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A converse to Fortin’s Lemma in Banach spaces

Alexandre Ern*  Jean-Luc Guermond†

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Abstract

We establish the converse of Fortin’s Lemma in Banach spaces. This result is useful to assert the existence of a Fortin operator once a discrete inf-sup condition has been proved. The proof uses a specific construction of a right-inverse of a surjective operator in Banach spaces. The key issue is the sharp determination of the stability constants.

1 Introduction

Let $V$ and $W$ be two complex Banach spaces equipped with the norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. We adopt the convention that dual spaces are denoted with primes and are composed of antilinear forms; complex conjugates are denoted by an overline. Let $a$ be a sesquilinear form on $V \times W$ (linear w.r.t. its first argument and antilinear w.r.t. its second argument). We assume that $a$ is bounded, i.e.,

$$\|a\| := \sup_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} < \infty,$$

and that the following inf-sup condition holds:

$$\alpha := \inf_{v \in V} \sup_{w \in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} > 0.$$ (2)

Here and in what follows, arguments in infima and suprema are implicitly assumed to be nonzero.

Assume that we have at hand two sequences of finite-dimensional subspaces $\{V_h\}_{h \in \mathcal{H}}$ and $\{W_h\}_{h \in \mathcal{H}}$ with $V_h \subset V$ and $W_h \subset W$ for all $h \in \mathcal{H}$, where the parameter $h$ typically refers to a family of underlying meshes. The spaces $V_h$

*Université Paris-Est, CERMICS (ENPC), 77455 Marne la Vallée Cedex 2, France; alexandre.ern@enpc.fr
†Department of Mathematics, Texas A&M University 3368 TAMU, College Station, TX 77843, USA; guermond@math.tamu.edu
and \( W_h \) are equipped with the norms of \( V \) and \( W \), respectively. A question of fundamental importance is to assert the following discrete inf-sup condition:

\[
\hat{\alpha}_h := \inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|v_h\|_V \|w_h\|_W} > 0.
\] (3)

The aim of this Note is to prove the following result.

**Theorem 1 (Fortin’s Lemma with converse)** Under the above assumptions, consider the following two statements:

(i) There exists a map \( \Pi_h : W \to W_h \) and a real number \( \gamma_{\Pi_h} > 0 \) such that

\[
a(v_h, \Pi_h w - w) = 0, \quad \text{for all} \quad (v_h, w) \in V_h \times W, \quad \text{and} \quad \gamma_{\Pi_h} \|\Pi_h w\|_W \leq \|w\|_W \quad \text{for all} \quad w \in W.
\]

(ii) The discrete inf-sup condition (3) holds.

Then, (i) \( \Rightarrow \) (ii) with \( \hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha \). Conversely, (ii) \( \Rightarrow \) (i) with \( \gamma_{\Pi_h} = \frac{\alpha}{\|a\|} \), and \( \Pi_h \) can be constructed to be idempotent. Moreover, \( \Pi_h \) can be made linear if \( W \) is a Hilbert space.

The statement (i) \( \Rightarrow \) (ii) in Theorem 1 is classical and is known in the literature as Fortin’s Lemma, see [5] and [1, Prop. 5.4.3]. It provides an effective tool to prove the discrete inf-sup condition (3) by constructing explicitly a Fortin operator \( \Pi_h \). We briefly outline a proof that (i) \( \Rightarrow \) (ii) for completeness. Assuming (i), we have

\[
\sup_{w_h \in W_h} \frac{|a(v_h, w_h)|}{\|w_h\|_W} \geq \sup_{w \in W} \frac{|a(v_h, \Pi_h w)|}{\|\Pi_h w\|_W} = \sup_{w \in W} \frac{|a(v_h, w)|}{\|\Pi_h w\|_W} \geq \gamma_{\Pi_h} \sup_{w \in W} \frac{|a(v_h, w)|}{\|w\|_W} \geq \gamma_{\Pi_h} \alpha \|v_h\|_V,
\]

since \( a \) satisfies (2) and \( V_h \subset V \). This proves (ii) with \( \hat{\alpha}_h \geq \gamma_{\Pi_h} \alpha \).

