference rule $R$.

If $\mathcal{S}$ is a set of $\mathcal{L}$–consecutions, this set $\mathcal{S}$ can be viewed as a logical language. Then, we call *sequent calculus for* $\mathcal{S}$ a proof system for $\mathcal{S}$. □

**Definition 10** (Truth, validity, logical consequence)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let $\mathcal{M} \in E$, $\varphi \in \mathcal{L}$ and $R, R'$ inference rule in $\mathcal{L}$. If $\Gamma$ is a set of formulas or inference rules, we write $\mathcal{M} \models \Gamma$ when for all $x \in \Gamma$, we have $\mathcal{M} \models x$. Then, we say that

- $\varphi$ is *true* (*satisfied*) at $\mathcal{M}$ or $\mathcal{M}$ is a *model* of $\varphi$ when $\mathcal{M} \models \varphi$;
- $\varphi$ is a *logical consequence* of $\Gamma$, denoted $\Gamma \models_{\mathsf{L}} \varphi$, when for all $\mathcal{M} \in E$, if $\mathcal{M} \models \Gamma$ then $\mathcal{M} \models \varphi$;
- $\varphi$ is *valid*, denoted $\models_{\mathsf{L}} \varphi$, when for all models $\mathcal{M} \in E$, we have $\mathcal{M} \models \varphi$;
- $R$ is *true* (*satisfied*) at $\mathcal{M}$ or $\mathcal{M}$ is a *model* of $R$, denoted $\mathcal{M} \models R$, when for all $(\varphi_1, \ldots, \varphi_l, \varphi) \in R$, if $\mathcal{M} \models \varphi_i$ for all $i \in \{1, \ldots, l\}$, then $\mathcal{M} \models \varphi$.
- $R$ is *equivalent* to $R'$, denoted $R \equiv R'$, when for all $\mathcal{M} \in E$, $\mathcal{M} \models R$ if, and only if, $\mathcal{M} \models R'$. $\square$

**Definition 11** (Soundness and completeness)**.** Let $\mathsf{L} = (\mathcal{L}, E, \models)$ be a logic. Let $\mathcal{P}$ be a proof system for $\mathcal{L}$. Then,

- $\mathcal{P}$ is *sound* for the logic $\mathsf{L}$ when for all $\varphi \in \mathcal{L}$, if $\vdash_{\mathcal{P}} \varphi$, then $\models_{\mathsf{L}} \varphi$.
- $\mathcal{P}$ is *(strongly) complete* for the logic $\mathsf{L}$ when for all $\varphi \in \mathcal{L}$ (and all $\Gamma \subseteq \mathcal{L}$), if $\models_{\mathsf{L}} \varphi$, then $\vdash_{\mathcal{P}} \varphi$ (resp. if $\Gamma \models_{\mathsf{L}} \varphi$, then $\Gamma \vdash_{\mathcal{P}} \varphi$). $\square$

**Definition 12** (Parameter, congruent parameter and principal formula)**.** A *parameter* in an inference rule is a structure (or formula) which is either held constant from premises to conclusion or which is introduced with no regard to its particular (formulas introduced by weakening are also parameters). A *principal* formulas in an inference rule is a non–parametric formula occurring in the conclusion. *Congruent* parameters in an inference rules are parameters that occur both in the premise(s) and the conclusion of that inference rule and that correspond to the same formula/structure. In our display calculi (like the display calulus $\mathsf{UL}$ of Figures 4 and 6), principal formulas are represented by Greek formulas $\varphi, \psi$ and parameters are denoted by the Latin letters $X, Y, Z$. Congruent parameters are denoted by the same Latin letter (be it $X, Y$ or $Z$). $\square$

See [10, 53] for more detailed explanations of the conditions $(C1) - (C8)$ listed below.

**Definition 13** (Proper display calculus and analytic inference rule)**.** An inference rule is *analytic* when it satisfies the following eight conditions $(C1) - (C8)$. A sequent calculus is a *proper display calculus* when each of its inference rules satisfies the following eight conditions $(C1) - (C8)$:

$(C1)$ *Preservation of formulas*. Each formula occurring in a premise of a rule is the subformula of some formula in the conclusion of that rule.

$(C2)$ *Shape-alikeness of parameters.* Congruent parameters in a rule are occurrences of the same structures.

$(C3)$ *Non-proliferation of parameters*. Each parameter of any rule is congruent to at most one parameter in the conclusion of that rule.

$(C4)$ *Position-alikeness of parameters*. Congruent parameters are either all antecedent or all consequent parts of their respective consecutions.

$(C5)$ *Display of principal constituents*. A principal formula of any rule is either the entire antecedent or the entire consequent of the conclusion of this rule.

$(C6)$ *Closure under substitution for consequent parts*. Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas which are consequent parts.

$(C7)$ *Closure under substitution for antecedent parts*. Each rule is closed under simultaneous substitution of arbitrary structures for congruent formulas which are antecedent parts.

$(C8)$ *Eliminability of matching principal formulas*. If there are inferences $\mathrm{inf}_1$ and $\mathrm{inf}_2$ with respective conclusions $(1)$ $X \vdash \varphi$ and $(2)$ $\varphi \vdash Y$ with $\varphi$ principal in both inferences, and if cut is applied to obtain $(3)$ $X \vdash Y$, then either $(3)$ is identical to $(1)$ or $(2)$, or there is a proof of $(3)$ from the premises of $\mathrm{inf}_1$ and $\mathrm{inf}_2$ in which every cut-formula of any application of cut is a proper formula of $\varphi$. $\square$

**Definition 14** (Displayable logic)**.** A logic $\mathsf{L} = (\mathcal{L}, E, \models)$ is *(properly) displayable* when there is a (proper) display calculus $\mathcal{P}$ which is sound and complete for $\mathsf{L}$. $\square$

## 4.2 A Generalized Modal Display Calculus

In this section, we introduce a display calculus for our update logic. It generalizes the modal display calculus of Wansing [52].

**Definition 15** (Display calculus $\mathsf{UL}(\mathsf{F})$)**.** Let $\mathsf{F} \subseteq \mathsf{Form}$. The *display calculus for* $\mathcal{C}\,(\mathbb{P}, F)$, denoted $\mathsf{UL}(\mathsf{F})$, is the display calculus containing the rules of Figure 6 mentioning the logical connectives of $\mathsf{F}$ and the rules of Figure 4 mentioning the structural connectives of $\mathsf{Struc}(\mathsf{F})$ (a double line means that the rule holds in both directions). When $\mathsf{F} = \mathsf{Form}$, the display calculus $\mathsf{UL}(\mathsf{F})$ is denoted $\mathsf{UL}$. In these rules, $U$ and $V$ can be empty structures and in that case $U, X$ denotes $X$. Moreover, in rule $\otimes^{\mathrm{i}}_K$ (for $i \in \{1, 2, 3\}$), the consequent of one of the premises can also be empty and in that case the consequent of the conclusion is also empty. For better readability, the brackets for binary connectives are omitted. $\square$

**Admissibility of the Cut Rule.** Theorem 1 below shows that $\mathsf{UL}$ is a *display* calculus: each antecedent (consequent) part of a consecution can be 'displayed' as the sole antecedent (resp. consequent) of a structurally equivalent consecution.

**Definition 16** (Antecedent and consequent part)**.** Let $X$ be a $\mathcal{S}\,(\mathbb{P}, \mathsf{Form})$–structure and let $Y$ be a substructure of $X$. We say that $Y$ *occurs positively in* $X$ if it is in the scope of an even number of $*$. Otherwise, if $Y$ is in the scope of an odd number of $*$ in $X$, we say that $Y$ *occurs negatively in* $X$. If $X \vdash Y$ is a $\mathcal{S}\,(\mathbb{P}, \mathsf{Form})$–consecution, then $X$ is called the antecedent and

*Classical Rules:*

$$\frac{U \vdash V}{U , X \vdash V}\ \mathsf{K} \qquad \frac{X , X \vdash U}{X \vdash U}\ \mathsf{WI} \qquad \frac{Y , X \vdash U}{X , Y \vdash U}\ \mathsf{CI} \qquad \frac{(X , Y) , Z \vdash U}{X , (Y , Z) \vdash U}\ \mathsf{B}^c$$

$$\frac{U \vdash Y}{U ,\ * Y \vdash}\ *_A \qquad \frac{U , Y \vdash}{U \vdash\ * Y}\ *_K$$

*Display Rules:*

$$\begin{array}{c} X \,{,}_{\mathrm{i}}\, Y \vdash Z \\ \hline\hline X \vdash * Y \,{,}_{\mathrm{j}}\, Z \\ \hline\hline Y \vdash Z \,{,}_{\mathrm{k}}\, * X \end{array} \qquad \begin{array}{c} Z \vdash X \,{,}_{\mathrm{i}}\, Y \\ \hline\hline * Y \,{,}_{\mathrm{j}}\, Z \vdash X \\ \hline\hline Z \,{,}_{\mathrm{k}}\, * X \vdash Y \end{array} \qquad \begin{array}{c} X \vdash \bullet_{\mathrm{j}} Y \\ \hline\hline \bullet_{\mathrm{j}} X \vdash Y \end{array}$$

*Cut Rule:*

$$\frac{U \vdash \varphi \quad \varphi \vdash V}{U \vdash V}$$

$(i, j, k) \in \{(0, 0, 0), (1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ and $j \in \mathbb{G}$

Figure 4: Display Calculus $\mathsf{UL}$: Structural Rules

$Y$ is called the consequent. Let $Z$ be a substructure of $X$ or $Y$. We say that $Z$ is an *antecedent part* of $X \vdash Y$ if $Z$ occurs positively in $X$ or negatively in $Y$. We say that $Z$ is a *consequent part* of $X \vdash Y$ if $Z$ occurs positively in $Y$ or negatively in $X$. $\square$

**Theorem 1** (Display Theorem [2])**.** *For each $\mathcal{S}(\mathbb{P}, \mathsf{Form})$–consecution $X \vdash Y$ and each antecedent part (respectively consequent part) $Z$ of $X \vdash Y$, if $X \vdash Y$ then there exists a $\mathcal{L}(\mathbb{P}, \mathsf{Form}, \mathsf{Struc})$–structure $W$ such that $Z \vdash W$ (respectively $W \vdash Z$).*

**Theorem 2** (Strong cut elimination [2])**.** *The display calculus $\mathsf{UL}$ is a* proper *display calculus. Hence, $\mathsf{UL}$ enjoys strong cut-elimination and therefore the cut rule is an admissible rule of $\mathsf{UL}$.*

# 5 Tense Logic as a Lingua Franca

Our second language is a multi-modal language. We break down the ternary relation $\mathcal{R}$ into two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ and define modalities that allow us to quantify at each point of the ternary relation over the next points or the previous ones. A similar approach was already followed in [23, 24]. This increases the expressivity of our language since we have more flexibility for combining the modal operators, in particular we can alternate existential modalities $\exists_1, \exists_2, \exists_1^-, \exists_2^-$ and universal modalities $\forall_1, \forall_2, \forall_1^-, \forall_2^-$.

$$w \xrightarrow[\mathcal{R}_1]{} v \xrightarrow[\mathcal{R}_2]{} u$$

*Axiom:*

$$\frac{}{p \vdash p}\ \mathsf{Id}$$

*Propositional Connectives:*

$$\frac{X \vdash *\varphi}{X \vdash \neg\varphi}\ \neg_K \qquad \frac{*\varphi \vdash X}{\neg\varphi \vdash X}\ \neg_A$$

$$\frac{X \vdash \varphi}{X \vdash \varphi \vee \psi}\ \vee_K^1 \qquad \frac{X \vdash \psi}{X \vdash \varphi \vee \psi}\ \vee_K^2 \qquad \frac{\varphi \vdash X \quad \psi \vdash X}{\varphi \vee \psi \vdash X}\ \vee_A$$

$$\frac{X \vdash \varphi \quad X \vdash \psi}{X \vdash \varphi \wedge \psi}\ \wedge_K \qquad \frac{\varphi \vdash X}{\varphi \wedge \psi \vdash X}\ \wedge_A^1 \qquad \frac{\psi \vdash X}{\varphi \wedge \psi \vdash X}\ \wedge_A^2$$

$$\frac{X\,, \varphi \vdash \psi}{X \vdash \varphi \to \psi}\ \to_K \qquad \frac{X \vdash \varphi \quad \psi \vdash Y}{\varphi \to \psi \vdash *X\,, Y}\ \to_A$$

*Modal Connectives:*

$$\frac{\bullet_{\mathrm{j}} X \vdash \varphi}{X \vdash \Box_j \varphi}\ \Box_K \qquad \frac{\varphi \vdash X}{\Box_j \varphi \vdash \bullet_{\mathrm{j}}\, X}\ \Box_A$$

$$\frac{X \vdash \varphi}{\bullet_{\mathrm{j}} X \vdash \Diamond_j^- \varphi}\ \Diamond_K^- \qquad \frac{\varphi \vdash \bullet_{\mathrm{j}}\, X}{\Diamond_j^- \varphi \vdash X}\ \Diamond_A^-$$

*Dual Modal Connectives:*

$$\frac{U \vdash *\bullet_{\mathrm{j}} * \varphi}{U \vdash \Box_j^- \varphi}\ \Box_K^- \qquad \frac{\varphi \vdash X}{\Box_j^- \varphi \vdash *\bullet_{\mathrm{j}} * X}\ \Box_A^-$$

$$\frac{X \vdash \varphi}{*\bullet_{\mathrm{j}} * X \vdash \Diamond_j \varphi}\ \Diamond_K \qquad \frac{*\bullet_{\mathrm{j}} * \varphi \vdash U}{\Diamond_j \varphi \vdash U}\ \Diamond_A$$

where $j \in \mathbb{G}$

Figure 5: Display Calculus $\mathsf{UL}$: Logical Rules for Propositional and Modal Connectives

*Substructural Connectives:*

$$\frac{X \vdash \varphi \quad Y \vdash \psi}{X \,,_{\mathrm{i}}\, Y \vdash \varphi \otimes_{\mathrm{i}} \psi}\ \otimes^{\mathrm{i}}_K \qquad \frac{\varphi \,,_{\mathrm{i}}\, \psi \vdash X}{\varphi \otimes_{\mathrm{i}} \psi \vdash X}\ \otimes^{\mathrm{i}}_A$$

$$\frac{X \,,_{\mathrm{i}}\, \varphi \vdash \psi}{X \vdash \varphi \supset_{\mathrm{j}} \psi}\ \supset^{\mathrm{j}}_K \qquad \frac{X \vdash \varphi \quad \psi \vdash Y}{\varphi \supset_{\mathrm{j}} \psi \vdash *X \,,_{\mathrm{j}}\, Y}\ \supset^{\mathrm{j}}_A$$

$$\frac{\varphi \,,_{\mathrm{i}}\, X \vdash \psi}{X \vdash \psi \subset_{\mathrm{k}} \varphi}\ \subset^{\mathrm{k}}_K \qquad \frac{\psi \vdash Y \quad X \vdash \varphi}{\psi \subset_{\mathrm{k}} \varphi \vdash Y \,,_{\mathrm{k}}\, *X}\ \subset^{\mathrm{k}}_A$$

*Dual Substructural Connectives:*

$$\frac{U \vdash \varphi \,,_{\mathrm{i}}\, \psi}{U \vdash \varphi \oplus_{\mathrm{i}} \psi}\ \oplus^{\mathrm{i}}_K \qquad \frac{\varphi \vdash X \quad \psi \vdash Y}{\varphi \oplus_{\mathrm{i}} \psi \vdash X \,,_{\mathrm{i}}\, Y}\ \oplus^{\mathrm{i}}_A$$

$$\frac{Y \vdash \psi \quad \varphi \vdash X}{*X \,,_{\mathrm{j}}\, Y \vdash \psi \prec_{\mathrm{j}} \varphi}\ \prec^{\mathrm{j}}_K \qquad \frac{\psi \vdash \varphi \,,_{\mathrm{k}}\, X}{\psi \prec_{\mathrm{j}} \varphi \vdash X}\ \prec^{\mathrm{j}}_A$$

$$\frac{\varphi \vdash X \quad Y \vdash \psi}{Y \,,_{\mathrm{k}}\, *X \vdash \varphi \succ_{\mathrm{k}} \psi}\ \succ^{\mathrm{k}}_K \qquad \frac{\psi \vdash \varphi \,,_{\mathrm{i}}\, X}{\varphi \succ_{\mathrm{k}} \psi \vdash X}\ \succ^{\mathrm{k}}_A$$

where $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$

Figure 6: Display Calculus $\mathsf{UL}$: Logical Rules for Substructural Connectives

**Notation 3.** *Let* $\mathit{Form}_t^0 := \left\{\neg, \Box_j, \Diamond_j, \Box_j^-, \Diamond_j^-, \vee, \wedge \mid j \in \mathbb{G}\right\}$ *and* $\mathit{Struc}_t^0 := \{*, \bullet_j, ,_0 \mid j \in \mathbb{G}\}$. *We will use the following logical connectives:*

$$\mathit{Form}_t := \mathit{Form}_t^0 \cup \left\{\forall_i, \exists_i, \forall_i^-, \exists_i^- \mid i \in \{0,1,2\}\right\}$$
$$\mathit{Struc}_t := \mathit{Struc}_t^0 \cup \{\bullet_i \mid i \in \{0,1,2\}\}$$

*If* $F \subseteq \mathit{Form}_t$*, the* structural connectives associated to $\mathsf{F}$*, denoted* $\mathit{Struc}(F)$*, is the set of structural connectives* $\{*, ,_0\}$ *together with* $\{\bullet_j\}$ *if* $F \cap \left\{\Box_j, \Diamond_j^-, \Diamond_j, \Box_j^- \mid j \in \mathbb{G}\right\} \neq \emptyset$ *and with* $\{\bullet_i\}$ *if* $F \cap \left\{\forall_i, \exists_i, \forall_i^-, \exists_i^- \mid i \in \{0,1,2\}\right\} \neq \emptyset$*. We denote by* $\mathcal{S}(\mathbb{P}, F)$ *the set of all* $\mathcal{S}(\mathbb{P}, F, \mathit{Struc}(F))$*–structures.*

## 5.1 Substructural and Restricted Tense Logics

In this section, we define substructural and restricted tense logics. Their semantics are based on the class of pointed substructural frames viewed as Kripke frames. The core idea is to break the ternary relation into two binary relations and to consider the relation $\sqsubseteq$ as a binary relation as well. Then, any substructural frame can be viewed as a multi-modal Kripke frame whose three of its binary relations satisfy specific properties.

