DESCRIPTION OF THE LACK OF COMPACTNESS IN ORLICZ SPACES AND APPLICATIONS

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ABSTRACT. In this paper, we investigate the lack of compactness of the Sobolev embedding of $H^1(\mathbb{R}^2)$ into the Orlicz space $L^{\phi_p}(\mathbb{R}^2)$ associated to the function ϕ_p defined by $\phi_p(s) := \mathrm{e}^{s^2} - \sum_{k=0}^{p-1} \frac{s^{2k}}{k!}\cdot$ We also undertake the study of a nonlinear wave equation with exponential growth where the Orlicz norm $\|.\|_{L^{\phi_p}}$ plays a crucial role. This study includes issues of global existence, scattering and qualitative study.

1. Introduction

1.1. **Critical $2D$ Sobolev embedding.** It is well known (see for instance [7]) that $H^1(\mathbb{R}^2)$ is continuously embedded in all Lebesgue spaces $L^q(\mathbb{R}^2)$ for $2 \le q < \infty$, but not in $L^\infty(\mathbb{R}^2)$. It is also known that (for more details, we refer the reader to [21])

$$H^1(\mathbb{R}^2) \hookrightarrow L^{\phi_p}(\mathbb{R}^2), \quad \forall p \in \mathbb{N}^*, \tag{1}$$

where $L^{\phi_p}(\mathbb{R}^2)$ denotes the Orlicz space associated to the function

$$\phi_p(s) = \mathrm{e}^{s^2} - \sum_{\mathrm{k}=0}^{\mathrm{p}-1} \frac{\mathrm{s}^{2\mathrm{k}}}{\mathrm{k}!}\cdot \tag{2}$$

The embedding (1) is a direct consequence of the following sharp Trudinger-Moser type inequalities (see [1, 20, 22, 26]):

Proposition 1.1.

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)}\le 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u(x)|^2} - 1\right) dx := \kappa < \infty, \tag{3}$$

and states as follows:

$$\|u\|_{L^{\phi_p}(\mathbb{R}^2)} \le \frac{1}{\sqrt{4\pi}} \|u\|_{H^1(\mathbb{R}^2)}, \tag{4}$$

where the norm $\|.\|_{L^{\phi_p}}$ is given by:

$$\|u\|_{L^{\phi_p}(\mathbb{R}^2)} = \inf\left\{ \lambda > 0, \int_{\mathbb{R}^2} \phi_p\left(\frac{|u(x)|}{\lambda}\right) dx \le \kappa \right\}.$$

Note that (4) follows from (3) and the following obvious inequality

$$\|u\|_{L^{\phi_p}(\mathbb{R}^2)} \le \|u\|_{L^{\phi_1}(\mathbb{R}^2)} \cdot$$

For our purpose, we shall resort to the following Trudinger-Moser inequality, the proof of which is postponed in the appendix.

Proposition 1.2. *Let $\alpha \in [0, 4\pi[$ and p an integer larger than 1. There is a constant $c(\alpha, p)$ such that*

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha|u(x)|^2} - \sum_{k=0}^{p-1} \frac{\alpha^k |u(x)|^{2k}}{k!} \right) dx \leq c(\alpha, p) \|u\|^{2p}_{L^{2p}(\mathbb{R}^2)}, \tag{5}$$

for all $u \in H^1(\mathbb{R}^2)$ satisfying $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$.

1.2. **Development on the lack of compactness of Sobolev embedding in the Orlicz space in the case $p = 1$.** In [3], [4] and [5], H. Bahouri, M. Majdoub and N. Masmoudi characterized the lack of compactness of $H^1(\mathbb{R}^2)$ into the Orlicz space $L^{\phi_1}(\mathbb{R}^2)$. To state their result in a clear way, let us recall some definitions.

Definition 1.3. *We shall designate by a scale any sequence (α_n) of positive real numbers going to infinity, a core any sequence (x_n) of points in $\mathbb{R}^2$ and a profile any function ψ belonging to the set*

$$\mathcal{P} := \Big\{ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s} ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0 \Big\}.$$

Given two scales (α_n), $(\tilde{\alpha}_n)$, two cores (x_n), $(\tilde{x}_n)$ and tow profiles ψ, $\tilde{\psi}$, we say that the triplets $\big((\alpha_n), (x_n), \psi\big)$ and $\big((\tilde{\alpha}_n), (\tilde{x}_n), \tilde{\psi}\big)$ are orthogonal if

$$\textit{either} \qquad \Big| \log\left(\tilde{\alpha}_n/\alpha_n\right) \Big| \to \infty,$$

or $\tilde{\alpha}_n = \alpha_n$ and

$$-\frac{\log|x_n - \tilde{x}_n|}{\alpha_n} \longrightarrow a \geq 0 \textit{ with } \psi \textit{ or } \tilde{\psi} \textit{ null for } s < a\,.$$

Remarks 1.4.

- *The profiles belong to the Hölder space $C^{\frac{1}{2}}$. Indeed, for any profile ψ and real numbers s and t, we have by Cauchy-Schwarz inequality*

$$|\psi(s) - \psi(t)| = \left| \int_s^t \psi'(\tau)\, d\tau \right| \leq \|\psi'\|_{L^2(\mathbb{R})} |s - t|^{\frac{1}{2}}.$$

- *Note also that (see [4])*

$$\frac{\psi(s)}{\sqrt{s}} \to 0 \quad \textit{as} \quad s \to 0 \quad \textit{and} \quad \textit{as} \quad s \to \infty. \tag{6}$$

The asymptotically orthogonal decomposition derived in [4] is formulated in the following terms:

Theorem 1.5. *Let (u_n) be a bounded sequence in $H^1(\mathbb{R}^2)$ such that*

$$u_n \rightharpoonup 0, \tag{7}$$

$$\limsup_{n\to\infty} \|u_n\|_{L^{\phi_1}} = A_0 > 0 \qquad \textit{and} \tag{8}$$

$$\lim_{R\to\infty} \limsup_{n\to\infty} \|u_n\|_{L^{\phi_1}(|x|>R)} = 0. \tag{9}$$

Then, there exist a sequence of scales $(\alpha_n^{(j)})$, a sequence of cores $(x_n^{(j)})$ and a sequence of profiles $(\psi^{(j)})$ such that the triplets $(\alpha_n^{(j)}, x_n^{(j)}, \psi^{(j)})$ are pairwise orthogonal and, up to a *subsequence extraction, we have for all $\ell \geq 1$,*

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)} \left(\frac{-\log|x - x_n^{(j)}|}{\alpha_n^{(j)}} \right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{L^{\phi_1}} \stackrel{\ell\to\infty}{\longrightarrow} 0. \tag{10}$$

Moreover, we have the following stability estimates

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)\prime}\|_{L^2}^2 + \|\nabla r_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n \to \infty. \tag{11}$$

Remarks 1.6.

- *It will be useful later on to point out that for any $q \geq 2$, we have*

$$\|g_n\|_{L^q} \stackrel{n\to\infty}{\longrightarrow} 0, \tag{12}$$

where g_n is the elementary concentration defined by

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\, \psi \left(\frac{-\log|x - x_n|}{\alpha_n} \right). \tag{13}$$

Since the Lebesgue measure is invariant under translations, we have

$$\|g_n\|_{L^q}^q = (2\pi)^{-\frac{q}{2}} (\alpha_n)^{\frac{q}{2}} \int_{\mathbb{R}^2} \left| \psi\Big(-\frac{\log|x|}{\alpha_n} \Big) \right|^q dx.$$

Performing the change of variable $s = -\frac{\log|x|}{\alpha_n}$ yields

$$\|g_n\|_{L^q}^q = (2\pi)^{1-\frac{q}{2}} (\alpha_n)^{\frac{q}{2}+1} \int_0^\infty \big|\psi(s)\big|^q \mathrm{e}^{-2\alpha_n s}\, ds.$$

Fix $\varepsilon > 0$. Then in view of (6), there exist two real numbers s_0 and S_0 such that $0 < s_0 < S_0$ and

$$|\psi(s)| \leq \varepsilon\sqrt{s}, \quad \forall\, s \in [0, s_0] \cup [S_0, \infty[.$$

This implies, by the change of variable $u = \alpha_n s$, that

$$\begin{aligned}(\alpha_n)^{\frac{q}{2}+1} \int_0^{s_0} |\psi(s)|^q\, \mathrm{e}^{-2\alpha_n s}\, ds \;&\leq\; \varepsilon^q \int_0^{\alpha_n s_0} u^{\frac{q}{2}} \mathrm{e}^{-2u}\, du \\ &\leq\; C_q\, \varepsilon^q.\end{aligned}$$

In the same way, we obtain

$$(\alpha_n)^{\frac{q}{2}+1} \int_{S_0}^\infty |\psi(s)|^q\, \mathrm{e}^{-2\alpha_n s}\, ds \;\leq\; C_q\, \varepsilon^q.$$

Finally, taking advantage of the continuity of ψ, we deduce that

$$\begin{aligned}(\alpha_n)^{\frac{q}{2}+1} \int_{s_0}^{S_0} |\psi(s)|^q\, \mathrm{e}^{-2\alpha_n s}\, ds \;&\lesssim\; (\alpha_n)^{\frac{q}{2}+1} \int_{s_0}^{S_0} \mathrm{e}^{-2\alpha_n s}\, ds \\ &\lesssim\; (\alpha_n)^{\frac{q}{2}} \left(\mathrm{e}^{-2\alpha_n s_0} - \mathrm{e}^{-2\alpha_n S_0} \right) \stackrel{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

which ends the proof of the assertion (12).

