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***Abstract*— We present a recursive formula for evaluating the per-flow throughput in a multirate system with elastic traffic, which is the analogue of the well-known Kaufman-Roberts formula used to evaluate the blocking probability in a multirate system with circuit traffic.**

***Index Terms*— Multirate system, elastic traffic, per-flow throughput, Kaufman-Roberts formula.**

I. INTRODUCTION

The performance of circuit-switched networks has been extensively studied in the 70's and 80's, preceding the development of Integrated Services Digital Networks (ISDN) [1]. A particularly useful result was derived independently by Kaufman [2] and Roberts [3] for evaluating the blocking probability in a link where circuits of different bit rates are multiplexed. In the present paper we derive the analogue of the so-called Kaufman-Roberts formula for a similar multirate system with *elastic* traffic. Such a system is representative of a link in a packet-switched network like the Internet where data flows are not blocked in case of congestion but experience lower bit rates. The maximum bit rate of each flow typically corresponds to the speed of the user access line, like 1 Mbit/s vs. 2 Mbit/s DSL access lines, as represented on the left of Figure 1.

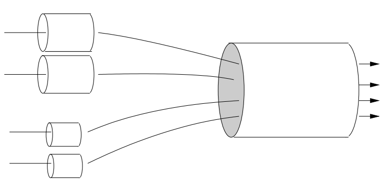

Fig. 1. A multirate system with elastic traffic

The performance of such a multirate system may be expressed in terms of flow throughput, the ratio of the mean flow size (in bits) to the mean flow duration. Flow throughput depends on traffic characteristics like the traffic intensity as well as on the way link capacity is shared between ongoing flows in case of congestion. We assume a balanced fair sharing, which ensures that flow throughput does not depend on *any* traffic characteristics except the traffic intensity [4]. In particular, the flow throughput is insensitive to the flow size distribution, in a similar way as the blocking probability in a circuit-switched network is insensitive to the holding time distribution. The only required assumption is that flows are generated within sessions, whose arrivals form a Poisson process. This has been identified as one of the rare invariants of Internet traffic [5]. The multirate system with balanced fair sharing was introduced in [4] and a recursive formula for evaluating the associated normalization constant was given in [6], allowing the numerical evaluation of flow throughput by means of differentiation. We give here a recursive formula to evaluate the flow throughput in a fully explicit way.

The model is described in the next section. The recursive formula is presented in Section III and applied to an example in Section IV. A proof of the recursive formula is given in the appendix.

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II. MODEL

Consider a link of capacity C shared by N flow classes. Class-i flows have a maximum bit rate c_i. Let λ_i be the arrival rate of class-i flows, σ_i their mean size (in bits). The traffic intensity of class i is defined by $a_i = \lambda_i \sigma_i$. It is expressed in bit/s and corresponds to the average volume of class i traffic offered to the system per unit of time. We denote by $\rho_i = a_i/C$ the link load due to class i. The overall link load is given by:

$$\rho \equiv \sum_{i=1}^{N} \rho_i.$$

Let x_i be the number of class-i flows. We denote by $x = (x_1, \dots, x_N)$ the system state, $c = (c_1, \dots, c_N)$ the vector of rate limits and $a = (a_1, \dots, a_N)$ the vector of traffic intensities. We denote by e_i the unit line vector with 1 in component i and 0 elsewhere.

We use the notation:

$$x! = \prod_{i=1}^{N} x_i!, \quad a^x = \prod_{i=1}^{N} a_i^{x_i}, \quad x.c = \sum_{i=1}^{N} x_i c_i.$$

The evolution of the system state x depends on the way link capacity C is shared between ongoing flows. We assume capacity is shared according to balanced fairness, which ensures that the stationary distribution of x does not depend on any traffic characteristics except the traffic intensity vector a [4]. The only assumption is that flows are generated within independent sessions, whose arrivals form a Poisson process.

