A Riesz type representation for lower semi-continuous, monotone, local functionals on $C_c(X)^+$

Ralph Chill

R. Chill, Institut für Analysis, Fachrichtung Mathematik, TU Dresden, 01062 Dresden, Germany

Mahamadi Warma

M. Warma, University of Puerto Rico, Faculty of Natural Sciences, Department of Mathematics (Rio Piedras Campus), PO Box 70377 San Juan PR 00936-8377 (USA)

Abstract

We prove a Riesz type representation theorem for lower semi-continuous, monotone, local functionals on $C_c(X)^+$, where X is a locally compact, separable, metric space.

Keywords: Lower semi-continuous functional, monotone functional, local functional, locally compact metric space, Riesz representation theorem.

2000 MSC: 28C15, 46E05, 28C99

1. Introduction

By the classical Riesz representation theorem, linear and continuous functionals on spaces of continuous functions are represented by regular Borel measures. More precisely, if X is a locally compact Hausdorff space, then every linear, continuous functional $\varphi : C_c(X) \rightarrow \mathbb{R}$ is of the form

$$\varphi(u) = \int_X u \, d\mu \quad (u \in C_c(X))$$

for some regular (locally bounded) Borel measure μ on X . Tikhonov & Arsenin [9] were perhaps the first to state a similar representation theorem for in general *nonlinear*, continuous functionals $\varphi : C([0, 1]) \rightarrow \mathbb{R}$ which are monotone and local; local means here that $\varphi(u+v) = \varphi(u) + \varphi(v)$ for all pairs u, v of continuous functions with disjoint support (in their article and several subsequent articles such functionals are called *additive* or *disjointly additive*). Tikhonov & Arsenin proved that such functionals are of the form

$$\varphi(u) = \int_0^1 B(x, u(x)) \, d\mu \quad (u \in C([0, 1]))$$

for some Borel measure μ and some function B which is measurable in the first variable and continuous and monotone in the second variable. Independently, Chacon & Friedman [3] and Friedman & Katz [6, 7] generalized the Riesz representation theorem in a slightly different way. Instead of monotone and local functionals they considered strongly additive functionals, first on $C([0, 1])$ and then on $C(X)$ where X is a compact, metric space, or, respectively, a compact Hausdorff space; for the definition of *strong additivity*, see [3, 6, 7]. For an account of these representation theorems, for representation theorems of Banach space valued functionals and other representation theorems on spaces of measurable functions we refer to Rao [8] and the references therein.

In this short note we state and prove a Riesz type representation theorem for *lower semi-continuous*, monotone, local functionals $\varphi : C_c(X)^+ \rightarrow [0, +\infty]$. Actually, our functionals satisfy an additional condition (see condition (3) below) which is between locality / additivity (as considered by Tikhonov and Arsenin) and strong additivity (as

considered by Chacon, Friedman and Katz). We think that a variant of the Tikhonov-Arsenin result or the Chacon-Friedman-Katz results for the class of lower semi-continuous functionals is interesting as such, but we point out that our main theorem is also motivated by the fact that lower semi-continuous functionals naturally arise in the theory of abstract gradient systems; see [2] for the general theory, and [4] for a particular application of the Riesz type representation theorem to nonlinear parabolic boundary value problems.

2. The result

Let (X, d) be a locally compact, separable, metric space, let $C_c(X)$ be the space of continuous functions with compact support in X (equipped with the usual topology, that is, the inductive limit topology, so that convergence of a sequence or net of elements in $C_c(X)$ means that all these elements have their support in one compact set and that they converge uniformly on X), and let $C_c(X)^+$ be the positive cone in $C_c(X)$. Given a functional $\varphi : C_c(X)^+ \rightarrow [0, +\infty]$, we call $D(\varphi) = \{u \in C_c(X)^+ : \varphi(u) < +\infty\}$ its *effective domain*. The *effective support* of the functional φ is the set

$$\text{supp}[\varphi] := X \setminus \{x \in X : \text{there exists a neighbourhood } U \text{ of } x \text{ such that for every } u \in D(\varphi) \text{ with } \text{supp}[u] \subseteq U \text{ one has } \varphi(u) = 0\}.$$

