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A converse to Fortin’s Lemma in Banach spaces

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Abstract

We establish the converse of Fortin’s Lemma in Banach spaces. This result is useful to assert the existence of a Fortin operator once a discrete inf-sup condition has been proved. The proof uses a specific construction of a right-inverse of a surjective operator in Banach spaces. The key issue is the sharp determination of the stability constants.

1 Introduction

Let $V$ and $W$ be two complex Banach spaces equipped with the norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. We adopt the convention that dual spaces are denoted with primes and are composed of antilinear forms; complex conjugates are denoted by an overline. Let $a$ be a sesquilinear form on $V \times W$ (linear w.r.t. its first argument and antilinear w.r.t. its second argument). We assume that $a$ is bounded, i.e.,

$$\|a\| := \sup_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} < \infty, \quad (1)$$

and that the following inf-sup condition holds:

$$\alpha := \inf_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} > 0. \quad (2)$$

Here and in what follows, arguments in infima and suprema are implicitly assumed to be nonzero.

Assume that we have at hand two sequences of finite-dimensional subspaces $\{V_h\}_{h \in \mathcal{H}}$ and $\{W_h\}_{h \in \mathcal{H}}$ with $V_h \subset V$ and $W_h \subset W$ for all $h \in \mathcal{H}$, where the parameter $h$ typically refers to a family of underlying meshes. The spaces $V_h$

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and $W_h$ are equipped with the norms of $V$ and $W$, respectively. A question of fundamental importance is to assert the following discrete inf-sup condition:

$$\hat{\alpha}_h := \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_W} > 0.$$  \hspace{1cm} (3)

The aim of this Note is to prove the following result.

**Theorem 1 (Fortin’s Lemma with converse)** Under the above assumptions, consider the following two statements:

(i) There exists a map $\Pi_h : W \to W_h$ and a real number $\gamma_{\Pi_h} > 0$ such that $a(v_h, \Pi_h w - w) = 0$, for all $(v_h, w) \in V_h \times W$, and $\gamma_{\Pi_h} \|\Pi_h w\|_W \leq \|w\|_W$ for all $w \in W$.

(ii) The discrete inf-sup condition (3) holds.

Then, (i) $\Rightarrow$ (ii) with $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$. Conversely, (ii) $\Rightarrow$ (i) with $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|}$, and $\Pi_h$ can be constructed to be idempotent. Moreover, $\Pi_h$ can be made linear if $W$ is a Hilbert space.

The statement (i) $\Rightarrow$ (ii) in Theorem 1 is classical and is known in the literature as Fortin’s Lemma, see [5] and [1, Prop. 5.4.3]. It provides an effective tool to prove the discrete inf-sup condition (3) by constructing explicitly a Fortin operator $\Pi_h$. We briefly outline a proof that (i) $\Rightarrow$ (ii) for completeness. Assuming (i), we have

$$\sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|w_h\|_W} \geq \sup_{w \in W} \frac{|a(v_h, \Pi_h w)|}{\|\Pi_h w\|_W} = \sup_{w \in W} \frac{|a(v_h, w)|}{\|\Pi_h w\|_W} \geq \gamma_{\Pi_h} \sup_{w \in W} \frac{|a(v_h, w)|}{\|w\|_W} \geq \gamma_{\Pi_h} \alpha \|v_h\|_V,$$

since $a$ satisfies (2) and $V_h \subset V$. This proves (ii) with $\hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha$.

The proof of the converse (ii) $\Rightarrow$ (i) is the main object of this Note. This property is useful when it is easier to prove the discrete inf-sup condition directly rather than constructing a Fortin operator. Another application of current interest is the analysis framework for discontinuous Petrov–Galerkin methods (dPG) recently proposed in [3] which includes the existence of a Fortin operator among its key assumptions. The proof of the converse is not so straightforward if one wishes to establish a sharp stability bound for $\Pi_h$, i.e., that indeed one can take $\gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|}$. Incidentally, we observe that there is a gap in the stability constant $\gamma_{\Pi_h}$ between the direct and converse statements, since the ratio of the two is equal to $\frac{\|a\|}{\|a\|}$ (which is independent of the discrete setting).

This Note is organized as follows. In Section 2, we establish a sharp bound on the stability of the right-inverse of surjective operators in Banach spaces. Since this result can be of independent theoretical interest, we present it in an infinite-dimensional setting. Then in Section 3, we prove the converse of Fortin’s Lemma. The proof is relatively simple once the sharp stability estimate from Section 2 is available.
2 Right-inverse of surjective Banach operators

Let $Y$ and $Z$ be two complex Banach spaces equipped with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Let $B : Y \to Z$ be a bounded linear map. The following result is a well-known consequence of Banach’s Open Mapping and Closed Range Theorems, see, e.g., [2, Thm. 2.20] or [4, Lem. A.36 & A.40].