**Notation 4.** *For all substructural frames* $F = (\mathcal{P}, S_1, \ldots S_m, \mathcal{R})$*, we introduce the following notations:*

$$\mathcal{R}_1 := \{(x, y) \in \mathcal{P} \times \mathcal{P} \mid \text{ there is } z \in \mathcal{P} \text{ such that } (x, y, z) \in \mathcal{R}\}$$
$$\mathcal{R}_2 := \{(x, y) \in \mathcal{P} \times \mathcal{P} \mid \text{ there is } z \in \mathcal{P} \text{ such that } (z, x, y) \in \mathcal{R}\}.$$

**Substructural Tense Logic**

**Definition 17** (Substructural tense logic)**.** If $F = (\mathcal{P}, S_1, \ldots S_m, \mathcal{R})$ is a substructural frame, a *valuation* for $F$ is a function $V : \mathbb{P} \to 2^P$. A *pointed multi-modal substructural model* is a pair $((F, V), w)$, where $(F, w)$ is a pointed substructural frame and $V$ is a valuation for $F$. It is also denoted $(\mathcal{M}, w)$ where $\mathcal{M} = (F, V)$. The class of all pointed multi-modal substructural models is denoted $\mathcal{E}^-$. We define a (canonical) tense logic based on $\mathcal{E}^-$.

- The *valuation relation* $\models \subseteq \mathcal{E}^- \times \mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ is defined inductively as follows. Let $(\mathcal{M}, w) \in \mathcal{F}$ be a pointed multi-modal substructural model and let $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form_t})$. The truth conditions for the connectives $\neg, \wedge, \vee$ are defined like in Definition 4. We define the truth conditions for the other connectives of $\mathsf{Form_t}$ as follows: for all $j \in \mathbb{G}$ and all

$i \in \{1,2\}$,

$$
\begin{array}{lll}
\mathcal{M}, w \models p & \text{iff} & p \in V(w) \\
\mathcal{M}, w \models \Box_j \varphi & \text{iff} & \text{for all } v \in \mathcal{P} \text{ such that } wS_j v, \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \Diamond_j \varphi & \text{iff} & \text{there is } v \in \mathcal{P} \text{ such that } wS_j v \text{ and } \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \Box_j^- \varphi & \text{iff} & \text{for all } v \in \mathcal{P} \text{ such that } vS_j w, \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \Diamond_j^- \varphi & \text{iff} & \text{there is } v \in \mathcal{P} \text{ such that } vS_j w \text{ and } \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \forall_i \varphi & \text{iff} & \text{for all } v \in \mathcal{P} \text{ such that } \mathcal{R}_i wv, \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \exists_i \varphi & \text{iff} & \text{there is } v \in \mathcal{P} \text{ such that } \mathcal{R}_i wv \text{ and } \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \forall_i^- \varphi & \text{iff} & \text{for all } v \in \mathcal{P} \text{ such that } \mathcal{R}_i vw, \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \exists_i^- \varphi & \text{iff} & \text{there is } v \in \mathcal{P} \text{ such that } \mathcal{R}_i vw \text{ and } \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \forall_0 \varphi & \text{iff} & \text{for all } v \in \mathcal{P} \text{ such that } w \sqsubseteq v, \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \exists_0 \varphi & \text{iff} & \text{there is } v \in \mathcal{P} \text{ such that } w \sqsubseteq v \text{ and } \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \forall_0^- \varphi & \text{iff} & \text{for all } v \in \mathcal{P} \text{ such that } v \sqsubseteq w, \mathcal{M}, v \models \varphi \\
\mathcal{M}, w \models \exists_0^- \varphi & \text{iff} & \text{there is } v \in \mathcal{P} \text{ such that } v \sqsubseteq w \text{ and } \mathcal{M}, v \models \varphi
\end{array}
$$

- We extend the scope of the evaluation relation $\models$ simultaneously in two different ways in order to also relate points to $\mathcal{S}(\mathbb{P}, \mathsf{Form_t})$–structures. The *antecedent evaluation relation* $\overset{A}{\models} \subseteq \mathcal{E}^- \times \mathcal{S}(\mathbb{P}, \mathsf{Form_t})$ and the *consequent evaluation relation* $\overset{K}{\models} \subseteq \mathcal{E}^- \times \mathcal{S}(\mathbb{P}, \mathsf{Form_t})$ are defined inductively as follows. The truth conditions for the formulas, the structural connectives $*, \bullet_{\mathrm{j}}$ and $,_{\mathrm{i}}$ (where $j \in \mathbb{G}$ and $i \in \{0,1,2,3\}$) are defined like in Definition 8. The truth conditions for the connectives $\bullet_{\mathrm{i}}$ are defined as follows: for all $i \in \{1,2\}$,

$$
\begin{array}{lll}
\mathcal{M}, w \overset{A}{\models} \bullet_{\mathrm{i}} X & \text{iff} & \text{there is } v \in \mathcal{M} \text{ such that } \mathcal{R}_i vw \\
 & & \text{and it holds that } \mathcal{M}, v \overset{A}{\models} X; \\
\mathcal{M}, w \overset{K}{\models} \bullet_{\mathrm{i}} X & \text{iff} & \text{for all } v \in \mathcal{M} \text{ such that } \mathcal{R}_i wv, \\
 & & \text{it holds that } \mathcal{M}, v \overset{K}{\models} X \\
\mathcal{M}, w \overset{A}{\models} \bullet_0 X & \text{iff} & \text{there is } v \in \mathcal{M} \text{ such that } v \sqsubseteq w \\
 & & \text{and it holds that } \mathcal{M}, v \overset{A}{\models} X; \\
\mathcal{M}, w \overset{K}{\models} \bullet_0 X & \text{iff} & \text{for all } v \in \mathcal{M} \text{ such that } w \sqsubseteq v, \\
 & & \text{it holds that } \mathcal{M}, v \overset{K}{\models} X.
\end{array}
$$

- We extend the scope of the relation $\models$ to also relate points to consecutions of $\mathcal{C}(\mathbb{P}, \mathsf{Form_t})$ like in Definition 8.

We extend these definitions to the class of pointed substructural *frames*. We define the *valuation relation* $\models \subseteq \mathcal{F} \times \mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ as follows: if $(F, w)$ is a pointed substructural frame and if $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form_t})$, then

$$F, w \models \varphi \quad \text{iff} \quad \text{for all valuations } V, \text{ it holds that } (F, V), w \models \varphi.$$

The triple $\left(\mathcal{L}(\mathbb{P}, \mathsf{Form_t}), \mathcal{F}, \models\right)$ is a (multi-modal) logic, called *substructural tense logic*. $\Box$

Note that a *valuation* is a function $V : P \to 2^{\mathbb{P}}$ which does not necessarily fulfill the Persistence condition of Definition 3.

| | |
|---|---|
| Propositional Logic | |
| $\Box(p \to q) \to (\Box p \to \Box q)$ | Distributivity |
| $p \to \Box\Diamond^- p$ | Converse |
| $p \to \Box^-\Diamond p$ | Converse$'$ |
| $\forall_0 p \to p$ | $\mathsf{T}_0$ |
| $\forall_0 p \to \forall_0\forall_0 p$ | $4_0$ |
| $\exists_1^- p \to \exists_2\exists_2^-\exists_1^- p$ | 3 |
| $\exists_2 p \to \exists_1^-\exists_1\exists_2 p$ | 3$'$ |
| $\exists_1^-\exists_0^- p \to \exists_0^-\exists_1^- p$ | 2- |
| $\exists_1\exists_0^- p \to \exists_0^-\exists_1 p$ | 2-$'$ |
| $\exists_2^-\exists_0^- p \to \exists_0\exists_2^- p$ | 2+ |
| $\exists_2\exists_0 p \to \exists_0^-\exists_2 p$ | 2+$'$ |
| $\Diamond_j^-\exists_0^- p \to \exists_0\Diamond_j^- p$ | 2+$_j$ |
| $\Diamond_j\exists_0 p \to \exists_0^-\Diamond_j p$ | 2+$'_j$ |
| $\dfrac{\varphi}{\Box\varphi} \qquad \dfrac{\varphi}{\Box^-\varphi}$ | Necessitation |

where $(\Box, \Diamond^-) \in \Big\{(\Box_j, \Diamond_j^-), (\forall_i, \exists_i^-) \mid j \in \mathbb{G}, i \in \{0,1,2\}\Big\}$,
$(\Box^-, \Diamond) \in \Big\{(\Box_j^-, \Diamond_j), (\forall_i^-, \exists_i) \mid j \in \mathbb{G}, i \in \{0,1,2\}\Big\}$,
and $j \in \mathbb{G}$.

Figure 7: Hilbert Calculus $\mathsf{K_t}$

*Structural Rules:*

$$\frac{\bullet_0 X \vdash U}{X \vdash U}\ \mathsf{T}_0 \qquad \frac{\bullet_0 X \vdash U}{\bullet_0 \bullet_0 X \vdash U}\ 4_0$$

$$\frac{* \bullet_2 * \bullet_2 \bullet_1 X \vdash U}{\bullet_1 X \vdash U}\ 3 \qquad \frac{\bullet_1 * \bullet_1 \bullet_2 * X \vdash U}{* \bullet_2 * X \vdash U}\ 3'$$

$$\frac{* \bullet_0 * \bullet_k X \vdash U}{\bullet_k \bullet_0 X \vdash U}\ 2+ \qquad \frac{\bullet_0 * \bullet_k * X \vdash U}{* \bullet_k \bullet_0 * X \vdash U}\ 2+'$$

$$\frac{\bullet_0 \bullet_1 X \vdash U}{\bullet_1 \bullet_0 X \vdash U}\ 2- \qquad \frac{* \bullet_1 * \bullet_0 X \vdash U}{\bullet_0 * \bullet_1 * X \vdash U}\ 2-'$$

*Display Rules:*

$$\frac{X \vdash \bullet_i Y}{\bullet_i X \vdash Y}\ \bullet_i$$

*Modal Connectives:*

$$\frac{\bullet_i X \vdash \varphi}{X \vdash \forall_i \varphi}\ \forall_K \qquad \frac{\varphi \vdash X}{\forall_i \varphi \vdash \bullet_i X}\ \forall_A$$

$$\frac{X \vdash \varphi}{\bullet_i X \vdash \exists_i^- \varphi}\ \exists_K^- \qquad \frac{\varphi \vdash \bullet_i X}{\exists_i^- \varphi \vdash X}\ \exists_A^-$$

*Dual Modal Connectives:*

$$\frac{U \vdash * \bullet_i * \varphi}{U \vdash \forall_i^- \varphi}\ \forall_K^- \qquad \frac{\varphi \vdash X}{\forall_i^- \varphi \vdash * \bullet_i * X}\ \forall_A^-$$

$$\frac{X \vdash \varphi}{* \bullet_i * X \vdash \exists_i \varphi}\ \exists_K \qquad \frac{* \bullet_i * \varphi \vdash U}{\exists_i \varphi \vdash U}\ \exists_A$$

where $i \in \{0, 1, 2\}$ and $k \in \mathbb{G} \cup \{2\}$

Figure 8: Additional Inference Rules $\Sigma_0$ for the Display Calculus $\mathsf{D_t}$

**Definition 18** (Hilbert and display calculi for tense logic)**.** The *Hilbert calculus for* $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$, denoted $\mathsf{K}_\mathsf{t}$, is defined in Figure 7. The *display calculus for* $\mathcal{C}(\mathbb{P}, \mathsf{Form}_t)$, denoted $\mathsf{D}_\mathsf{t}$, is the display calculus containing the rules of Figure 4, 5 and 8 which mention only the logical connectives of $\mathsf{Form}_\mathsf{t}$ or the corresponding structural connectives. $\square$

**Definition 19.** A *pointed substructural Kripke model* $(\mathcal{M}, w) = (W, \sqsubseteq, S_1, \ldots S_m, \mathcal{R}_1, \mathcal{R}_2, V, w)$ is a pointed Kripke model where $S_1, \ldots S_m$ are binary accessibility relations (as defined in Definition 2) and $\sqsubseteq, \mathcal{R}_1, \mathcal{R}_2$ are binary relations over $W$ that moreover satisfy the following conditions: for all $w, v \in P$,

1. if $\mathcal{R}_1 wv$ then there is $u \in P$ such that $\mathcal{R}_2 vu$;
2. if $\mathcal{R}_2 vw$ then there is $u \in P$ such that $\mathcal{R}_1 uv$;
3. $\mathcal{R}_1$ is a negative two-place accessibility relation;
4. $\mathcal{R}_2$ is a positive two-place accessibility relation;
5. $\sqsubseteq$ is a reflexive, transitive and antisymmetric binary relation on $P$.

A *pointed substructural Kripke frame* is a pointed Kripke frame without valuation. The class of all pointed substructural Kripke frames is denoted $\mathcal{K}^-$. The class of pointed substructural Kripke frames whose relation $\sqsubseteq$ is only reflexive and transitive (and not necessarily antisymmetric) is denoted $\mathcal{K}^{S4}$. The language $\mathcal{L}(\mathbb{P}, \mathsf{Form}_\mathsf{t})$ is interpreted canonically over $\mathcal{K}^-$ by means of a valuation relation, also denoted $\models$. $\square$

**Lemma 3.** *Let $\mathcal{F}_0$ be a class of pointed substructural frames and let $\mathcal{K}_0^-$ be its (canonically) associated class of pointed substructural Kripke frames. Then, the sets of validities of $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{K}_0^-, \models)$ and $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}_0, \models)$ are the same.*

*Proof.* It is due to the fact that any pointed substructural Kripke model can be mapped to a pointed substructural frame satisfying the same formulas of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_\mathsf{t})$, and vice versa. $\square$

**Lemma 4.** *Let $\mathcal{K}_0^{S4}$ be a class of pointed substructural Kripke frames whose relation $\sqsubseteq$ is reflexive and transitive but* not necessarily *antisymmetric and which is defined by a set $S_t$ of primitive formulas of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$ or $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ (see Definition 32) or by a set of analytic inference rules of $\mathcal{C}_t$. Let $\mathcal{K}_0^-$ be the subclass of frames of $\mathcal{K}_0^{S4}$ whose relation $\sqsubseteq$ is also antisymmetric (and therefore a partial order). Then, the sets of validities of $\left(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{K}_0^{S4}, \models\right)$ and $\left(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{K}_0^-, \models\right)$ are the same.*

*Proof.* Since the sets of validities with respect to pointed frames or pointed models coincide, it suffices to prove that every pointed *model* of $\mathcal{K}_0^{S4}$ can be transfomed into a pointed *model* of $\mathcal{K}_0^-$ that satisfy the same formulas of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_\mathsf{t})$. We resort to the techniques of *unraveling* [11, Section 4.5] and *product update*. Because we deal with tense logics, we need to make explicit the converse relations $\sqsubseteq^-, S_1^-, \ldots, S_m^-, \mathcal{R}_1^-$ and $\mathcal{R}_2^-$. Adapted to our setting, the *unraveling of* $(\mathcal{M}, w) = (W, \sqsubseteq, S_1, \ldots S_m, \mathcal{R}_1, \mathcal{R}_2, \sqsubseteq^-, S_1^-, \ldots, S_m^-, \mathcal{R}_1^-, \mathcal{R}_2^-, V, w)$ is the pointed Kripke model $\left(\vec{\mathcal{M}}, \vec{w}\right) := \left(\vec{W}, \vec{\sqsubseteq}, \vec{S}_1, \ldots \vec{S}_m, \vec{\mathcal{R}}_1, \vec{\mathcal{R}}_2, \vec{V}, \vec{w}\right)$ defined as follows:

- $\overrightarrow{W}$ is the set of sequences of points $(w, w_1, \ldots, w_n)$, denoted $\overrightarrow{w}$, such that $w_1, \ldots, w_n \in W$ and there are $R_1, \ldots, R_n \in \{\sqsubseteq, S_1, \ldots, S_m, \mathcal{R}_1, \mathcal{R}_2, \sqsubseteq^-, S_1^-, \ldots, S_m^-, \mathcal{R}_1^-, \mathcal{R}_2^-\}$ such that $wR_1w_1R_2\ldots R_nw_n$;
- for all $R \in \{\sqsubseteq, S_1, \ldots, S_m, \mathcal{R}_1, \mathcal{R}_2, \sqsubseteq^-, S_1^-, \ldots, S_m^-, \mathcal{R}_1^-, \mathcal{R}_2^-\}$, we set $\overrightarrow{R}\overrightarrow{w}\overrightarrow{v}$ when there is some $v \in W$ such that $Rw_nv$ (where $\overrightarrow{w} = (w_1, \ldots, w_n)$);
- $\overrightarrow{V}(p) := \left\{(w, w_1, \ldots, w_n) \in \overrightarrow{W} \mid w_n \in V(p)\right\}$;
- $\overrightarrow{w} := (w)$.

