A converse to Fortin’s Lemma in Banach spaces
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Abstract

We establish the converse of Fortin’s Lemma in Banach spaces. This result is useful to assert the existence of a Fortin operator once a discrete inf-sup condition has been proved. The proof uses a specific construction of a right-inverse of a surjective operator in Banach spaces. The key issue is the sharp determination of the stability constants.

1 Introduction

Let $V$ and $W$ be two complex Banach spaces equipped with the norms $\| \cdot \|_V$ and $\| \cdot \|_W$, respectively. We adopt the convention that dual spaces are denoted with primes and are composed of antilinear forms; complex conjugates are denoted by an overline. Let $a$ be a sesquilinear form on $V \times W$ (linear w.r.t. its first argument and antilinear w.r.t. its second argument). We assume that $a$ is bounded, i.e.,

$$\|a\| := \sup_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} < \infty,$$  \hspace{1cm} (1)

and that the following inf-sup condition holds:

$$\alpha := \inf_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} > 0.$$  \hspace{1cm} (2)

Here and in what follows, arguments in infima and suprema are implicitly assumed to be nonzero.

Assume that we have at hand two sequences of finite-dimensional subspaces $\{V_h\}_{h \in H}$ and $\{W_h\}_{h \in H}$ with $V_h \subset V$ and $W_h \subset W$ for all $h \in H$, where the parameter $h$ typically refers to a family of underlying meshes. The spaces $V_h$...
and $W_h$ are equipped with the norms of $V$ and $W$, respectively. A question of fundamental importance is to assert the following discrete inf-sup condition:

\[ \hat{\alpha}_h := \inf_{v_h \in V_h, w_h \in W_h} \sup_{v_h \in V} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_W} > 0. \]  

(3)

The aim of this Note is to prove the following result.

**Theorem 1 (Fortin’s Lemma with converse)** Under the above assumptions, consider the following two statements:

(i) There exists a map $\Pi_h : W \to W_h$ and a real number $\gamma_{\Pi_h} > 0$ such that

\[ a(v_h, \Pi_h w - w) = 0, \]

for all $(v_h, w) \in V_h \times W$, and $\gamma_{\Pi_h} \|\Pi_h w\|_W \leq \|w\|_W$ for all $w \in W$.

(ii) The discrete inf-sup condition (3) holds.

Then, (i) $\Rightarrow$ (ii) with $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$. Conversely, (ii) $\Rightarrow$ (i) with $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|_{\alpha}}$, and $\Pi_h$ can be constructed to be idempotent. Moreover, $\Pi_h$ can be made linear if $W$ is a Hilbert space.

The statement (i) $\Rightarrow$ (ii) in Theorem 1 is classical and is known in the literature as Fortin’s Lemma, see [5] and [1, Prop. 5.4.3]. It provides an effective tool to prove the discrete inf-sup condition (3) by constructing explicitly a Fortin operator $\Pi_h$. We briefly outline a proof that (i) $\Rightarrow$ (ii) for completeness. Assuming (i), we have

\[ \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|w_h\|_W} \geq \sup_{w \in W} \frac{|a(v_h, \Pi_h w)|}{\|\Pi_h w\|_W} = \sup_{w \in W} \frac{|a(v_h, w)|}{\|\Pi_h w\|_W} \geq \gamma_{\Pi_h} \sup_{w \in W} \frac{|a(v_h, w)|}{\|w\|_W} \geq \gamma_{\Pi_h} \alpha \|v_h\|_V, \]

since $a$ satisfies (2) and $V_h \subset V$. This proves (ii) with $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$.

The proof of the converse (ii) $\Rightarrow$ (i) is the main object of this Note. This property is useful when it is easier to prove the discrete inf-sup condition directly rather than constructing a Fortin operator. Another application of current interest is the analysis framework for discontinuous Petrov–Galerkin methods (dPG) recently proposed in [3] which includes the existence of a Fortin operator among its key assumptions. The proof of the converse is not so straightforward if one wishes to establish a sharp stability bound for $\Pi_h$, i.e., that indeed one can take $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|_{\alpha}}$. Incidentally, we observe that there is a gap in the stability constant $\gamma_{\Pi_h}$ between the direct and converse statements, since the ratio of the two is equal to $\frac{\|a\|_{\alpha}}{\|a\|_1}$ (which is independent of the discrete setting).

This Note is organized as follows. In Section 2, we establish a sharp bound on the stability of the right-inverse of surjective operators in Banach spaces. Since this result can be of independent theoretical interest, we present it in an infinite-dimensional setting. Then in Section 3, we prove the converse of Fortin’s Lemma. The proof is relatively simple once the sharp stability estimate from Section 2 is available.
2 Right-inverse of surjective Banach operators

Let \( Y \) and \( Z \) be two complex Banach spaces equipped with the norms \( \|\cdot\|_Y \) and \( \|\cdot\|_Z \), respectively. Let \( B : Y \to Z \) be a bounded linear map. The following result is a well-known consequence of Banach’s Open Mapping and Closed Range Theorems, see, e.g., [2, Thm. 2.20] or [4, Lem. A.36 & A.40].