We say that the functional φ is *monotone* if for every $u, v \in C_c(X)^+$

$$u \leq v \quad \Rightarrow \quad \varphi(u) \leq \varphi(v), \quad (1)$$

and we call it *local* if for every $u, v \in C_c(X)^+$

$$\text{supp}[u] \cap \text{supp}[v] = \emptyset \quad \Rightarrow \quad \varphi(u + v) = \varphi(u) + \varphi(v). \quad (2)$$

Theorem 1. *For every functional $\varphi : C_c(X)^+ \rightarrow [0, +\infty]$ the following assertions (I) and (II) are equivalent.*

(I) *The functional φ is lower semi-continuous, monotone, local and for every $u, v \in D(\varphi)$, one has $u \vee v, u \wedge v \in D(\varphi)$ and*

$$\varphi(u \vee v) + \varphi(u \wedge v) \leq \varphi(u) + \varphi(v) \quad (3)$$

where $u \vee v$ (resp. $u \wedge v$) denote the (pointwise) supremum (resp. infimum) of the functions u and v .

(II) *There exist a finite, regular Borel measure μ on X with $\text{supp}[\mu] \subseteq \text{supp}[\varphi]$ and a function $B : X \times \mathbb{R} \rightarrow [0, +\infty]$ such that,*

- (i) *$B(\cdot, s)$ is a measurable function for each s ,*
- (ii) *$B(x, 0) = 0$ and $B(x, \cdot)$ is monotone and lower semi-continuous for μ -a.e. $x \in X$,*
- (iii) *$\varphi(u) = \int_X B(x, u(x)) \, d\mu$ for every $u \in D(\varphi)$.*

Remark 2. (a) *We point out that the representing measure μ and function B are not unique. For example, given a representing measure μ and a representing function B , and given any Borel measurable weight $w : X \rightarrow \mathbb{R}_+$ which is bounded from above and from below (away from zero), the weighted measure $w d\mu$ and the function B/w represent φ , too.*

(b) *On the other hand, the proof of the Theorem shows that for any pair φ_1, φ_2 of lower semi-continuous, monotone, local functionals satisfying the inequality (3) one can find a common representing measure μ with $\text{supp}[\mu] \subseteq \text{supp}[\varphi_1] \cup \text{supp}[\varphi_2]$ and two representing functions B_1 and B_2 satisfying the conditions (II) (i)-(iii) of the Theorem. It suffices to take for example $\mu = \mu_1 + \mu_2$, if μ_1 and μ_2 are the representing measures the existence of which is guaranteed by the Theorem.*

3. Proof of the Theorem

We start by proving the implication in the Theorem which is relatively straightforward.

PROOF (PROOF OF THE THEOREM, (II) $\Rightarrow$ (I)). Assume that Assertion (II) holds. Note that for every $u \in C_c(X)^+$ the function $B(\cdot, u(\cdot))$ is measurable due to assumptions (II)(i) and (II)(ii). In particular, the integral in (II)(iii) is well-defined. The monotonicity of the function $B(x, \cdot)$ (for μ -almost every $x \in X$, assumption (II)(i)) and the monotonicity of the integral imply that the functional φ is monotone. We show that it is lower semi-continuous. Let $(u_n) \subseteq C_c(X)^+$ be a sequence which converges to $u \in C_c(X)^+$. Then u_n converges to u everywhere. Since, for μ -almost every $x \in X$, the function $B(x, \cdot)$ is lower semi-continuous, we obtain $B(x, u(x)) \leq \liminf_{n \rightarrow \infty} B(x, u_n(x))$ for μ -almost every $x \in X$. Using Fatou's lemma, we therefore obtain that

$$\varphi(u) = \int_X B(x, u) \, d\mu \leq \int_X \liminf_{n \rightarrow \infty} B(x, u_n(x)) \, d\mu \leq \liminf_{n \rightarrow \infty} \int_X B(x, u_n(x)) \, d\mu = \liminf_{n \rightarrow \infty} \varphi(u_n).$$

Hence, φ is lower semi-continuous.

