Large deviations of a velocity jump process with a Hamilton-Jacobi approach

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Abstract

We study a random process on $\mathbb{R}^n$ moving in straight lines and changing randomly its velocity at random exponential times. We focus more precisely on the Kolmogorov equation in the hyperbolic scale $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$, with $\varepsilon > 0$, before proceeding to a Hopf-Cole transform, which gives a kinetic equation on a potential. We show convergence as $\varepsilon \to 0$ of the potential towards the viscosity solution of a Hamilton-Jacobi equation $\partial_t \varphi + H(\nabla_x \varphi) = 0$ where the hamiltonian may lack $\mathcal{C}^1$ regularity, which is quite unseen in this type of studies.

Résumé

Grandes déviations pour un processus à sauts de vitesse avec une approche Hamilton-Jacobi. Nous nous intéressons à un processus aléatoire sur $\mathbb{R}^n$ qui alterne des phases de mouvements rectilignes uniformes et change de vitesse à des temps exponentiels. Nous étudions plus précisément l'équation de Kolmogorov après rééchelonnement hyperbolique $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$, $\varepsilon > 0$, puis nous effectuons une transformée de Hopf-Cole qui nous donne une équation cinétique suivie par un potentiel. Nous montrons la convergence pour $\varepsilon \to 0$ de ce potentiel vers la solution de viscosités d'une équation de Hamilton-Jacobi $\partial_t \varphi + H(\nabla_x \varphi) = 0$ où le hamiltonien peut présenter une singularité $\mathcal{C}^1$, ce qui est assez inédit dans ce type d'études.

Version française abrégée

Nous nous donnons une densité de probabilité $M \in L^1(\mathbb{R}^n)$ et nous notons V son support. Nous supposons que V est compact et que 0 appartient à l'intérieur de l'enveloppe convexe de V, que l'on note $\mathrm{Conv}(V)$. Pour $p \in \mathbb{R}^n$, nous notons $\mu(p) = \max\{v \cdot p \mid v \in \mathrm{Conv}(V)\}$. Nous étudions le mouvement de particules dans $\mathbb{R}^n$ suivant le processus de Markov déterministe par morceaux défini comme suit : une particule donnée se déplace de manière rectiligne uniforme avec une vitesse $v \in V$ tirée aléatoirement en suivant la loi de probabilité $M(v')\,dv'$. À des temps exponentiels de paramètre 1, la particule change de direction en tirant une nouvelle vitesse tirée selon la loi $M(v')\,dv'$. Afin d'étudier des résultats de larges déviations du processus similairement aux techniques développées dans [3]-[7], nous nous intéressons à l'équation de Chapman-Kolmogorov forward suivie par la densité de particules après un rééchelonement hyperbolique $(t,x,v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$, $\varepsilon > 0$:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(M(v)\rho^\varepsilon - f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V.$$

Nous étudions plus particulièrement l'équation vérifiée par un potentiel φ^ε obtenu après passage par une transformée de Hopf-Cole : $f^\varepsilon(t,x,v) = M(v)\, e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}$. Nous cherchons alors une éventuelle limite pour φ^ε. Nous procédons à un développement WKB : $\varphi^\varepsilon = \varphi + \varepsilon\eta$, ce qui amène, en posant $p = \nabla_x \varphi$ et $H = -\partial_t \varphi$, à la résolution d'un problème spectral dans l'espace des mesures positives : chercher (H,Q) un couple valeur/vecteur propres associé à l'opérateur $Q \mapsto (v \cdot p - 1)\,Q + \int_V M'Q'dv'$. On obtient une équation de Hamilton-Jacobi $\partial_t \varphi + H(\nabla_x \varphi) = 0$. Pour $n = 1$ et $M \geq \delta > 0$ sur son support, le vecteur propre Q a une densité et conduit à un hamiltonien H défini par l'équation implicite

$$\int_V \frac{M(v)}{1 + H(p) - v \cdot p}\, dv = 1.$$

La positivité de Q garantit que $H(p) \geq \mu(p) - 1$. En dimension supérieure toutefois, et même si $M \geq \delta > 0$, cette équation peut ne pas avoir de solution $H(p)$ lorsque p devient grand. Cela se manifeste pour le vecteur propre par une concentration de la mesure Q autour des valeurs v qui annulent $1 + H(p) - v \cdot p$, ce qui force $H(p) = \mu(p) - 1$. Cette transition entraîne une singularité $\mathcal{C}^1$ du hamiltonien.