- *Setting*

$$g_n^{(j)}(x) := \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\;\psi^{(j)}\left(\frac{-\log|x-x_n^{(j)}|}{\alpha_n^{(j)}}\right) \tag{14}$$

the elementary concentration involved in Decomposition (10)*, we recall that it was proved in* [5] *that*

$$\|g_n^{(j)}\|_{L^{\phi_1}} \overset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi^{(j)}(s)|}{\sqrt{s}}$$

and

$$\|\sum_{j=1}^{\ell} g_n^{(j)}\|_{L^{\phi_1}} \overset{n\to\infty}{\longrightarrow} \sup_{1\le j\le \ell}\left(\lim_{n\to\infty}\|g_n^{(j)}\|_{L^{\phi_1}}\right), \tag{15}$$

in the case when the scales $(\alpha_n^{(j)})_{1\le j\le \ell}$ *are pairwise orthogonal. Note that Property* (15) *does not necessarily remain true in the case when we have the same scales and the pairwise orthogonality of the couples* $\big((x_n^{(j)}),\psi^{(j)}\big)$ *(see Lemma* 3.6 *in* [4]*).*

1.3. **Study of the lack of compactness of Sobolev embedding in the Orlicz space in the case $p>1$.** Our first goal in this paper is to describe the lack of compactness of the Sobolev embedding (1) for $p>1$. Our result states as follows:

Theorem 1.7. *Let $p>1$ be an integer and (u_n) be a bounded sequence in $H^1(\mathbb{R}^2)$ such that*

$$u_n \rightharpoonup 0, \tag{16}$$

$$\limsup_{n\to\infty}\|u_n\|_{L^{\phi_p}} = A_0 > 0 \qquad \textit{and} \tag{17}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\|u_n\|_{L^{\phi_p}(|x|>R)} = 0. \tag{18}$$

Then, there exist a sequence of scales $(\alpha_n^{(j)})$, a sequence of cores $(x_n^{(j)})$ and a sequence of profiles $(\psi^{(j)})$ such that the triplets $(\alpha_n^{(j)}, x_n^{(j)}, \psi^{(j)})$ are pairwise orthogonal in the sense of Definition 1.3 and, up to a subsequence extraction, we have for all $\ell \ge 1$,

$$u_n(x) = \sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\;\psi^{(j)}\left(\frac{-\log|x-x_n^{(j)}|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \tag{19}$$

with $\limsup_{n\to\infty}\|\mathrm{r}_n^{(\ell)}\|_{L^{\phi_p}} \overset{\ell\to\infty}{\longrightarrow} 0$*. Moreover, we have the following stability estimates*

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell}\|\psi^{(j)\prime}\|_{L^2}^2 + \|\nabla r_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n\to\infty. \tag{20}$$

Remarks 1.8.

- *Arguing as in* [5]*, we can easily prove that*

$$\|g_n\|_{L^{\phi_p}} \overset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{4\pi}}\max_{s>0}\frac{|\psi(s)|}{\sqrt{s}}, \tag{21}$$

where

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\;\psi\left(\frac{-\log|x-x_n|}{\alpha_n}\right).$$

Indeed setting $L = \liminf_{n\to\infty} \|g_n\|_{L^{\phi_p}}$, *we have for fixed* $\varepsilon > 0$ *and* n *sufficiently large (up to subsequence extraction)*

$$\int_{\mathbb{R}^2} \Big(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{L+\varepsilon}\right|^2} - \sum_{k=0}^{p-1} \frac{|g_n(x+x_n)|^{2k}}{(L+\varepsilon)^{2k} k!}\Big)\, dx \leq \kappa.$$

Therefore,

$$\int_{\mathbb{R}^2} \Big(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{L+\varepsilon}\right|^2} - 1\Big)\, dx \lesssim \kappa + \sum_{k=1}^{p-1} \|g_n\|_{L^{2k}}^{2k}. \tag{22}$$

Since

$$\int_{\mathbb{R}^2} \Big(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{L+\varepsilon}\right|^2} - 1\Big)\, dx = 2\pi \int_0^{+\infty} \alpha_n \mathrm{e}^{2\alpha_n s\left[\frac{1}{4\pi(L+\varepsilon)^2}\left(\frac{\psi(s)}{\sqrt{s}}\right)^2 - 1\right]}\, ds - \pi,$$

we obtain in view of (12) *and* (22) *that*

$$\int_0^{+\infty} \alpha_n \mathrm{e}^{2\alpha_n s\left[\frac{1}{4\pi(L+\varepsilon)^2}\left(\frac{\psi(s)}{\sqrt{s}}\right)^2 - 1\right]}\, ds \leq C,$$

for some absolute constant C *and for* n *large enough. Using the fact that* ψ *is a continuous function, we deduce that*

$$L + \varepsilon \geq \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}},$$

which ensures that

$$L \geq \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}\cdot$$

To end the proof of (21), *it suffices to establish that for any* $\delta > 0$

$$\int_{\mathbb{R}^2} \Big(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{\lambda}\right|^2} - \sum_{k=0}^{p-1} \frac{|g_n(x+x_n)|^{2k}}{(\lambda)^{2k} k!}\Big)\, dx \overset{n\to\infty}{\longrightarrow} 0,$$

where $\lambda = \frac{1+\delta}{\sqrt{4\pi}} \max_{s>0} \dfrac{|\psi(s)|}{\sqrt{s}}\cdot$ *Since*

$$\int_{\mathbb{R}^2} \Big(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{\lambda}\right|^2} - \sum_{k=0}^{p-1} \frac{|g_n(x+x_n)|^{2k}}{(\lambda)^{2k} k!}\Big)\, dx \leq \int_{\mathbb{R}^2} \Big(\mathrm{e}^{\left|\frac{g_n(x+x_n)}{\lambda}\right|^2} - 1\Big)\, dx,$$

the result derives immediately from Proposition 1.15 *in* [5], *which achieves the proof of the result.*

- *Applying the same lines of reasoning as in the proof of Proposition 1.19 in* [4], *we obtain the following result:*

Proposition 1.9. *Let* $\big((\alpha_n^{(j)}), (x_n^{(j)}), \psi^{(j)}\big)_{1\leq j\leq \ell}$ *be a family of triplets of scales, cores and profiles such that the scales are pairwise orthogonal. Then for any integer* p *larger than* 1, *we have*

$$\Big\|\sum_{j=1}^{\ell} g_n^{(j)}\Big\|_{L^{\phi_p}} \overset{n\to\infty}{\longrightarrow} \sup_{1\leq j\leq \ell} \Big(\lim_{n\to\infty} \|g_n^{(j)}\|_{L^{\phi_p}}\Big)\ ,$$

where the functions $g_n^{(j)}$ *are defined by* (14).

As we will see in Section 2, it turns out that the heart of the matter in the proof of Theorem 1.7 is reduced to the following result concerning the radial case:

Theorem 1.10. *Let p be an integer strictly larger than 1 and (u_n) be a bounded sequence in $H^1_{rad}(\mathbb{R}^2)$ such that*

$$u_n \rightharpoonup 0 \qquad \textit{and} \tag{23}$$

$$\limsup_{n\to\infty} \|u_n\|_{L^{\phi_p}} = A_0 > 0. \tag{24}$$

Then, there exist a sequence of pairwise orthogonal scales $(\alpha_n^{(j)})$ and a sequence of profiles $(\psi^{(j)})$ such that up to a subsequence extraction, we have for all $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{L^{\phi_p}} \overset{\ell\to\infty}{\longrightarrow} 0. \tag{25}$$

Moreover, we have the following stability estimates

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)\prime}\|_{L^2}^2 + \|\nabla r_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n \to \infty.$$

Remarks 1.11.

- *Compared with the analogous result concerning the Sobolev embedding of $H^1_{rad}(\mathbb{R}^2)$into L^{ϕ_1} established in* [5]*, the hypothesis of compactness at infinity is not required. This is justified by the fact that $H^1_{rad}(\mathbb{R}^2)$ is compactly embedded in $L^q(\mathbb{R}^2)$ for any $2 < q < \infty$ which implies that*

$$\lim_{n\to\infty} \|u_n\|_{L^q(\mathbb{R}^2)} = 0, \quad \forall\, 2 < q < \infty. \tag{26}$$

- *In view of Proposition 1.9, Theorem 1.10 yields to*

$$\|u_n\|_{L^{\phi_p}} \to \sup_{j\geq 1} \left(\lim_{n\to\infty} \|g_n^{(j)}\|_{L^{\phi_p}}\right),$$

  which implies that the first profile in Decomposition (25) *can be chosen such that up to extraction*

$$A_0 := \limsup_{n\to\infty} \|u_n\|_{L^{\phi_p}} = \lim_{n\to\infty} \left\| \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \psi^{(1)}\left(-\frac{\log|x|}{\alpha_n^{(1)}}\right) \right\|_{L^{\phi_p}}. \tag{27}$$

Note that the description of the lack of compactness in other critical Sobolev embeddings was achieved in [8, 10, 14] and has been at the origin of several prospectus. Among others, one can mention [2, 6, 9, 11, 19].