We already know that $(\overrightarrow{\mathcal{M}}, \overrightarrow{w})$ satisfies the same formulas of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_{\mathsf{t}})$ as $(\mathcal{M}, w)$ by [11, Lemma 4.53] and [11, Proposition 2.14]. Yet, this is not over. On the one hand, the binary relation $\overrightarrow{\sqsubseteq}$ of this unraveled model is reflexive, transitive and antisymmetric. On the other hand, the other conditions of the other relations may not be necessarily preserved by the unraveling (they are spelled out in Definition 19). However, these other conditions are satisfied by the original pointed Kripke model $(\mathcal{M}, w)$. Hence, we define a third Kripke model $(\mathcal{M}', w') = (W', \sqsubseteq', S_1', \ldots S_m', \mathcal{R}_1', \mathcal{R}_2', V', w')$ which combines all the desired conditions in a single model by means of a product update, as follows:

- $W' := \left\{\left(x, \overrightarrow{x}\right) \mid x \in W, \overrightarrow{x} \in \overrightarrow{W} \text{ and for all } p \in \mathbb{P},\ x \in V(p) \text{ iff } \overrightarrow{x} \in \overrightarrow{V}(p)\right\}$;
- for all $R \in \{\sqsubseteq, S_1, \ldots, S_m, \mathcal{R}_1, \mathcal{R}_2, \sqsubseteq^-, S_1^-, \ldots, S_m^-, \mathcal{R}_1^-, \mathcal{R}_2^-\}$, we set $R'(x, \overrightarrow{x})(y, \overrightarrow{y})$ when $Rxy$ and $\overrightarrow{R}\overrightarrow{x}\overrightarrow{y}$;
- $(x, \overrightarrow{x}) \in V'(p)$ iff $x \in V(p)$.
- $w' := (w, \overrightarrow{w})$.

Now, we check that the conditions of Definition 19 are indeed fulfilled. Clearly, $\sqsubseteq''$ is a partial order (reflexive, transitive and antisymmetric). Let $R$ be a positive two-place accessibility relation of $\{S_1, \ldots, S_m, \mathcal{R}_2\}$. Assume that $R'(w, \overrightarrow{w})(v, \overrightarrow{v})$ and $(v, \overrightarrow{v}) \sqsubseteq' (v', \overrightarrow{v'})$. Then we have both that $Rwv$, $v \sqsubseteq v'$ $(*)$ and $\overrightarrow{R}\overrightarrow{w}\overrightarrow{v}$, $\overrightarrow{v} \overrightarrow{\sqsubseteq} \overrightarrow{v'}$, by definition of $R'$ and $\sqsubseteq'$. By definition of $(*)$ and because $R$ is positive, there is $w' \in W$ such that $w' \sqsubseteq w$ and $Rw'v'$. Let us define $\overrightarrow{w'} := (\overrightarrow{w}, w')$ and let us consider the point $(w', \overrightarrow{w'})$, which belongs to $W'$ by definition. Because $Rw'v'$, we have that $\overrightarrow{R}\overrightarrow{w'}\overrightarrow{v'}$ and therefore also that $R'(w', \overrightarrow{w'})(v', \overrightarrow{v'})$ by definition of $R'$. Moreover, because $w' \sqsubseteq w$, we have that $\overrightarrow{w'} \overrightarrow{\sqsubseteq} \overrightarrow{w}$ and therefore also that $(w', \overrightarrow{w'}) \sqsubseteq' (w, \overrightarrow{w})$ by definition of $\sqsubseteq'$. This partly proves that $R'$ is positive, we need to prove the second part of the definition. Assume that $R'(w, \overrightarrow{w})(v, \overrightarrow{v})$ and $(w', \overrightarrow{w'}) \sqsubseteq' (w, \overrightarrow{w})$. Then we have both that $Rwv$, $w' \sqsubseteq w$ $(*)$ and $\overrightarrow{R}\overrightarrow{w}\overrightarrow{v}$, $\overrightarrow{w'} \overrightarrow{\sqsubseteq} \overrightarrow{w}$, by definition of $R'$ and $\sqsubseteq'$. By definition of $(*)$ and because

$R$ is positive, there is $v' \in W$ such that $v \sqsubseteq v'$ and $Rw'v'$. Let us define $\overrightarrow{v'} := (\overrightarrow{v}, v')$ and let us consider the point $(v', \overrightarrow{v'})$, which belongs to $W'$ by definition. Because $Rw'v'$, we have that $\overrightarrow{R'}\overrightarrow{w'}\overrightarrow{v'}$ and therefore also that $R'(w', \overrightarrow{w'})(v', \overrightarrow{v'})$ by definition of $R'$. Moreover, because $v \sqsubseteq v'$, we have that $\overrightarrow{v} \overrightarrow{\sqsubseteq} \overrightarrow{v'}$ and therefore also that $(v, \overrightarrow{v}) \sqsubseteq' (v', \overrightarrow{v'})$ by definition of $\sqsubseteq'$. This proves the second part of the definition of a *positive* two-place accessibility relation. The proof that $\mathcal{R}'_1$ is a *negative* two-place accessiblity relation is completely similar. $\square$

**Theorem 5** (Soundness and completeness)**.** *The proof systems $\mathsf{K_t}$ and $\mathsf{D_t}$ are sound and complete for the logics $\left(\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{F}, \models\right)$ and $\left(\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{F}, \models\right)$ respectively.*

*Proof.* The result follows from the fact that the additional axioms $\mathsf{T}_0$, $\mathsf{4}_0$, 3, 3', 2+, 2+', 2+$_j$, 2+$_j$', 2- and 2-' are in fact primitive formulas of $\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right)$. Their first-order translations correspond to the conditions of Definition 19, except for the condition of antisymmetry. Their corresponding inference rules obtained by applying Kracht's algorithm are the structural rules of $\mathsf{D_t}$. Then, by [22, Theorems 16-17], $\mathbf{DLM}_{\mathrm{t}} + \Sigma_0$ is sound and complete for the logic $\left(\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{K}^{S4}, \models\right)$, where $\mathbf{DLM}_{\mathrm{t}}$ is the multi-modal version based on $\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right)$ of the display calculus **DLM** defined in [22]. However, by Lemma 4, the sets of validities of $\left(\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{K}^{S4}, \models\right)$ and $\left(\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{K}^{-}, \models\right)$ are the same, and, by Lemma 3, the sets of validities of $\left(\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{K}^{-}, \models\right)$ and $\left(\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{F}, \models\right)$ are also the same. So, $\mathbf{DLM}_{\mathrm{t}} + \Sigma_0$ is sound and complete for the logic $\left(\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{F}, \models\right)$. Since every inference rule of $\mathbf{DLM}_{\mathrm{t}} + \Sigma_0$ is derivable in $\mathsf{D_t}$ and vice versa, we have that $\mathsf{D_t}$ is also sound and complete for the logic $\left(\mathcal{C}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{F}, \models\right)$. $\square$

### Restricted Tense Logic

**Definition 20** (Restricted tense logic)**.** The *restricted tense language* $\mathcal{L}_\mathsf{t}$ and the restricted tense sets of structures $\mathcal{S}_\mathsf{t}^A$ and $\mathcal{S}_t^K$ are defined inductively by the following grammars in BNF:

$$\begin{array}{lllcccccccc}
\mathcal{L}_\mathsf{t}: & \varphi & ::= & \exists_0^- p & \mid & \neg\varphi & \mid & (\varphi \wedge \varphi) & \mid & \Diamond_j\varphi \mid \Diamond_j^-\varphi & \mid \\
& & & \exists_1(\varphi \wedge \exists_2\varphi) & \mid & (\exists_1^-\varphi \wedge \exists_2\varphi) & \mid & \exists_2^-(\varphi \wedge \exists_1^-\varphi) & & & \\
\\
\mathcal{S}_\mathsf{t}^A: & X & ::= & \varphi & \mid & *X & \mid & (X, X) & \mid & \bullet_i X & \mid \\
& & & \bullet_2(\bullet_1 X, X) & \mid & (\bullet_1 X,\ * \bullet_2 * X) & \mid & * \bullet_1 *(X,\ * \bullet_2 * X) & & & \\
\\
\mathcal{S}_t^K: & Y & ::= & \varphi & \mid & *Y & \mid & (Y, Y) & \mid & \bullet_i X & \mid \\
& & & \bullet_1(Y,\ \bullet_2 Y) & \mid & (* \bullet_1 *Y,\ \bullet_2 Y) & \mid & * \bullet_2 *(Y,\ * \bullet_1 * Y) & & &
\end{array}$$

where $j$ ranges over $\mathbb{G}$ and, in $\mathcal{S}_\mathsf{t}^A$ and $\mathcal{S}_t^K$, $\varphi$ ranges over $\mathcal{L}_\mathsf{t}$ and $i$ ranges over $\mathbb{G} \cup \{0, 1, 2\}$. We denote by $\mathcal{C}_\mathsf{t}$ the set of consecutions of the form $X \vdash Y$, $X \vdash$ or $\vdash Y$, where $X \in \mathcal{S}_\mathsf{t}^A$ and $Y \in \mathcal{S}_t^K$. The triple $(\mathcal{C}_\mathsf{t}, \mathcal{F}, \models)$ is a logic, called the *restricted tense logic*. $\square$

## 5.2 Update Logic and Restricted Tense Logic: Equal Expressiveness

We will use extensively the following translations $\tau_1$ and $\tau_1^-$ between update and tense formulas.

**Definition 21** (Translations $\tau_1$ and $\tau_1^-$: formulas)**.** We define the mapping $\tau_1 : \mathcal{L}(\mathbb{P}, \mathsf{Form}) \to \mathcal{L}_\mathrm{t}$ inductively as follows:

$$
\begin{array}{lcllcl}
\tau_1(p) & := & \exists_0^- p & & & \\
\\
\tau_1(\varphi \to \psi) & := & \neg(\tau_1(\varphi) \wedge \neg\tau_1(\psi)) & \tau_1(\neg\varphi) & := & \neg\tau_1(\varphi) \\
\tau_1(\varphi \wedge \psi) & := & \tau_1(\varphi) \wedge \tau_1(\psi) & \tau_1(\varphi \vee \psi) & := & \neg(\neg\tau_1(\varphi) \wedge \neg\tau_1(\psi)) \\
\\
\tau_1(\Box_j\varphi) & := & \Box_j\tau_1(\varphi) & \tau_1(\Diamond_j\varphi) & := & \Diamond_j\tau_1(\varphi) \\
\tau_1(\Diamond_j^-\varphi) & := & \Diamond_j^-\tau_1(\varphi) & \tau_1(\Box_j^-\varphi) & := & \Box_j^-\tau_1(\varphi) \\
\\
\tau_1(\chi \otimes_1 \psi) & := & \exists_1(\tau_1(\chi) \wedge \exists_2\tau_1(\psi)) & \tau_1(\chi \oplus_1 \psi) & := & \neg\exists_1(\neg\tau_1(\chi) \wedge \exists_2\neg\tau_1(\psi)) \\
\tau_1(\chi \supset_1 \psi) & := & \neg\exists_1(\tau_1(\chi) \wedge \exists_2\neg\tau_1(\psi)) & \tau_1(\chi \succ_1 \psi) & := & \exists_1(\neg\tau_1(\chi) \wedge \exists_2\tau_1(\psi)) \\
\tau_1(\chi \subset_1 \psi) & := & \neg\exists_1(\neg\tau_1(\chi) \wedge \exists_2\tau_1(\psi)) & \tau_1(\chi \prec_1 \psi) & := & \exists_1(\tau_1(\chi) \wedge \exists_2\neg\tau_1(\psi)) \\
\\
\tau_1(\varphi \otimes_3 \chi) & := & \exists_2^-(\tau_1(\chi) \wedge \exists_1^-\tau_1(\varphi)) & \tau_1(\varphi \oplus_3 \chi) & := & \neg\exists_2^-(\neg\tau_1(\chi) \wedge \exists_1^-\neg\tau_1(\varphi)) \\
\tau_1(\varphi \subset_3 \chi) & := & \neg\exists_2^-(\tau_1(\chi) \wedge \exists_1^-\neg\tau_1(\varphi)) & \tau_1(\varphi \prec_3 \chi) & := & \exists_2^-(\neg\tau_1(\chi) \wedge \exists_1^-\tau_1(\varphi)) \\
\tau_1(\varphi \supset_3 \chi) & := & \neg\exists_2^-(\neg\tau_1(\chi) \wedge \exists_1^-\tau_1(\varphi)) & \tau_1(\varphi \succ_3 \chi) & := & \exists_2^-(\tau_1(\chi) \wedge \exists_1^-\neg\tau_1(\varphi)) \\
\\
\tau_1(\psi \otimes_2 \varphi) & := & \exists_1^-\tau_1(\varphi) \wedge \exists_2\tau_1(\psi) & \tau_1(\psi \oplus_2 \varphi) & := & \neg(\exists_1^-\neg\tau_1(\varphi) \wedge \exists_2\neg\tau_1(\psi)) \\
\tau_1(\psi \subset_2 \varphi) & := & \neg(\exists_1^-\neg\tau_1(\varphi) \wedge \exists_2\tau_1(\psi)) & \tau_1(\psi \prec_2 \varphi) & := & \exists_1^-\tau_1(\varphi) \wedge \exists_2\neg\tau_1(\psi) \\
\tau_1(\psi \supset_2 \varphi) & := & \neg(\exists_1^-\tau_1(\varphi) \wedge \exists_2\neg\tau_1(\psi)) & \tau_1(\psi \succ_2 \varphi) & := & \exists_1^-\neg\tau_1(\varphi) \wedge \exists_2\tau_1(\psi)
\end{array}
$$

We define the mapping $\tau_1^- : \mathcal{L}_\mathrm{t} \to \mathcal{L}(\mathbb{P}, \mathsf{Form})$ inductively as follows:

$$
\begin{array}{lcllcl}
\tau_1^-(\exists_0^- p) & := & p & & & \\
\\
\tau_1^-(\varphi \wedge \psi) & := & \tau_1^-(\varphi) \wedge \tau_1^-(\psi) & \tau_1^-(\neg\varphi) & := & \neg\tau_1^-(\varphi) \\
\\
\tau_1^-(\Box_j\varphi) & := & \Box_j\tau_1^-(\varphi) & \tau_1^-(\Diamond_j^-\varphi) & := & \Diamond_j^-\tau_1^-(\varphi) \\
\\
\tau_1^-(\exists_1(\varphi \wedge \exists_2\psi)) & := & \tau_1^-(\varphi) \otimes_1 \tau_1^-(\psi) & \tau_1^-(\exists_1^-\varphi \wedge \exists_2\psi) & := & \tau_1^-(\varphi) \otimes_2 \tau_1^-(\psi) \\
\tau_1^-(\exists_2^-(\varphi \wedge \exists_1^-\varphi)) & := & \tau_1^-(\varphi) \otimes_3 \tau_1^-(\psi) & & &
\end{array}
$$

□

*Remark* 1. If we deal with *flat* point sets in substructural frames (that is with point sets of the form $(P, =)$) then the translation $\tau_1$ for propositional letters is simply $\tau_1(p) := p$. In that case, the tense modalities $\exists_0, \exists_0^-, \forall_0, \forall_0^-$ can be removed from our translations.

**Proposition 6** (Equal expressiveness of update logic and restricted tense logic: formulas)**.** *Restricted tense logic is as expressive as update logic. More precisely, for all pointed substructural frames $(F, w) \in \mathcal{F}$ and all $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form})$, all $\varphi_t \in \mathcal{L}_t$, it holds that*

$$F, w \Vdash \varphi \ \textit{ iff } \ F, w \models \tau_1(\varphi) \qquad\qquad F, w \models \varphi_t \ \textit{ iff } \ F, w \Vdash \tau_1^-(\varphi_t) \tag{3}$$

*Hence, for all $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form})$ and all $\varphi_t \in \mathcal{L}_t$,*

$$\tau_1^-(\tau_1(\varphi)) \;=\; \varphi \qquad\qquad \tau_1(\tau_1^-(\varphi_t)) \;=\; \varphi_t$$

*Proof.* We only prove Expression (3). Every valuation $V$ of a substructural frame $F$ can be extended into an interpretation $V^{\sqsubseteq}$ of that very substructural frame $F$ such that for all $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form})$, $(F, V), w \models \tau_1(\varphi)$ iff $(F, V^{\sqsubseteq}), w \models \tau_1(\varphi)$ $(*)$. Indeed, if $V$ is such a valuation, it suffices to define $V^{\sqsubseteq}(p)$ for all $p \in \mathbb{P}$ as follows: $V^{\sqsubseteq}(p) := \{w \in \mathcal{P} \mid$ there is $v \in \mathcal{P}$ such that $v \sqsubseteq w$ and $v \in V(p)\}$. Moreover, for all interpretation functions $\mathcal{I}$ of $F$ (which is also a valuation of $F$), we can easily prove that for all $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form})$, $(F, \mathcal{I}), w \models \tau_1(\varphi)$ iff $(F, \mathcal{I}), w \Vdash \varphi$ $(**)$. Thus, we have that $F, w \models \tau_1(\varphi)$ iff for all valuations $V$, $(F, V), w \models \tau_1(\varphi)$ by definition, iff for all interpretations $\mathcal{I}$ we have that $(F, \mathcal{I}), w \models \tau_1(\varphi)$ by $(*)$, iff for all interpretations $V^{\sqsubseteq}$ we have that $(F, V^{\sqsubseteq}), w \Vdash \varphi$ by $(**)$, iff $F, w \Vdash \varphi$ by definition. This proves the first item of Expression (3), the second item being proved similarly. □

Then, we lift these translations $\tau_1$ and $\tau_1^-$ to structures and consecutions.