Nous démontrons la convergence de φ^ε vers φ, où φ est solution de viscosité [4] de l'équation de Hamilton-Jacobi en utilisant la méthode de la fonction test perturbée [6].

1 Introduction

We continue the work initiated in [1]-[2]. Let $M \in L^1(\mathbb{R}^n)$ be a probability density function. We suppose that the support of M, which we denote V, is compact and that 0 belongs to the interior of $\mathrm{Conv}(V)$, the convex hull of V. We denote by $|\cdot|$ the euclidian norm in $\mathbb{R}^n$ and by $\cdot$ the canonical scalar product. For $p \in \mathbb{R}^n$, we define

$$\mu(p) := \max\{v \cdot p \mid v \in \mathrm{Conv}(V)\}, \tag{1}$$

$\mathrm{Arg}\mu(p) := \{v \in \mathrm{Conv}(V) \mid v \cdot p = \mu(p)\}$ and $\mathrm{Sing}(M) := \left\{p \in \mathbb{R}^n, \quad \int_V \frac{M(v)}{\mu(p) - v \cdot p} dv \leq 1\right\}$.

We focus on the motion dynamics in $\mathbb{R}^n$ of particles given by the following piecewise deterministic Markov process: a particle moves successively in straight lines with velocity v, chosen randomly with probability distribution $M(v')\,dv'$. At random exponential times (with parameter 1), the particle changes its velocity, choosing randomly a new velocity with distribution $M(v')\,dv'$. The Chapman-Kolmogorov forward equation associated to the probability density function $f(t,x,v)$ of this process is given by:

$$\partial_t f + v \cdot \nabla_x f = M\rho - f, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V, \tag{2}$$

where $\rho(t,x) = \int_V f(t,x,v)\,dv$. In order to investigate large deviation principles for the process, one can study the large scale hyperbolic limit $(t,x) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$ with $\varepsilon > 0$. In this scale, the kinetic equation (2) reads:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}\left(M\rho^\varepsilon - f^\varepsilon\right), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V. \tag{3}$$

Then, we perform the following Hopf-Cole transformation: $f^\varepsilon(t,x,v) = M(v)\,e^{-\frac{\varphi^\varepsilon(t,x,v)}{\varepsilon}}$, where we expect the potential φ^ε to become independent of v as $\varepsilon \to 0$. Such techniques have already been studied for a more general case of Markov process with a finite discrete set of states in [3] and, from a probabilistic point of view, in [7]. Here, assume that the initial condition is well-prepared, i.e. it does not depend on v and ε: $\varphi^\varepsilon(0,x,v) = \varphi_0(x)$. The equation satisfied by φ^ε reads

$$\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon = \int_V M(v')\left(1 - e^{\frac{\varphi^\varepsilon - \varphi'^\varepsilon}{\varepsilon}}\right) dv', \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^n \times V. \tag{4}$$

As in [8], the limit potential satisfy a Hamilton-Jacobi equation. Surprisingly enough, our Hamiltonian may lack $\mathcal{C}^1$ regularity as we will show in Proposition 2.1.

Theorem 1.1 *Under the previous assumptions, φ^ε converges locally uniformly on $\mathbb{R}_+ \times \mathbb{R}^n \times V$ toward φ, where φ does not depend on v. Moreover, φ is the viscosity solution of the following Hamilton-Jacobi equation:*

$$\partial_t \varphi(t,x) + H(\nabla_x \varphi(t,x)) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{5}$$

where the hamiltonian H is given as follows: if $p \in \mathrm{Sing}(M)$, then $H(p) = \mu(p) - 1$. Else, $H(p)$ is uniquely determined by the following formula:

$$\int_V \frac{M(v)}{1 + H(p) - v \cdot p} dv = 1. \tag{6}$$

2 Identification of the hamiltonian

In order to identify the limit $\varepsilon \to 0$ of the equation (4) we perform the formal WKB expansion: $\varphi^\varepsilon(t,x,v) = \varphi(t,x) + \varepsilon\eta(t,x,v)$, where φ and η are to be determined. Plugging this ansatz into the kinetic formulation (4), we get, taking the formal limit $\varepsilon \to 0$:

$$\partial_t \varphi + v \cdot \nabla_x \varphi = 1 - \int_V M(v')\,e^{\eta - \eta'} dv'.$$