Lemma 2 (Surjectivity) The following three statements are equivalent:

(i) $B : Y \to Z$ is surjective.

(ii) $B^* : Z^* \to Y^*$ is injective and $\text{im}(B^*)$ is closed in $Y^*$.

(iii) The following holds:

$$\inf_{z' \in Z^*} \frac{\|B^* z'\|_{Y^*}}{\|z'\|_{Z^*}} = \inf_{z' \in Z^*, y \in Y^*} \frac{|(B^* z', y)_{Y^*, Y}|}{\|z'\|_{Z^*} \|y\|_Y} =: \beta > 0. \quad (4)$$

Let us now turn to the main result of this section. To motivate the result, assume that (4) holds; then $B$ is surjective and thus admits a bounded right-inverse. The crucial question is whether the stability of this right-inverse can be formulated using precisely the constant $\beta > 0$ from (4).

Lemma 3 (Right inverse) Assume that (4) holds and that $Y$ is reflexive. Then there is a right-inverse map $B^! : Z \to Y$ such that

$$\forall z \in Z, \quad (B \circ B^!)(z) = z \quad \text{and} \quad \beta \|B^! z\|_Y \leq \|z\|_Z. \quad (5)$$

Moreover, this right-inverse map $B^!$ is linear if $Y$ is a Hilbert space.

Proof Parts of this result can be found in [4, Lem. A.42]; for completeness, we present a proof. Owing to Lemma 2, $B^*$ is injective and $R := \text{im}(B^*)$ is closed in $Y^*$. Since the operator $B^*$ is injective, it admits a left-inverse linear map $B^{*1} : R \to Z'$ such that $(B^{*1} \circ B^*)(z') = z'$ for all $z' \in Z'$. Moreover, the inf-sup condition (4) implies that $\|B^{*1} y\|_{Z'} \leq \beta^{-1} \|y\|_{Y^*}$ for all $y \in R$. Consider now the adjoint $B^{*1*} : Z'' \to R'$. Let $E^\text{lin}_{RY'}$, be the Hahn–Banach extension operator that extends antilinear forms over $R \subset Y'$ into antilinear forms over $Y'$ (see [2, Prop. 11.23]); $E^\text{lin}_{RY'}$ maps from $R'$ to $Y''$. Let $J_Y$ (resp., $J_Z$) be the canonical isometry from $Y$ to $Y''$ (resp., $Z$ to $Z''$), and observe that $J_Y$ is an isomorphism since $Y$ is assumed to be reflexive. Let us set

$$B^! := J_Y^{-1} \circ E^\text{lin}_{RY'} \circ B^{*1*} \circ J_Z : Z \to Y, \quad (6)$$

and let us verify that $B^!$ satisfies the expected properties. We have, for all $(z', z) \in Z' \times Z$,

$$\langle z', B(B^!(z)) \rangle_{Z', Z} = \langle B^*(z'), B^!(z) \rangle_{Y', Y} = \langle J_Y(B^!(z)), B^*(z') \rangle_{Y'', Y'} = \langle E^\text{lin}_{RY'}(B^{*1*}(J_Z z)), B^*(z') \rangle_{Y'', Y'} = \langle B^{*1*}(J_Z z), B^*(z') \rangle_{R', R'}$$

$$= \langle J_Z z, B^{*1*} B^*(z') \rangle_{Z'', Z'} = \langle J_Z z, z' \rangle_{Z'', Z'} = \langle z', z \rangle_{Z', Z}.$$
where we have used that $B^*z' \in R$ to pass from the first to the second line. This shows that $(B \circ B^*)(z) = z$. Moreover, since $J_Y$ is an isometry and the extension operator $E_R^{\mathcal{W}Y'}$ preserves the norm, we observe that, for all $z \in Z$,

$$||B^*z||_Y = ||B^*z(Jz)||_R = \sup_{z' \in Z} \frac{||B^*z(Jz)||_R}{\|B^*z'\|_Y},$$

$$= \sup_{z' \in Z} \frac{||Jz||_R}{\|B^*z'\|_Y} \leq \sup_{z' \in Z} \frac{\|z\|_Z}{\|B^*z'\|_Y} \|z\|_Z.$$

We conclude from (4) that $\beta ||B^*z||_Y \leq ||z||_Z$. Finally, if $Y$ is a Hilbert space, we can consider the orthogonal complement of $R$ in $Y'$ (recall that $R$ is a closed subspace of $Y'$) and write $Y' = R \oplus R^\perp$. Then, the Hahn–Banach extension operator $E_{R^{\perp}Y'}$ in (6) can be replaced by the linear map $E_{R^{\perp}Y'}$ such that, for all $\phi \in R^{\perp}$, $(E_{R^{\perp}Y'}\phi, y)_Y' = \langle \phi, r \rangle_R \in R$ for all $y' \in Y'$ with $y' = r + r^\perp$, $r \in R$, $r^\perp \in R^\perp$.