Let $u, v \in D(\varphi) \subseteq C_c(X)^+$. Clearly, $u \vee v, u \wedge v \in C_c(X)^+$. From the equality

$$\begin{aligned} \varphi(u \vee v) + \varphi(u \wedge v) &= \int_X B(x, u \vee v) \, d\mu + \int_X B(x, u \wedge v) \, d\mu \\ &= \int_{\{u \leq v\}} B(x, v) \, d\mu + \int_{\{u \leq v\}} B(x, u) \, d\mu \\ &\quad + \int_{\{u > v\}} B(x, u) \, d\mu + \int_{\{u > v\}} B(x, v) \, d\mu \\ &= \int_X B(x, u) \, d\mu + \int_X B(x, v) \, d\mu \\ &= \varphi(u) + \varphi(v) \end{aligned}$$

we obtain that $u \vee v, u \wedge v \in D(\varphi)$ and (3) holds (even with equality). The fact that φ is local follows immediately from condition (3) by noticing that $\varphi(u + v) = \varphi(u \vee v)$ and $0 = \varphi(0) = \varphi(u \wedge v)$ for every $u, v \in C_c(X)^+$ with disjoint support.

To prove the converse implication (I) $\Rightarrow$ (II), we proceed stepwise, in the form of several lemmas.

Throughout the following, we denote the Borel sets of X by $\mathcal{B}$ and the class of compact subsets of X is denoted by $\mathcal{K}$. We assume also that the functional φ satisfies the condition (I) in the Theorem.

For $\delta > 0$ and every subset $K \subseteq X$ we define

$$K^\delta := \{x \in X : d(x, K) \leq \delta\}.$$

With this definition, for every compact subset $K \in \mathcal{K}$ we define

$$\mathcal{R}(K) := \{\rho \in C_c(X)^+ : \text{there exists } \delta > 0 \text{ such that } \rho \geq 1 \text{ on } K^\delta\}.$$

Definition 3. For each $u \in D(\varphi)$ we define a nonnegative set function $\mu_u(\cdot)$ on $\mathcal{K}$ by setting

$$\mu_u(K) = \inf_{\rho \in \mathcal{R}(K)} \varphi(u\rho) \quad (K \in \mathcal{K}). \quad (4)$$

We remark that for every $\rho \in \mathcal{R}(K)$ one has $\rho \wedge 1 \in \mathcal{R}(K)$. Therefore, in the definition of $\mu_u(K)$ it suffices to take the infimum over all functions $\rho \in \mathcal{R}(K)$ which are $= 1$ on some K^δ . In particular, by the monotonicity of φ , $\mu_u(K) \leq \varphi(u) < +\infty$ for every compact $K \subseteq X$.

Lemma 4 (Finite additivity). Let $u \in D(\varphi)$, and let $K_1, K_2 \subseteq X$ be two compact sets such that $K_1 \cap K_2 = \emptyset$. Then

$$\mu_u(K_1 \cup K_2) = \mu_u(K_1) + \mu_u(K_2).$$

PROOF. Let $K_1, K_2 \subseteq X$ be two compact sets such that $K_1 \cap K_2 = \emptyset$. Then $d(K_1, K_2) =: r > 0$. We can find two functions $\rho_i \in \mathcal{R}(K_i)$ ($i = 1, 2$) such that $0 \leq \rho_i \leq 1$ and $\text{supp}[\rho_1] \cap \text{supp}[\rho_2] = \emptyset$; think of functions $\rho_i = \rho_i(x)$ which depend in fact only on the distance $\text{dist}(x, K_i)$.

Let $\rho \in \mathcal{R}(K_1 \cup K_2)$. Since $\text{supp}[u\rho\rho_i] \subseteq \text{supp}[\rho_i]$, it follows that $\text{supp}[u\rho\rho_1] \cap \text{supp}[u\rho\rho_2] = \emptyset$. The monotonicity and locality of φ implies

$$\begin{aligned}\varphi(u\rho) &\geq \varphi(u\rho(\rho_1 + \rho_2)) \\ &= \varphi(u\rho\rho_1) + \varphi(u\rho\rho_2) \\ &\geq \mu_u(K_1) + \mu_u(K_2).\end{aligned}$$

Since $\rho \in \mathcal{R}(K_1 \cup K_2)$ was arbitrary, this implies $\mu_u(K_1 \cup K_2) \geq \mu_u(K_1) + \mu_u(K_2)$.

Now let $\rho'_i \in \mathcal{R}(K_i)$ ($i = 1, 2$). Then, again by monotonicity and locality,

$$\begin{aligned}\varphi(u\rho'_1) + \varphi(u\rho'_2) &\geq \varphi(u\rho'_1\rho_1) + \varphi(u\rho'_2\rho_2) \\ &= \varphi(u(\rho'_1\rho_1 + \rho'_2\rho_2)) \\ &\geq \mu_u(K_1 \cup K_2).\end{aligned}$$

Since $\rho'_i \in \mathcal{R}(K_i)$ ($i = 1, 2$) were arbitrary, this implies $\mu_u(K_1) + \mu_u(K_2) \geq \mu_u(K_1 \cup K_2)$.