Let us write $p = \nabla_x \varphi$ and $H = -\partial_t \varphi$. The equation for $Q = e^{-\eta}$ is the following spectral problem: $HQ = (v \cdot p - 1)\,Q + \int_V M(v')\,Q(v')\,dv'$. The positivity of Q yields $H \geq v \cdot p - 1$ for all $v \in V$ hence $H \geq \mu(p) - 1$. Suppose $H > \mu(p) - 1$. Then, $1 + H - vp > 0$ for all $v \in V$ and $Q(v) = \dfrac{\int_V M(v')\,Q(v')\,dv'}{1 + H - v \cdot p}$. Integrating against M with respect to v, we obtain the following problem: find H such that $\int_V \frac{M(v)}{1+H-v\cdot p} dv = 1$. If $p \in \mathrm{Sing}(M)^c$, by monotonicity, such H exists and is unique. Equation (6), however, does not have a solution for $p \in \mathrm{Sing}\,(M)$, so we necessarily have $H = \mu\,(p) - 1$. Then, a possible solution of the spectral problem is the positive measure $Q = \frac{dv}{\mu(p) - v \cdot p} + \alpha\,(p)\,\delta_w$ where $\alpha\,(p) = 1 - \int_V \frac{M(v)}{\mu(p) - vp} dv \geq 0$ and δ_w is the Dirac measure centered in $w \in \mathrm{Arg}\mu\,(p) \cap V$.

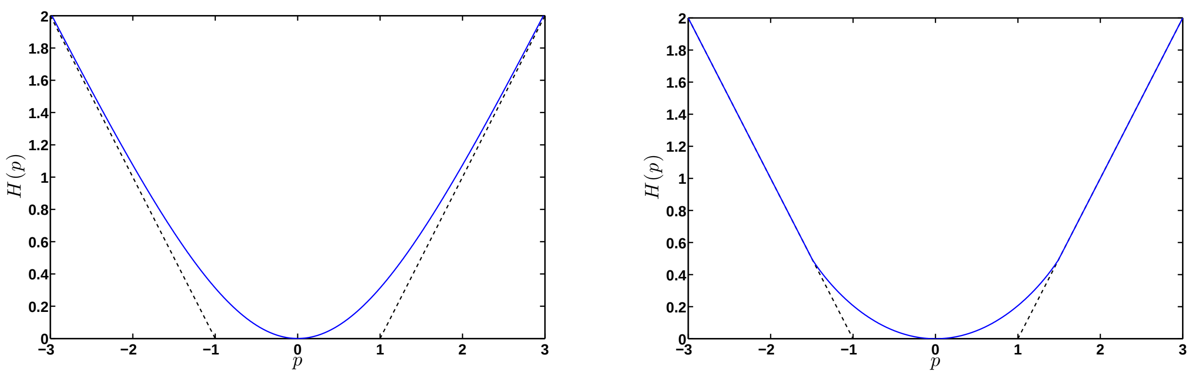
Here is an example where $\mathrm{Sing}\,(M) \neq \emptyset$:

Example 1 Let $n > 1$ and $M = \omega_n^{-1}.\mathbb{1}_{\overline{B(0,1)}}$ where ω_n is the Lebesgue measure of the n-dimensional unit ball. Then, $\mathrm{Sing}\,(M) = B\left(0, \frac{n}{n-1}\right)^c$. Indeed, for $p = |p| \cdot e_1$, we have $\mu\,(p) = |p|$ and $v \cdot p = |p|\,v_1$ hence

$$\int_V \frac{M\,(v)}{\mu\,(p) - v \cdot p} dv = \frac{1}{|p|\,\omega_n} \int_{B(0,1)} \frac{1}{1 - v_1} dv = \frac{\omega_{n-1}}{|p|\,\omega_n} \int_{-1}^{1} \frac{\left(1 - v_1^2\right)^{\frac{n-1}{2}}}{1 - v_1} dv_1 = \frac{1}{|p|} \times \frac{n}{n-1}.$$

