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Empirical comparison of two methods for the Bayesian update of the parameters of probability distributions in a two-level hybrid probabilistic-possibilistic uncertainty framework for risk assessment

N. Pedroni¹, E. Zio², A. Pasanisi³, M. Couplet⁴

Abstract

In this paper, we address the issue of updating in a Bayesian framework, the possibilistic representation of the epistemically-uncertain parameters of (aleatory) probability distributions, as new information (e.g., data) becomes available. Two approaches are considered: the first is based on a purely possibilistic counterpart of the classical, well-grounded probabilistic Bayes' theorem; the second relies on the hybrid combination of (i) Fuzzy Interval Analysis (FIA) to process the uncertainty described by possibility distributions and (ii) repeated Bayesian updating of the uncertainty represented by probability distributions.

The feasibility of the two methods is shown on a literature case study involving the risk-based design of a flood protection dike.

Keywords: hierarchical two-level uncertainty, Bayesian update, possibility distributions, fuzzy interval analysis, possibilistic Bayes theorem, flood protection dike.

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21

22 **Introduction**

23 We consider a framework of uncertainty representation with two hierarchical levels [Limbourg and
24 de Rocquigny, 2010], in which risk analysis models of aleatory (i.e., random) events (e.g., failures)
25 contain parameters (e.g., probabilities, failure rates, ...) that are epistemically-uncertain, i.e., known
26 with poor precision due to lack of knowledge and information. Traditionally, both types of
27 uncertainty are represented by probability distributions [Apostolakis and Kaplan, 1981;
28 Apostolakis, 1990; NUREG-CR-6850, 2005; USNRC, 2002 and 2009; NASA, 2010] and Bayes'
29 rule is useful for updating the (probabilistic) epistemic uncertainty representation as new
30 information (e.g., data) becomes available [Bernardo and Smith, 1994; Siu and Kelly, 1998;
31 Lindley, 2000 and 2006; Bedford and Cooke, 2001; Atwood et al., 2003; Kelly and Smith, 2009 and
32 2011; Pasanisi et al., 2012].

33 However, in some situations, insufficient knowledge, information and data impair a probabilistic
34 representation of epistemic uncertainty. A number of alternative representation frameworks have
35 been proposed for such cases [Aven, 2010 and 2011; Aven and Steen, 2010; Aven and Zio, 2011;
36 Flage et al., 2009; Beer et al., 2013b and 2014b; Zhang et al., 2013], e.g., e.g., fuzzy set theory [Klir
37 and Yuan, 1995], fuzzy probabilities [Buckley, 2005; Beer, 2009b; Pannier et al., 2013], random set
38 theory [Molchanov, 2005], evidence theory [Ferson et al., 2003 and 2004; Helton et al., 2007 and
39 2008; Sentz and Ferson, 2002; Le Duy et al., 2013; Sallak et al., 2013], possibility theory (that can
40 be considered a special case of evidence theory) [Baudrit and Dubois, 2006; Baudrit et al., 2006 and
41 2008; Dubois, 2006; Dubois and Prade, 1988], probability bound analysis using probability boxes
42 (p-boxes) [Ferson and Ginzburg, 1996; Crespo et al., 2013; Mehl, 2013], interval analysis [Ferson
43 and Hajagos, 2004; Ferson and Tucker, 2006; Ferson et al., 2007 and 2010; Jalal-Kamali and
44 Kreinovich, 2013; Muscolino and Sofi, 2013; Zhang et al., 2013] and interval probabilities
45 [Weichselberger, 2000]; notice that most of these theories can be included within the general
46 common framework of *imprecise probabilities* [Kuznetsov, 1991; Walley, 1991; Kozine and