**Definition 22** (Translations $\tau_1$ and $\tau_1^-$: structures and consecutions)**.** We define the translations $t_1 : \mathcal{S}(\mathbb{P}, \mathsf{Form}) \to \mathcal{S}_t^A$ and $t_2 : \mathcal{S}(\mathbb{P}, \mathsf{Form}) \to \mathcal{S}_t^K$ inductively as follows: for all $j \in \mathbb{G}$,

$$\begin{array}{lll|lll}
t_1(\varphi) & := & \tau_1(\varphi) & t_2(\varphi) & := & \tau_1(\varphi) \\
t_1(\bullet_j X) & := & \bullet_j t_1(X) & t_2(\bullet_j X) & := & \bullet_j t_2(X) \\
t_1(*X) & := & *t_2(X) & t_2(*X) & := & *t_1(X) \\
t_1(X, Y) & := & t_1(X), t_1(Y) & t_2(X, Y) & := & t_2(X), t_2(Y) \\
t_1(X \,,_1\, Y) & := & * \bullet_1 *(t_1(X),\ * \bullet_2 * t_1(Z)) & t_2(X \,,_1\, Y) & := & \bullet_1(t_2(X),\ \bullet_2\, t_2(Y)) \\
t_1(X \,,_2\, Y) & := & \bullet_1 t_1(X),\ * \bullet_2 * t_1(Y) & t_2(X \,,_2\, Y) & := & * \bullet_1 * t_2(X),\ \bullet_2\, t_2(Y) \\
t_1(X \,,_3\, Y) & := & \bullet_2(\bullet_1 t_1(X), t_1(Y)) & t_2(X \,,_3\, Y) & := & * \bullet_2 *(t_2(Y),\ * \bullet_1 * t_2(X))
\end{array}$$

We extend the translations to consecutions. We define the translation $\tau_1 : \mathcal{C}(\mathbb{P}, \mathsf{Form}) \to \mathcal{C}_t$ as follows: for all $X \vdash Y \in \mathcal{C}(\mathbb{P}, \mathsf{Form})$, we set

$$\tau_1(X \vdash Y) := t_1(X) \vdash t_2(Y).$$

We define the translations $t_1^- : \mathcal{S}_t^A \to \mathcal{S}(\mathbb{P}, \mathsf{Form})$ and $t_2^- : \mathcal{S}_t^K \to \mathcal{S}(\mathbb{P}, \mathsf{Form})$ inductively as follows: for all $j \in \mathbb{G}$,

$$\begin{array}{lll|lll}
t_1^-(\varphi) & := & \tau_1^-(\varphi) & t_2^-(\varphi) & := & \tau_1^-(\varphi) \\
t_1^-(*X) & := & *t_2^-(X) & t_2^-(*X) & := & *t_1^-(X) \\
t_1^-(\bullet_j X) & := & \bullet_j t_1^-(X) & t_2^-(\bullet_j X) & := & \bullet_j t_2^-(X) \\
t_1^-(X, Y) & := & t_1^-(X), t_1^-(Y) & t_2^-(X, Y) & := & t_2^-(X), t_2^-(Y) \\
t_1^-(\bullet_2(\bullet_1 X, Y)) & := & t_1^-(X) \,,_3\, t_1^-(Y) & t_2^-(* \bullet_2 *(X,\ * \bullet_1 * Y)) & := & t_2^-(X) \,,_3\, t_2^-(Y) \\
t_1^-(\bullet_1 X,\ * \bullet_2 * Y) & := & t_1^-(X) \,,_2\, t_1^-(Y) & t_2^-(* \bullet_1 * X,\ \bullet_2\, Y) & := & t_2^-(X) \,,_2\, t_2^-(Y) \\
t_1^-(* \bullet_1 *(X,\ * \bullet_2 * Y)) & := & t_1^-(X) \,,_1\, t_1^-(Y) & t_2^-(\bullet_1(X,\ \bullet_2\, Y)) & := & t_2^-(X) \,,_1\, t_2^-(Y)
\end{array}$$

We extend the translations to consecutions. We define the translation $\tau_1^- : \mathcal{C}_t \to \mathcal{C}(\mathbb{P}, \mathsf{Form})$ as follows: for all $X \vdash Y \in \mathcal{C}_t$, we set

$$\tau_1^-(X \vdash Y) := t_1^-(X) \vdash t_2^-(Y).$$

□

**Proposition 7** (Equal expressiveness of update logic and restricted tense logic)**.**

- *We have that* $\left(\mathcal{S}^A_t, \mathcal{F}, \models^A\right) \equiv \left(\mathcal{S}(\mathbb{P}, \textsf{Form}), \mathcal{F}, \Vdash^A\right)$ *and* $\left(\mathcal{S}^K_t, \mathcal{F}, \models^K\right) \equiv \left(\mathcal{S}(\mathbb{P}, \textsf{Form}), \mathcal{F}, \Vdash^K\right)$. *More precisely, for all pointed substructural frames* $(F, w)$ *and all* $X \in \mathcal{S}(\mathbb{P}, \textsf{Form})$*, all* $X_t \in \mathcal{S}^A_t$*, all* $Y_t \in \mathcal{S}^K_t$*, it holds that*

$$\begin{array}{lll} F, w \Vdash^A X \ \text{ iff } \ F, w \models^A t_1(X) & \qquad & F, w \models^A X_t \ \text{ iff } \ F, w \Vdash^A t_1^-(X_t) \\ F, w \Vdash^K X \ \text{ iff } \ F, w \models^K t_1(X) & \qquad & F, w \models^K Y_t \ \text{ iff } \ F, w \Vdash^K t_1^-(Y_t) \end{array}$$

- *We have that* $\left(\mathcal{C}_t, \mathcal{F}, \models\right) \equiv \left(\mathcal{C}\left(\mathbb{P}, \textsf{Form}\right), \mathcal{F}, \Vdash\right)$. *More precisely, for all pointed substructural frames* $(F, w)$ *and all* $X \vdash Y \in \mathcal{C}\left(\mathbb{P}, \textsf{Form}\right)$*, all* $X_t \vdash Y_t \in \mathcal{C}\left(\mathbb{P}, \textsf{Form}_t\right)$*, it holds that*

$$F, w \Vdash X \vdash Y \ \text{ iff } \ F, w \models \tau_1(X \vdash Y) \quad F, w \models X_t \vdash Y_t \ \text{ iff } \ F, w \Vdash \tau_1^-(X_t \vdash Y_t)$$

*Proof.* By induction on the structures $X \in \mathcal{S}(\mathbb{P}, \textsf{Form})$ and $X_t \in \mathcal{S}^A_t, Y_t \in \mathcal{S}^K_t$ and on the consecutions $X \vdash Y \in \mathcal{C}\left(\mathbb{P}, \textsf{Form}\right)$ and $X_t \vdash Y_t \in \mathcal{C}_t$, using Proposition 6 for the base case. $\square$

Finally, we lift these translations $\tau_1$ and $\tau_1^-$ to inference rules.

**Definition 23** (Translations $\tau_1$ and $\tau_1^-$: inference rules)**.** Let $R$ be an inference rule in $\mathcal{C}\left(\mathbb{P}, \textsf{Form}\right)$ and let $R_t$ be an inference rule in $\mathcal{C}\left(\mathbb{P}, \textsf{Form}_t\right)$. We define the inference rules $\tau_1(R)$ in $\mathcal{C}\left(\mathbb{P}, \textsf{Form}_t\right)$ and $\tau_1(R)$ in $\mathcal{C}\left(\mathbb{P}, \textsf{Form}\right)$ as follow:

$$\begin{aligned} \tau_1(R) &:= \{(\tau_1(C_1), \ldots, \tau_1(C_l), \tau_1(C_{l+1})) \mid (C_1, \ldots, C_l, C_{l+1}) \in R\} \\ \tau_1^-(R_t) &:= \left\{\left(\tau_1^-(C_1), \ldots, \tau_1^-(C_l), \tau_1^-(C_{l+1})\right) \mid (C_1, \ldots, C_l, C_{l+1}) \in R_t\right\}. \end{aligned}$$

If $\Sigma$ is a set of inference rules, then $\tau_1(\Sigma) := \{\tau_1(R) \mid R \in \Sigma\}$ and $\tau_1^-(\Sigma) := \left\{\tau_1^-(R) \mid R \in \Sigma\right\}$. $\square$

**Example 4.** Below are the display rules $\textsf{dr}_1, \textsf{dr}_2, \textsf{dr}_3, \textsf{dr}_4$ of $\textsf{UL}$, for $(i, j, k) = (3, 1, 2)$, and their respective translations $\tau_1(\textsf{dr}_1), \tau_1(\textsf{dr}_2), \tau_1(\textsf{dr}_3)$ and $\tau_1(\textsf{dr}_4)$ in $\textsf{D}_t$.

$$\begin{array}{c} X \,,_3\, Y \vdash Z \\ \hline\hline X \vdash * Y \,,_1\, Z \\ \hline\hline Y \vdash Z \,,_2\, * X \end{array} \begin{array}{l} \textsf{dr}_1 \\ \textsf{dr}_2 \end{array} \qquad\qquad \begin{array}{c} Z \vdash X \,,_3\, Y \\ \hline\hline *Y \,,_1\, Z \vdash X \\ \hline\hline Z \,,_2\, * X \vdash Y \end{array} \begin{array}{l} \textsf{dr}_3 \\ \textsf{dr}_4 \end{array}$$

$$\begin{array}{c} \bullet_2(\bullet_1 X, Y) \vdash Z \\ \hline\hline X \vdash \bullet_1 (*Y, \ \bullet_2 Z) \\ \hline\hline Y \vdash \bullet_2 Z, \ * \bullet_1 X \end{array} \begin{array}{l} \tau_1(\textsf{dr}_1) \\ \tau_1(\textsf{dr}_2) \end{array} \qquad\qquad \begin{array}{c} Z \vdash * \bullet_2 * Y, \ * \bullet_1 * X \\ \hline\hline * \bullet_1 * (*Y, \ * \bullet_2 * Z) \vdash X \\ \hline\hline \bullet_1 * X, \ * \bullet_2 * Z \vdash Y \end{array} \begin{array}{l} \tau_1(\textsf{dr}_3) \\ \tau_1(\textsf{dr}_4) \end{array}$$

$\square$

**Proposition 8.** *Let* $R$ *be an analytic inference rule in* $\mathcal{C}\left(\mathbb{P}, \textsf{Form}\right)$ *and let* $R_t$ *be an inference rule in* $\mathcal{C}_t$*, both in the special form of Expression* (9)*. Then,*

$$\tau_1^-\left(\tau_1\left(R\right)\right) \equiv R \qquad\qquad \tau_1\left(\tau_1^-\left(R_t\right)\right) = R_t$$

*Proof.* It follows from Propositions 6 and 7. $\square$

## 6 Correspondence Theory for Tense Logics

In this section, we recall the main results of correspondence theory adapted to our (restricted) substructural tense logic. Roughly speaking, *correspondence theory* investigates to what extent specific properties of accessibility relations can be reformulated in terms of the validity of specific formulas. Correspondence theory addresses the following kinds of questions: when does the truth of a given (modal or tense) formula in a frame corresponds to a first-order property in this frame ? (Sahlqvist correspondence theorem); and when does the validity of a (modal or tense) formula on a class of frames corresponds to the fact that this class of frames satisfies a specific first-order property (and vice versa) ? (Sahlqvist and Kracht theorems) (see [11, 49] for more details on correspondence theory for modal and tense logic). Sahlqvist and inductive formulas are modal formulas which have a first-order correspondent (every inductive formula is equivalent to a Sahlqvist formula with converse tense modality). These first-order correspondents are called *Kracht formulas* (see [11, Section 3.6] for a formal definition).

### 6.1 Preliminary Definitions

**Definition 24** (Languages $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$, $\mathcal{L}_{\text{SOL}}(\mathcal{R}_1, \mathcal{R}_2)$ and $\mathcal{L}_{\text{FOL}}(\mathcal{R})$, $\mathcal{L}_{\text{SOL}}(\mathcal{R})$)**.**

- The *(binary) first-order substructural frame language*, denoted $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ (resp. $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$), is the first-order language that has the identity symbol with $m+1$ binary relations $S_1, \dots, S_m, \sqsubseteq$ and a ternary relation $\mathcal{R}$ (resp. with $m+3$ binary relations $S_1, \dots, S_m$, $\mathcal{R}_1, \mathcal{R}_2$ and $\sqsubseteq$).
- The *(binary) second-order substructural frame language*, denoted $\mathcal{L}_{\text{SOL}}(\mathcal{R})$ (resp. $\mathcal{L}_{\text{SOL}}(\mathcal{R}_1, \mathcal{R}_2)$), is the second-order language obtained by augmenting $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ (resp. $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$) with a collection of monadic predicate variables $P_1, P_2, \dots$ associated to each $p_1, p_2, \dots \in \mathbb{P}$.

The satisfaction relations $\models_{\text{FOL}}$ and $\models_{\text{SOL}}$ for the languages $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$, $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ and $\mathcal{L}_{\text{SOL}}(\mathcal{R}_1, \mathcal{R}_2)$, $\mathcal{L}_{\text{SOL}}(\mathcal{R})$ respectively on the class of all substructural frames $\mathcal{F}^-$ are defined as usual (see [31] for example). $\square$

**Notation 5.** *Like in [22], we introduce the following abbreviations: for all $k \in \mathbb{G} \cup \{1, 2\}$,*

$$\begin{array}{ll}
(\forall y \rhd_k x)\alpha(y) := \forall y(S_k xy \to \alpha(y)) & (\forall y \lhd_k x)\alpha(y) := \forall y(S_k yx \to \alpha(y)) \\
(\exists y \rhd_k x)\alpha(y) := \exists y(S_k xy \wedge \alpha(y)) & (\exists y \lhd_k x)\alpha(y) := \exists y(S_k yx \wedge \alpha(y)) \\
(\forall y \rhd_0 x)\alpha(y) := \forall y(x \sqsubseteq y \to \alpha(y)) & (\forall y \lhd_0 x)\alpha(y) := \forall y(y \sqsubseteq x \to \alpha(y)) \\
(\exists y \rhd_0 x)\alpha(y) := \exists y(x \sqsubseteq y \wedge \alpha(y)) & (\exists y \lhd_0 x)\alpha(y) := \exists y(y \sqsubseteq x \wedge \alpha(y))
\end{array}$$

*Besides, we also introduce the following abbreviations: for all $i \in \{1, 2, 3\}$,*

$$\mathcal{R}^i xyz \quad \textit{iff} \quad \sigma_i(x, y, z) \in \mathcal{R} \tag{4}$$

$$(\forall yz \rhd^i x)\alpha(y, z) := \forall yz(\mathcal{R}^i xyz \to \alpha(y, z)) \quad (\exists yz \rhd^i x)\alpha(y, z) := \exists yz(\mathcal{R}^i xyz \wedge \alpha(y, z))$$

*We call the constructs* $(\forall y \rhd_k x), (\forall y \lhd_k x), (\forall yz \rhd^i x)$ *and* $(\exists y \rhd_k x), (\exists y \lhd_k x), (\exists yz \rhd^i x)$ restricted universal (resp. existential) quantifiers *(where* $k \in \mathbb{G} \cup \{0, 1, 2\}$ *and* $i \in \{1, 2, 3\}$*). They are generally denoted* $(\forall y \rhd x)$*,* $(\forall y \lhd x)$*,* $(\forall yz \rhd x)$*,* $(\forall yz \lhd x)$ *and* $(\exists yz \rhd x)$*,* $(\exists y \lhd x)$*,* $(\exists y \rhd x)$*,* $(\exists yz \lhd x)$*.*

*Remark* 2. The restricted quantifiers $(\forall y \rhd x), (\forall y \lhd x)$ and $(\exists y \rhd x), (\exists y \lhd x)$ of Notation 5 could be defined like our notations $(\forall yz \rhd x), (\forall yz \lhd x)$ and $(\exists yz \rhd x), (\exists yz \lhd x)$ if we introduced the following notations: for all $i \in \{1, 2\}$,

$$S^i xy \quad \text{iff} \quad \tau_i(x, y) \in S \tag{5}$$

where $\tau_1(x, y) = (x, y)$ and $\tau_2(x, y) = (y, x)$. Expression 5 is the binary counterpart of Expression 4. Now, let us define for all $i \in \{1, 2\}$,

$$(\forall y \rhd^i x)\alpha(y) := \forall y(S^i xy \to \alpha(y)) \qquad (\exists y \rhd^i x)\alpha(y) := \exists y(S^i xy \wedge \alpha(y)).$$

Then we have that $(\forall y \rhd^1 x)\alpha(y) = (\forall y \rhd x)\alpha(y)$, $(\forall y \rhd^2 x)\alpha(y) = (\forall y \lhd x)\alpha(y)$ and $(\exists y \rhd^1 x)\alpha(y) = (\exists y \rhd x)\alpha(y)$, $(\exists y \rhd^2 x)\alpha(y) = (\exists y \lhd x)\alpha(y)$.

**Definition 25** (Definability and local substructural frame correspondence)**.** Let $(\mathcal{L}, \mathcal{F}_0, \models)$ be a logic defined on a class $\mathcal{F}_0$ of substructural frames. Let $\varphi \in \mathcal{L}$, let $\Sigma$ be a set of inference rules in $\mathcal{L}$ and let $\Theta(x)$ be a (set of) formula(s) of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ (or of $\mathcal{L}_{\text{FOL}}(\mathcal{R})$) such that $x$ is supposed to be the only free variable of $\Theta$.