By rotational invariance, we conclude that $\mathrm{Sing}\,(M) = B\left(0, \frac{n}{n-1}\right)^c$. The Figure 1 gives illustrations of the hamiltonian and μ as functions of the radius of p, in the cases $n = 1$ and $n = 3$. In the cases $n = 3$ we can see the $\mathcal{C}^1$ singularity where $|p| = \frac{3}{2}$.

Figure 1 – Blue plain lines : Hamiltonian for $n = 1, 3$ and $M = \omega_n^{-1}.\mathbb{1}_{\overline{B(0,1)}}$. Black dotted lines : $|p| \mapsto \mu\,(p) - 1$. Lignes pleines bleues : Hamiltonien pour $n = 1, 3$ et $M = \omega_n^{-1}.\mathbb{1}_{\overline{B(0,1)}}$. Lignes noires en pointillés : $|p| \mapsto \mu\,(p) - 1$.



Proposition 2.1 *The following properties hold:*
(i) The set $\mathrm{Sing}\,(M)^c$ is convex.
(ii) The function H is continuous and convex.
(iii) If $\mathrm{Sing}\,(M) \neq \emptyset$, then H is not $\mathcal{C}^1$. More precisely, ∇H has a jump discontinuity at $\partial \mathrm{Sing} M$.

Proof Let us first notice that μ is positively 1-homogeneous. Moreover, it is convex since it is a supremum of linear functions.

(i) Let $p, q \in \mathrm{Sing}\,(M)^c$ with $p \neq q$. Since μ is convex, we have for all $\tau \in [0,1]$

$$I\,(\tau) := \int_V \frac{M\,(v)}{\mu\,(p) - v \cdot p + \tau\,(\mu\,(q) - \mu\,(p) - v \cdot (q - p))} dv \quad \leq \quad \int_V \frac{M\,(v)}{\mu\,((1-\tau)\,p + \tau q) - v \cdot ((1-\tau)\,p + \tau q)} dv.$$

Moreover, $I\,(0), I\,(1) > 1$ and I is differentiable on $[0,1]$ with

$$\partial_\tau I\,(\tau) \quad = \quad \int_V \frac{M\,(v)}{\left(\mu\,(p) - v \cdot p + \tau\,(\mu\,(q) - \mu\,(p) - v \cdot (q - p))\right)^2} \left(\mu\,(p) - \mu\,(q) - v \cdot (p - q)\right) dv.$$

It is clear that the sign of $\partial_\tau I$ does not change hence $I\,(\tau) > 1$, which proves (i).

(ii) We refer to [2] to prove that H is $\mathcal{C}^2$ and strictly convex on $\mathrm{Sing}\,(M)^c$ and that

$$\int_V \frac{M\,(v)}{\left(1 + H\,(q) - v \cdot q\right)^2} \left(\nabla H\,(q) - v\right) dv = 0, \quad \forall q \in \mathrm{Sing}\,(M)^c. \tag{7}$$

In particular, $\nabla H\,(q) \in \mathrm{Conv}\,(V)$ for all $q \in \mathrm{Sing}\,(M)^c$. It is easy to see that H is continuous in the interior of $\mathrm{Sing}\,(M)$. To show continuity of H on $\partial \mathrm{Sing}\,(M)$, let $(p_m)_m$ converge to $p \in \partial \mathrm{Sing}\,(M) \subset \mathrm{Sing}\,(M)$. If we can extract a subsequence $(p_{m_l})_l \subset \mathrm{Sing}(M)$, then $H(p_{m_l}) = \mu(p_{m_l}) - 1 \underset{l\to\infty}{\longrightarrow} \mu(p) - 1 = H(p)$. If not, then $p_m \in \mathrm{Sing}(M)^c$ for m large enough and $1 = \int_V \frac{M(v)}{1+H(p_m)-v\cdot p_m} dv < \int_V \frac{M(v)}{\mu(p_m)-v\cdot p_m} dv$. Taking the limit, we get by dominated convergence $1 = \int_V \lim_{m\to\infty} \frac{M(v)}{1+H(p_m)-v\cdot p_m} dv \le \int_V \frac{M(v)}{\mu(p)-v\cdot p_m} dv \le 1$ hence $H(p_m) \underset{m\to\infty}{\longrightarrow} \mu(p) - 1 = H(p)$.