47 Filimonov, 2000; Kozine and Utkin, 2002; Coolen and Utkin, 2007; Beer and Ferson, 2013; Beer et
 48 al., 2013a; Blockley, 2013; Reid, 2013; Sankararaman and Mahadevan, 2013].
 49 In this paper, we adopt possibility distributions to describe epistemic uncertainty and address the
 50 issue of updating, in a Bayesian framework, the possibilistic representation of the epistemically-
 51 uncertain parameters of (aleatory) probability distributions. We take two approaches of literature.
 52 The first is based on a purely possibilistic counterpart of the classical, well-grounded probabilistic
 53 Bayes' theorem: it requires the construction of a possibilistic likelihood function which is used to
 54 revise the prior possibility distributions of the uncertain parameters (determined on the basis of a
 55 priori subjective knowledge and/or data) [Dubois and Prade, 1997; Lapointe and Bobee, 2000]. This
 56 approach has been already applied by the authors for updating possibility distributions in [Pedroni
 57 et al., 2014]. The second is a hybrid probabilistic-possibilistic method that relies on the use of
 58 Fuzzy Probability Density Functions (FPDFs), i.e., PDFs with possibilistic (fuzzy) parameters. It is
 59 based on the combination of (i) Fuzzy Interval Analysis (FIA) to process the uncertainty described
 60 by possibility distributions and (ii) repeated Bayesian updating of the uncertainty represented by
 61 probability distributions [Beer, 2009a; Stein and Beer, 2011; Stein et al., 2013; Beer et al., 2014a].
 62 The objective (and the main contribution of the paper) is to compare the effectiveness of the two
 63 methods. To the best of the authors' knowledge, this is the first time that the above mentioned
 64 techniques are systematically compared with reference to risk assessment problems where hybrid
 65 uncertainty is separated into two hierarchical levels. To keep the analysis simple and retain a clear
 66 view of each step, the investigations are carried out with respect to a simple literature case study
 67 involving the risk-based design of a flood protection dike [Pasanisi et al., 2009; Limbourg and de
 68 Rocquigny, 2010]. In addition, different numerical indicators (e.g., cumulative distributions,
 69 exceedance probabilities, percentiles, ...) are considered to perform a fair, quantitative comparison
 70 between the methods and evaluate their rationale and appropriateness in relation to risk analysis.
 71 Other methods have been proposed in the literature to revise, in a Bayesian framework, non-
 72 probabilistic representations of epistemic uncertainty [Ferson, 2005]. In [Viertl, 1996, 1997, 1999,

2008a,b and 2011; Viertl and Hareter, 2004a,b; Viertl and Hule, 1991] a modification of the Bayes' theorem is presented to account for the presence of fuzzy data and fuzzy prior PDFs: the approach is similar to that employed by [Beer, 2009a; Stein and Beer, 2011; Stein et al., 2013] and considered in this paper. In [Smets, 1993] a Generalized Bayes Theorem (GBT) is proposed within the framework of evidence theory: this approach is applied by [Le-Duy et al., 2011] to update the estimates of the failure rates of mechanical components in the context of nuclear Probabilistic Risk Assessment (PRA). Finally, in [Walley, 1996; Bernard, 2005; Masegosa and Moral, 2014] Imprecise Dirichlet Models (IDMs) are proposed for objective statistical inference from multinomial data. In the IDM, prior or posterior uncertainty about a parameter is described by a set of Dirichlet distributions, and inferences about events are summarized by lower and upper probabilities. This model has been extended by [Quaeghebeur and de Cooman, 2005] to generalized Bayesian inference from canonical exponential families and by [Walter and Augustin, 2009] with the aim of handling prior-data conflicts.

The remainder of the paper is organized as follows. First, the representation of aleatory (probabilistic) and epistemic (possibilistic) uncertainties in a “two-level” framework is provided; then, the two methods employed in this paper for the Bayesian update of the possibilistic parameters of aleatory probability distributions are described in details; after that, the case study concerning the risk-based design of a flood protection dike is presented; in the following Section, the methods described are applied to the case study: the results obtained are discussed and the two methods are synthetically compared; finally, some conclusions are drawn in the last Section.

Representation of aleatory and epistemic uncertainties in a two-level framework: fuzzy random variables

In all generality, we consider an uncertain variable Y , whose uncertainty is described by the Probability Distribution Function (PDF) $p^Y(y|\theta)$, where $\theta = \{\theta_1, \theta_2, \dots, \theta_m, \dots, \theta_p\}$ is the vector of the corresponding internal parameters. In a two-level framework, the parameters θ are themselves