Balanced fairness

Let $\phi_i(x)$ be the bit rate of class-i flows in state x, which is equally shared between these flows. As shown in [4], the balanced fair sharing is given by:

$$\phi_i(x) = \frac{\Phi(x - e_i)}{\Phi(x)}, \quad x_i > 0,$$

where Φ is the so-called balance function, recursively defined by $\Phi(x) = 0$ if $x_i < 0$ for some i,

$$\Phi(x) = \frac{1}{x!c^x} \quad \text{if } x \geq 0,\ x.c \leq C, \tag{1}$$

and

$$\Phi(x) = \frac{1}{C} \sum_{i=1}^{N} \Phi(x - e_i) \quad \text{otherwise.} \tag{2}$$

A stationary measure of the system state is given by:

$$\chi(x) = \Phi(x) a^x, \tag{3}$$

whose sum is finite if and only if $\rho < 1$. The steady-state probability that the system is in state x then follows by normalization:

$$\pi(x) = \frac{\chi(x)}{\sum_y \chi(y)}. \tag{4}$$

Flow throughput

We are interested in the flow throughput, the ratio of the mean flow size to the mean flow duration. By Little's law, the mean duration of a class-i flow is equal to $\mathrm{E}[x_i]/\lambda_i$. We deduce the flow throughput of class i:

$$\gamma_i = \frac{\lambda_i \sigma_i}{\mathrm{E}[x_i]} = \frac{a_i}{\mathrm{E}[x_i]}.$$

Using (4), we get:

$$\gamma_i = \frac{a_i}{\sum_x x_i \pi(x)} = a_i \frac{\sum_x \chi(x)}{\sum_x x_i \chi(x)}. \tag{5}$$

III. Recursive formula

We assume as for the Kaufman-Roberts formula that the link capacity C and the rate limits $c_1, \ldots, c_N$ are integers. Note that this is not a restrictive assumption since this system is equivalent to a unit capacity link with rate limits $c_1/C, \ldots, c_N/C$, which may approximate any multirate system. For any integer n, let:

$$p(n) = \sum_{x:x.c=n} \chi(x)$$

and for $i = 1, \ldots, N$,

$$q_i(n) = \sum_{x:x.c=n} x_i \chi(x).$$

In view of (5), the flow throughput of class i is given by:

$$\gamma_i = a_i \frac{\sum_{n \geq 0} p(n)}{\sum_{n \geq 0} q_i(n)}. \tag{6}$$

The recursive formula consists of two steps. First, we evaluate $p(n)$ and $q_i(n)$ for $n = 1, \ldots, C$ in a similar way as in the Kaufman-Roberts formula. Second, we evaluate the infinite sums:

$$\bar{p} = \sum_{n>C} p(n) \quad \text{and} \quad \bar{q}_i = \sum_{n>C} q_i(n).$$

First step

For $n = 1, \ldots, C$, we have

$$p(n) = \sum_{j=1}^{N} \frac{a_j}{n} p(n - c_j), \tag{7}$$

with $p(0) = 1$ and $p(n) = 0$ for all $n < 0$, and

$$q_i(n) = \frac{a_i}{n} p(n - c_i) + \sum_{j=1}^{N} \frac{a_j}{n} q_i(n - c_j), \tag{8}$$

with $q_i(n) = 0$ for all $n \leq 0$.

Second step

We have

$$\bar{p} = \sum_{i=1}^{N} \frac{\rho_i \bar{p}_i}{1 - \rho}, \tag{9}$$

with

$$\bar{p}_i = \sum_{C - c_i < n \leq C} p(n),$$

and

$$\bar{q}_i = \rho_i \frac{\bar{p}_i + \bar{p}}{1 - \rho} + \sum_{j=1}^{N} \frac{\rho_j \bar{q}_{ij}}{1 - \rho}. \tag{10}$$

with

$$\bar{q}_{ij} = \sum_{C - c_j < n \leq C} q_i(n).$$

Equation (7) is exactly the Kaufman-Roberts formula. Equations (8), (9) and (10) are proven in the Appendix.

Complexity

The proposed recursive formula is *scalable* in the number of classes. Its complexity is the same as that of the Kaufman-Roberts formula, namely $O(CN)$.

IV. Application

Consider a link of capacity $C = 10$ shared by $N = 10$ classes with respective rate limits $c_1 = 1, c_2 = 2, \ldots, c_{10} = 10$ and equal traffic intensities:

$$a_1 = a_2 = \ldots = a_{10} = \frac{\rho C}{N}.$$

Figure 2 gives the flow throughput γ_i of each class i as a function of the link load ρ. At low load, the system is most often empty so that the instantaneous bit rate of any ongoing flow of class i is most often maximum and equal to c_i. As a result, the flow throughput of class i is close to the rate limit c_i. At high load, on the other hand, the number of competing flows is so large that the instantaneous bit rate of each flow is very low. The flow throughput is then close to zero for all classes.