1.4. **Layout of the paper.** Our paper is organized as follows: in Section 2, we establish the algorithmic construction of the decomposition stated in Theorem 1.7. Then, we study in Section 3 a nonlinear two-dimensional wave equation with the exponential nonlinearity $u\,\phi_p(\sqrt{4\pi}u)$. Firstly, we prove the global well-posedness and the scattering in the energy space both in the subcritical and critical cases, and secondly we compare the evolution of this equation with the evolution of the solutions of the free Klein-Gordon equation in the same space.
We mention that C will be used to denote a constant which may vary from line to line. We also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some absolute constant C and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. For simplicity, we shall also still denote by (u_n) any

subsequence of (u_n) and designate by $\circ(1)$ any sequence which tends to 0 as n goes to infinity.

2. Proof of Theorem 1.7

2.1. **Strategy of the proof.** The proof of Theorem 1.7 uses in a crucial way capacity arguments and is done in three steps: in the first step, we begin by the study of u_n^* the symmetric decreasing rearrangement of u_n. This led us to establish Theorem 1.10. In the second step, by a technical process developed in [4], we reduce ourselves to one scale and extract the first core $(x_n^{(1)})$ and the first profile $\psi^{(1)}$ which enables us to extract the first element $\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\psi^{(1)}\left(\frac{-\log|x-x_n^{(1)}|}{\alpha_n^{(j)}}\right)$. The third step is devoted to the study of the remainder term. If the limit of its Orlicz norm is null we stop the process. If not, we prove that this remainder term satisfies the same properties as the sequence we start with which allows us to extract a second elementary concentration concentrated around a second core $(x_n^{(2)})$. Thereafter, we establish the property of orthogonality between the first two elementary concentrations and finally we prove that this process converges.

2.2. **Proof of Theorem 1.10.** The main ingredient in the proof of Theorem 1.10 consists to extract a scale and a profile ψ such that

$$\|\psi'\|_{L^2(\mathbb{R})} \geq C A_0, \tag{28}$$

where C is a universal constant. To go to this end, let us for a bounded sequence (u_n) in $H^1_{rad}(\mathbb{R}^2)$ satisfying the assumptions (23) and (24), set $v_n(s) = u_n(\mathrm{e}^{-s})$. Combining (26) with the following well-known radial estimate:

$$|u(r)| \leq \frac{C}{r^{\frac{1}{p+1}}}\|u\|_{L^{2p}}^{\frac{p}{p+1}}\|\nabla u\|_{L^2}^{\frac{1}{p+1}}$$

where $r=|x|$, we infer that

$$\lim_{n\to\infty}\|v_n\|_{L^\infty(]-\infty,M])} = 0, \quad \forall M\in\mathbb{R}. \tag{29}$$

This gives rise to the following result:

Proposition 2.1. *For any $\delta>0$, we have*

$$\sup_{s\geq 0}\left(\Big|\frac{v_n(s)}{A_0-\delta}\Big|^2 - s\right) \to \infty, \quad n\to\infty. \tag{30}$$

Proof. We proceed by contradiction. If not, there exists $\delta>0$ such that, up to a subsequence extraction

$$\sup_{s\geq 0, n\in\mathbb{N}}\left(\Big|\frac{v_n(s)}{A_0-\delta}\Big|^2 - s\right) \leq C < \infty. \tag{31}$$

On the one hand, thanks to (29) and (31), we get by virtue of Lebesgue theorem

$$\begin{aligned}\int_{|x|<1}\left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\delta}|^2} - \sum_{k=0}^{p-1}\frac{|u_n(x)|^{2k}}{(A_0-\delta)^{2k}k!}\right)dx \;&\leq\; \int_{|x|<1}\left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\delta}|^2}-1\right)dx\\ &\leq\; 2\pi\int_0^\infty\left(\mathrm{e}^{|\frac{v_n(s)}{A_0-\delta}|^2}-1\right)\mathrm{e}^{-2s}\,ds \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

On the other hand, using Property (29) and the simple fact that for any positive real number M, there exists a finite constant $C_{M,p}$ such that

$$\sup_{|t|\leq M}\left(\frac{\mathrm{e}^{t^2}-\sum_{k=0}^{p-1}\frac{t^{2k}}{k!}}{t^{2p}}\right)<C_{M,p},$$

we deduce in view of (26) that

$$\int_{|x|\geq 1}\left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\delta}|^2}-\sum_{k=0}^{p-1}\frac{|u_n(x)|^{2k}}{(A_0-\delta)^{2k}k!}\right)dx\lesssim\|u_n\|_{L^{2p}}^{2p}\to 0\,.$$

Consequently,

$$\limsup_{n\to\infty}\|u_n\|_{L^{\phi_p}}\leq A_0-\delta,$$

which is in contradiction with Hypothesis (24). □

An immediate consequence of the previous proposition is the following corollary whose proof is identical to the proof of Corollaries 2.4 and 2.5 in [5].

Corollary 2.2. *Under the above notations, there exists a sequence $(\alpha_n^{(1)})$ in $\mathbb{R}_+$ tending to infinity such that*

$$4\left|\frac{v_n(\alpha_n^{(1)})}{A_0}\right|^2-\alpha_n^{(1)}\overset{n\to\infty}{\longrightarrow}\infty \tag{32}$$

and for n sufficiently large, there exists a positive constant C such that

$$\frac{A_0}{2}\sqrt{\alpha_n^{(1)}}\leq|v_n(\alpha_n^{(1)})|\leq C\sqrt{\alpha_n^{(1)}}+\circ(1). \tag{33}$$

Now, setting

$$\psi_n(y)=\sqrt{\frac{2\pi}{\alpha_n^{(1)}}}v_n(\alpha_n^{(1)}y),$$

we obtain along the same lines as in Lemma 2.6 in [5] the following result:

Lemma 2.3. *Under notations of Corollary 2.2, there exists a profile $\psi^{(1)}\in\mathcal{P}$ such that, up to a subsequence extraction*

$$\psi_n'\rightharpoonup(\psi^{(1)})' \text{ in } L^2(\mathbb{R}) \quad \text{and} \quad \|(\psi^{(1)})'\|_{L^2}\geq\sqrt{\frac{\pi}{2}}A_0.$$

To achieve the proof of Theorem 1.10, let us consider the remainder term

$$r_n^{(1)}(x)=u_n(x)-g_n^{(1)}(x), \tag{34}$$

where

$$g_n^{(1)}(x)=\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\psi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right).$$

By straightforward computations, we can easily prove that $(r_n^{(1)})$ is bounded in $H^1_{rad}(\mathbb{R}^2)$ and satisfies the hypothesis (23) together with the following property:

$$\lim_{n\to\infty}\|\nabla r_n^{(1)}\|^2_{L^2(\mathbb{R}^2)}=\lim_{n\to\infty}\|\nabla u_n\|^2_{L^2(\mathbb{R}^2)}-\left\|(\psi^{(1)})'\right\|^2_{L^2(\mathbb{R})}. \tag{35}$$

Let us now define $A_1 = \limsup\limits_{n\to\infty}\|r_n^{(1)}\|_{L^{\phi_p}}$. If $A_1 = 0$, we stop the process. If not, arguing as above, we prove that there exist a scale $(\alpha_n^{(2)})$ satisfying the statement of Corollary 2.2 with A_1 instead of A_0 and a profile $\psi^{(2)}$ in $\mathcal{P}$ such that

$$r_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(2)}}{2\pi}}\psi^{(2)}\left(\frac{-\log|x|}{\alpha_n^{(2)}}\right) + r_n^{(2)}(x),$$

with $\|(\psi^{(2)})'\|_{L^2} \geq \sqrt{\frac{\pi}{2}}A_1$ and

$$\lim_{n\to\infty}\|\nabla r_n^{(2)}\|^2_{L^2(\mathbb{R}^2)} = \lim_{n\to\infty}\|\nabla r_n^{(1)}\|^2_{L^2(\mathbb{R}^2)} - \left\|(\psi^{(2)})'\right\|^2_{L^2(\mathbb{R})}.$$

Moreover, as in [5] we can show that $(\alpha_n^{(1)})$ and $(\alpha_n^{(2)})$ are orthogonal. Finally, iterating the process, we get at step ℓ

$$u_n(x) = \sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x),$$

with

$$\limsup_{n\to\infty}\|r_n^{(\ell)}\|^2_{H^1} \lesssim 1 - A_0^2 - A_1^2 - \cdots - A_{\ell-1}^2\,,$$

which implies that $A_\ell \to 0$ as $\ell \to \infty$ and ends the proof of the theorem.