Lemma 5 (Monotonicity). *Let $u \in D(\varphi)$ and let $K_1, K_2 \subseteq X$ be two compact sets such that $K_1 \subseteq K_2$. Then*

$$\mu_u(K_1) \leq \mu_u(K_2).$$

PROOF. This follows immediately from the definition of μ_u and the monotonicity of φ .

Lemma 6. *Let $u \in D(\varphi)$ and let $K_1, K_2 \subseteq X$ be two compact sets. Then*

$$\mu_u(K_1 \cup K_2) + \mu_u(K_1 \cap K_2) \leq \mu_u(K_1) + \mu_u(K_2).$$

PROOF. Let $\rho_i \in \mathcal{R}(K_i)$ ($i = 1, 2$). Then, by assumption (3),

$$\begin{aligned}\varphi(u\rho_1) + \varphi(u\rho_2) &\geq \varphi(u(\rho_1 \vee \rho_2)) + \varphi(u(\rho_1 \wedge \rho_2)) \\ &\geq \mu_u(K_1 \cup K_2) + \mu_u(K_1 \cap K_2).\end{aligned}$$

Since $\rho_i \in \mathcal{R}(K_i)$ ($i = 1, 2$) were arbitrary, this implies $\mu_u(K_1) + \mu_u(K_2) \geq \mu_u(K_1 \cup K_2) + \mu_u(K_1 \cap K_2)$.

Lemma 7 (Outer regularity). *Let $u \in D(\varphi)$, let (K_m) be a decreasing sequence of compact subsets of X , and let $K := \bigcap_m K_m$. Then*

$$\lim_{m \rightarrow \infty} \mu_u(K_m) = \mu_u(K). \quad (5)$$

PROOF. First, the monotonicity of μ_u (Lemma 5) implies that $\lim_{m \rightarrow \infty} \mu_u(K_m) \geq \mu_u(K)$. In order to prove the converse inequality, observe that for every $\delta, \delta' > 0$ with $\delta > \delta'$ there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ one has $K_m^{\delta'} \subseteq K^\delta$ (here we use that (K_m) is decreasing and $K = \bigcap_m K_m$). In particular, for every $\rho \in \mathcal{R}(K)$ there exists $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$ one has $\rho \in \mathcal{R}(K_m)$. As a consequence

$$\varphi(u\rho) \geq \mu_u(K_m) \quad \text{for every } m \geq m_0,$$

or,

$$\varphi(u\rho) \geq \lim_{m \rightarrow \infty} \mu_u(K_m).$$

Since this inequality holds for every $\rho \in \mathcal{R}(K)$, this proves $\mu_u(K) \geq \lim_{m \rightarrow \infty} \mu_u(K_m)$.

Lemma 8. *For every $u \in D(\varphi)$ the set function μ_u can be uniquely extended to a finite, regular Borel measure on X (denoted again by μ_u in the following). Moreover, $\text{supp}[\mu_u] \subseteq \text{supp}[\varphi]$.*

PROOF. By Lemmas 4, 5, 6 and 7, μ_u is a regular content on $\mathcal{H}$. The fact that μ_u extends to a regular Borel measure (which we denote again by μ_u) follows from standard measure theory, including the theory of measures on topological spaces (see, for example, [1, Kapitel I §3, Kapitel IV], [5, §416, and in particular Corollary 416M]). Note that $\mu_u(X) = \varphi(u) < +\infty$ for $u \in D(\varphi)$. The inclusion $\text{supp}[\mu_u] \subseteq \text{supp}[\varphi]$ is a straightforward consequence of the definition of μ_u and the definition of the effective support of φ .

Lemma 9 (Monotonicity of μ). *Let $u, v \in D(\varphi)$ be such that $u \leq v$. Then $\mu_u \leq \mu_v$.*

PROOF. The monotonicity of φ and the definition of the measures μ_u and μ_v imply $\mu_u(K) \leq \mu_v(K)$ for every compact subset $K \subseteq X$. The claim then follows from the inner regularity of μ_u and μ_v .