The proof of the converse (ii) \( \Rightarrow \) (i) is the main object of this Note. This property is useful when it is easier to prove the discrete inf-sup condition directly rather than constructing a Fortin operator. Another application of current interest is the analysis framework for discontinuous Petrov–Galerkin methods (dPG) recently proposed in [3] which includes the existence of a Fortin operator among its key assumptions. The proof of the converse is not so straightforward if one wishes to establish a sharp stability bound for \( \Pi_h \), i.e., that indeed one can take \( \gamma_{\Pi_h} = \frac{\hat{\alpha}_h}{\|a\|} \). Incidentally, we observe that there is a gap in the stability constant \( \gamma_{\Pi_h} \) between the direct and converse statements, since the ratio of the two is equal to \( \frac{\|a\|}{\alpha} \) (which is independent of the discrete setting).

This Note is organized as follows. In Section 2, we establish a sharp bound on the stability of the right-inverse of surjective operators in Banach spaces. Since this result can be of independent theoretical interest, we present it in an infinite-dimensional setting. Then in Section 3, we prove the converse of Fortin’s Lemma. The proof is relatively simple once the sharp stability estimate from Section 2 is available.
2 Right-inverse of surjective Banach operators

Let $Y$ and $Z$ be two complex Banach spaces equipped with the norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. Let $B : Y \to Z$ be a bounded linear map. The following result is a well-known consequence of Banach’s Open Mapping and Closed Range Theorems, see, e.g., [2, Thm. 2.20] or [4, Lem. A.36 & A.40].

Lemma 2 (Surjectivity) The following three statements are equivalent:

(i) $B : Y \to Z$ is surjective.

(ii) $B^* : Z' \to Y'$ is injective and $\text{im}(B^*)$ is closed in $Y'$.

(iii) The following holds:
\[
\inf_{z' \in Z'} \frac{\|B^*z'\|_{Y'}}{\|z'\|_{Z'}} = \inf_{z' \in Z'} \sup_{y \in Y} \frac{|\langle B^*z', y \rangle_{Y', Y}|}{\|y\|_Y} =: \beta > 0.
\]

Let us now turn to the main result of this section. To motivate the result, assume that (4) holds; then $B$ is surjective and thus admits a bounded right-inverse. The crucial question is whether the stability of this right-inverse can be formulated using precisely the constant $\beta > 0$ from (4).

Lemma 3 (Right inverse) Assume that (4) holds and that $Y$ is reflexive. Then there is a right-inverse map $B^\dagger : Z \to Y$ such that
\[
\forall z \in Z, \quad (B \circ B^\dagger)(z) = z \quad \text{and} \quad \beta \|B^\dagger z\|_Y \leq \|z\|_Z.
\]

Moreover, this right-inverse map $B^\dagger$ is linear if $Y$ is a Hilbert space.