2.3. **Extraction of the cores and profiles.** This step is performed as the proof of Theorem 1.16 in [4]. We sketch it here briefly for the convenience of the reader. Let u_n^* be the symmetric decreasing rearrangement of u_n. Since $u_n^* \in H^1_{rad}(\mathbb{R}^2)$ and satisfies the assumptions of Theorem 1.10, we infer that there exist a sequence $(\alpha_n^{(j)})$ of pairwise orthogonal scales and a sequence of profiles $(\varphi^{(j)})$ such that, up to subsequence extraction,

$$u_n^*(x) = \sum_{j=1}^{\ell}\sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\varphi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty}\|\mathrm{r}_n^{(\ell)}\|_{L^{\phi_p}} \overset{\ell\to\infty}{\longrightarrow} 0.$$

Besides, in view of (27), we can assume that

$$A_0 = \lim_{n\to\infty}\left\|\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\varphi^{(1)}\left(-\frac{\log|x|}{\alpha_n^{(1)}}\right)\right\|_{L^{\Phi_p}}.$$

Now to extract the cores and profiles, we shall firstly reduce to the case of one scale according to Section 2.3 in [4], where a suitable truncation of u_n was introduced. Then assuming that

$$u_n^*(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\,\varphi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right),$$

we apply the strategy developed in Section 2.4 in [4] to extract the cores and the profiles. This approach is based on capacity arguments: to carry out the extraction process of mass concentrations, we prove by contradiction that if the mass responsible for the lack of compactness of the Sobolev embedding in the Orlicz space is scattered, then the energy used would exceed that of the starting sequence. This main point can be formulated on the following terms:

Lemma 2.4 (Lemma 2.5 in [4])**.** *There exist $\delta_0 > 0$ and $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$ there exists x_n such that*

$$\frac{|E_n \cap B(x_n, \mathrm{e}^{-\mathrm{b}\alpha_{\mathrm{n}}^{(1)}})|}{|E_n|} \geq \delta_0 A_0^2, \tag{36}$$

where $E_n := \{x \in \mathbb{R}^2; |u_n(x)| \geq \sqrt{2\alpha_n^{(1)}}(1 - \frac{\varepsilon_0}{10})A_0\}$ with $0 < \varepsilon_0 < \frac{1}{2}$, $B(x_n, \mathrm{e}^{-\mathrm{b}\,\alpha_{\mathrm{n}}^{(1)}})$ designates the ball of center x_n and radius $\mathrm{e}^{-\mathrm{b}\,\alpha_{\mathrm{n}}^{(1)}}$ with $b = 1 - 2\varepsilon_0$ and $|.|$ denotes the Lebesgue measure.

Once extracting the first core $(x_n^{(1)})$ making use of the previous lemma, we focus on the extraction of the first profile. For that purpose, we consider the sequence

$$\psi_n(y, \theta) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}} v_n(\alpha_n^{(1)} y, \theta),$$

where $v_n(s, \theta) = (\tau_{x_n^{(1)}} u_n)(\mathrm{e}^{-\mathrm{s}} \cos \theta, \mathrm{e}^{-\mathrm{s}} \sin \theta)$ and $(x_n^{(1)})$ satisfies

$$\frac{|E_n \cap B(x_n, \mathrm{e}^{-(1-2\varepsilon_0)\alpha_{\mathrm{n}}^{(1)}}|}{|E_n|} \geq \delta_0 A_0^2.$$

Taking advantage of the invariance of Lebesgue measure under translations, we deduce that

$$\begin{aligned} \|\nabla u_n\|_{L^2}^2 &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^{2\pi} |\partial_y \psi_n(y, \theta)|^2 dy d\theta \\ &+ \frac{\alpha_n^{(1)}}{2\pi} \int_{\mathbb{R}} \int_0^{2\pi} |\partial_\theta \psi_n(y, \theta)|^2 dy d\theta. \end{aligned}$$

Since the scale $\alpha_n^{(1)}$ tends to infinity and the sequence (u_n) is bounded in $H^1(\mathbb{R}^2)$, this implies that up to a subsequence extraction $\partial_\theta \psi_n \underset{n\to\infty}{\to} 0$ and $\partial_y \psi_n \underset{n\to\infty}{\to} g$ in $L^2(\mathbb{R} \times [0, 2\pi])$, where g only depends on the variable y. Thus introducing the function

$$\psi^{(1)}(y) = \int_0^y g(\tau) d\tau,$$

we obtain along the same lines as in Proposition 2.8 in [4] the following result:

Proposition 2.5. *The function $\psi^{(1)}$ belongs to the set of profiles $\mathcal{P}$. Besides for any $y \in \mathbb{R}$, we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \psi_n(y, \theta)\, d\theta \to \psi^{(1)}(y), \tag{37}$$

as n tends to infinity and there exists an absolute constant C such that

$$\|\psi^{(1)\prime}\|_{L^2} \geq C\, A_0. \tag{38}$$

2.4. **End of the proof.** To achieve the proof of the theorem, we argue exactly as in Section 2.5 in [4] by iterating the process exposed in the previous section. For that purpose, we set

$$r_n^{(1)}(x) = u_n(x) - g_n^{(1)}(x),$$

where

$$g_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \psi^{(1)} \left(-\frac{\log |x - x_n^{(1)}|}{\alpha_n^{(1)}} \right).$$

One can easily check that the sequence $(r_n^{(1)})$ weakly converges to 0 in $H^1(\mathbb{R}^2)$. Moreover, since $\psi^{(1)}_{|]-\infty,0]} = 0$, we have for any $R \geq 1$

$$\|r_n^{(1)}\|_{L^{\Phi_p}(|x-x_n^{(1)}|\geq R)} = \|u_n\|_{L^{\Phi_p}(|x-x_n^{(1)}|\geq R)}. \tag{39}$$

But by assumption, the sequence (u_n) is compact at infinity in the Orlicz space L^{Φ_p}. Thus the core $(x_n^{(1)})$ is bounded in $\mathbb{R}^2$, which ensures in view of (39) that $(r_n^{(1)})$ satisfies the hypothesis of compactness at infinity (18). Finally, taking advantage of the weak convergence of $(\partial_y \psi_n)$ to $\psi^{(1)'}$ in $L^2(y,\theta)$ as n goes to infinity, we get

$$\lim_{n\to\infty} \|\nabla r_n^{(1)}\|_{L^2}^2 = \lim_{n\to\infty} \|\nabla u_n^{(1)}\|_{L^2}^2 - \|\psi^{(1)'}\|_{L^2}^2.$$

Now, let us define $A_1 := \limsup_{n\to\infty} \|r_n^{(1)}\|_{L^{\Phi_p}}$. If $A_1 = 0$, we stop the process. If not, knowing that $(r_n^{(1)})$ verifies the assumptions of Theorem 1.7, we apply the above reasoning, which gives rise to the existence of a scale $(\alpha_n^{(2)})$, a core $(x_n^{(2)})$ satisfying the statement of Lemma 2.4 with A_1 instead of A_0 and a profile $\psi^{(2)}$ in $\mathcal{P}$ such that

$$r_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(2)}}{2\pi}}\, \psi^{(2)}\left(-\frac{\log|x-x_n^{(2)}|}{\alpha_n^{(2)}}\right) + r_n^{(2)}(x),$$

with $\|\psi^{(2)'}\|_{L^2} \geq C\, A_1$ and

$$\lim_{n\to\infty} \|\nabla r_n^{(2)}\|_{L^2}^2 = \lim_{n\to\infty} \|\nabla r_n^{(1)}\|_{L^2}^2 - \|\psi^{(2)'}\|_{L^2}^2.$$

Arguing as in [4], we show that the triplets $\big(\alpha_n^{(1)}, x_n^{(1)}, \psi^{(1)}\big)$ and $\big(\alpha_n^{(2)}, x_n^{(2)}, \psi^{(2)}\big)$ are orthogonal in the sense of Definition 1.3 and prove that the process of extraction of the elementary concentration converges. This ends the proof of Decomposition (10). The orthogonality equality (11) derives immediately from Proposition 2.10 in [4]. The proof of Theorem 1.7 is then achieved.

3. Nonlinear wave equation

3.1. **Statement of the results.** In this section, we investigate the initial value problem for the following nonlinear wave equation:

$$\begin{cases} \Box u + u + u\left(\mathrm{e}^{4\pi u^2} - \displaystyle\sum_{k=0}^{p-1} \frac{(4\pi)^k u^{2k}}{k!}\right) = 0, \\ \\ u(0) = u_0 \in H^1(\mathbb{R}^2), \quad \partial_t u(0) = u_1 \in L^2(\mathbb{R}^2), \end{cases} \tag{40}$$

where $p \geq 1$ is an integer, $u = u(t,x)$ is a real-valued function of $(t,x) \in \mathbb{R}\times\mathbb{R}^2$ and $\Box = \partial_t^2 - \Delta$ is the wave operator.

Let us recall that in [17, 18], the authors proved the global well-posedness for the Cauchy problem (40) when $p = 1$ and the scattering when $p = 2$ in the subcritical and critical cases (i.e when the energy is less or equal to some threshold). Note also that in [24, 25], M. Struwe constructed global smooth solutions to (40) with smooth data of arbitrary size in the case $p = 1$.

Formally, the solutions of the Cauchy problem (40) satisfy the following conservation law:

$$\begin{aligned} E_p(u,t) &:= \|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4\pi}\left\| \mathrm{e}^{4\pi u(t)^2} - 1 - \sum_{k=2}^{p} \frac{(4\pi)^k}{k!} u(t)^{2k} \right\|_{L^1} \quad (41) \\ &= E_p(u,0) := E_p^0. \end{aligned}$$

This conducts us, as in [17], to define the notion of criticality in terms of the size of the initial energy E_p^0 with respect to 1.