Lemma 10. *Let $(u_n) \subseteq D(\varphi)$ and $u \in D(\varphi)$ be such that $u_n \leq u$ and $\lim_{n \rightarrow \infty} u_n = u$ in $C_c(X)$. Then*

$$\lim_{n \rightarrow \infty} \mu_{u_n}(G) = \mu_u(G) \quad \text{for every } G \in \mathcal{B}. \quad (6)$$

PROOF. By Lemma 9, the domination $u_n \leq u$ implies $\mu_u(G) - \mu_{u_n}(G) \geq 0$ for every $G \in \mathcal{B}$. Hence, for every $G \in \mathcal{B}$,

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} (\mu_u(G) - \mu_{u_n}(G)) \\ &\leq \limsup_{n \rightarrow \infty} (\mu_u(G) - \mu_{u_n}(G) + \mu_u(G^c) - \mu_{u_n}(G^c)) \\ &= \limsup_{n \rightarrow \infty} (\mu_u(X) - \mu_{u_n}(X)) \\ &= \limsup_{n \rightarrow \infty} (\varphi(u) - \varphi(u_n)) \\ &\leq 0, \end{aligned}$$

where in the last inequality we have used the lower semi-continuity of φ . The preceding chain of inequalities implies the claim.

Lemma 11. *For every $u \in D(\varphi)$ one has $\mu_u(\{u = 0\}) = 0$.*

PROOF. First, for every compact set $K \subseteq X \setminus \text{supp}[u]$ one has $\mu_u(K) = 0$. This follows from the definition of μ_u , from the fact that $X \setminus \text{supp}[u]$ is open, and from $\varphi(0) = 0$.

Now, we can find an increasing sequence $(u_n) \subseteq D(\varphi)$ such that $\lim_{n \rightarrow \infty} u_n = u$ in $C_c(X)$ and $u_n = 0$ in a neighborhood of $\{u = 0\}$. In particular, $\{u = 0\} \subseteq X \setminus \text{supp}[u_n]$. By the preceding argument, and by interior regularity, we have $\mu_{u_n}(\{u = 0\}) = 0$ for every n . By Lemma 10, this implies $\mu_u(\{u = 0\}) = 0$.

Lemma 12. *Let $u, v \in D(\varphi)$. Then $\mu_u(G) \leq \mu_v(G)$ for every Borel set $G \subseteq \{u \leq v\}$.*

PROOF. *Step 1:* We first prove the inequality $\mu_u(G) \leq \mu_v(G)$ for every Borel set $G \subseteq \{u < v\}$. By continuity of u and v , the set $\{u < v\}$ is open. Hence, for every compact set $K \subseteq \{u < v\}$ there exists $\delta > 0$ such that $K^\delta \subseteq \{u < v\}$. It is then easy to see that the monotonicity of φ and the definition of the measures μ_u and μ_v imply $\mu_u(K) \leq \mu_v(K)$ for every compact subset $K \subseteq G$. The inequality $\mu_u(G) \leq \mu_v(G)$ for Borel sets $G \subseteq \{u < v\}$ then follows from the inner regularity of μ_u and μ_v .

Step 2: Let now G be a Borel set in $\{u \leq v \text{ and } 0 < v\}$. Let $(\lambda_n) \subseteq \mathbb{R}_+$ be a sequence such that $\lambda_n < 1$ and $\lim_{n \rightarrow \infty} \lambda_n = 1$. Then, for every n one has

$$\{u \leq v \text{ and } 0 < v\} \subseteq \{\lambda_n u < v\}.$$

Hence, by Step 1, for every n ,

$$\mu_{\lambda_n u}(G) \leq \mu_v(G).$$

Clearly, $\lambda_n u \leq u$ and $\lim_{n \rightarrow \infty} \lambda_n u = u$ in $C_c(X)$. Hence, by Lemma 10,

$$\mu_u(G) = \lim_{n \rightarrow \infty} \mu_{\lambda_n u}(G) \leq \mu_v(G).$$

Step 3: Finally, let G be an arbitrary Borel set in $\{u \leq v\}$. Then, by Step 2 and by Lemma 11,

$$\begin{aligned}\mu_u(G) &= \mu_u(G \cap \{0 < v\}) \\ &\leq \mu_v(G \cap \{0 < v\}) \\ &= \mu_v(G),\end{aligned}$$

which is the claim.