Proof Parts of this result can be found in [4, Lem. A.42]; for completeness, we present a proof. Owing to Lemma 2, $B^*$ is injective and $R := \text{im}(B^*)$ is closed in $Y'$. Since the operator $B^*$ is injective, it admits a left-inverse linear map $B^{*1} : R \to Z'$ such that $(B^{*1} \circ B^*)(z') = z'$ for all $z' \in Z'$. Moreover, the inf-sup condition (4) implies that $\|B^{*1}y'\|_{Z'} \leq \beta^{-1}\|y'\|_{Y'}$ for all $y' \in R$. Consider now the adjoint $B^{*1*} : Z'' \to R'$. Let $E_{RY'}^{\text{ant}}$, be the Hahn–Banach extension operator that extends antilinear forms over $R \subset Y'$ into antilinear forms over $Y'$ (see [2, Prop. 11.23]); $E_{RY'}^{\text{ant}}$, maps from $R'$ to $Y''$. Let $J_Y$ (resp., $J_Z$) be the canonical isometry from $Y$ to $Y''$ (resp., $Z$ to $Z''$), and observe that $J_Y$ is an isomorphism since $Y$ is assumed to be reflexive. Let us set
\[
B^\dagger := J_Y^{-1} \circ E_{RY'}^{\text{ant}} \circ B^{*1*} \circ J_Z : Z \to Y,
\]
and let us verify that $B^\dagger$ satisfies the expected properties. We have, for all $(z', z) \in Z' \times Z$,
\[
\langle z', B(B^\dagger(z)) \rangle_{Z', Z} = \langle B^*z', B^\dagger(z) \rangle_{Y', Y} = \langle J_Y(B^\dagger(z)), B^*z' \rangle_{Y'', Y'} = \langle E_{RY'}^{\text{ant}}(J_Z B^\dagger(z)), B^*z' \rangle_{Y'', Y'} = \langle B^*z' \rangle_{Y', Y'} = \langle B^{*1*}(J_Z z), (B^{*1*}z')_{R', R} \rangle
\]
\[
= \langle (J_Z z), B^{*1}(B^*z') \rangle_{Z'', Z'} = \langle (J_Z z), z' \rangle_{Z'', Z'} = \langle z', z \rangle_{Z', Z}.
\]
where we have used that $B^*z' \in R$ to pass from the first to the second line. This shows that $(B \circ B^1)(z) = z$. Moreover, since $J_Y$ is an isometry and the extension operator $E^{\text{ext}}_{R \oplus Y''}$ preserves the norm, we observe that, for all $z \in Z$,

$$||B^1z||_Y = ||B^{\ast \ast}(J_Z z)||_R = \sup_{z' \in Z'} \frac{||(B^{\ast \ast}(J_Z z), B^*z')_R||_R}{||B^*z'||_Y},$$

$$= \sup_{z' \in Z'} \frac{|(J_Z z, z')_Z|}{||B^*z'||_Y} \leq \sup_{z' \in Z'} ||z'||_Z \leq ||z||_Z.$$  

We conclude from (4) that $\beta ||B^1z||_Y \leq ||z||_Z$. Finally, if $Y$ is a Hilbert space, we can consider the orthogonal complement of $R$ in $Y'$ (recall that $R$ is a closed subspace of $Y'$) and write $Y' = R \oplus R^\perp$. Then, the Hahn–Banach extension operator $E^{\text{ext}}_{R \oplus Y''}$ in (6) can be replaced by the linear map $E^{\text{ext}}_{R \oplus Y''}$ such that, for all $\phi \in R'$, $(E^{\text{ext}}_{R \oplus Y''}\phi, y')_{Y''} = (\phi, r)_R$ for all $y' \in Y'$ with $y' = r + r^\perp$, $r \in R$, $r^\perp \in R^\perp$. \hfill \Box

3 Proof of the converse in Theorem 1

Let $A_h : V_h \to W'_h$ be the operator defined by $(A_h v_h, w_h)_{W'_h, W_h} := a(v_h, w_h)$ for all $(v_h, w_h) \in V_h \times W'_h$. We identify $V''_h$ with $V_h$ and $W''_h$ with $W_h$ (since these spaces are finite-dimensional). We consider the adjoint operator $A^*_h : W'_h \to V'_h$, and identify $A^{\ast \ast}_h$ with $A_h$. We apply Lemma 3 to $Y := W_h$, $Z := V'_h$, and $B := A^*_h$. Owing to the discrete inf-sup condition (3), we infer that (4) holds with $\beta = \bar{\alpha}_h$. Therefore, there exists a right-inverse map $A_h^{\dagger} : V'_h \to W_h$ such that, for all $\theta_h \in V'_h$, $(A_h^* \circ A_h^{\dagger})(\theta_h) = \theta_h$ and $\bar{\alpha}_h ||A_h^{\dagger}\theta_h||_W \leq ||\theta_h||_{V'_h}$. Let us now set