\[ \square \]

3 Proof of the converse in Theorem 1

Let $A_h : V_h \to W_h$ be the operator defined by $(A_h v_h, w_h)_{W_h} := a(v_h, w_h)$ for all $(v_h, w_h) \in V_h \times W_h$. We identify $V_h''$ with $V_h$ and $W_h''$ with $W_h$ (since these spaces are finite-dimensional). We consider the adjoint operator $A_h^\ast : W_h \to V_h''$, and identify $A_h^\ast$ with $A_h$. We apply Lemma 3 to $Y := W_h$, $Z := V_h''$, and $B := A_h$. Owing to the discrete inf-sup condition (3), we infer that (4) holds with $\beta = \alpha_h$. Therefore, there exists a right-inverse map $A_h^\perp : V_h' \to W_h$ such that, for all $\theta_h \in V_h'$, $(A_h \circ A_h^\perp)(\theta_h) = \theta_h$ and $\alpha_h \|A_h^\perp \theta_h\|_W \leq \|\theta_h\|_{V_h'}$. Let us now set

$$\Pi_h := A_h^\perp \circ \Theta : W \to V_h,$$

with the linear map $\Theta : W \to V_h'$ such that, for all $w \in W$, $(\Theta(w), v_h)_{V_h', V_h} := a(v_h, w)$ for all $v_h \in V_h$. We then infer that

$$a(v_h, \Pi_h(w)) = (A_h v_h, A_h^\ast (\Theta(w)))_{W_h', V_h} = (A_h^\ast (A_h^\ast (\Theta(w))), v_h)_{V_h', V_h},$$

$$= (\Theta(w), v_h)_{V_h', V_h} = a(v_h, w),$$

which establishes that $a(v_h, \Pi_h(w) - w) = 0$ for all $w \in W$. Moreover,

$$\alpha_h \|\Pi_h(w)\|_W = \alpha_h \|A_h^\ast (\Theta(w))\|_W \leq \|\Theta(w)\|_{V_h'} \leq \|a\| \|w\|_W,$$

which proves that $\frac{\alpha_h}{\|a\|} \|\Pi_h(w)\| \leq \|w\|_W$. In addition, we observe that

$$\langle \Theta(A_h^\ast (\theta_h)), v_h \rangle_{V_h', V_h} = (A_h v_h, A_h^\ast (\Theta(w)))_{W_h', W_h},$$

$$= (\Theta(A_h^\ast (\theta_h)), v_h)_{V_h', V_h} = \langle \theta_h, v_h \rangle_{V_h', V_h},$$

for all $v_h \in V_h$, which proves that $\Theta(A_h^\ast (\theta_h)) = \theta_h$ for all $\theta_h \in V_h'$. As a result, $\Pi_h(\Theta(w)) = A_h^\ast (\Theta(\theta_h)) = A_h^\ast (\Theta(w)) = \Pi_h(w)$, i.e., $\Pi_h$ is
idempotent. Finally, if $W$ is a Hilbert space, the right-inverse map $A_h^{*†}$ is linear by Lemma 3, and so is the operator $\Pi_h$ defined from (7).

Remark 1 (Value of $\gamma_{\Pi_h}$) Without the use of Lemma 3, one only knows that $A_h^{*}$ has a stable right-inverse, but a stability bound for this right-inverse is not available. In particular, if the discrete inf-sup condition (3) holds uniformly with respect to $h$, i.e., if there is $\hat{\alpha}_0 > 0$ such that $\hat{\alpha}_h \geq \hat{\alpha}_0$ for all $h \in \mathcal{H}$, then a uniform stability bound for $\Pi_h$ is $\gamma_{\Pi_h} \geq \gamma_{\Pi_0} = \hat{\alpha}_0 \|a\|$ for all $h \in \mathcal{H}$.

Remark 2 (Linearity) Even in the case of Banach spaces, the linearity of the map $\Pi_h$ can be asserted if one has at hand a stable decomposition $W_h = \ker(A_h^{*}) \oplus K_h$ such that there is $\kappa_h > 0$ such that the induced projector $\pi_{K_h} : W_h \to K_h$ satisfies $\kappa_h \|\pi_{K_h} w_h\|_W \leq \|w_h\|_W$ for all $w_h \in W_h$ (this property holds in the Hilbertian setting with $\kappa_h = 1$). Then, one can adapt the reasoning at the end of the proof of Lemma 3 to build a stable, linear right-inverse map $A_h^{*†}$. The mild price to be paid is that the stability constant of $\Pi_h$ now becomes $\gamma_{\Pi_h} = \frac{\kappa_h \hat{\alpha}_h \|a\|}{\|\cdot\|}$.

References