- We say that $\Theta(x)$ *defines* a class of pointed substructural frames $\mathcal{F}_0$ or that $\mathcal{F}_0$ is *defined by* $\Theta(x)$ when the following holds:

$$\mathcal{F}_0 = \big\{(F, w) \mid (F, w) \in \mathcal{F} \text{ and } F \models_{\text{FOL}} \Theta[w]\big\}$$

- We say that $\varphi$ (resp. $\Sigma$) and $\Theta(x)$ are *local substructural frame correspondents* when for all pointed substructural frames $(F, w) \in \mathcal{F}$,

$$\begin{aligned} F, w \models \varphi \quad &\text{iff} \quad F \models_{\text{FOL}} \Theta[w] \\ F, w \models \Sigma \quad &\text{iff} \quad F \models_{\text{FOL}} \Theta[w] \end{aligned}$$

where $F \models_{\text{FOL}} \Theta[w]$ means that $\Theta(x)$ is interpreted in $F$ with respect to an assignment that assigns $w$ to the free variable $x$. □

## 6.2 Correspondence Theory for Substructural Tense Logic

### Correspondence for Axioms

In this section, we deal with the problem of how specific properties of accessibility relations can be expressed in terms of the validity of specific kinds of formulas. Every formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ can be translated in $\mathcal{L}_{\text{SOL}}(\mathcal{R}_1, \mathcal{R}_2)$ by the *standard translation*:

**Definition 26** (Standard translation)**.** For all (free) variables $x$, we define the *standard translation* $ST_x : \mathcal{L}(\mathbb{P}, \mathsf{Form_t}) \to \mathcal{L}_{\mathrm{SOL}}(\mathcal{R}_1, \mathcal{R}_2)$ inductively as follows:

$$\begin{array}{lll} ST_x(p) & := & P(x) \\ ST_x(\varphi \wedge \psi) & := & ST_x(\varphi) \wedge ST_x(\psi) \\ ST_x(\neg\varphi) & := & \neg ST_x(\varphi) \\ ST_x(\Diamond\varphi) & := & (\exists y \rhd x) ST_y(\varphi) \\ ST_x(\Box\varphi) & := & (\forall y \rhd x) ST_y(\varphi) \\ ST_x(\Diamond^-\varphi) & := & (\exists y \lhd x) ST_y(\varphi) \\ ST_x(\Box^-\varphi) & := & (\forall y \lhd x) ST_y(\varphi) \end{array}$$

where $(\Box, \Diamond, \rhd) \in \{(\Box_j, \Diamond_j, \rhd_j), (\forall_i, \exists_i, \rhd_i) \mid j \in \mathbb{G}, i \in \{0,1,2\}\}$, $(\Box^-, \Diamond^-, \lhd) \in \{(\Box_j^-, \Diamond_j^-, \lhd_j), (\forall_i^-, \exists_i^-, \lhd_i) \mid j \in \mathbb{G}, i \in \{0,1,2\}\}$ and $P$ is a monadic predicate variable. □

Then, one can easily show that if $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ holds in a pointed substructural frame $(F, w)$, then $ST_x(\varphi)$ holds as well in $(F, w)$:

**Fact 1.** *Let $(F, w)$ be a pointed substructural frame and let $\varphi \in \mathcal{L}(\mathbb{P}, \mathsf{Form_t})$. Then,*

$$F, w \models \varphi \quad \textit{iff} \quad F \models_{\mathrm{FOL}} ST_x(\varphi)[w]$$

*where $F \models_{\mathrm{FOL}} ST_x(\varphi)[w]$ means that $ST_x(\varphi)$ is interpreted in $F$ with respect to an assignment that assigns $w$ to the free variable $x$.*

One would want instead to have an equivalent *first-order* formula of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ and not a *second-order* formula of $\mathcal{L}_{\mathrm{SOL}}(\mathcal{R}_1, \mathcal{R}_2)$. This cannot be the case in general and we therefore introduce a fragment of the formulas of $\mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ called the *Sahlqvist fragment* which consists of formulas for which we can always find a first-order equivalent. Larger fragments exist, such as the class of *inductive formulas* [19], but they will not be needed here.

**Definition 27** (Boxed atom, positive and negative formula, Sahlqvist formula)**.**

- A *boxed atom* is a formula of the form $L_1 \dots L_k p$ where $L_1, \dots, L_k \in \{\Box_j, \Box_j^-, \forall_i, \forall_i^- \mid j \in \mathbb{G}, i \in \{0,1,2\}\}$ (they are not necessarily distinct). In the case where $k = 0$, the boxed atom $L_1 \dots L_k p$ is just the proposition letter $p$.
- An occurrence of a proposition letter $p$ in a formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ is a *positive* occurrence if it is in the scope of an even number of negation signs; it is a *negative* occurrence if it is in the scope of an odd number of negation signs. A formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ is *positive in p (negative in p)* if all occurrences of $p$ in $\varphi$ are positive (negative). A formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ is *positive (negative)* if it is positive (negative) in all proposition letters occurring in it.
- A *Sahlqvist antecedent* is a formula built up from boxed atoms, negative formulas and the connectives $\left\{\top, \bot, \Diamond_j, \Diamond_j^-, \exists_i, \exists_i^- \mid j \in \mathbb{G}, i \in \{0,1,2\}\right\}$ of $\mathsf{Form_t}$. A *Sahlqvist implication* is an implication $\varphi \to \psi$ in which $\psi$ is a positive formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ and $\varphi$ is a Sahlqvist antecedent of $\mathcal{L}(\mathbb{P}, \mathsf{Form_t})$. A *Sahlqvist formula* of $\mathcal{L}(\mathbb{P}, \mathsf{Form_t})$ is a formula built up from Sahlqvist implications by freely applying boxes and conjunctions, and by applying disjunctions only between formulas that do not share any proposition letters. □

**Definition 28** (Translations $\tau_3$ and $\tau_3^-$)**.** The *Sahlqvist algorithm* for Sahlqvist formulas of the form $\varphi \to \psi$, denoted $\tau_3$, is defined below. The *Kracht algorithm*, denoted $\tau_3^-$, can be found in [11, p. 169–178].

*Step 0: Translate the formula $\varphi \to \psi$ in $\mathcal{L}_{\mathrm{SOL}}(\mathcal{R}_1, \mathcal{R}_2)$ (Definition 26).*

*Step 1: Pull out diamonds and pre-process.*

We can rewrite the second-order translation of $\varphi \to \psi$ into a conjonction of formulas of the form

$$\forall P_1 \ldots \forall P_n \forall x_1 \ldots \forall x_m (REL \wedge BOXAT \wedge NEG \to ST_x(\psi)) \tag{6}$$

where $REL$ is a conjunction of atomic first-order statements of the form $Rxy$ corresponding to occurrences of diamonds and other existential modalities, $BOXAT$ is a conjunction of (translations of) boxed atoms, and $NEG$ is a conjunction of (translation of) negative formulas. We must show that each formula of the form displayed in Formula (6) has a first-order equivalent. This is done by using the equivalence

$$((\alpha \wedge NEG) \to \beta) \leftrightarrow (\alpha \to (\beta \vee \neg NEG))$$

where $\neg NEG$ is the *positive* formula that arises by negating the negative formula $NEG$. Using this equivalence we can rewrite Formula (6) to obtain a formula of the form

$$\forall P_1 \ldots \forall P_n \forall x_1 \ldots \forall x_m (REL \wedge BOXAT \to POS) \tag{7}$$

*Step 2: Read off instances.*

Let $P$ be a unary predicate occurring in Formula (7), and let $\pi_1(x_{i_1}), \ldots, \pi_k(x_{i_k})$ be all the (translations of the) boxed atoms in the antecedent of Formula (7) in which the predicate $P$ occurs: $\pi_j(x_{i_j}) := \Box_{j_1} \ldots \Box_{j_l} p$. Observe that every $\pi_j(x_{i_j})$ is of the form $\forall y(R_{\beta_j} x_{i_j} y \to Py)$, where $\beta_j := j_1 \ldots j_l$ and $R_{\beta_j} xy$ is an abbreviation for

$$\exists y_{j_1}(R_{j_1} x y_{j_1} \wedge \exists y_{j_2}(R_{j_2} y_{j_1} y_{j_2} \wedge \ldots \wedge \exists y_{j_l-1}(R_{j_l-1} y_{j_l-2} y_{j_l-1} \wedge R_{j_l} y_{j_l-1} y) \ldots))$$

If $l = 0$, $\beta$ is the empty sequence $\epsilon$; in this case the formula $R_\epsilon x_{i_j} y$ should be read as $x_{i_j} = y$. Let $Py_{i_{k+1}}, \ldots, Py_{i_{k+l}}$ be all the occurrences of the predicate $P$ in the antecedent of Formula (7). Define

$$\sigma(P) := \lambda u.(R_{\beta_1} x_{i_1} u \vee \ldots \vee R_{\beta_k} x_{i_k} u \vee y_{i_{k+1}} = u \vee \ldots \vee y_{k+l} = u)$$

$\sigma(P_1), \ldots, \sigma(P_n)$ form the *minimal* instances making the antecedent $BOXAT$ true.

We can assume that every unary predicate $P$ that occurs in the consequent of Formula (7) also occurs in the antecedent of Formula (7). Otherwise Formula (7) is positive and we can substitute $\lambda u.u \neq u$ for $P$: we will obtain an equivalent formula without occurrences of $P$.

*Step 3: Instantiating.*

We now use the formulas of the form $\sigma(P)$ found in Step 2 as instantiations; we substitute $\sigma(P)$ for each occurrence of $P$ in the first-order matrix of Formula (7). This results in a formula of the form

$$[\sigma(P_1)/P_1, \ldots, \sigma(P_n)/P_n]\forall x_1 \ldots \forall x_m(REL \wedge BOXAT \rightarrow POS).$$

Now, there are no occurrences of monadic second-order variables in $REL$. Furthermore, observe that by our choice of the substitution instances $\sigma(P)$, the formula $[\sigma(P_1)/P_1], \ldots, \sigma(P_n)/P_n]BOXAT$ will be trivially true. So after carrying out these substitutions we end up with a formula that is equivalent to one of the form

$$\forall x_1 \ldots \forall x_m(REL \rightarrow [\sigma(P_1)/P_1, \ldots, \sigma(P_n)/P_n]POS). \tag{8}$$

Formula (8) is a first-order formula involving only $=$ and the relation symbol $S$. It remains to show that Formula (8) is equivalent to Formula (6). The implication from Formula (6) to Formula (8) is simply an instantiation. For the implication from Formula (8) to Formula (6), let $\mathcal{M}$ be a substructural model and assume that $\mathcal{M} \models \forall x_1 \ldots \forall x_m(REL \rightarrow [\sigma(P_1)/P_1, \ldots, \sigma(P_n)/P_n]POS)$ and $\mathcal{M} \models REL \wedge BOXAT[ww_1 \ldots w_m]$. We need to show that $\mathcal{M} \models POS[ww_1 \ldots w_m]$. First of all, it follows from the above assumptions that $\mathcal{M} \models [\sigma(P_1)/P_1, \ldots, \sigma(P_n)/P_n]POS[ww_1 \ldots w_m]$. As $POS$ is positive, it is upwards monotone in all unary predicates occurring in it, so it suffices to show that for all $P \in \{P_1, \ldots, P_n\}$, it holds that $\mathcal{M} \models \forall y(\sigma(P)(y) \rightarrow Py)[ww_1 \ldots w_m]$. If $\mathcal{M} \models \sigma(P)(y)$, then either (1) $\mathcal{M} \models R_{\beta_j} x_{i_j} y$ or (2) $\mathcal{M} \models y_{i_j} = y$, and we have to show that $\mathcal{M} \models Py$. In the first case (1), this entails that $\mathcal{M} \models \forall y(R_{\beta_j} x_{i_j} y \rightarrow Py)$ also holds. Hence, we can conclude that $\mathcal{M} \models Py$. In the second case (2), because $\mathcal{M} \models Py_{i_j}$, $y_{i_j} = y$ and $\mathcal{M}$ is a substructural model, we have that $\mathcal{M} \models Py$. $\square$

**Theorem 9** (Correspondence)**.** *Let $\varphi \in \mathcal{L}(\mathbb{P}, \mathit{Form}_t)$ be a Sahlqvist formula. Then, $\varphi$ and $\tau_3(\varphi)$ are local substructural frame correspondents.*

*Proof.* It suffices to apply the Sahlqvist algorithm to $\varphi$. The result then follows from [11, Th. 3.54]. $\square$

**Theorem 10** (Canonicity)**.** *Let $S \subseteq \mathcal{L}(\mathbb{P}, \mathit{Form}_t)$ be a set of Sahlqvist formulas. Then, the proof system $K_t + \Sigma$ is sound and strongly complete for the logic $\big(\mathcal{L}(\mathbb{P}, \mathit{Form}_t), \mathcal{F}_0, \models\big)$, where $\mathcal{F}_0$ is the class of pointed substructural frames defined by the set of first-order formulas $\tau_3(S) := \{\tau_3(\varphi) \mid \varphi \in S\}$.*

*Proof.* The result follows from [11, Th. 4.42]. $\square$

**Example 5.** We did not impose any particular restriction upon the accessibility relation $S_j$. So, there is no reason that it captures the notion of knowledge and belief as they are usually represented in epistemic and doxastic logics [1, 15, 32]. Usually, the notion of knowledge is captured by the fact that the epistemic accessibility relations $S_j$ are (at least) reflexive and transitive. We say that a substructural frame $F$ is *transitive* when for all accessibility relations $S_j$, for all $w, v, u \in F$, if $S_j wv$ and $S_j vu$ then $S_j wu$. We say that $F$ is *reflexive* when for all accessibility relations $S_j$, for all $w \in F$, $S_j ww$. We denote by $\mathcal{F}_4$ (resp. $\mathcal{F}_\mathsf{T}$) the class of transitive (resp. reflexive and transitive) pointed substructural frames. Let us consider the following axiom schemata: for all $j \in \mathbb{G}$,

$$\Diamond_j \Diamond_j \varphi \to \Diamond_j \varphi \qquad \mathsf{4} \qquad\qquad \varphi \to \Diamond_j \varphi \qquad \mathsf{T}$$

Axiom schemata $\mathsf{4}$ and $\mathsf{T}$ are Sahlqvist axioms. Their first-order translations are $\forall w \forall v \forall u (S_j wv \wedge S_j vu \to S_j wu)$ and $\forall w S_j ww$. So, by Theorem 10, the Hilbert calculus $\mathsf{K_t} + \{\mathsf{4}\}$ (resp. $\mathsf{K_t} + \{\mathsf{T}\}$) is sound and strongly complete for $\left(\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{F}_4, \models\right)$ (resp. $\left(\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right), \mathcal{F}_\mathsf{T}, \models\right)$). $\square$

**Correspondence for Inference Rules**

We recall the main results of Kracht [22], adapted to our substructural tense logic.

**Definition 29** (Primitive formulas of $\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right)$)**.** A *primitive formula of* $\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right)$ is a formula of $\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right)$ of the form $\varphi \to \psi$, where $\varphi, \psi \in \mathcal{L}\left(\mathbb{P}, \mathsf{F}\right)$ with $\mathsf{F} := \left\{\top, \wedge, \vee, \Diamond_j, \Diamond_j^-, \exists_i, \exists_i^- \mid j \in \mathbb{G}, i \in \{1, 2\}\right\}$ and such that $\varphi$ contains each propositional variable at most once. $\square$

Note that a primitive tense formula is a Sahlqvist formula.

**Definition 30** (Translations $\tau_2$ and $\tau_2^-$)**.** Let $R$ be an analytic inference rule in $\mathcal{C}\left(\mathbb{P}, \mathsf{Form}\right)$. It can be represented in this special form ($U, V, U_1, \ldots, U_n$ can be empty structures):

$$\frac{U_1 \vdash V \quad \ldots \quad U_n \vdash V}{U \vdash V}\ R \tag{9}$$

We define the primitive formula of $\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right)$ associated to $R$ as follows where each structure variable in $U, U_1, \ldots, U_n$ has been uniformly replaced by a propositional letter:

$$\tau_2\left(R\right) := t_1(U) \to t_1\left(U_1\right) \vee \ldots \vee t_1\left(U_n\right)$$

The converse algorithm $\tau_2^-$ can be found in [22, p. 106-107]. It transforms any primitive formula of $\mathcal{L}\left(\mathbb{P}, \mathsf{Form_t}\right)$ into an analytic inference rule in $\mathcal{C}\left(\mathbb{P}, \mathsf{Form}\right)$ which has the same deductive power as the formula. $\square$

**Definition 31** (Inherently universal variable)**.** We say that an occurrence of a variable $y$ in a formula $\alpha$ is *inherently universal* if either $y$ is free, or else $y$ is bound by a restricted universal quantifier which is not in the scope of a (restricted) existential quantifier. $\square$

**Definition 32** (Primitive formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$)**.**

- A *Kracht formula of* $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ is built up from atomic formulas of the form $x = y$, $x \sqsubseteq y$, $S_j xy$ (where $j \in \mathbb{G}$), $\mathcal{R}_1 xy$ or $\mathcal{R}_2 xy$ with the help of $\wedge, \vee$ and the restricted quantifiers in such a way that in a subformula $x = y$, $x \sqsubseteq y$, $S_j xy$, $\mathcal{R}_1 xy$ or $\mathcal{R}_2 xy$ at least one of $x$ or $y$ is hereditary universal.

- A *primitive formula of* $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ is a Kracht formula of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ in which no universal quantifier is in the scope of an existential quantifier. $\square$

**Theorem 11.** *Let $\mathcal{F}_0$ be a class of pointed substructural frames. There exists a finite set of analytic inference rules $\Sigma_t$ in $\mathcal{C}_t$ such that $\mathsf{D}_t + \Sigma_t$ is sound and complete for the logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}_0, \models)$ if, and only if, there exists a finite set $\Theta(x)$ of primitive formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ with one free variable $x$ that defines $\mathcal{F}_0$. Moreover, $\Sigma_t$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma_t$, as follows: $\Theta(x) := \tau_3(\tau_2(\Sigma_t))$ and $\Sigma_t := \tau_2^-(\tau_3^-(\Theta(x)))$.*

*Proof.* As shown in the proof of Theorem 5, $\mathbf{DLM}_t + \Sigma_0$ is sound and complete for the logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}, \models)$, where $\mathbf{DLM}_t$ is the multi-modal version based on $\mathcal{C}(\mathbb{P}, \mathsf{Form}_t)$ of the display calculus **DLM** defined in [22]. Moreover, by Theorem 5, $\mathsf{D}_t$ is also sound and complete for the same logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}, \models)$. Thus, for any class of substructural frames $\mathcal{F}_0$, it holds that $\mathsf{D}_t + \Sigma_t$ is sound and complete for the logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}_0, \models)$ iff $\mathbf{DLM}_t + \Sigma_0 + \Sigma_t$ is sound and complete for the logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}_0, \models)$. So, by Lemma 3, $\mathsf{D}_t + \Sigma_t$ is sound and complete for the logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}_0, \models)$ iff $\mathbf{DLM}_t + \Sigma_0 + \Sigma_t$ is sound and complete for the logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{K}_0^-, \models)$ (we recall that $\mathcal{K}_0^-$ is the class of pointed substructural Kripke frames associated to $\mathcal{F}_0$). The result then follows from [22, Theorems 16-17]. $\square$

**Proposition 12** ([22])**.** *Let $R$ be an analytic inference rule in $\mathcal{C}(\mathbb{P}, \mathsf{Form}_t)$, let $\varphi$ be a primitive formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$ and let $(F, w)$ be a pointed substructural frame. Then, we have $F, w \models R$ if, and only if, $F, w \models \tau_2(R)$, and, vice versa, we have $F, w \models \varphi$ if, and only if, $F, w \models \tau_2^-(\varphi)$. Moreover,*

$$\models \tau_2\left(\tau_2^-(\varphi)\right) \leftrightarrow \varphi \qquad\qquad \tau_2^-(\tau_2(R)) = R$$

## 6.3 Correspondence Theory for Restricted Tense Logic

In this section, we adapt the results of the previous section (in particular Theorem 11) to our restricted tense logic. We proceed in two steps. In Section 6.3, we provide a translation between primitive formulas of $\mathcal{L}_t$ and specific first-order formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ based only on binary relations. In Section 6.3, we provide a translation between the formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ and other first-order formulas where the two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ have been combined and replaced by the ternary relation $\mathcal{R}$. We show in Theorem 17 that these two translations preserve the expected properties.