Definition 3.1. *The Cauchy problem* (40) *is said to be subcritical if*

$$E_p^0 < 1.$$

It is said to be critical if $E_p^0 = 1$ *and supercritical if* $E_p^0 > 1$.

We shall prove the following result:

Theorem 3.2. *Assume that* $E_p^0 \le 1$. *Then the Cauchy problem* (40) *has a unique global solution* u *in the space*

$$\mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^2)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^2)).$$

Moreover, $u \in L^4(\mathbb{R}, \mathcal{C}^{1/4})$ *and scatters.*

3.2. **Technical tools.** The proof of Theorem 3.2 is based on a priori estimates. This requires the control of the nonlinear term

$$F_p(u) := u\left(\mathrm{e}^{4\pi u^2} - \sum_{k=0}^{p-1} \frac{(4\pi)^k u^{2k}}{k!}\right) \quad (42)$$

in $L_t^1(L_x^2)$. To achieve our goal, we will resort to Strichartz estimates for the 2D Klein-Gordon equation. These estimates, proved in [15], state as follows:

Proposition 3.3. *Let* $T > 0$ *and* $(q,r) \in [4,\infty] \times [2,\infty]$ *an admissible pair, i.e*

$$\frac{1}{q} + \frac{2}{r} = 1.$$

Then,

$$\|v\|_{L^q([0,T],\mathrm{B}^1_{r,2}(\mathbb{R}^2))} \lesssim \Big[\|v(0)\|_{H^1(\mathbb{R}^2)} + \|\partial_t v(0)\|_{L^2(\mathbb{R}^2)} + \|\Box v + v\|_{L^1([0,T],L^2(\mathbb{R}^2))}\Big], \quad (43)$$

where $\mathrm{B}^1_{r,2}(\mathbb{R}^2)$ *stands for the usual inhomogeneous Besov space (see for example* [12] *or* [23] *for a detailed exposition on Besov spaces).*

Noticing that $(q,r) = (4, 8/3)$ is an admissible pair and recalling that

$$\mathrm{B}^1_{8/3,2}(\mathbb{R}^2) \hookrightarrow \mathcal{C}^{1/4}(\mathbb{R}^2),$$

we deduce that

$$\|v\|_{L^4([0,T],\mathcal{C}^{1/4}(\mathbb{R}^2))} \lesssim \Big[\|v(0)\|_{H^1(\mathbb{R}^2)} + \|\partial_t v(0)\|_{L^2(\mathbb{R}^2)} + \|\Box v + v\|_{L^1([0,T],L^2(\mathbb{R}^2))}\Big]. \quad (44)$$

To control the nonlinear term $F_p(u)$ in $L_t^1(L_x^2)$, we will make use of the following logarithmic inequalities proved in [16, Theorem 1.3].

Proposition 3.4. *For any $\lambda > \frac{2}{\pi}$ and any $0 < \mu \leq 1$, a constant $C_{\lambda,\mu} > 0$ exists such that for any function u in $H^1(\mathbb{R}^2) \cap \mathcal{C}^{1/4}(\mathbb{R}^2)$, we have*

$$\|u\|_{L^\infty}^2 \leq \lambda \|u\|_\mu^2 \log\left(C_{\lambda,\mu} + \frac{2\|u\|_{\mathcal{C}^{1/4}}}{\|u\|_\mu}\right), \tag{45}$$

where $\|u\|_\mu^2 := \|\nabla u\|_{L^2}^2 + \mu^2 \|u\|_{L^2}^2$.

3.3. **Proof of Theorem 3.2.** The proof of this result, divided into three steps, is inspired from the proofs of Theorems 1.8, 1.11, 1.12 in [17] and Theorem 1.3 in [18].

3.3.1. *Local existence.* Let us start by proving the local existence to the Cauchy problem (40). To do so, we use a standard fixed-point argument and introduce for any nonnegative time T the following space:

$$\mathcal{E}_T = \mathcal{C}([0,T], H^1(\mathbb{R}^2)) \cap \mathcal{C}^1([0,T], L^2(\mathbb{R}^2)) \cap L^4([0,T], \mathcal{C}^{1/4}(\mathbb{R}^2))$$

endowed with the norm

$$\|u\|_T = \sup_{0\leq t\leq T}\Big[\|u(t)\|_{H^1} + \|\partial_t u(t)\|_{L^2}\Big] + \|u\|_{L^4([0,T],\mathcal{C}^{1/4})}.$$

For a positive time T and a positive real number δ, we denote by $\mathcal{E}_T(\delta)$ the ball in the space $\mathcal{E}_T$ of radius δ and centered at the origin. On this ball, we define the map Φ by

$$v \longmapsto \Phi(v) = \widetilde{v},$$

where

$$\Box \widetilde{v} + \widetilde{v} = -F_p(v + v_0), \quad \widetilde{v}(0) = \partial_t \widetilde{v}(0) = 0$$

and v_0 is the solution of the free Klein-Gordon equation

$$\Box v_0 + v_0 = 0, \quad v_0(0) = u_0, \quad \textit{and} \quad \partial_t v_0(0) = u_1.$$

Now, the goal is to show that if δ and T are small enough, then the map Φ is well-defined from $\mathcal{E}_T(\delta)$ into itself and it is a contraction. To prove that Φ is well-defined, it suffices in view of the Strichartz estimates (43) to estimate $F_p(v+v_0)$ in the space $L^1([0,T], L^2(\mathbb{R}^2))$. Arguing as in [17] and using the Hölder inequality and the Sobolev embedding, we obtain for any $\epsilon > 0$

$$\begin{aligned}\int_{\mathbb{R}^2} |F_p(v+v_0)|^2\, dx &\leq \int_{\mathbb{R}^2} |F_1(v+v_0)|^2\, dx \\ &\lesssim \|v+v_0\|_{H^1}^2\, \mathrm{e}^{4\pi\|v+v_0\|_{L^\infty}^2} \left\|\mathrm{e}^{4\pi(v+v_0)^2} - 1\right\|_{L^{1+\epsilon}}.\end{aligned}$$

Note that the assumption $E_p^0 \leq 1$ implies that $\|\nabla u_0\|_{L^2} < 1$. Hence, we can choose $\mu > 0$ such that $\|u_0\|_\mu < 1$. Since v_0 is continuous in time, there exist a time T_0 and a constant $0 < c < 1$ such that for any t in $[0, T_0]$ we have

$$\|v_0(t)\|_\mu \leq c.$$

According to Proposition 3.4, we infer that

$$\mathrm{e}^{4\pi\|v+v_0\|_{L^\infty}^2} \lesssim \left(1 + \frac{\|v+v_0\|_{\mathcal{C}^{1/4}}}{\delta + c}\right)^{8\eta},$$

for some $0 < \eta < 1$. Besides, applying the Trudinger-Moser inequality (5) for $p = 1$, the fact that

$$4\pi(1+\epsilon)(\delta+c)^2 \longrightarrow 4\pi c < 4\pi \quad \text{as } \epsilon,\, \delta \to 0 \quad \text{and} \quad \left\|\nabla\left(\frac{v+v_0}{\delta+c}\right)\right\|_{L^2} \leq 1$$

ensures that

$$\begin{aligned}\left\|\mathrm{e}^{4\pi(v+v_0)^2}-1\right\|_{L^{1+\epsilon}}^{1+\epsilon} &\leq C_\epsilon\left\|\mathrm{e}^{4\pi(1+\epsilon)(v+v_0)^2}-1\right\|_{L^1}\\ &\leq C_{\epsilon,\delta}\|v+v_0\|_{L^2}^2\\ &\leq C_{\epsilon,\delta}(1+\|u_0\|_{H^1}+\|u_1\|_{L^2})^2.\end{aligned}$$

Therefore, for any $0<T\leq T_0$, we obtain that

$$\|F_p(v+v_0)\|_{L^1([0,T],L^2(\mathbb{R}^2))}\lesssim T^{1-\eta}(1+\|u_0\|_{H^1}+\|u_1\|_{L^2})^{4\eta}.$$

Now, to prove that Φ is a contraction (at least for T small), let us consider two elements v_1 and v_2 in $\mathcal{E}_T(\delta)$. Notice that, for any $\epsilon>0$,

$$\begin{aligned}|F_p(v_1+v_0)-F_p(v_2+v_0)| &= |v_1-v_2|(1+8\pi\overline{v}^2)\left(\mathrm{e}^{4\pi\overline{v}^2}-\sum_{k=0}^{p-2}\frac{(4\pi)^k\overline{v}^{2k}}{k!}\right)\\ &\leq C_\epsilon|v_1-v_2|\left(\mathrm{e}^{4\pi(1+\epsilon)\overline{v}^2}-1\right),\end{aligned}$$

where $\overline{v}=(1-\theta)(v_0+v_1)+\theta(v_0+v_2)$, for some $\theta=\theta(t,x)\in[0,1]$. Using a convexity argument, we get

$$\begin{aligned}|F_p(v_1+v_0)-F_p(v_2+v_0)| &\leq C_\epsilon\left|(v_1-v_2)\left(\mathrm{e}^{4\pi(1+\epsilon)(v_1+v_0)^2}-1\right)\right|\\ &+ C_\epsilon\left|(v_1-v_2)\left(\mathrm{e}^{4\pi(1+\epsilon)(v_2+v_0)^2}-1\right)\right|.\end{aligned}$$