We now show that H is convex by proving that it is a maximum of convex functions:

$$H(p) \quad = \quad \max\left(\sup\left\{\nabla H(q)\cdot(p-q) + H(q) \mid q \in \mathrm{Sing}(M)^c\right\}, \mu(p) - 1\right), \quad \forall p \in \mathbb{R}^n. \tag{8}$$

In $\mathrm{Sing}(M)^c$, (8) holds by convexity of H and $H(p) > \mu(p) - 1$. Let $p \in \mathrm{Sing}(M)$ and $q \in \mathrm{Sing}(M)^c$. By convexity of $\mathrm{Sing}(M)^c \ni 0$, there exists a unique $\lambda \in (0,1]$ such that $\lambda p \in \partial\mathrm{Sing}(M)$. For all $\tau \in [0,1]$, we set $\omega_1(\tau) := \mu(\tau p) - 1 = \tau\mu(p) - 1$ and $\omega_2(\tau) := \nabla H(q)\cdot(\tau p - q) + H(q)$. By continuity of H, $\mu(\lambda p) - 1 = H(\lambda p) \ge \nabla H(q)(\lambda p - q) + H(q)$ hence $\omega_1(\lambda) \ge \omega_2(\lambda)$. Moreover, ω_1 and ω_2 are both differentiable and $\partial_\tau\omega_1(\tau) = \mu(p) \ge \nabla H(q)\cdot p = \partial_\tau\omega_2(\tau)$ since $\nabla H(q) \in \mathrm{Conv}(V)$. Hence, $\omega_1(1) \ge \omega_2(1)$, which ends the proof of (ii).

(iii) Suppose $\mathrm{Sing}(M) \neq \emptyset$ and H is $\mathcal{C}^1$. Since $H + 1 = \mu$ is positive homogeneous of degree 1 on $\mathrm{Sing}(M)$ and since $\lambda p \in \mathrm{Sing}(M)$ for all $\lambda \ge 1$ and $p \in \mathrm{Sing}(M)$, we know that $\nabla H(p)\cdot p = H(p) + 1 = \mu(p)$ for all $p \in \mathrm{Sing}(M) \subset \mathrm{Sing}(M)$ hence $p\cdot(\nabla H(p) - v) \ge 0$, for all $v \in V$, the inequality being strict on a neighborhood of 0. Then,

$$p\cdot\int_V \frac{M(v)}{(1+H(p)-v\cdot p)^2}(\nabla H(p) - v)\,dv > 0, \quad \forall p \in \partial\mathrm{Sing}(M). \tag{9}$$

By continuity, equations (7) and (9) are contradictory.

□

3 Proof of Theorem 1.1

Let $\varphi_0 \in W^{1,\infty}(\mathbb{R}^n)$. We refer to Proposition 2.1 in [2] to prove that the Cauchy Problem (4) with initial condition φ_0 has a unique solution $\varphi^\varepsilon \in W^{1,\infty}$ which is locally (in t) uniformly (in ε, x and v) bounded in norm $W^{1,\infty}$. In particular, let us mention that

$$0 \le \varphi^\varepsilon(t,\cdot,\cdot) \le \|\varphi_0\|_\infty, \quad \|\nabla_v\varphi^\varepsilon(t,\cdot,\cdot)\|_\infty \le \|\nabla_x\varphi_0\|_\infty. \tag{10}$$

Using the Arzelá-Ascoli theorem, we extract a locally uniformly converging subsequence. We denote by φ the limit. The function φ does not depend on v since $\int_V M(v)\, e^{\frac{\varphi^\varepsilon - \varphi'^\varepsilon}{\varepsilon}} dv$ is uniformly bounded on $[0,T]\times\mathbb{R}^n\times V$ for all $T > 0$. We use the perturbed test function method [6] to show that φ is a viscosity solution of (5). Theorem 1.1 will follow by uniqueness of the solution [5].