#### Translations between Prototypic Formulas of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$ and $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$

**Definition 33** (Prototypic formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$)**.** A *prototypic formula of* $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$ is a primitive formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$ which belongs moreover to $\mathcal{L}_t$. $\square$

**Definition 34** (Inherently universal variable)**.** We say that an occurrence of a variable $y$ in a formula $\alpha$ is *inherently universal* if either $y$ is free, or else $y$ is bound by a restricted universal quantifier which is not in the scope of a restricted existential quantifier. $\square$

**Definition 35** (Prototypic formulas of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ and $\mathcal{L}_{\text{FOL}}(\mathcal{R})$)**.**

- A *Kracht formula of* $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ is built up from atomic formulas of the form $x = y$, $x \sqsubseteq y$, $S_j xy$ (where $j \in \mathbb{G}$), $\mathcal{R}xyz$ with the help of $\wedge, \vee$ and the restricted quantifiers in such a way that in a subformula $x = y$, $x \sqsubseteq y$, $S_j xy$ at least one of $x$ and $y$ is hereditary universal and in a subformula $\mathcal{R}xyz$ either $y$ is hereditary universal or $x$ and $z$ are both hereditary universal (inclusive or).
- A *prototypic formula of* $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ is a Kracht formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ in which no universal quantifier is in the scope of an existential quantifier.

The set of prototypic formulas of $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ is denoted $\mathcal{L}_P(\mathcal{R})$.

- A *Kracht formula of* $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ is built up from atomic formulas of the form $x = y$, $x \sqsubseteq y$, $S_j xy$ (where $j \in \mathbb{G}$), $\mathcal{R}_1 xy$, $\mathcal{R}_2 xy$ with the help of $\wedge, \vee$ and the restricted quantifiers in such a way that in a subformula $x \sqsubseteq y$, $\mathcal{R}_1 xy$, $\mathcal{R}_2 xy$, $S_j xy$ or $x = y$, at least one of $x$ and $y$ is hereditary universal.
- A *prototypic formula of* $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ is a primitive formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ such that:

  *(P1)* Every $(\forall y \rhd_1 x)$ is immediately followed by a $(\forall z \rhd_2 y)$;

  *(P2)* Every $(\forall x \lhd_1 y)$ is immediately preceded by a $(\forall y \lhd_2 z)$ or immediately preceded or followed by a $(\forall z \rhd_2 y)$ (exclusive or);

  *(P3)* Every $(\forall z \rhd_2 y)$ is immediately preceded by a $(\forall y \rhd_1 x)$ or immediately preceded or followed by a $(\forall x \lhd_1 y)$ (exclusive or);

  *(P4)* Every $(\forall y \lhd_2 z)$ is immediately followed by a $(\forall x \lhd_1 y)$;

  *(P5)* Every $(\exists y \rhd_1 x)$ is immediately followed by a $(\exists z \rhd_2 y)$;

  *(P6)* Every $(\exists x \lhd_1 y)$ is either immediately preceded by a $(\exists y \lhd_2 z)$ or immediately followed or preceded by a $(\exists z \rhd_2 y)$ (exclusive or);

  *(P7)* Every $(\exists z \rhd_2 y)$ is either immediately preceded by a $(\exists y \rhd_1 x)$ or immediately followed or preceded by a $(\exists x \lhd_1 y)$ (exclusive or);

  *(P8)* Every $(\exists y \lhd_2 z)$ is immediately followed by a $(\exists x \lhd_1 y)$.

  $(\mathcal{R})$ Every atomic formula $\mathcal{R}_1 xy$ or $\mathcal{R}_2 xy$ which is not in a restricted quantifer appears under the form $\mathcal{R}_1 xy \wedge \mathcal{R}_2 yz$ or $\mathcal{R}_1 zx \wedge \mathcal{R}_2 xy$ (for some $z$) respectively.

The set of prototypic formulas of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ is denoted $\mathcal{L}_P(\mathcal{R}_1, \mathcal{R}_2)$. $\square$

If $\mathcal{R}_1 xy$ and $\mathcal{R}_2 yz$ are two atomic formulas of a primitive formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$, then, by definition of a primitive formula, if $y$ is not hereditary universal, both $x$ and $z$ must be hereditary universal. Now, every atomic formula $\mathcal{R}xyz$ in a formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ translates into such a pair of atomic formulas $\mathcal{R}_1 xy$ and $\mathcal{R}_2 yz$ in a prototypic formula of $\mathcal{L}_P(\mathcal{R})$. So, this explains why in the definition of a Kracht formula, in every subformula $\mathcal{R}xyz$, either $y$ is hereditary universal or $x$ and $z$ are both hereditary universal.

**Lemma 13.** *If $\varphi$ is a prototypic formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$, then $\tau_3(\varphi)$ is equivalent on the class of pointed substructural frames to a prototypic formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$. Vice versa, if $\Theta(x)$ is a prototypic formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$, then $\tau_3^-(\Theta(x))$ is a prototypic formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$.*

*Proof.* A prototypic formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$ is a specific instance of primitive formulas of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$. So, $\tau_3(\varphi)$ is a primitive formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$. Moreover, because $\varphi \in \mathcal{L}_t$, its standard translation $ST_x(\varphi)$ is such that it fulfills the 8 conditions *P1–P8*. It turns out that the Sahlqvist algorithm preserves these conditions *P1–P8*. Hence the formula $\tau_3(\varphi)$ fulfills the 8 conditions *P1–P8* as well, and $\tau_3(\varphi)$ is a prototypic formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$. Vice versa, the version of the algorithm $\tau_3^-$ that we use in Section 8.2 is such that the formula $\tau_3^-(\varphi)$ obtained from a prototypic formula $\varphi$ of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ via this algorithm belongs to $\mathcal{L}_t$, besides being a primitive formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$. Hence, $\tau_3^-(\varphi)$ is a prototypic tense formula. □

**Proposition 14** ([11, 22])**.** *Let $\varphi$ be a prototypic formula of $\mathcal{L}(\mathbb{P}, \mathsf{Form}_t)$ and let $\Theta(x)$ be a prototypic formula of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$. Then,*

$$\models \tau_3\left(\tau_3^-(\Theta(x))\right) \leftrightarrow \Theta(x) \qquad\qquad \models \tau_3^-(\tau_3(\varphi)) \leftrightarrow \varphi$$

**Translations between Prototypic Formulas of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ and $\mathcal{L}_{\text{FOL}}(\mathcal{R})$**

**Definition 36** (Translations $\tau_4$ and $\tau_4^-$)**.** We define the translation $\tau_4 : \mathcal{L}_P(\mathcal{R}_1, \mathcal{R}_2) \to \mathcal{L}_P(\mathcal{R})$ and its converse $\tau_4^- : \mathcal{L}_P(\mathcal{R}) \to \mathcal{L}_P(\mathcal{R}_1, \mathcal{R}_2)$ by the following correspondence, which relates atomic formulas and each restricted quantifer of $\mathcal{L}_{\text{FOL}}(\mathcal{R})$ to one or two pairs of restricted quantifiers of $\mathcal{L}_{\text{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$, and vice versa.

$$\begin{array}{lll}
(\exists yz \rhd^1 x) & :=: & (\exists y \rhd_1 x)(\exists z \rhd_2 y) \\
(\exists yz \rhd^2 x) & :=: & \begin{cases} (\exists y \rhd_2 x)(\exists z \lhd_1 x) \\ (\exists z \lhd_1 x)(\exists y \rhd_2 x) \end{cases} \\
(\exists yz \rhd^3 x) & :=: & (\exists z \lhd_2 x)(\exists y \lhd_1 z) \\
(\forall yz \rhd^1 x) & :=: & (\forall y \rhd_1 x)(\forall z \rhd_2 y) \\
(\forall yz \rhd^2 x) & :=: & \begin{cases} (\forall y \rhd_2 x)(\forall z \lhd_1 x) \\ (\forall z \lhd_1 x)(\forall y \rhd_2 x) \end{cases} \\
(\forall yz \rhd^3 x) & :=: & (\forall z \lhd_2 x)(\forall y \lhd_1 z) \\
\mathcal{R}xyz & :=: & \mathcal{R}_1 xy \wedge \mathcal{R}_2 yz
\end{array}$$

For the restricted quantifiers $(\exists yz \rhd^2 x)$ and $(\forall yz \rhd^2 x)$, note that two translations are possible. □

**Proposition 15.** *The logics $(\mathcal{L}_P(\mathcal{R}_1, \mathcal{R}_2), \mathcal{F}, \models_{\text{FOL}})$ and $(\mathcal{L}_P(\mathcal{R}), \mathcal{F}, \models_{\text{FOL}})$ are equally expressive. More precisely, for all $\varphi \in \mathcal{L}_P(\mathcal{R}_1, \mathcal{R}_2)$ and all $(F, w) \in \mathcal{F}$, we have $F, w \models \varphi$ if, and only if, $F, w \models \tau_4(\varphi)$, and, vice versa, for all $\psi \in \mathcal{L}_P(\mathcal{R})$, we have $F, w \models \psi$ if, and only if, $\mathcal{M}, w \models \tau_4^-(\psi)$. Hence,*

$$\models \tau_4\left(\tau_4^-(\psi)\right) \leftrightarrow \psi \qquad\qquad \models \tau_4^-(\tau_4(\varphi)) \leftrightarrow \varphi$$

*Proof.* By induction on the number of restricted quantifiers in the given formula of $\mathcal{L}_P(\mathcal{R}_1, \mathcal{R}_2)$ or $\mathcal{L}_P(\mathcal{R})$. $\square$

**Theorem 16.** *Let $R$ be an analytic inference rule in $\mathcal{C}_t$. Then, $R$ and $\tau_4(\tau_3(\tau_2(R)))$ are local substructural frame correspondents.*

*Proof.* It follows from Theorem 9 and Propositions 12 and 15. $\square$

**Theorem 17.** *Let $\mathcal{F}_0$ be a a class of pointed substructural frames. There exists a finite set of analytic inference rules $\Sigma_t$ in $\mathcal{C}_t$ such that $\mathsf{D}_t + \Sigma_t$ is sound and complete for the logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}_0, \models)$ if, and only if, there exists a finite set $\Theta(x)$ of prototypic formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$ that defines $\mathcal{F}_0$. Moreover, $\Sigma_t$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma_t$, as follows: $\Theta(x) := \tau_4(\tau_3(\tau_2(\Sigma_t)))$ and $\Sigma_t := \tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)$.*

*Proof.* Let $\Sigma_t$ be a finite set of analytic inference rules in $\mathcal{C}_t$. Then, by Theorem 11, $\mathsf{D}_t + \Sigma_t$ is sound and complete for $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}_0, \models)$ if, and only if, $\mathcal{F}_0$ can be defined by some finite set $\Theta_0(x)$ of primitive formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2)$ computable as follows: $\Theta_0(x) := \tau_3(\tau_2(\Sigma_t))$. However, by Proposition 15, $\Theta_0(x)$ is equivalent on the class of pointed substructural frames to a finite set of prototypic formulas $\Theta(x) := \tau_4(\Theta_0(x))$ of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$. Hence, $\mathsf{D}_t + \Sigma_t$ is sound and complete for $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_t), \mathcal{F}_0, \models)$ if, and only if, $\mathcal{F}_0$ can be defined by some finite set $\Theta(x)$ of prototypic formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$. Moreover, again using Proposition 15, $\Sigma_t$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma_t$, as follows: $\Theta(x) := \tau_4(\tau_3(\tau_2(\Sigma_t)))$ and $\Sigma_t := \tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)$. $\square$

# 7 Correspondence Theory for Update Logic

The shape of update logic combined with correspondence results for tense logics will allow us to characterize the class of properly displayable logics which extend update logic. Update logic being a generalization of the non-associative Lambek calculus, this will subsequently provide a characterization of all properly displayable logics without (truth) constants that extend the non-associative Lambek calculus.

## 7.1 From Inference Rules to First-order Frame Conditions

**Lemma 18.** *An inference rule $R$ in $\mathcal{C}(\mathbb{P}, \mathsf{Form})$ (resp. $\mathcal{C}_t$) is analytic if, and only if, $\tau_1(R)$ (resp. $\tau_1^-(R)$) is analytic in $\mathcal{C}_t$ (resp. $\mathcal{C}(\mathbb{P}, \mathsf{Form})$).*

*Proof.* It suffices to observe that the inductive steps in the translations $\tau_1$ and $\tau_1^-$ in Definition 22, when viewed as inference rules (the input is viewed as the premise and the output is viewed as the conclusion of the rule, and vice versa), fulfill conditions $(C1)-(C8)$. $\square$

**Lemma 19.** *Let $\Sigma$ be a set of analytic inference rules in $\mathcal{C}(\mathbb{P}, \mathsf{Form})$ and let $X \vdash Y \in \mathcal{C}(\mathbb{P}, \mathsf{Form})$. If $X \vdash Y$ is provable in $\mathsf{UL} + \Sigma$ then $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_t + \tau_1(\Sigma)$. Moreover, the proof of $\tau_1(X \vdash Y)$ in $\mathsf{D}_t + \tau_1(\Sigma)$ is effectively computable from the proof of $X \vdash Y$ in $\mathsf{UL} + \Sigma$.*

*Proof.* We first prove the left to right direction. If $X \vdash Y$ is provable in $\mathsf{UL} + \Sigma$, then $X \vdash Y$ is valid in $(\mathcal{C}(\mathbb{P}, \mathsf{Form}), \mathcal{F}_0, \Vdash)$, where $\mathcal{F}_0$ is the class of pointed substructural frames defined by the inference rules of $\Sigma$. Therefore, by Proposition 7, $\tau_1(X \vdash Y)$ is valid in $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_\mathsf{t}), \mathcal{F}_0, \models)$, where $\mathcal{F}_0$ is also the class of pointed substructural frames defined by the inference rules of $\tau_1(\Sigma)$. Moreover, by Lemma 18, $\tau_1(\Sigma)$ is a set of analytic inference rules and, by Theorem 16, it locally corresponds to the set of prototypic formulas $\tau_4(\tau_3(\tau_2(\tau_1(\Sigma))))$ of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$. So, $\mathcal{F}_0$ is also the class of pointed substructural frames defined by the set of prototypic formulas $\tau_4(\tau_3(\tau_2(\tau_1(\Sigma))))$ of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$. Hence, by Theorem 17, $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\mathsf{t} + \tau_1(\Sigma)$.