$$\Pi_h := A_h^{\dagger} \circ \Theta : W \to W_h,$$

with the linear map $\Theta : W \to V'_h$ such that, for all $w \in W$, $(\Theta(w), v_h)_{V'_h, V_h} := a(v_h, w)$ for all $v_h \in V_h$. We then infer that

$$a(v_h, \Pi_h(w)) = \langle A_h v_h, A_h^{\dagger}(\Theta(w)) \rangle_{W'_h, W_h} = \langle A_h^* (A_h^{\dagger}(\Theta(w))), v_h \rangle_{V'_h, V_h} = (\Theta(w), v_h)_{V'_h, V_h} = a(v_h, w),$$

which establishes that $a(v_h, \Pi_h(w) - w) = 0$ for all $w \in W$. Moreover,

$$\bar{\alpha}_h ||\Pi_h(w)||_W = \bar{\alpha}_h ||A_h^{\dagger}(\Theta(w))||_W \leq ||\Theta(w)||_{V'_h} \leq ||a||_W ||w||_W,$$

which proves that $\frac{\bar{\alpha}_h}{||a||} ||\Pi_h(w)||_W \leq ||w||_W$. In addition, we observe that

$$\langle \Theta(A_h^{\dagger}(\theta_h)), v_h \rangle_{V'_h, V_h} = \langle A_h v_h, A_h^{\dagger}(\theta_h) \rangle_{W'_h, W_h} = \langle A_h^* (A_h^{\dagger}(\theta_h)), v_h \rangle_{V'_h, V_h} = \langle \theta_h, v_h \rangle_{V'_h, V_h},$$

for all $v_h \in V_h$, which proves that $\Theta(A_h^{\dagger}(\theta_h)) = \theta_h$ for all $\theta_h \in V'_h$. As a result, $\Pi_h(\Pi_h(w)) = A_h^{\dagger}(\Theta \circ A_h^{\dagger}(\Theta(w))) = A_h^{\dagger}(\Theta(w)) = \Pi_h(w)$, i.e., $\Pi_h$ is
idempotent. Finally, if $W$ is a Hilbert space, the right-inverse map $A_h^\dagger$ is linear by Lemma 3, and so is the operator $\Pi_h$ defined from (7).

Remark 1 (Value of $\gamma_{\Pi_h}$) Without the use of Lemma 3, one only knows that $A_h^*$ has a stable right-inverse, but a stability bound for this right-inverse is not available. In particular, if the discrete inf-sup condition (3) holds uniformly with respect to $h$, i.e., if there is $\hat{\alpha}_0 > 0$ such that $\hat{\alpha}_h \geq \hat{\alpha}_0$ for all $h \in \mathcal{H}$, then a uniform stability bound for $\Pi_h$ is $\gamma_{\Pi_h} \geq \gamma_{\Pi_0} = \frac{\hat{\alpha}_0}{\|a\|}$ for all $h \in \mathcal{H}$.

Remark 2 (Linearity) Even in the case of Banach spaces, the linearity of the map $\Pi_h$ can be asserted if one has at hand a stable decomposition $W_h = \ker(A_h^*) \oplus K_h$ such that there is $\kappa_h > 0$ such that the induced projector $\pi_{K_h} : W_h \to K_h$ satisfies $\kappa_h \|\pi_{K_h} w_h\|_W \leq \|w_h\|_W$ for all $w_h \in W_h$ (this property holds in the Hilbertian setting with $\kappa_h = 1$). Then, one can adapt the reasoning at the end of the proof of Lemma 3 to build a stable, linear right-inverse map $A_h^\dagger$. The mild price to be paid is that the stability constant of $\Pi_h$ now becomes $\gamma_{\Pi_h} = \frac{\kappa_h \hat{\alpha}_h}{\|a\|}.$

References