Lemma 2 (Surjectivity) The following three statements are equivalent:

(i) \( B : Y \to Z \) is surjective.

(ii) \( B^* : Z' \to Y' \) is injective and \( \text{im}(B^*) \) is closed in \( Y' \).

(iii) The following holds:

\[
\inf_{z' \in Z'} \frac{\|B^* z'\|_{Y'}}{\|z'\|_{Z'}} = \inf_{z' \in Z'} \sup_{y' \in Y'} \frac{|\langle B^* z', y' \rangle_{Y' Y}|}{\|z'\|_{Z'} \|y'\|_Y} =: \beta > 0. \tag{4}
\]

Let us now turn to the main result of this section. To motivate the result, assume that (4) holds; then \( B \) is surjective and thus admits a bounded right-inverse. The crucial question is whether the stability of this right-inverse can be formulated using precisely the constant \( \beta > 0 \) from (4).

Lemma 3 (Right inverse) Assume that (4) holds and that \( Y \) is reflexive. Then there is a right-inverse map \( B^: \) \( \to Y \) such that

\[
\forall z \in Z, \quad (B \circ B^)(z) = z \quad \text{and} \quad \beta \|B^z\|_Y \leq \|z\|_Z. \tag{5}
\]

Moreover, this right-inverse map \( B^\) is linear if \( Y \) is a Hilbert space.

Proof Parts of this result can be found in [4, Lem. A.42]; for completeness, we present a proof. Owing to Lemma 2, \( B^* \) is injective and \( R := \text{im}(B^*) \) is closed in \( Y' \). Since the operator \( B^* \) is injective, it admits a left-inverse linear map \( B^\dagger : R \to Z' \) such that \( (B^\dagger \circ B^*)(z') = z' \) for all \( z' \in Z' \). Moreover, the inf-sup condition (4) implies that \( \|B^\dagger y'\|_{Z'} \leq \beta^{-1} \|y'\|_{Y'} \) for all \( y' \in R \). Consider now the adjoint \( B^{\dagger*} : Z'' \to R' \). Let \( E^\text{lin}_{RY} \) be the Hahn–Banach extension operator that extends antilinear forms over \( R \subset Y' \) into antilinear forms over \( Y' \) (see [2, Prop. 11.23]); \( E^\text{lin}_{RY'} \) maps from \( R' \) to \( Y'' \). Let \( J_Y \) (resp., \( J_Z \)) be the canonical isometry from \( Y \) to \( Y'' \) (resp., \( Z \) to \( Z'' \)), and observe that \( J_Y \) is an isomorphism since \( Y \) is assumed to be reflexive. Let us set

\[
B^\dagger := J_Y^{-1} \circ E^\text{lin}_{RY'} \circ B^{\dagger*} \circ J_Z : Z \to Y, \tag{6}
\]

and let us verify that \( B^\dagger \) satisfies the expected properties. We have, for all \( (z', z) \in Z' \times Z \),

\[
\langle z', B(B^\dagger(z)) \rangle_{Z', Z} = \langle B^* z', B^\dagger(z) \rangle_{Y', Y} = \langle J_Y(B^\dagger(z)), B^* z' \rangle_{Y'', Y'} = \langle E^\text{lin}_{RY''}(B^{\dagger*}(J_Z z)), B^* z' \rangle_{Y'', Y'} = \langle B^{\dagger*}(J_Z z), B^* z' \rangle_{R'', R'} = \langle J_Z z, B^{\dagger*} B^* z' \rangle_{Z'', Z'} = \langle J_Z z, z' \rangle_{Z'', Z'} = \langle z', z \rangle_{Z', Z}.
\]
where we have used that $B^* z' \in R$ to pass from the first to the second line. This shows that $(B \circ B^1)(z) = z$. Moreover, since $J_Y$ is an isometry and the extension operator $E_{R Y''}$ preserves the norm, we observe that, for all $z \in Z$,

$$\|B^1 z\|_Y = \|B^* (J_z z)\|_R = \sup_{z' \in Z'} \frac{|(B^* (J_z z), B^* z') R : R|}{\|B^* z'\|_Y},$$

$$= \sup_{z' \in Z'} \frac{|(J_z z, z') Z', Z'|}{\|B^* z'\|_Y} \leq \sup_{z' \in Z'} \|z'\|_Z = \|\tilde{z}\|_Z.$$

We conclude from (4) that $\beta \|B^1 z\|_Y \leq \|z\|_Z$. Finally, if $Y$ is a Hilbert space, we can consider the orthogonal complement of $R$ in $Y'$ (recall that $R$ is a closed subspace of $Y'$) and write $Y' = R \oplus R'^\perp$. Then, the Hahn–Banach extension operator $E_{R Y''}$ in (6) can be replaced by the linear map $E_{R Y'}$, such that, for all $\phi \in R', (E_{R Y'} \phi, y') Y', Y' = (\phi, r) R, r$ for all $y' \in Y'$ with $y' = r + r'^\perp$, $r \in R, r'^\perp \in R'^\perp$. □