In the remainder of this note, fix $(w_n) \subseteq D(\varphi)$ a dense subset of $D(\varphi)$ (this is possible since the space $C_c(X)$ is separable by assumption and [10]). Then we define a measure μ on $\mathcal{B}$ by

$$\mu(G) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu_{w_n}(G)}{1 + \mu_{w_n}(X)} \quad (G \in \mathcal{B}). \quad (7)$$

It is clear that μ is a finite, regular Borel measure on X .

Lemma 13. *Every measure μ_u ($u \in D(\varphi)$) is absolutely continuous with respect to μ .*

PROOF. Let $G \in \mathcal{B}$ be such that $\mu(G) = 0$. It follows from the definition of μ and the positivity of μ_{w_n} that $\mu_{w_n}(G) = 0$ for every $n \in \mathbb{N}$.

Now let $u \in D(\varphi)$. There exists a subsequence (w_{n_k}) such that $\lim_{k \rightarrow \infty} w_{n_k} = u$ in $C_c(X)$. Define $u_k := w_{n_k} \wedge u$. Then $u_k \leq w_{n_k}$ and Lemma 9 implies that $\mu_{u_k}(G) = 0$. Moreover, $u_k \leq u$ and $\lim_{k \rightarrow \infty} u_k = u$ in $C_c(X)$. From this and Lemma 10 we obtain $\mu_u(G) = 0$. Hence, μ_u is absolutely continuous with respect to μ .

By the preceding lemma and by the Radon-Nikodym theorem, for every $u \in D(\varphi)$ there exists a Borel measurable function $B_u = B_u(x)$ such that

$$\mu_u(G) = \int_G B_u(x) d\mu(x) \quad (G \in \mathcal{B}).$$

Lemma 14. *Let $u_n, u, v \in D(\varphi)$. Then:*

- (a) *If $u_n \leq u$ and $\lim_{n \rightarrow \infty} u_n = u$ in $C_c(X)$, then $\lim_{n \rightarrow \infty} B_{u_n} = B_u$ μ -almost everywhere.*
- (b) *$B_u = 0$ μ -almost everywhere on $\{u = 0\}$.*
- (c) *$B_u \leq B_v$ μ -almost everywhere on $\{u \leq v\}$.*

PROOF. This lemma is an immediate consequence of Lemma 10, Lemma 11 and Lemma 12.

For every $x \in X$ we define the sets $W(x) := \{w_n(x) : n \in \mathbb{N}\}$ and $I(x) := \{u(x) : u \in D(\varphi)\}$. Clearly, $W(x) \subseteq I(x)$. Then, for every $x \in X$ and every $s \in \mathbb{R}_+$ we define

$$B(x, s) = \begin{cases} \sup_n B_{w_n}(x) 1_{\{w_n < s\}}(x) & \text{if } s \in I(x), \\ +\infty & \text{if } s \notin I(x). \end{cases}$$

Lemma 15. *The function $B : X \times \mathbb{R}_+ \rightarrow [0, +\infty]$ defined above satisfies the hypotheses II (i) and II (ii) of the Theorem. Moreover,*

- (iv) *for every $u \in D(\varphi)$ one has $B(\cdot, u(\cdot)) = B_u(\cdot)$ μ -almost everywhere on X .*

PROOF. First, for every $s \in \mathbb{R}_+$ the set $\{x \in X : s \in I(x)\} = \{x \in X : s \leq \sup_n w_n(x)\}$ is a Borel set. From this and from the definition of B one obtains that for every $s \in \mathbb{R}_+$ the function $B(\cdot, s)$ is measurable. Thus, B satisfies hypothesis II (i) of the Theorem.

Second, it follows readily from the definition of B that $B(x, 0) = 0$ for every $x \in X$ (since the sets $\{w_n < 0\}$ are empty and therefore $1_{\{w_n < 0\}} = 0$ for every n). Moreover, since the sets $\{w_n < s\}$ are increasing with $s \in \mathbb{R}_+$,

the function $B(x, \cdot)$ is monotone for every $x \in X$. Finally, for every $x \in X$, every $s \in I(x) \setminus \{0\}$ and every $\varepsilon > 0$ there exists, by definition of the supremum, n such that $w_n(x) < s$ and $B(x, s) - \varepsilon \leq B_{w_n}(x) \leq B(x, s)$. This implies $B(x, s) - \varepsilon \leq B(x, s') \leq B(x, s)$ for every $w_n(x) < s' \leq s$. As a consequence, $B(x, \cdot)$ is lower semi-continuous for every $x \in X$. Thus, B satisfies hypothesis II (ii) of the Theorem.