3.1 Subsolution procedure

Let $\psi \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n)$ be a test function such that $\varphi - \psi$ has a local strict maximum at (t^0, x^0). We want to show that ψ is a subsolution of (5). If $\nabla_x\psi(t^0,x^0) \in \mathrm{Sing}(M)^c$, then we refer to [2].

Suppose now that $\nabla_x\psi(t^0,x^0) \in \mathrm{Sing}(M)$. Let $w \in \mathrm{Arg}\mu(\nabla_x\psi(t^0,x^0)) \cap V$. Then, $w\cdot\nabla_x\psi(t^0,x^0) = \mu(\nabla_x\psi(t^0,x^0))$. The uniform convergence of φ^ε toward φ ensures that the function $(t,x) \mapsto \varphi^\varepsilon(t,x,w) - \psi(t,x)$ has a local maximum at a point $(t^\varepsilon, x^\varepsilon)$ satisfying $(t^\varepsilon,x^\varepsilon) \to (t^0,x^0)$, as $\varepsilon \to 0$. We then have:

$$\partial_t\psi(t^\varepsilon,x^\varepsilon) + w\cdot\nabla_x\psi(t^\varepsilon,x^\varepsilon) = \partial_t\varphi^\varepsilon(t^\varepsilon,x^\varepsilon) + w\cdot\nabla_x\varphi^\varepsilon(t^\varepsilon,x^\varepsilon) = 1 - \int_V M(v')\, e^{\frac{\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,w) - \varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v')}{\varepsilon}} dv' \le 1.$$

Passing to the limit $\varepsilon \to 0$, we get $\partial_t\psi(t^0,x^0) + \mu(\nabla_x\psi(t^0,x^0)) \le 1$. We conclude that φ is a viscosity subsolution of (5).

3.2 Supersolution procedure

Let $\psi \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n)$ be a test function such that $\varphi - \psi$ has a local strict minimum at (t^0, x^0). We want to show that ψ is a supersolution of (5). If $\nabla_x\psi(t^0,x^0) \in \mathrm{Sing}(M)^c$, then we refer to [2].

Suppose now that $\nabla_x \psi\left(t^0, x^0\right) \in \operatorname{Sing}(M)$. Then, $\nabla_x \psi\left(t^0, x^0\right) \neq 0$ because $0 \in \operatorname{Sing}(M)^c$. We suppose without loss of generality that the minimum of $\varphi - \psi$ is global and that $\varphi\left(t^0, x^0\right) - \psi\left(t^0, x^0\right) = 0$. Let $\psi^\varepsilon := \psi - C\left(t - t^0\right)^2 + \varepsilon \eta$ with $C > 0$ yet to be determined and

$$\eta(v) := \ln\left(\mu\left(\nabla_x \psi\left(t^0, x^0\right)\right) - v \cdot \nabla_x \psi\left(t^0, x^0\right)\right).$$

Then, η is a continuous function on $D(\eta) = V \setminus \operatorname{Arg}\mu\left(\nabla_x \psi\left(t^0, x^0\right)\right)$ and, for all $w \in \operatorname{Arg}\mu\left(\nabla_x \psi\left(t^0, x^0\right)\right) \cap V$, we have $\lim_{v \to w} \eta(v) = -\infty$. Moreover, η is bounded from below on all compact sets yielding the uniform convergence $\psi^\varepsilon \to \psi$ on all compact sets of $D(\eta)$. Finally, $\int_V M(v') e^{-\eta(v')} dv' \leq 1$ since $\nabla_x \psi\left(t^0, x^0\right) \in \operatorname{Sing}(M)$.