3 Proof of the converse in Theorem 1

Let $A_h : V_h \to W'_h$ be the operator defined by $(A_h v_h, w'_h) W'_h, W_h := a(v_h, w_h)$ for all $(v_h, w_h) \in V_h \times W'_h$. We identify $V'_h$ with $V_h$ and $W'_h$ with $W_h$ (since these spaces are finite-dimensional). We consider the adjoint operator $A^*_h : W'_h \to V'_h$, and identify $A^*_h$ with $A_h$. We apply Lemma 3 to $Y := W_h, Z := V'_h$, and $B := A^*_h$. Owing to the discrete inf-sup condition (3), we infer that (4) holds with $\beta = \tilde{\alpha}_h$. Therefore, there exists a right-inverse map $A^\dagger_h : V'_h \to W_h$ such that, for all $\theta_h \in V'_h, (A^\dagger_h \circ A^*_h)(\theta_h) = \theta_h$ and $\tilde{\alpha}_h \|A^\dagger_h \theta_h\|_W \leq \|\theta_h\|_{V'_h}$. Let us now set

$$\Pi_h := A^\dagger_h \circ \Theta : W \to W_h,$$

with the linear map $\Theta : W \to V'_h$ such that, for all $w \in W, (\Theta(w), v_h) V'_h, V_h := a(v_h, w)$ for all $v_h \in V_h$. We then infer that

$$a(v_h, \Pi_h(w)) = (A_h v_h, A^\dagger_h (\Theta(w))) W'_h, W_h = (A^*_h (A^\dagger_h (\Theta(w))), v_h) V'_h, V_h = (\Theta(w), v_h) V'_h, V_h = a(v_h, w),$$

which establishes that $a(v_h, \Pi_h(w) - w) = 0$ for all $w \in W$. Moreover,

$$\tilde{\alpha}_h \|\Pi_h(w)\|_W = \tilde{\alpha}_h \|A^\dagger_h (\Theta(w))\|_W \leq \|\Theta(w)\| V'_h \leq \|a\|_W \|w\|_W,$$

which proves that $\|a\|_W \|\Pi_h(w)\|_W \leq \|w\|_W$. In addition, we observe that

$$\langle \Theta(A^\dagger_h (\theta_h)), v_h \rangle_{V'_h, V_h} = (A^*_h (A^\dagger_h (\theta_h)) W'_h, W_h = (A^*_h (A^\dagger_h (\theta_h)), v_h) V'_h, V_h = (\theta_h, v_h) V'_h, V_h,$n

for all $v_h \in V_h$, which proves that $\Theta(A^\dagger_h (\theta_h)) = \theta_h$ for all $\theta_h \in V'_h$. As a result, $\Pi_h(\Pi_h(w)) = A^\dagger_h (\Theta \circ A^*_h (\Theta(w))) = A^\dagger_h (\Theta(w)) = \Pi_h(w)$, i.e., $\Pi_h$ is
idempotent. Finally, if \( W \) is a Hilbert space, the right-inverse map \( A_h^{\dagger} \) is linear by Lemma 3, and so is the operator \( \Pi_h \) defined from (7).

Remark 1 (Value of \( \gamma_{\Pi_h} \)) Without the use of Lemma 3, one only knows that \( A_h^* \) has a stable right-inverse, but a stability bound for this right-inverse is not available. In particular, if the discrete inf-sup condition (3) holds uniformly with respect to \( h \), i.e., if there is \( \alpha_0 > 0 \) such that \( \alpha_h \geq \alpha_0 \) for all \( h \in \mathcal{H} \), then a uniform stability bound for \( \Pi_h \) is \( \gamma_{\Pi_h} \geq \gamma_{\Pi_0} = \frac{\alpha_h}{\|a\|} \) for all \( h \in \mathcal{H} \).

Remark 2 (Linearity) Even in the case of Banach spaces, the linearity of the map \( \Pi_h \) can be asserted if one has at hand a stable decomposition \( W_h = \ker(A_h^*) \oplus K_h \) such that there is \( \kappa_h > 0 \) such that the induced projector \( \pi_{K_h} : W_h \to K_h \) satisfies \( \kappa_h \| \pi_{K_h} w_h \|_W \leq \| w_h \|_W \) for all \( w_h \in W_h \) (this property holds in the Hilbertian setting with \( \kappa_h = 1 \)). Then, one can adapt the reasoning at the end of the proof of Lemma 3 to build a stable, linear right-inverse map \( A_h^{\dagger} \). The mild price to be paid is that the stability constant of \( \Pi_h \) now becomes \( \gamma_{\Pi_h} = \frac{\kappa_h \alpha_h}{\|a\|} \).

References