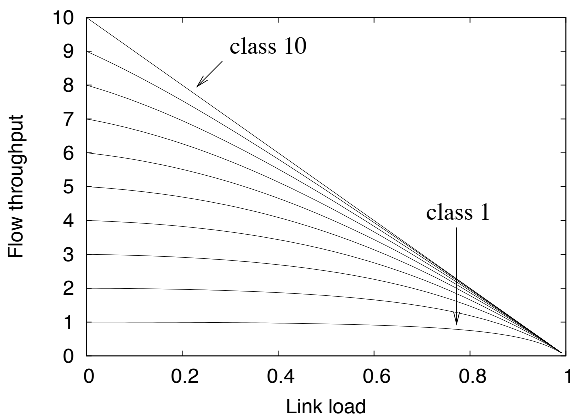

Fig. 2. Flow throughput in a multirate system with link capacity $C = 10$, rate limits $c_1 = 1, \dots, c_{10} = 10$ and equal traffic intensities.

APPENDIX I
PROOF OF THE FIRST STEP

Equation (8) can be proven in similar way as the Kaufman-Roberts formula. We have for all $n \geq 0$:

$$nq_i(n) = \sum_{x:x.c=n} x_i \left(\sum_{j=1}^{N} x_j c_j \right) \chi(x).$$

Using (1) and (3), we get for $n = 1, \dots, C$:

$$\begin{aligned} nq_i(n) &= \sum_{x:x.c=n} x_i \sum_{j:x_j>0} a_j \chi(x - e_j) \\ &= \sum_{j \neq i} \sum_{x:x.c=n-c_j} x_i a_j \chi(x) \\ &\quad + \sum_{x:x.c=n-c_i} (x_i + 1) a_i \chi(x) \\ &= a_i p(n - c_i) + \sum_{j=1}^{N} a_j q_i(n - e_j). \end{aligned}$$

APPENDIX II
PROOF OF THE SECOND STEP

It follows from (2) and (3) that:

$$\begin{aligned} \bar{p} &= \sum_{x:x.c>C} \Phi(x) a^x \\ &= \sum_{x:x.c>C} \frac{1}{C} \sum_{i=1}^{N} \Phi(x - e_i) a^x \\ &= \sum_{i=1}^{N} \frac{a_i}{C} \sum_{x:x.c>C} \Phi(x - e_i) a^{x - e_i} \\ &= \sum_{i=1}^{N} \rho_i (\bar{p}_i + \bar{p}), \end{aligned}$$

from which (9) easily follows.

The proof of (10) is similar:

$$\begin{aligned} \bar{q}_i &= \sum_{x:x.c>C} x_i \Phi(x) a^x \\ &= \sum_{x:x.c>C} \frac{x_i}{C} \sum_{j=1}^{N} \Phi(x - e_j) a^x \\ &= \sum_{j=1}^{N} \frac{a_j}{C} \sum_{x:x.c>C} x_i \Phi(x - e_j) a^{x - e_j} \\ &= \rho_i (\bar{p}_i + \bar{p}) + \sum_{j=1}^{N} \rho_j (\bar{q}_{ij} + \bar{q}_i). \end{aligned}$$

REFERENCES

[1] F. P. Kelly, “Blocking probabilities in large circuit-switched networks,” *Adv. in Applied Prob.*, vol. 18, 1986, pp. 473–505.

[2] J. S. Kaufman, “Blocking in a shared resource environment,” *IEEE Trans. Commun.*, vol. 29, 1981, pp. 1474–1481.

[3] J. W. Roberts, “A service system with heterogeneous user requirement,” in *Performance of Data Communications Systems and Their Applications*, G. Pujolle, Ed. Amsterdam, The Netherlands: North-Holland, 1981, pp. 423–431.

[4] T. Bonald, A. Proutière, “Insensitive bandwidth sharing in data networks,” *Queueing Systems*, vol. 44, 2003, pp. 69–100.

[5] V. Paxson and S. Floyd, “Difficulties in Simulating the Internet,” *IEEE/ACM Trans. on Networking*, vol. 9, 2001, pp. 392–403.

[6] T. Bonald, A. Proutière, J. Roberts, J. Virtamo, “Computational aspects of balanced fairness,” in *Proc. of ITC 18*, 2003, pp. 801–810.