Third, we show the property (iv). Let $u \in D(\varphi)$. By Lemma 14 (c), there exists a set A_u of μ -measure zero such that for every n and every $x \in \{w_n \leq u\} \setminus A_u$ one has $B_{w_n}(x) \leq B_u(x)$. As a consequence, $B(\cdot, u(\cdot)) \leq B_u(\cdot)$ μ -almost everywhere on X . In order to show the converse inequality we first note that, by Lemma 14 (b), $B_u(\cdot) = 0 = B(x, 0)$ μ -almost everywhere on $\{u = 0\}$. Hence, it remains to show that the inequality $B(\cdot, u(\cdot)) \geq B_u(\cdot)$ holds almost everywhere on $\{u > 0\}$.

Let $(\lambda_m) \subseteq \mathbb{R}_+$ be a sequence such that $\lambda_m < 1$ and $\lim_{m \rightarrow \infty} \lambda_m = 1$. Then, by Lemma 14 (c), there exists a set A'_u of μ -measure zero such that for every n, m and every $x \in \{\lambda_m u \leq w_n\} \setminus A'_u$ one has $B_{\lambda_m u}(x) \leq B_{w_n}(x)$. Since $\lambda_m u < u$ on $\{u > 0\}$ and since $W(x)$ is dense in $I(x)$ (for every $x \in X$), this implies $B_{\lambda_m u}(x) \leq B(x, u(x))$ for every m and every $x \in \{u > 0\}$. Since $B_{\lambda_m u} \rightarrow B_u$ μ -almost everywhere on X by Lemma 14 (a), we thus obtain the remaining inequality $B_u(\cdot) \leq B(\cdot, u(\cdot))$ on $\{u > 0\}$.

PROOF (PROOF OF THE THEOREM, (I) $\Rightarrow$ (II)). Let $(w_n) \subseteq D(\varphi)$ be dense in $D(\varphi)$ and let μ be the Borel measure and $B : X \times \mathbb{R}_+ \rightarrow [0, +\infty]$ be the function defined above. It follows from Lemma 8, that $\text{supp}[\mu] \subseteq \text{supp}[\varphi]$. By Lemma 15, the function B satisfies the hypotheses II (i) and (ii) of the Theorem, and by property (iv) in Lemma 15, for every $u \in D(\varphi)$ we have $B(\cdot, u(\cdot)) = B_u(\cdot)$ μ -almost everywhere on X . By the definition of B_u this means $\varphi(u) = \int_X B_u(x) d\mu(x) = \int_X B(x, u(x)) d\mu(x)$ for every $u \in D(\varphi)$. This just shows that B satisfies hypothesis II (iii) of the Theorem. The Theorem is completely proved.

References

- [1] H. Bauer, *Maß- und Integrationstheorie*, Walter de Gruyter, Berlin, New York, 1990.
- [2] H. Brezis, *Opérateurs maximaux monotones*. North-Holland, Amsterdam (1973).
- [3] R. V. Chacon and N. Friedman. Additive functionals. *Arch. Ration. Mech. Anal.* 18 (1965), 230–240.
- [4] R. Chill and M. Warma. Dirichlet and Neumann boundary conditions for the p -Laplace operator: What is in between? *Proc. Royal Soc. Edinburgh Sect. A* 142 (2012), 975–1002.
- [5] D. H. Fremlin, *Measure theory. Vol. 4, Topological measure spaces. Part I*, Torres Fremlin, Colchester, 2006, Corrected second printing of the 2003 original.
- [6] N. Friedman and M. Katz. A representation theorem for additive functionals. *Arch. Ration. Mech. Anal.* 21 (1966), 49–57.
- [7] N. Friedman and M. Katz. On additive functionals. *Proc. Amer. Math. Soc.* 21 (1969), 557–561.
- [8] M. M. Rao Local functionals. *Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979)*, pp. 484–496, *Lecture Notes in Math.*, 794, Springer, Berlin, 1980.
- [9] A. N. Tihonov and V. Ja. Arsenin, *On some non-linear functionals*, *Mat. Sb. (N.S.)* **65 (107)** (1964), 512–521.
- [10] S. Warner, *The topology of compact convergence on continuous function spaces*, *Duke Math. J.* **25** (1958), 265–282.