This implies, in view of Strichartz estimates (44), that

$$\begin{aligned}\|\Phi(v_1)-\Phi(v_2)\|_T &\lesssim \|F_p(v_1+v_0)-F_p(v_2+v_0)\|_{L^1([0,T],L^2(\mathbb{R}^2))}\\ &\leq C_\epsilon\int_0^T\left\|(v_1-v_2)\left(\mathrm{e}^{4\pi(1+\epsilon)(v_1+v_0)^2}-1\right)\right\|_{L^2}dt\\ &+ C_\epsilon\int_0^T\left\|(v_1-v_2)\left(\mathrm{e}^{4\pi(1+\epsilon)(v_2+v_0)^2}-1\right)\right\|_{L^2}dt,\end{aligned}$$

which leads along the same lines as above to

$$\|\Phi(v_1)-\Phi(v_2)\|_T \lesssim T^{1-(1+\epsilon)\eta}(1+\|u_0\|_{H^1}+\|u_1\|_{L^2})^{4(1+\epsilon)\eta}\|v_1-v_2\|_T.$$

If the parameter ϵ is small enough, then $(1+\epsilon)\eta<1$ and therefore, for T small enough, Φ is a contraction map. This implies the uniqueness of the solution in $v_0+\mathcal{E}_T(\delta)$.
Now, we shall prove the uniqueness in the energy space. The idea here is to establish that, if $u=v_0+v$ is a solution of (40) in $\mathcal{C}([0,T],H^1(\mathbb{R}^2))\cap\mathcal{C}^1([0,T],L^2(\mathbb{R}^2))$, then necessarily $v\in\mathcal{E}_T(\delta)$ at least for T small. Starting from the fact that v satisfies

$$\Box v+v=-F_p(v+v_0),\quad v(0)=\partial_t v(0)=0,$$

we are reduced, thanks to the Strichartz estimates (43), to control the term $F_p(v+v_0)$ in the space $L^1([0,T],L^2(\mathbb{R}^2))$. But $|F_p(v+v_0)|\leq|F_1(v+v_0)|$, which leads to the result arguing exactly as in [17].

3.3.2. *Global existence.* In this section, we shall establish that our solution is global in time both in subcritical and critical cases. Firstly, let us notice that the assumption $E_p^0 \leq 1$ implies that $\|\nabla u_0\|_{L^2(\mathbb{R}^2)} < 1$, which ensures in view of Section 3.3.1 the existence of a unique maximal solution u defined on $[0, T^*)$ where $0 < T^* \leq \infty$ is the lifespan time of u. We shall proceed by contradiction assuming that $T^* < \infty$. In the subcritical case, the conservation law (41) implies that

$$\sup_{t\in(0,T^*)} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} < 1.$$

Let then $0 < s < T^*$ and consider the following Cauchy problem:

$$\Box v + v + F_p(v) = 0, \quad v(s) = u(s), \quad \text{and} \quad \partial_t v(s) = \partial_t u(s). \tag{46}$$

As in the first step of the proof, a fixed-point argument ensures the existence of $\tau > 0$ and a unique solution v to (46) on the interval $[s, s+\tau]$. Noticing that τ does not depend on s, we can choose s close to T^* such that $T^* - s < \tau$. So, we can prolong the solution u after the time T^*, which is a contradiction.

In the critical case, we cannot apply the previous argument because it is possible that the following concentration phenomenon holds:

$$\limsup_{t\to T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = 1. \tag{47}$$

In fact, we shall show that (47) cannot hold in this case. To go to this end, we argue as in the proof of Theorem 1.12 in [17]. Firstly, since the first equation of the Cauchy problem (40) is invariant under time translation, we can assume that $T^* = 0$ and that the initial time is $t = -1$. Similarly to [17, Proposition 4.2, Corollary 4.4], it follows that the maximal solution u satisfies

$$\limsup_{t\to 0^-} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)} = 1, \tag{48}$$

$$\lim_{t\to 0^-} \|u(t)\|_{L^2(\mathbb{R}^2)} = 0, \tag{49}$$

$$\lim_{t\to 0^-} \int_{|x-x^*|\leq -t} |\nabla u(t,x)|^2 \, dx = 1, \quad \text{and} \tag{50}$$

$$\forall t < 0, \quad \int_{|x-x^*|\leq -t} e_p(u)(t,x)\, dx = 1, \tag{51}$$

for some $x^* \in \mathbb{R}^2$, where $e_p(u)$ denotes the energy density defined by

$$e_p(u)(t,x) := (\partial_t u)^2 + |\nabla u|^2 + \frac{1}{4\pi}\left(\mathrm{e}^{4\pi u^2} - 1 - \sum_{k=2}^{p} \frac{(4\pi)^k u^{2k}}{k!}\right).$$

Without loss of generality, we can assume that $x^* = 0$, then multiplying the equation of the problem (40) respectively by $\partial_t u$ and u, we obtain formally

$$\partial_t e_p(u) - div_x(2\partial_t u \nabla u) = 0, \tag{52}$$

$$\partial_t(u\partial_t u) - div_x(u\nabla u) + |\nabla u|^2 - |\partial_t u|^2 + u^2 \mathrm{e}^{4\pi u^2} - \sum_{k=1}^{p-1} \frac{(4\pi)^k u^{2k+2}}{k!} = 0. \tag{53}$$

Integrating the conservation laws (52) and (53) over the backward truncated cone

$$K_S^T := \Big\{(t,x) \in \mathbb{R}\times\mathbb{R}^2 \;\text{ such that } S \leq t \leq T \text{ and } |x| \leq -t\Big\}$$

for $S < T < 0$, we get

$$\int_{B(-T)} e_p(u)(T,x)\,dx - \int_{B(-S)} e_p(u)(S,x)\,dx \tag{54}$$

$$= \frac{-1}{\sqrt{2}} \int_{M_S^T} \left[\left| \partial_t u \frac{x}{|x|} + \nabla u \right|^2 + \frac{1}{4\pi} \left(\mathrm{e}^{4\pi u^2} - 1 - \sum_{k=2}^{p} \frac{(4\pi)^k u^{2k}}{k!} \right) \,dx\,dt \right],$$

$$\int_{B(-T)} \partial_t u(T) u(T)\,dx - \int_{B(-S)} \partial_t u(S) u(S)\,dx + \frac{1}{\sqrt{2}} \int_{M_S^T} \left(\partial_t u + \nabla u . \frac{x}{|x|} \right) u\,dx\,dt \tag{55}$$

$$+ \int_{K_S^T} \left(|\nabla u|^2 - |\partial_t u|^2 + u^2 \mathrm{e}^{4\pi u^2} - \sum_{k=1}^{p-1} \frac{(4\pi)^k u^{2k+2}}{k!} \right) \,dx\,dt = 0,$$

where $B(r)$ is the ball centered at 0 and of radius r and

$$M_S^T := \Big\{ (t,x) \in \mathbb{R} \times \mathbb{R}^2 \ \text{ such that } S \leq t \leq T \text{ and } |x| = -t \Big\}.$$

According to (51) and (54), we infer that

$$\int_{M_S^T} \left[\left| \partial_t u \frac{x}{|x|} + \nabla u \right|^2 + \frac{1}{4\pi} \left(\mathrm{e}^{4\pi u^2} - 1 - \sum_{k=2}^{p} \frac{(4\pi)^k u^{2k}}{k!} \right) \right] \,dx\,dt = 0.$$

This implies, using (55) and Cauchy-Schwarz inequality, that

$$\int_{B(-T)} \partial_t u(T) u(T)\,dx - \int_{B(-S)} \partial_t u(S) u(S)\,dx \tag{56}$$

$$+ \int_{K_S^T} \left(|\nabla u|^2 - |\partial_t u|^2 + u^2 \mathrm{e}^{4\pi u^2} - \sum_{k=1}^{p-1} \frac{(4\pi)^k u^{2k+2}}{k!} \right) \,dx\,dt = 0,$$

By virtue of Identities (48) and (49) and the conservation law (41), it can be seen that

$$\partial_t u(t) \underset{t \to 0}{\longrightarrow} 0 \quad \text{in } L^2(\mathbb{R}^2), \tag{57}$$

which ensures by Cauchy-Schwarz inequality that

$$\int_{B(-T)} \partial_t u(T) u(T)\,dx \to 0. \tag{58}$$

Letting $T \to 0$ in (56), we deduce from (58) and the fact that $u^2 \mathrm{e}^{4\pi u^2} - \displaystyle\sum_{k=1}^{p-1} \frac{(4\pi)^k u^{2k+2}}{k!}$ is positive

$$- \int_{B(-S)} \partial_t u(S) u(S)\,dx \leq - \int_{K_S^0} |\nabla u|^2\,dx\,dt + \int_{K_S^0} |\partial_t u|^2\,dx\,dt. \tag{59}$$

Multiplying Inequality (59) by the positive number $-\frac{1}{S}$, we infer that

$$\int_{B(-S)} \partial_t u(S) \frac{u(S)}{S}\,dx \leq \frac{1}{S} \int_{K_S^0} |\nabla u|^2\,dx\,dt - \frac{1}{S} \int_{K_S^0} |\partial_t u|^2\,dx\,dt. \tag{60}$$

Now, Identity (57) leads to

$$\lim_{S \to 0^-} \frac{1}{S} \int_{K_S^0} |\partial_t u|^2\,dx\,dt = 0. \tag{61}$$

Moreover, using (50), it is clear that

$$\lim_{S\to 0^-}\frac{1}{S}\int_{K_S^0}|\nabla u|^2\,dx\,dt=-1. \tag{62}$$

Finally, since

$$\frac{u(S)}{S}=\frac{1}{S}\int_0^S\partial_t u(\tau)\,d\tau,$$

then $(\frac{u(S)}{S})$ is bounded in $L^2(\mathbb{R}^2)$ and hence

$$\lim_{S\to 0^-}\int_{B(-S)}\partial_t u(S)\frac{u(S)}{S}\,dx=0. \tag{63}$$

The identities (61), (62) and (63) yield a contradiction in view of (60). This achieves the proof of the global existence in the critical case.