For the left to right direction, it suffices to prove that every inference rule $R$ of $\mathsf{UL}$ is such that $\tau_1(R)$ is derivable in $\mathsf{D}_\mathsf{t}$. We only prove it for the display rules for the case where $(i, j, k) = (3, 1, 2)$. In the derivations below, we write $\mathsf{csr}$ when one or more classical (structural) rules are applied.

$$
\begin{array}{cl}
\tau_1(X \,,_3\, Y \vdash Z) & \\ \hline\hline
\bullet_2(\bullet_1 t_1(X), t_1(Y)) \vdash t_2(Z) & \mathsf{Def} \\ \hline\hline
\bullet_1 t_1(X), t_1(Y) \vdash \bullet_2\, t_2(Z) & \bullet_2 \\ \hline\hline
\bullet_1 t_1(X) \vdash *t_1(Y),\ \bullet_2\, t_2(Z) & *_K \\ \hline\hline
t_1(X) \vdash \bullet_1\, (*t_1(Y),\ \bullet_2\, t_2(Z)) & \bullet_1 \\ \hline\hline
\tau_1(X \vdash *Y \,,_1\, Z) & \mathsf{Def}
\end{array}
\qquad
\begin{array}{cl}
\tau_1(X \,,_3\, Y \vdash Z) & \\ \hline\hline
\bullet_2(\bullet_1 t_1(X), t_1(Y)) \vdash t_2(Z) & \mathsf{Def} \\ \hline\hline
\bullet_1 t_1(X), t_1(Y) \vdash \bullet_2\, t_2(Z) & \bullet_2 \\ \hline\hline
t_1(Y) \vdash \bullet_2\, t_2(Z),\ *\bullet_1 t_1(X) & *_K \\ \hline\hline
t_1(Y) \vdash \bullet_2\, t_2(Z),\ *\bullet_1 * t_2(*X) & \mathsf{csr} \\ \hline\hline
\tau_1(Y \vdash Z \,,_2\, *X) & \mathsf{Def}
\end{array}
$$

$$
\begin{array}{cl}
\tau_1(Z \vdash X \,,_3\, Y) & \\ \hline\hline
t_1(Z) \vdash *\bullet_2 * (t_2(Y),\ *\bullet_1 * t_2(X)) & \mathsf{Def} \\ \hline\hline
\bullet_2 * (t_2(Y),\ *\bullet_1 * t_2(X)) \vdash *t_1(Z) & *_A, *_K \\ \hline\hline
*(t_2(Y),\ *\bullet_1 * t_2(X)) \vdash \bullet_2 *\tau_1(Z) & \bullet_2 \\ \hline\hline
*\bullet_2 *t_1(Z) \vdash t_2(Y),\ *\bullet_1 * t_2(X) & \mathsf{csr} \\ \hline\hline
\bullet_1 * t_2(X),\ *\bullet_2 * t_1(Z) \vdash t_2(Y) & \mathsf{csr} \\ \hline\hline
\bullet_1 t_1(*X),\ *\bullet_2 * t_1(Z) \vdash t_2(Y) & \mathsf{csr} \\ \hline\hline
\tau_1(Z \,,_2\, *X \vdash Y) & \mathsf{Def}
\end{array}
\qquad
\begin{array}{cl}
\tau_1(Z \vdash X \,,_3\, Y) & \\ \hline\hline
t_1(Z) \vdash *\bullet_2 * (t_2(Y),\ *\bullet_1 * t_2(X)) & \mathsf{Def} \\ \hline\hline
\bullet_2 * (t_2(Y),\ *\bullet_1 * t_2(X)) \vdash *t_1(Z) & *_A, *_K \\ \hline\hline
*(t_2(Y),\ *\bullet_1 * t_2(X)) \vdash \bullet_2 *\tau_1(Z) & \bullet_2 \\ \hline\hline
*\bullet_2 *t_1(Z) \vdash t_2(Y),\ *\bullet_1 * t_2(X) & \mathsf{csr} \\ \hline\hline
*t_2(Y),\ *\bullet_2 * t_1(Z) \vdash *\bullet_1 * t_2(X) & \mathsf{csr} \\ \hline\hline
\bullet_1 * t_2(X) \vdash *(*t_2(Y),\ *\bullet_2 * t_1(Z)) & \bullet_1 \\ \hline\hline
*t_2(X) \vdash \bullet_1 *(*t_2(Y),\ *\bullet_2 * t_1(Z)) & \mathsf{csr} \\ \hline\hline
*\bullet_1 *(*t_2(Y),\ *\bullet_2 * t_1(Z)) \vdash t_2(X) & \mathsf{csr} \\ \hline\hline
\tau_1(*Y \,,_2\, Z \vdash X) & \mathsf{Def}
\end{array}
$$

$\square$

**Theorem 20** (Correspondence for inference rules)**.** *Let $\Sigma$ be a finite set of analytic inference rules in $\mathcal{C}(\mathbb{P}, \mathsf{Form})$. Then, $\Sigma$ and $\Theta(x) := \tau_3(\tau_2(\tau_1(\Sigma)))$ are local substructural frame correspondents.*

*Proof.* From Theorem 9 and Propositions 6 and 12, it follows directly that if $\Sigma$ is a finite set of analytic inference rules in $\mathcal{C}(\mathbb{P}, \mathsf{Form})$, then $\Sigma$ and $\Theta(x) := \tau_3(\tau_2(\tau_1(\Sigma)))$ are local substructural frame correspondents. $\square$

**Theorem 21** (Canonicity for inference rules)**.** *Let $\mathsf{F} \subseteq \mathsf{Form}$ and let $\Sigma$ be a finite set of analytic inference rules in $\mathcal{C}(\mathbb{P}, \mathsf{F})$. Then, $\mathsf{UL}(\mathsf{F}) + \Sigma$ is sound and complete for the logic $\left(\mathcal{C}(\mathbb{P}, \mathsf{F}), \mathcal{F}_0, \Vdash\right)$, where $\mathcal{F}_0$ is the class of pointed substructural frames defined by the set of prototypic formulas $\Theta(x) := \tau_4\left(\tau_3\left(\tau_2(\tau_1(\Sigma))\right)\right)$ of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$.*

*Proof.* We first prove completeness. Let $X \vdash Y \in \mathcal{C}(\mathbb{P}, \mathsf{F})$ and assume that $X \vdash Y$ is valid in $\left(\mathcal{C}(\mathbb{P}, \mathsf{F}), \mathcal{F}_0, \Vdash\right)$. Then, $\tau_1(X \vdash Y)$ is valid in $\left(\mathcal{C}(\mathbb{P}, \mathsf{Form}_\mathsf{t}), \mathcal{F}_0, \models\right)$ by Proposition 7. Moreover, $\tau_1(\Sigma)$ is a set of analytic inference rules by Lemma 18. So, by Theorem 17, $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\mathsf{t} + \tau_1(\Sigma)$. Thus, $X \vdash Y$ is provable in $\mathsf{UL} + \Sigma$ by Lemma 19, and therefore also in $\mathsf{UL}(\mathsf{F}) + \Sigma$ (we recall that $\mathsf{UL} + \Sigma$ admits the cut rule and therefore any proof of $X \vdash Y$ in $\mathsf{UL} + \Sigma$ will resort to logical rules where only the connectives of $X \vdash Y$ occur). Now, we prove soundness. Assume that $X \vdash Y$ is provable in $\mathsf{UL}(\mathsf{F}) + \Sigma$. Then, by Lemma 19, $\tau_1(X \vdash Y)$ is provable in $\mathsf{D}_\mathsf{t} + \tau_1(\Sigma)$. Moreover, $\tau_1(\Sigma)$ is a set of analytic inference rules by Lemma 18. So, by Theorem 17, $\tau_1(X \vdash Y)$ is valid in $\left(\mathcal{C}(\mathbb{P}, \mathsf{Form}_\mathsf{t}), \mathcal{F}_0, \models\right)$. And finally, $X \vdash Y$ is valid in $\left(\mathcal{C}(\mathbb{P}, \mathsf{F}), \mathcal{F}_0, \Vdash\right)$ by Proposition 7. □

**Corollary 1.** *The display calculus $\mathsf{UL}$ is sound and complete for the logic $\left(\mathcal{C}(\mathbb{P}, \mathsf{Form}), \mathcal{F}, \Vdash\right)$.*

On the one hand, a more general result than Corollary 1 is proved in [2], namely the *strong* completeness of $\mathsf{UL}$ and some of its extension with respect to substructural *models* (which implies completeness with respect to frames). On the other hand, Theorem 21 extends the (weak) completeness result to all the extensions of $\mathsf{UL}$ with analytic inference rules and this result is systematic, whereas the completeness result of [2] was proved on a case by case basis from the completeness results of [39].

## 7.2 From First-order Frame Conditions to Inference Rules

**Theorem 22.** *Let $\mathcal{F}_0$ be a class of pointed substructural frames defined by a set $\Theta(x)$ of prototypic formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$ with one free variable $x$. Then, the logic $(\mathcal{C}(\mathbb{P}, \mathsf{Form}), \mathcal{F}_0, \Vdash)$ is properly displayable. Moreover, the set of analytic inference rules $\Sigma$ of the proper display calculus which is sound and complete for $(\mathcal{C}(\mathbb{P}, \mathsf{Form}), \mathcal{F}_0, \Vdash)$ is effectively computable from $\Theta(x)$ as follows: $\Sigma := \tau_1^-\left(\tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)\right)$.*

*Proof.* By Theorem 17, $\mathsf{D}_\mathsf{t} + \Sigma_\mathsf{t}$ is sound and complete for $(\mathcal{C}(\mathbb{P}, \mathsf{Form}_\mathsf{t}), \mathcal{F}_0, \models)$, where $\Sigma_\mathsf{t} := \tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)$. Now, $\Sigma_\mathsf{t} = \tau_1\left(\tau_1^-(\Sigma_\mathsf{t})\right)$ by Lemma 8, and $\tau_1^-(\Sigma_\mathsf{t})$ is a set of analytic inference rules in $\mathcal{C}(\mathbb{P}, \mathsf{Form})$. So, by Theorem 21, $\mathsf{UL} + \tau_1^-(\Sigma_\mathsf{t})$ is sound and complete for the logic $\left(\mathcal{C}(\mathbb{P}, \mathsf{Form}), \mathcal{F}_0, \Vdash\right)$, where $\mathcal{F}_0$ is the class of pointed substructural frames defined by the following set of prototypic formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$:

$$\tau_4\left(\tau_3\left(\tau_2\left(\tau_1\left(\tau_1^-\left(\tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)\right)\right)\right)\right)\right)$$

which is equivalent to $\Theta(x)$ on the class of pointed substructural frames, by Propositions 8, 12, 14, 15. □

### 7.3 Characterization Theorem

Here is the main theorem of this report.

**Theorem 23.** *Let $\mathsf{F} \subseteq \mathsf{Form}$. A logic $(\mathcal{C}(\mathbb{P},\mathsf{F}), \mathcal{F}_0, \Vdash)$ based on a class $\mathcal{F}_0$ of pointed substructural frames is properly displayable by a set of analytic inference rules $\Sigma$ if, and only if, $\mathcal{F}_0$ is defined by some finite set $\Theta(x)$ of prototypic formulas of $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$ with one free variable. Moreover, $\Sigma$ is effectively computable from $\Theta(x)$ and, vice versa, $\Theta(x)$ is effectively computable from $\Sigma$, as follows: $\Theta(x) := \tau_4(\tau_3(\tau_2(\tau_1(\Sigma))))$ and $\Sigma := \tau_1^-\left(\tau_2^-\left(\tau_3^-\left(\tau_4^-(\Theta(x))\right)\right)\right)$ (see Figure 9).*

*Proof.* It follows straightforwardly from Theorems 21 and 22. $\square$

Theorem 23 shows that a logic extending update logic is properly displayable if, and only if, the class of pointed substructural frames on which it is defined can be defined by some finite set of prototypic first-order formulas. In that case, we have at our disposal algorithms to compute the prototypic first-order formulas defining the class of pointed substructural frames that correspond to the structural rules of the proper display calculus and, vice versa, we also have at our disposal algorithms to compute the structural rules of the display calculus that correspond to the prototypic first-order formulas defining the class of pointed substructural frames. These different algorithms are listed in Figure 9.

*Remark* 3. Our results also hold if we consider 'plain' substructural frames instead of *pointed* substructural frames and if validity is defined with respect to classes of (plain) substructural frames instead of classes of *pointed* substructural frames. This is because Sahlqvist's and Kracht's results hold both with respect to the so-called global *and* local correspondence.

## 8 Examples of Correspondence Translations

In this section, we provide examples of correspondence translations that use the algorithms $\tau_4, \tau_3, \tau_2, \tau_1$ and $\tau_4^-, \tau_3^-, \tau_2^-, \tau_1^-$ defined in the previous sections. The algorithm $\tau_3^-$ that we use is different from Kracht's algorithm and has been defined specifically for the kind of prototypic (in fact primitive) formulas that we consider.

In the rules below, we use the structural connective $\,,_3\,$ because its semantics corresponds to the semantics of the usual structural connective of substructural logic, often denoted ";" [39]. This obviously does not preclude ourselves to apply our algorithms to the other structural connectives like $\,,_1\,$ and $\,,_2\,$.

### 8.1 From Inference Rules to First-order Frame Conditions

We execute the algorithms $\tau_4, \tau_3, \tau_2, \tau_1$ on three classical inference rules: $\mathsf{K}$, $\mathsf{WI}$ and $\mathsf{CI}$.

**Inference Rule $\mathsf{K}$**

$$\frac{X \vdash U}{X \,,_3\, Y \vdash U}\ \mathsf{K}$$

$$
\begin{tikzcd}
\text{Analytic inference rules in } \mathcal{C}\,(\mathbb{P}, \mathsf{Form}) \text{ [Def. 13]} \arrow[d, bend left, "\tau_1 \quad \text{[Def. 21,23]}"] \\
\text{Analytic inference rules in } \mathcal{C}\,(\mathbb{P}, \mathsf{Form_t}) \text{ [Def. 13]} \arrow[u, bend left, "\text{[Def. 21,23]} \quad \tau_1^-"] \arrow[d, bend left, "\tau_2 \quad \text{[Def. 30], [22, p. 107-109]}"] \\
\text{Prototypic formulas of } \mathcal{L}\,(\mathbb{P}, \mathsf{Form_t}) \text{ [Def. 33]} \arrow[u, bend left, "\text{[Def. 30], [22, p. 106-107]} \quad \tau_2^-"] \arrow[d, bend left, "\tau_3 \quad \text{[Def. 28], [45]}"] \\
\text{Prototypic formulas of } \mathcal{L}_{\mathrm{FOL}}(\mathcal{R}_1, \mathcal{R}_2) \text{ [Def. 35]} \arrow[u, bend left, "\text{[Def. 28], [21]} \quad \tau_3^-"] \arrow[d, bend left, "\tau_4 \quad \text{[Def. 36]}"] \\
\text{Prototypic formulas of } \mathcal{L}_{\mathrm{FOL}}(\mathcal{R}) \text{ [Def. 35]} \arrow[u, bend left, "\text{[Def. 36]} \quad \tau_4^-"]
\end{tikzcd}
$$

Figure 9: Translations from analytic inference rules to first-order frame conditions, and vice versa

*Algorithm* $\tau_1$*:*

1. $\dfrac{\tau_1\left(X \vdash U\right)}{\tau_1\left(X \,,_3\, Y \vdash U\right)}$
2. $\dfrac{X \vdash U}{\bullet_2(\bullet_1 X \,, Y) \vdash U}$

*Algorithm* $\tau_2$*:*

1. $t_1(\bullet_2(\bullet_1 p, q)) \to t_1(p)$
2. $\exists_2^-((\exists_1^- \exists^- p) \wedge \exists^- q) \to \exists^- p$

*Algorithm* $\tau_3$*:*

1. $\forall PQ\left(\left(\exists y\left(\mathcal{R}_2 yx \wedge \left(\exists y'\left(y' \sqsubseteq y \wedge Q(y')\right) \wedge \exists z\left(\mathcal{R}_1 zy \wedge \exists z'\left(z' \sqsubseteq z \wedge P(z')\right)\right)\right)\right)\right) \to \exists x'\left(x' \sqsubseteq x \wedge P(x')\right)\right)$
2. $\forall PQ\left(\left(\exists yzy'z'\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge Q(y') \wedge P(z')\right)\right) \to \exists x'\left(x' \sqsubseteq x \wedge P(x')\right)\right)$
3. We take $\sigma(P) := \lambda u.z' = u$ and $\sigma(Q) := \lambda u.y' = u$
4. $\left(\exists yzy'z'\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge y' = y' \wedge z' = z'\right)\right) \to \left(\exists x'\left(x' \sqsubseteq x \wedge z' = x'\right)\right)$
5. $\left(\exists yzy'z'\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z\right)\right) \to z' \sqsubseteq x$
6. $\forall yzy'z'\left(\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z\right) \to z' \sqsubseteq x\right)$
7. $(\forall y \lhd_2 x)(\forall z \lhd_1 y)(\forall y' \lhd_0 y)(\forall z' \lhd_0 z)z' \sqsubseteq x$

*Algorithm* $\tau_4$*:*

1. $(\forall yz \rhd^3 x)(\forall y' \lhd_0 y)(\forall z' \lhd_0 z)z' \sqsubseteq x$

Finally, translated into plain $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$, we obtain:

$$\forall yzy'z'\left(\mathcal{R}zyx \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \to z' \sqsubseteq x\right) \tag{10}$$

If $\mathcal{R}$ is plump, then Expression (10) is equivalent to Expression (11) (for the direction (10) to (11), it suffices to take $y' = y$ and $z' = z$ and the direction (11) to (10) holds because we have $\mathcal{R}z'y'x$ since $\mathcal{R}$ is plump)

$$\forall yz\,(\mathcal{R}zyx \to z \sqsubseteq x) \tag{11}$$

Condition (11) is indeed the condition given in [39, Table 11.1, p. 250].

**Inference Rule** WI

$$\frac{X \,,_3\, X \vdash U}{X \vdash U}\ \mathsf{WI}$$

*Algorithm* $\tau_1$*:*

1. $\dfrac{\tau_1\left(X \,,_3\, X \vdash U\right)}{\tau_1\left(X \vdash U\right)}$
2. $\dfrac{\bullet_2(\bullet_1 X\,,\, X) \vdash U}{X \vdash U}$

*Algorithm* $\tau_2$*:*

1. $t_1(X) \to t_1(\bullet_2(\bullet_1 X, X))$
2. $\exists^- p \to \exists_2^-(\exists_1^- \exists^- p \wedge \exists^- p)$

*Algorithm* $\tau_3$*:*

1. $\forall P\left(\exists x'\left(x' \sqsubseteq x \wedge P(x')\right) \to\right.$
   $\left.\exists y\left(\mathcal{R}_2 yx \wedge \left(\exists y'\left(y' \sqsubseteq y \wedge P(y')\right) \wedge \exists z\left(\mathcal{R}_1 zy \wedge \exists z'\left(z' \sqsubseteq z \wedge P(z')\right)\right)\right)\right)\right)$
2. $\forall P\left(\exists x'\left(x' \sqsubseteq x \wedge P(x')\right) \to\right.$
   $\left.\exists yzy'z'\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge P(y') \wedge P(z')\right)\right)$
3. $\forall P\left(\forall x'\left(x' \sqsubseteq x \wedge P(x') \to\right.\right.$
   $\left.\left.\exists yzy'z'\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge P(y') \wedge P(z')\right)\right)\right)$
4. We take $\sigma(P) := \lambda u.x' = u$
5. $\forall x'\left(x' \sqsubseteq x \to \exists yzy'z'\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge x' = y' \wedge x' = z'\right)\right)$
6. $(\forall x' \lhd_0 x)(\exists y \lhd_2 x)(\exists z \lhd_1 y)(x' \sqsubseteq y \wedge x' \sqsubseteq z)$

*Algorithm* $\tau_4$*:*

1. $(\forall x' \lhd_0 x)(\exists yz \rhd^3 x)(x' \sqsubseteq y \wedge x' \sqsubseteq z)$

Finally, translated into plain $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$, we obtain:

$$\forall x'\left(x' \sqsubseteq x \to \exists yz\left(\mathcal{R}zyx \wedge x' \sqsubseteq y \wedge x' \sqsubseteq z\right)\right) \tag{12}$$

If $\mathcal{R}$ is plump, then Expression (12) is equivalent to Expression (13) (for the direction (12) to (13), take $x' = x$ and we obtain $\mathcal{R}zyx \wedge x \sqsubseteq z \wedge x \sqsubseteq y$, so $\mathcal{R}xxx$ because $\mathcal{R}$ is plump; for the direction (13) to (12), take $y = x'$ and $z = x'$):

$$\forall x'(x' \sqsubseteq x \to \mathcal{R}x'x'x) \tag{13}$$

Condition (13) is slightly different from the condition given in [39, Table 11.1, p. 250], which is the following:

$$\mathcal{R}xxx \tag{14}$$

This difference can be explained by the fact that our first-order correspondents are *local* first-order correspondent, which means that they have to be evaluated on *pointed* substructural frames, whereas the first-order correspondent in [39] are *global*, which means that they have to be evaluated on *all* the points of the substructural frames. As noted in Remark 3, all our results also hold if we consider *global* correspondence and plain substructural frames instead of *pointed* substructural frames. In that case, we do have that Expression 13 is equivalent to Expression (14), because $\forall x \left(\forall x'(x' \sqsubseteq x \to \mathcal{R}x'x'x)\right)$ is equivalent to $\forall x \mathcal{R}xxx$.