The function $\varphi - \left(\psi - C\left(t - t^0\right)^2\right)$ has a global strict minimum at $\left(t^0, x^0\right)$. The first inequality (10) ensures that the function $\varphi^\varepsilon - \psi^\varepsilon$ has a local minimum at a point $(t^\varepsilon, x^\varepsilon, v^\varepsilon) \in \mathbb{R}_+ \times \mathbb{R}^n \times D(\eta)$. As V compact, we can extract a subsequence $(v^\varepsilon)_\varepsilon$, without relabelling, such that $v^\varepsilon \to v^0$, as $\varepsilon \to 0$.

If $v^0 \in V \setminus \operatorname{Arg}\mu(p)$, then there exists a compact $A \subset D(\eta)$ such that $v^0 \in A$ and the uniform convergence of ψ^ε towards ψ on A guarantees that $(t^\varepsilon, x^\varepsilon) \to \left(t^0, x^0\right)$, as $\varepsilon \to 0$. We then get at point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$,

$$\begin{aligned}
\partial_t \psi - 2C\left(t^\varepsilon - t^0\right) + v^\varepsilon \cdot \nabla_x \psi = \partial_t \psi^\varepsilon + v^\varepsilon \cdot \nabla_x \psi^\varepsilon = \partial_t \varphi^\varepsilon + v^\varepsilon \cdot \nabla_x \varphi^\varepsilon &= 1 - \int_V M' e^{\frac{\varphi^\varepsilon - \varphi'^\varepsilon}{\varepsilon}} dv' \\
&\geq 1 - \int_V M(v') e^{\eta(v^\varepsilon) - \eta(v')} dv'.
\end{aligned}$$

We take the limit $\varepsilon \to 0$:

$$\partial_t \psi\left(t^0, x^0\right) + v^0 \cdot \nabla_x \psi\left(t^0, x^0\right) \geq 1 - e^{\eta(v^0)} \int_V M(v') e^{-\eta(v')} dv' \geq 1 - e^{\eta(v^0)}.$$

By construction, for all $v, v' \in D(\eta)$, we have $e^{\eta(v)} - e^{\eta(v')} = (v' - v) \cdot \nabla_x \psi\left(t^0, x^0\right)$ hence, for all $v \in D(\eta)$,we have $\partial_t \psi\left(t^0, x^0\right) + v \cdot \nabla_x \psi\left(t^0, x^0\right) \geq 1 - e^{\eta(v)}$. Let $w \in V \cap \operatorname{Arg}\mu\left(\nabla_x \psi\left(t^0, x^0\right)\right)$. Since $\operatorname{Arg}\mu\left(\nabla_x \psi\left(t^0, x^0\right)\right)$ is a null-set, V is dense in $\operatorname{Arg}\mu\left(\nabla_x \psi\left(t^0, x^0\right)\right)$. Taking the limit $v \to w$, we get: $\partial_t \psi\left(t^0, x^0\right) + \mu\left(\nabla_x \psi\left(t^0, x^0\right)\right) \geq 1$.

If $v^0 \in V \cap \operatorname{Arg}\mu(p)$, we still have $(t^\varepsilon, x^\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \left(t^0, x^0\right)$ thanks to the following lemma:

Lemma 3.1 *For $C = 4\|\varphi_0\|_\infty$, we have $\lim_{\varepsilon \to 0} \varepsilon \eta(v^\varepsilon) = 0$.*

Proof of Lemma 3.1 We have $\varphi^\varepsilon(t, x, v) - \varphi(t, x) \geq -2\|\varphi_0\|_\infty$ by (10) and $\varphi(t, x) - \psi(t, x) \geq 0$ hence

$$\varphi^\varepsilon(t, x, v) - \psi^\varepsilon(t, x, v) \geq -2\|\varphi_0\|_\infty + C\left(t - t^0\right)^2 - \varepsilon \eta(v), \quad \forall \varepsilon > 0.$$

Moreover,

$$\varphi^\varepsilon\left(t^0, x^0, v\right) - \psi^\varepsilon\left(t^0, x^0, v\right) = \varphi^\varepsilon\left(t^0, x^0, v\right) - \varphi\left(t^0, x^0\right) - \varepsilon \eta(v) \leq 2\|\varphi_0\|_\infty - \varepsilon \eta(v).$$