3.3.3. *Scattering.* Our concern now is to prove that, in the subcritical and critical cases, the solution of the equation (40) approaches a solution of a free wave equation when the time goes to infinity. Using the fact that

$$|F_p(u)|\leq|F_2(u)|,\quad\forall p\geq 2, \tag{64}$$

we can apply the arguments used in [18]. More precisely, in the subcritical case the key point consists to prove that there exists an increasing function $C:[0,1[\longrightarrow[0,\infty[$ such that for any $0\leq E<1$, any global solution u of the Cauchy problem (40) with $E_p(u)\leq E$ satisfies

$$\|u\|_{X(\mathbb{R})}\leq C(E), \tag{65}$$

where $X(\mathbb{R})=L^8(\mathbb{R},L^{16}(\mathbb{R}^2))$. Now, denoting by

$$E^*:=\sup\{0\leq E<1;\ \sup_{E_p(u)\leq E}\|u\|_{X(\mathbb{R})}<\infty\},$$

and arguing as in [18, Lemma 4.1], we can show that Inequality (65) is satisfied if $E_p(u)$ is small, which implies that $E^*>0$. Now our goal is to prove that $E^*=1$. To do so, let us proceed by contradiction and assume that $E^*<1$. Then, for any $E\in]E^*,1[$ and any $n>0$, there exists a global solution u to (40) such that $E_p(u)\leq E$ and $\|u\|_{X(\mathbb{R})}>n$. By time translation, one can reduce to

$$\|u\|_{X(]0,\infty[)}>\frac{n}{2}. \tag{66}$$

Along the same lines as the proof of Proposition 5.1 in [18], we can show taking advantage of (64) that if E is close enough to E^*, then n cannot be arbitrarily large which yields a contradiction and ends the proof of the result in the subcritical case.
The proof of the scattering in the critical case is done as in Section 6 in [18] once we observed Inequality (64). It is based on the notion of concentration radius $r_\epsilon(t)$ introduced in [18].

Remark 3.5. *Lower order nonlinear terms become more difficult when we look for global decay properties of the solutions. In* [18]*, the authors avoid this problem by subtracting the cubic part from the nonlinearity $F_p(u)$ for $p=1$, which is the lower critical power for the scattering problem in $\mathbb{R}^2$.*

3.4. **Qualitative study.** In this section we shall investigate the feature of solutions of the two-dimensional nonlinear Klein-Gordon equation (40) taking into account the different regimes. As in [5], the approach that we adopt here is the one introduced by P. Gérard in [13] which consists in comparing the evolution of oscillations and concentration effects displayed by sequences of solutions of the nonlinear Klein-Gordon equation (40) and solutions of the linear Klein-Gordon equation

$$\Box v + v = 0. \tag{67}$$

More precisely, let (φ_n, ψ_n) be a sequence of data in $H^1 \times L^2$ supported in some fixed ball and satisfying

$$\varphi_n \rightharpoonup 0 \quad \text{in } H^1, \quad \psi_n \rightharpoonup 0 \quad \text{in } L^2, \tag{68}$$

such that

$$E_p^n \leq 1, \quad n \in \mathbb{N} \tag{69}$$

where E_p^n stands for the energy of (φ_n, ψ_n) given by

$$E_p^n = \|\psi_n\|_{L^2}^2 + \|\nabla\varphi_n\|_{L^2}^2 + \frac{1}{4\pi}\left\|\mathrm{e}^{4\pi\varphi_n^2} - 1 - \sum_{k=2}^{p}\frac{(4\pi)^k}{k!}\varphi_n^{2k}\right\|_{L^1},$$

and let us consider (u_n) and (v_n) the sequences of finite energy solutions of (40) and (67) such that

$$(u_n, \partial_t u_n)(0) = (v_n, \partial_t v_n)(0) = (\varphi_n, \psi_n).$$

Arguing as in [13], the notion of linearizability is defined as follows:

Definition 3.6. *Let T be a positive time. We shall say that the sequence (u_n) is linearizable on $[0,T]$, if*

$$\sup_{t\in[0,T]} E_c(u_n - v_n, t) \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

where $E_c(w,t)$ denotes the kinetic energy defined by:

$$E_c(w,t) = \int_{\mathbb{R}^2}\left[|\partial_t w|^2 + |\nabla_x w|^2 + |w|^2\right](t,x)\,dx.$$

For any time slab $I \subset \mathbb{R}$, we shall denote

$$\|v\|_{\mathrm{ST}(I)} := \sup_{(q,r)\ \text{admissible}} \|v\|_{L^q(I;\mathrm{B}^1_{r,2}(\mathbb{R}^2))}\,.$$

By interpolation argument, this Strichartz norm is equivalent to

$$\|v\|_{L^\infty(I;H^1(\mathbb{R}^2))} + \|v\|_{L^4(I;\mathrm{B}^1_{8/3,2}(\mathbb{R}^2))}\,.$$

As $\mathrm{B}^1_{r,2}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ for all $r \leq p < \infty$ (and $r \leq p \leq \infty$ if $r > 2$), it follows that

$$\|v\|_{L^q(I;L^p)} \lesssim \|v\|_{\mathrm{ST}(I)}, \quad \frac{1}{q} + \frac{2}{p} \leq 1\,. \tag{70}$$

As in [5], in the subcritical case, i.e $\limsup_{n\to\infty} E_p^n < 1$, the nonlinearity does not induce any effect on the behavior of the solutions. But, in the critical case i.e $\limsup_{n\to\infty} E_p^n = 1$, it turns out that a nonlinear effect can be produced. More precisely, we have the following result:

Theorem 3.7. *Let T be a strictly positive time. Then*

(1) *If $\limsup_{n\to\infty} E_p^n < 1$, the sequence (u_n) is linearizable on $[0,T]$.*

(2) *If* $\limsup_{n\to\infty} E_p^n = 1$*, the sequence* (u_n) *is linearizable on* $[0,T]$ *provided that the sequence* (v_n) *satisfies*

$$\limsup_{n\to\infty} \|v_n\|_{L^\infty([0,T];L^{\Phi_p})} < \frac{1}{\sqrt{4\pi}}. \tag{71}$$

Proof. The proof of Theorem 3.7 is similar to the one of Theorem 3.3 and 3.5 in [5]. Denoting by $w_n = u_n - v_n$, it is clear that w_n is the solution of the nonlinear wave equation

$$\Box w_n + w_n = -F_p(u_n)$$

with null Cauchy data.
Under energy estimate, we obtain

$$\|w_n\|_T \lesssim \|F_p(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))},$$

where $\|w_n\|_T^2 \stackrel{\text{def}}{=} \sup_{t\in[0,T]} E_c(w_n,t)$. Therefore, it suffices to prove in the subcritical and critical cases that

$$\|F_p(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))} \longrightarrow 0 \quad \text{as} \quad n\to\infty. \tag{72}$$

Let us begin by the subcritical case. Our goal is to prove that the nonlinear term does not affect the behavior of the solutions. By hypothesis, there exists some nonnegative real ρ such that $\limsup_{n\to\infty} E_p^n = 1-\rho$. The main point for the proof is based on the following lemma, the proof of which is similar to the proof of Lemma 3.16 in [5] once we observed that

$$|F_p(u)| \le |F_1(u)|, \quad \forall p \ge 1.$$

Lemma 3.8. *For every* $T>0$ *and* $E_p^0 < 1$*, there exists a constant* $C(T,E_p^0)$*, such that every solution* u *of the nonlinear Klein-Gordon equation* (40) *of energy* $E_p(u) \le E_p^0$*, satisfies*

$$\|u\|_{L^4([0,T];\mathcal{C}^{1/4})} \le C(T,E_p^0). \tag{73}$$

Now to establish (72), it suffices to prove that the sequence $(F_p(u_n))$ is bounded in $L^{1+\epsilon}([0,T],L^{2+\epsilon}(\mathbb{R}^2))$ for some nonnegative ϵ and converges to 0 in measure in $[0,T]\times\mathbb{R}^2$. This can done exactly as in [5] using the fact that $|F_p(u_n)| \le |F_1(u_n)|$.