**Inference Rule CI**

$$\frac{Y \,,_3\, X \vdash U}{X \,,_3\, Y \vdash U}\ \mathsf{CI}$$

*Algorithm $\tau_1$:*

1. $\dfrac{\tau_1\left(Y \,,_3\, X \vdash U\right)}{\tau_1\left(X \,,_3\, Y \vdash U\right)}$
2. $\dfrac{\bullet_2(\bullet_1 Y, X) \vdash U}{\bullet_2(\bullet_1 X, Y) \vdash U}$

*Algorithm $\tau_2$:*

1. $t_1(\bullet_2(\bullet_1 X, Y)) \to t_1(\bullet_2(\bullet_1 Y, X))$
2. $\exists_2^-(\exists_1^-\exists^- p \wedge \exists^- q) \to \exists_2^-(\exists_1^-\exists^- q \wedge \exists^- p)$

*Algorithm $\tau_3$:*

1. $\forall PQ \left(\exists y \left(\mathcal{R}_2 yx \wedge \exists y' \left(y' \sqsubseteq y \wedge Q(y')\right) \wedge \exists z \left(\mathcal{R}_1 zy \wedge \exists z' \left(z' \sqsubseteq z \wedge P(z')\right)\right)\right) \to$
   $\exists \left(\mathcal{R}_2 tx \wedge \exists t' \left(t' \sqsubseteq t \wedge P(t')\right) \wedge \exists u \left(\mathcal{R}_1 ut \wedge \exists u' \left(u' \sqsubseteq u \wedge Q(u)\right)\right)\right)\right)$
2. $\forall PQ \left(\exists yzy'z' \left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge Q(y') \wedge P(z')\right) \to\right.$
   $\left.\exists tut'u' \left(\mathcal{R}_2 tx \wedge \mathcal{R}_1 ut \wedge t' \sqsubseteq t \wedge u' \sqsubseteq u \wedge P(t') \wedge Q(u')\right)\right)$
3. $\forall PQ \forall yzy'z' \left(\left(\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z \wedge Q(y') \wedge P(z')\right)\right) \to\right.$
   $\left.\exists tut'u' \left(\mathcal{R}_2 tx \wedge \mathcal{R}_1 ut \wedge t' \sqsubseteq t \wedge u' \sqsubseteq u \wedge P(t') \wedge Q(u')\right)\right)$
4. We take $\sigma(P) := \lambda u.z' = u$ and $\sigma(Q) := \lambda u.y' = u$
5. $\forall yzy'z' \left(\left(\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z\right)\right) \to\right.$
   $\left.\exists tut'u' \left(\mathcal{R}_2 tx \wedge \mathcal{R}_1 ut \wedge t' \sqsubseteq t \wedge u' \sqsubseteq u \wedge z' = t' \wedge y' = u'\right)\right)$
6. $\forall yzy'z' \left(\left(\left(\mathcal{R}_2 yx \wedge \mathcal{R}_1 zy \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z\right)\right) \to \exists tu \left(\mathcal{R}_2 tx \wedge \mathcal{R}_1 ut \wedge z' \sqsubseteq t \wedge y' \sqsubseteq u\right)\right)$
7. $(\forall y \lhd_2 x)(\forall z \lhd_1 y)(\forall y' \lhd_0 y)(\forall z' \lhd_0 z)(\exists t \lhd_2 x)(\exists u \lhd_1 t)(z' \sqsubseteq t \wedge y' \sqsubseteq u)$

*Algorithm $\tau_4$:*

1. $(\forall yz \rhd^3 x)(\forall y' \lhd_0 y)(\forall z' \lhd_0 z)(\exists tu \rhd^3 x)(z' \sqsubseteq t \wedge y' \sqsubseteq u)$

Finally, translated into plain $\mathcal{L}_{\mathrm{FOL}}(\mathcal{R})$, we obtain:

$$\forall yzy'z' \left(\left(\left(\mathcal{R}zyx \wedge y' \sqsubseteq y \wedge z' \sqsubseteq z\right)\right) \to \exists tu \left(\mathcal{R}utx \wedge z' \sqsubseteq t \wedge y' \sqsubseteq u\right)\right) \tag{15}$$

If $\mathcal{R}$ is plump, then Expression (15) is equivalent to Expression (16) (for the direction (15) to (16), take $y' = y$ and $z' = z$; for the direction (16) to (15), take $u = y$ and $t = z'$)

$$\forall yz \left(\mathcal{R}zyx \to \mathcal{R}yzx\right) \tag{16}$$

Condition (11) is indeed the condition given in [39, Table 11.1, p. 250].

## 8.2 From First-order Frame Conditions to Inference Rules

We execute the algorithms $\tau_4^-, \tau_3^-, \tau_2^-, \tau_1^-$ on the corresponding first-order conditions of the inference rules mMP, mWI and $\mathsf{B}^c$. These first-order conditions are taken from [39, Table 11.1, p. 250] and it will turn out that our algorithms yield the same inference rules as the ones given in [39, Table 11.1, p. 250]. The algorithm $\tau_3^-$ that we use is different from Kracht's algorithm and has been defined specifically for the kind of prototypic (in fact primitive) formulas that we consider.

#### First-order Frame Condition of Inference Rule $\mathsf{B}^c$

$$\forall yzwu \left(\left(\mathcal{R}yzu \wedge \mathcal{R}xuw\right) \to \exists t \left(\mathcal{R}xyt \wedge \mathcal{R}tzw\right)\right) \tag{$\mathsf{B}^c$}$$

*Algorithm $\tau_4^-$:*

1. $(\forall wu \rhd^3 x)(\forall yz \rhd^3 u)(\exists tz' \rhd^3 x)(\mathcal{R}wyt \wedge z' = z)$
2. $(\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z)(\exists z' \lhd_2 x)(\exists t \lhd_1 z') \left(\mathcal{R}_1 wy \wedge \mathcal{R}_2 yt \wedge z' = z\right)$

*Algorithm $\tau_3^-$:*

1. $\forall PQR \left((\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z) \left(\left(P(w) \wedge Q(y) \wedge R(z)\right) \to (\exists z' \lhd_2 x)(\exists t \lhd_1 z')(\exists y' \lhd_2 t)(\exists w' \lhd_1 y') \left(P(w') \wedge Q(y') \wedge R(z')\right)\right)\right)$
2. $\forall PQR \left((\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z) \left(\left(P(w) \wedge Q(y) \wedge R(z)\right) \to (\exists z' \lhd_2 x) \left(R(z') \wedge (\exists t \lhd_1 z')(\exists y' \lhd_2 t) \left(Q(y') \wedge (\exists w' \lhd_1 y') \wedge P(w')\right)\right)\right)\right)$
3. $\forall PQR \left((\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z) \left(\left(P(w) \wedge Q(y) \wedge R(z)\right) \to (\exists z' \lhd_2 x) \left(R(z') \wedge (\exists t \lhd_1 z')(\exists y' \lhd_2 t) \left(Q(y') \wedge (\exists w' \lhd_1 y') \wedge P(w')\right)\right)\right)\right)$
4. $\forall PQR \left((\forall u \lhd_2 x)(\forall w \lhd_1 u)(\forall z \lhd_2 u)(\forall y \lhd_1 z) \left(\left(P(w) \wedge Q(y) \wedge R(z)\right) \to ST_x \left(\exists_2^- \left(r \wedge \exists_1^- \left(\exists_2^- \left(q \wedge \exists_1^- p\right)\right)\right)\right)\right)\right)$

5. $\forall PQR\,(((\exists u \lhd_2 x)\,((\exists z \lhd_2 u)\,(R(z) \wedge (\exists y \lhd_1 z)Q(y)) \wedge (\exists w \lhd_1 u)P(w))) \rightarrow$
   $ST_x\left(\exists_2^-\left(r \wedge \exists_1^-\left(\exists_2^-\left(q \wedge \exists_1^- p\right)\right)\right)\right))$
6. $\forall PQR\left(ST_x\left(\exists_2^-\left(\left(\exists_2^-\left(r \wedge \exists_1^- q\right)\right) \wedge \exists_1^- p\right)\right) \rightarrow\right.$
   $\left.ST_x\left(\exists_2^-\left(r \wedge \exists_1^-\left(\exists_2^-\left(q \wedge \exists_1^- p\right)\right)\right)\right)\right)$
7. $\forall PQR\left(ST_x\left(\exists_2^-\left(\left(\exists_2^-\left(r \wedge \exists_1^- q\right)\right) \wedge \exists_1^- p\right) \rightarrow \exists_2^-\left(r \wedge \exists_1^-\left(\exists_2^-\left(q \wedge \exists_1^- p\right)\right)\right)\right)\right)$
8. $\exists_2^-\left(\left(\exists_2^-\left(r \wedge \exists_1^- q\right)\right) \wedge \exists_1^- p\right) \rightarrow \exists_2^-\left(r \wedge \exists_1^-\left(\exists_2^-\left(q \wedge \exists_1^- p\right)\right)\right)$

*Algorithm $\tau_2^-$:*

1. $\dfrac{\exists_2^-\left(r \wedge \exists_1^-\left(\exists_2^-\left(q \wedge \exists_1^- p\right)\right)\right) \vdash U}{\exists_2^-\left(\left(\exists_2^-\left(r \wedge \exists_1^- q\right)\right) \wedge \exists_1^- p\right) \vdash U}$
2. $\dfrac{\bullet_2\,(Z,\ \bullet_1\,(\bullet_2\,(Y,\ \bullet_1\,X))) \vdash U}{\bullet_2\,((\bullet_2\,(Z,\ \bullet_1\,Y))\,,\ \bullet_1\,X) \vdash U}$

*Algorithm $\tau_1^-$:*

1. $\dfrac{\tau_1^-\left(\bullet_2\,(Z,\ \bullet_1\,(\bullet_2\,(Y,\ \bullet_1\,X))) \vdash U\right)}{\tau_1^-\left(\bullet_2\,((\bullet_2\,(Z,\ \bullet_1\,Y))\,,\ \bullet_1\,X) \vdash U\right)}$
2. $\dfrac{(X\ ,_3\ Y)\ ,_3\ Z \vdash U}{X\ ,_3\ (Y\ ,_3\ Z) \vdash U}\ \mathsf{B}^c$

**First-order Frame Condition of Inference Rule mMP**

$$\forall x'y'z'(S_j x'x \wedge \mathcal{R}z'y'x' \rightarrow \exists zy(\mathcal{R}zyx \wedge S_j z'z \wedge S_j y'y) \qquad \text{(mMP)}$$

*Algorithm $\tau_4^-$:*

1. $(\forall x' \lhd_j x)(\forall z'y' \rhd^3 x')(\exists zy \rhd^3 x)(S_j z'z \wedge S_j y'y)$
2. $(\forall x' \lhd_j x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')(\exists y \lhd_2 x)(\exists z \lhd_1 y)(S_j z'z \wedge S_j y'y)$

*Algorithm $\tau_3^-$:*

1. $\forall PQ\left((\forall x' \lhd_j x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')\left(P(y') \wedge Q(z') \rightarrow\right.\right.$
   $(\exists y \lhd_2 x)(\exists z \lhd_1 y)(\exists t \lhd_j y)(\exists u \lhd_j z)(P(t) \wedge Q(u))))$
2. $\forall PQ\left((\forall x' \lhd_j x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')\left(P(y') \wedge Q(z') \rightarrow\right.\right.$
   $(\exists y \lhd_2 x)(\exists z \lhd_1 y)\,((\exists u \lhd_j z)P(t) \wedge (\exists t \lhd_j y)Q(u))))$
3. $\forall PQ\left((\forall x' \lhd_j x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')\left(P(y') \wedge Q(z') \rightarrow\right.\right.$
   $(\exists y \lhd_2 x)(\exists z \lhd_1 y)\,((\exists u \lhd_j z)P(t) \wedge (\exists t \lhd_j y)Q(u))))$
4. $\forall PQ\left((\forall x' \lhd_j x)(\forall y' \lhd_2 x')(\forall z' \lhd_1 y')\left(P(y') \wedge Q(z') \rightarrow\right.\right.$
   $\left.\left.ST_x\left(\exists_2^-(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- q)\right)\right)\right)$
5. $\forall PQ\left((\exists x' \lhd_j x)(\exists y' \lhd_2 x')(\exists z' \lhd_1 y')(P(y') \wedge Q(z')) \rightarrow\right.$
   $\left.ST_x\left(\exists_2^-(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- q)\right)\right)$

6. $\forall PQ\left((\exists x' \lhd_j x)(\exists y' \lhd_2 x')\left(P(y') \wedge (\exists z' \lhd_1 y')Q(z')\right) \rightarrow\right.$
   $\left. ST_x\left(\exists_2^-(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- q)\right)\right)$
7. $\forall PQ\left(ST_x\left(\Diamond_j^- \exists_2^-\left(p \wedge \exists_1^- q\right)\right) \rightarrow ST_x\left(\exists_2^-(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- q)\right)\right)$
8. $\forall PQ\left(ST_x\left(\Diamond_j^- \exists_2^-\left(p \wedge \exists_1^- q\right) \rightarrow \exists_2^-(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- q)\right)\right)$
9. $\Diamond_j^- \exists_2^-\left(p \wedge \exists_1^- q\right) \rightarrow \exists_2^-(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- q)$.

*Algorithm $\tau_2^-$:*

1. $\dfrac{\exists_2^-(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- p) \vdash U}{\Diamond_j^- \exists_2^-\left(p \wedge \exists_1^- q\right) \vdash U}$
2. $\dfrac{\bullet_2(\bullet_j Y\,,\ \bullet_1 \bullet_j X) \vdash U}{\bullet_j \bullet_2(Y\,,\ \bullet_1 X) \vdash U}$

*Algorithm $\tau_1^-$:*

1. $\dfrac{\tau_1^-\left(\bullet_2(\bullet_j Y\,,\ \bullet_1 \bullet_j X) \vdash U\right)}{\tau_1^-\left(\bullet_j \bullet_2(Y\,,\ \bullet_1 X) \vdash U\right)}$
2. $\dfrac{\bullet_j X\ ,_3\ \bullet_j Y \vdash U}{\bullet_j(X\ ,_3\ Y) \vdash U}$ mMP

**First-order Frame Condition of Inference Rule mWI**

$$\forall y\,(S_j yx \rightarrow \exists x_1 x_2\,(\mathcal{R} x_1 x_2 x \wedge S_j y x_1 \wedge S_j y x_2)) \tag{mWI}$$

*Algorithm $\tau_4^-$:*

1. $(\forall y \lhd_j x)(\exists x_1 x_2 \rhd^3 x)(S_j y x_1 \wedge S_j y x_2)$
2. $(\forall y \lhd_j x)(\exists x_1 \lhd_2 x)(\exists x_2 \lhd_1 x_1)(S_j y x_1 \wedge R_j y x_2)$

*Algorithm $\tau_3^-$:*

1. $\forall P\,((\forall y \lhd_j x)\,(P(y) \rightarrow$
   $(\exists x_1 \lhd_2 x)(\exists x_2 \lhd_1 x_1)\,((\exists y_1 \lhd_j x_1)P(y_1) \wedge (\exists y_2 \lhd_j x_2)P(y_2))))$
2. $\forall P\,((\forall y \lhd_j x)\,(P(y) \rightarrow$
   $(\exists x_1 \lhd_2 x)\,((\exists y_1 \lhd_j x_1)P(y_1) \wedge (\exists x_2 \lhd_1 x_1)(\exists y_2 \lhd_j x_2)P(y_2))))$
3. $\forall P\left((\forall y \lhd_j x)\left(P(y) \rightarrow ST_x\left(\exists_2^-\left(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- p\right)\right)\right)\right)$
4. $\forall P\left((\exists y \lhd_j x)P(y) \rightarrow ST_x\left(\exists_2^-\left(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- p\right)\right)\right)$
5. $\forall P\left(ST_x(\Diamond_j^- p) \rightarrow ST_x\left(\exists_2^-\left(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- p\right)\right)\right)$

6. $\forall P\left(ST_x\left(\Diamond_j^- p \to \exists_2^-\left(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- p\right)\right)\right)$

7. $\Diamond_j^- p \to \exists_2^-\left(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- p\right)$

*Algorithm $\tau_2^-$:*

1. $\dfrac{\exists_2^-\left(\Diamond_j^- p \wedge \exists_1^- \Diamond_j^- p\right) \vdash U}{\Diamond_j^- p \vdash U}$

2. $\dfrac{\bullet_2\left(\bullet_j X\, ,\ \bullet_1 \bullet_j X\right) \vdash U}{\bullet_j X \vdash U}$

*Algorithm $\tau_1^-$:*

1. $\dfrac{\tau_1^-\left(\bullet_2\left(\bullet_j X\, ,\ \bullet_1 \bullet_j X\right) \vdash U\right)}{\tau_1^-\left(\bullet_j X \vdash U\right)}$

2. $\dfrac{\bullet_j X \ ,_3\ \bullet_j X \vdash U}{\bullet_j X \vdash U}$ mWI

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