Since $C = 4\|\varphi_0\|_\infty$, we have $\varphi^\varepsilon(t, x, v) - \psi^\varepsilon(t, x, v) > \varphi^\varepsilon\left(t^0, x^0, v\right) - \psi^\varepsilon\left(t^0, x^0, v\right)$ for all $t > t^0 + 1$ and, thus, the minimum of $\varphi^\varepsilon - \psi^\varepsilon$ cannot be attained for $t > t^0 + 1$ hence $t^\varepsilon \leq t^0 + 1$ for all $\varepsilon > 0$. At point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ we have:

$$\nabla_v \varphi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon) = \nabla_v \psi^\varepsilon(t^\varepsilon, x^\varepsilon, v^\varepsilon) = \varepsilon \nabla_v \eta(v^\varepsilon) = -\frac{\varepsilon \nabla_x \psi\left(t^0, x^0\right)}{\mu\left(\nabla_x \psi\left(t^0, x^0\right)\right) - v^\varepsilon \cdot \nabla_x \psi\left(t^0, x^0\right)}.$$

The second estimation (10) yields $\|\nabla_v \varphi^\varepsilon(t^\varepsilon, \cdot, \cdot)\|_\infty \leq t^\varepsilon \|\nabla_x \varphi_0\|_\infty \leq \left(t^0 + 1\right) \|\nabla_x \varphi_0\|_\infty$ hence

$$\begin{aligned}
\frac{\varepsilon}{(t^0 + 1)\|\nabla_x \varphi_0\|_\infty} \left|\nabla_x \psi\left(t^0, x^0\right)\right| &\leq \mu\left(\nabla_x \psi\left(t^0, x^0\right)\right) - v^\varepsilon \cdot \nabla_x \psi\left(t^0, x^0\right), \\
\Longrightarrow \varepsilon K \geq \varepsilon \eta(v^\varepsilon) &\geq \varepsilon \ln\left(\frac{\varepsilon}{(t^0 + 1)\|\nabla_x \varphi_0\|_\infty} \left|\nabla_x \psi\left(t^0, x^0\right)\right|\right),
\end{aligned}$$

and $\varepsilon \eta(v^\varepsilon) \to 0$ as $\varepsilon \to 0$.

□

Thanks to Lemma 3.1, the function $(t,x) \mapsto \psi^\varepsilon(t,x,v^\varepsilon) = \psi(t,x) - 4\|\varphi_0\|_\infty \left(t-t^0\right)^2 + \varepsilon\eta(v^\varepsilon)$ converges uniformly towards $(t,x) \mapsto \psi(t,x) - 4\|\varphi_0\|_\infty \left(t-t^0\right)^2$ and has a local minimum at $(t^\varepsilon, x^\varepsilon)$ satisfying $(t^\varepsilon, x^\varepsilon) \to \left(t^0, x^0\right)$, as $\varepsilon \to 0$. At point $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$, we have:

$$\partial_t \psi^\varepsilon + v^\varepsilon \cdot \nabla_x \psi^\varepsilon = \partial_t \varphi^\varepsilon + v^\varepsilon \cdot \nabla_x \varphi^\varepsilon = 1 - \int_V M(v') e^{\frac{\varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v^\varepsilon) - \varphi^\varepsilon(t^\varepsilon,x^\varepsilon,v')}{\varepsilon}} dv'.$$

The minimal property of $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$ implies at this point:

$$\begin{aligned}\partial_t \psi(t^\varepsilon, x^\varepsilon) - 8\|\varphi_0\|_\infty \left(t^\varepsilon - t^0\right) + v^\varepsilon \cdot \nabla_x \psi(t^\varepsilon, x^\varepsilon) = \partial_t \psi^\varepsilon + v^\varepsilon \cdot \nabla_x \psi^\varepsilon &\geq 1 - \int_V M(v') e^{\eta(v^\varepsilon) - \eta\left(v'\right)} dv' \\ &\geq 1 - e^{\eta(v^\varepsilon)}.\end{aligned}$$

Passing to the limit $\varepsilon \to 0$, we get $\partial_t \psi\left(t^0, x^0\right) + \mu\left(\nabla_x \psi\left(t^0, x^0\right)\right) \geq 1$. We conclude that φ is a viscosity supersolution of (5).

□

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