Let us now prove (72) in the critical case. For that purpose, let $T>0$ and assume that

$$L := \limsup_{n\to\infty} \|v_n\|_{L^\infty([0,T];L^{\Phi_p})} < \frac{1}{\sqrt{4\pi}}. \tag{74}$$

Applying Taylor's formula, we obtain

$$F_p(u_n) = F_p(v_n+w_n) = F_p(v_n) + F_p'(v_n)\,w_n + \frac{1}{2}\,F_p''(v_n+\theta_n\,w_n)\,w_n^2,$$

for some $0\le\theta_n\le 1$. Strichartz estimates (43) yield

$$\|w_n\|_{\mathrm{ST}([0,T])} \lesssim I_n + J_n + K_n,$$

where

$$\begin{aligned} I_n &= \|F_p(v_n)\|_{L^1([0,T];L^2(\mathbb{R}^2))}, \\ J_n &= \|F_p'(v_n)\,w_n\|_{L^1([0,T];L^2(\mathbb{R}^2))}, \quad \text{and} \\ K_n &= \|F_p''(v_n+\theta_n\,w_n)\,w_n^2\|_{L^1([0,T];L^2(\mathbb{R}^2))}. \end{aligned}$$

As in [5], we have

$$\begin{array}{rcl} I_n & \underset{n\to\infty}{\longrightarrow} & 0 \quad \text{and} \\ J_n & \leq & \varepsilon_n \|w_n\|_{ST([0,T])}, \end{array}$$

where $\varepsilon_n \to 0$. Besides, provided that

$$\limsup_{n\to\infty} \|w_n\|_{L^\infty([0,T];H^1)} \leq \frac{1 - L\sqrt{4\pi}}{2}, \tag{75}$$

we get

$$K_n \leq \varepsilon_n \|w_n\|^2_{ST([0,T])}, \quad \varepsilon_n \to 0.$$

Since $\|w_n\|_{ST([0,T])} \lesssim I_n + \varepsilon_n \|w_n\|^2_{ST([0,T])}$, wet obtain by bootstrap argument

$$\|w_n\|_{ST([0,T])} \lesssim \varepsilon_n,$$

which ends the proof of the result. □

4. Appendix: Proof of Proposition 1.2

The proof uses in a crucial way the rearrangement of functions (for a complete presentation and more details, we refer the reader to [20]). By virtue of density arguments and the fact that for any function $f \in H^1(\mathbb{R}^2)$ and f^* the rearrangement of f, we have

$$\begin{array}{rcl} \|\nabla f\|_{L^2} & \geq & \|\nabla f^*\|_{L^2}, \\ \|f\|_{L^p} & = & \|f^*\|_{L^p}, \\ \|f\|_{L^{\phi_p}} & = & \|f^*\|_{L^{\phi_p}}, \end{array}$$

one can reduce to the case of a nonnegative radially symmetric and non-increasing function u belonging to $\mathcal{D}(\mathbb{R}^2)$. With this choice, let us introduce the function

$$w(t) = (4\pi)^{\frac{1}{2}} u(|x|), \quad \text{where} \quad |x| = \mathrm{e}^{-\frac{t}{2}}.$$

It is then obvious that the functions $w(t)$ and $w'(t)$ are nonnegative and satisfy

$$\begin{array}{rcl} \displaystyle\int_{\mathbb{R}^2} |\nabla u(x)|^2\, dx & = & \displaystyle\int_{-\infty}^{+\infty} |w'(t)|^2\, dt, \\ \displaystyle\int_{\mathbb{R}^2} |u(x)|^{2p}\, dx & = & \displaystyle\frac{1}{4^p\, \pi^{p-1}} \int_{-\infty}^{+\infty} |w(t)|^{2p}\, \mathrm{e}^{-t}\, dt \\ \displaystyle\int_{\mathbb{R}^2} \Big(\mathrm{e}^{\alpha|u(x)|^2} - \sum_{k=0}^{p-1} \frac{\alpha^k |u(x)|^{2k}}{k!}\Big) dx & = & \displaystyle\pi \int_{-\infty}^{+\infty} \Big(\mathrm{e}^{\frac{\alpha}{4\pi}|w(t)|^2} - \sum_{k=0}^{p-1} \frac{\alpha^k |w(t)|^{2k}}{(4\pi)^k k!}\Big) \mathrm{e}^{-t}\, dt. \end{array}$$

So we are reduced to prove that for all $\beta \in [0,1[$, there exists $C_\beta \geq 0$ so that

$$\int_{-\infty}^{+\infty} \Big(\mathrm{e}^{\beta|w(t)|^2} - \sum_{k=0}^{p-1} \frac{\beta^k |w(t)|^{2k}}{k!}\Big) \mathrm{e}^{-t} dt \leq C(\beta,p) \int_{-\infty}^{+\infty} |w(t)|^{2p} \mathrm{e}^{-t}\, dt, \quad \forall\, \beta \in [0,1[,$$

when $\int_{-\infty}^{+\infty} |w'(t)|^2 dt \leq 1$. For that purpose, let us set

$$T_0 = \sup\{t \in \mathbb{R},\ w(t) \leq 1\}.$$

The existence of a real number t_0 such that $w(t_0) = 0$ ensures that the set $\{t \in \mathbb{R},\ w(t) \leq 1\}$ is non empty. Then

$$T_0 \in]-\infty, +\infty].$$

Knowing that w is nonnegative and increasing function, we deduce that

$$w :]-\infty, T_0] \longrightarrow [0,1].$$

Therefore, observing that $\mathrm{e}^s - \sum_{k=0}^{p-1} \frac{s^k}{k!} \leq c_p\, s^p\, \mathrm{e}^s$ for any nonnegative real s, we obtain

$$\int_{-\infty}^{T_0} \Big(\mathrm{e}^{\beta|w(t)|^2} - \sum_{k=0}^{p-1} \frac{\beta^k |w(t)|^{2k}}{k!}\Big)\mathrm{e}^{-t} dt \leq c_p \beta^p\, \mathrm{e}^{\beta} \int_{-\infty}^{T_0} |w(t)|^{2p} \mathrm{e}^{-t} dt.$$

To estimate the integral on $[T_0, +\infty[$, let us first notice that in view of the definition of T_0, we have for all $t \geq T_0$

$$\begin{aligned} w(t) &= w(T_0) + \int_{T_0}^{t} w'(\tau) d\tau \\ &\leq w(T_0) + (t - T_0)^{\frac{1}{2}} \Big(\int_{T_0}^{+\infty} w'(\tau)^2 d\tau\Big)^{\frac{1}{2}} \\ &\leq 1 + (t - T_0)^{\frac{1}{2}}. \end{aligned}$$

Thus, using the fact that for any $\varepsilon > 0$ and any $s \geq 0$, we have

$$(1 + s^{\frac{1}{2}})^2 \leq (1+\varepsilon)s + 1 + \frac{1}{\varepsilon} = (1+\varepsilon)s + C_\varepsilon,$$

we infer that for for any $\varepsilon > 0$ and all $t \geq T_0$

$$|w(t)|^2 \leq (1+\varepsilon)(t - T_0) + C_\varepsilon. \tag{76}$$

Now β being fixed in $[0, 1[$, let us choose $\varepsilon > 0$ so that $\beta(1+\varepsilon) < 1$. Then by virtue of (76)

$$\begin{aligned} \int_{T_0}^{+\infty} \Big(\mathrm{e}^{\beta|w(t)|^2} - \sum_{k=0}^{p-1} \frac{\beta^k |w(t)|^{2k}}{k!}\Big)\mathrm{e}^{-t}\, dt &\leq \int_{T_0}^{+\infty} \mathrm{e}^{\beta|w(t)|^2} \mathrm{e}^{-t}\, dt \\ &\leq \frac{\mathrm{e}^{\beta C_\varepsilon - T_0}}{1 - \beta(1+\varepsilon)}. \end{aligned}$$

But

$$\mathrm{e}^{-T_0} = \int_{T_0}^{+\infty} \mathrm{e}^{-t}\, dt \leq \int_{T_0}^{+\infty} |w(t)|^{2p}\, \mathrm{e}^{-t}\, dt,$$

which gives rise to

$$\int_{T_0}^{+\infty} \Big(\mathrm{e}^{\beta|w(t)|^2} - \sum_{k=0}^{p-1} \frac{\beta^k |w(t)|^{2k}}{k!}\Big)\mathrm{e}^{-t} dt \leq \frac{\mathrm{e}^{\beta C_\varepsilon}}{1 - \beta(1+\varepsilon)} \int_{T_0}^{\infty} |w(t)|^{2p} \mathrm{e}^{-t}\, dt.$$

Choosing $C(\beta, p) = \max\Big(c_p \mathrm{e}^{\beta} \beta^p, \dfrac{\mathrm{e}^{\beta C_\varepsilon}}{1 - \beta(1+\varepsilon)}\Big)$ ends the proof of the proposition.

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Université de Tunis El Manar, Faculté des Sciences de Tunis, LR03ES04 Équations aux dérivées partielles et applications, 2092 Tunis, Tunisie
E-mail address: *abenyed08@gmail.com*

Université de Tunis El Manar, Faculté des Sciences de Tunis, Laboratoire équations aux dérivées partielles (LR03ES04), 2092 Tunis, Tunisie
E-mail address: *zghal-khalil@hotmail.fr*