98 affected by epistemic uncertainty [Limbourg and de Rocquigny, 2010]. In the present work, we
 99 describe these uncertainties by the (generally joint) possibility distribution $\pi^\theta(\theta)$ (it is
 100 straightforward to notice that in case the internal parameters $\{\theta_1, \theta_2, \dots, \theta_m, \dots, \theta_p\}$ are independent,
 101 then $\pi^\theta(\theta)$ is simply represented by a group of P separate, marginal and independent (i.e., non-
 102 interactive) possibility distributions, i.e., $\pi^\theta(\theta) = \{\pi^{\theta_1}(\theta_1), \pi^{\theta_2}(\theta_2), \dots, \pi^{\theta_m}(\theta_m), \dots, \pi^{\theta_p}(\theta_p)\}$). A
 103 random variable Y with possibilistic parameters θ is a particular case of a Fuzzy Random Variable
 104 (FRV), i.e., of a random variable whose values are not real, but rather fuzzy numbers [Féron, 1976;
 105 Kwakernaak, 1978; Puri and Ralescu, 1986; Baudrit et al., 2008; Couso and Sanchez, 2008]. The
 106 corresponding Fuzzy Probability Distribution Function (FPDF) is here indicated as $\tilde{p}^Y(y|\theta)$.
 107 For clarification by way of example, we may consider the generic uncertain variable Y described
 108 by a Gumbel PDF, i.e., $Y \sim p^Y(y|\theta) = \text{Gum}(\theta) = \text{Gum}(\theta_1, \theta_2) = \text{Gum}(\gamma, \delta) = p^Y(y|\gamma, \delta)$.
 109 Parameter $\delta = \theta_2$ (i.e., the scale paramter) is a fixed point-wise value ($\delta = \theta_2 = 100$), whereas
 110 parameter $\gamma = \theta_1$ (i.e., the location parameter) is epistemically-uncertain. By hypothesis, the only
 111 information available on $\gamma = \theta_1$ is that it is defined on interval $[a_\gamma, b_\gamma] = [900, 1300]$ and its most
 112 likely value is $c_\gamma = 1100$. Notice that such information is *not sufficient* for assigning a *single specific*
 113 probability distribution to describe the epistemic uncertainty in parameter $\gamma = \theta_1$. In facts, such
 114 scarce information is actually *compatible* with a *variety* of probability distributions (e.g., truncated
 115 normal, lognormal, triangular, ...). To address this issue, this limited state of knowledge about
 116 $\gamma = \theta_1$ is here described by a triangular possibility distribution $\pi^\gamma(\gamma)$ with core $c_\gamma = 1100$ and
 117 support $[a_\gamma, b_\gamma] = [900, 1300]$ (Fig. 1, top left column) [Baudrit and Dubois, 2006]. Indeed, this
 118 representation is *coherent* with the *information available*, as it can be demonstrated that such
 119 possibility distribution “encodes” the *family* of *all* the probability distributions with mode $c_\gamma = 1100$
 120 and support $[a_\gamma, b_\gamma] = [900, 1300]$ (obviously, this does *not* mean that the triangular possibility
 121 distribution is the *only one* with these characteristics, i.e., the only one able to encode such a

122 probability family). In other words, *one single* possibility distribution generates in practice a
 123 “*bundle*” of probability distributions with mode $c_\gamma = 1100$ and support $[a_\gamma, b_\gamma] = [900, 1300]$. The
 124 reader is referred to [Baudrit and Dubois, 2006; Couso et al., 2001; Dubois et al., 2004] for further
 125 technical details and a formal proof of these statements.

126 In order to provide an additional *practical* interpretation of the possibility distribution $\pi^\gamma(\gamma)$ of
 127 $\gamma = \theta_1$, we can define its α -cut sets $A_\alpha^\gamma = \{ \gamma : \pi^\gamma(\gamma) \geq \alpha \}$, with $0 \leq \alpha \leq 1$. For example, $A_{0.5}^\gamma =$
 128 $[1000, 1200]$ is the set of γ values for which the possibility function is greater than or equal to 0.5
 129 (dashed segment in Fig. 1, top left column). Notice that the α -cut set A_α^γ of parameter γ can be
 130 interpreted also as the $(1 - \alpha) \cdot 100\%$ Confidence Interval (CI) for γ , i.e., the interval such that
 131 $P[\gamma \in A_\alpha^\gamma] \geq 1 - \alpha$. For example, $A_0^\gamma = [900, 1300]$ is the $(1 - 0) \cdot 100\% = 100\%$ CI for γ , i.e., the
 132 interval that contains the “true” value of γ with certainty (solid segment in Fig. 1, top left column);
 133 $A_{0.5}^\gamma = [1000, 1200]$ ($\subset A_0^\gamma$) is the $(1 - 0.5) \cdot 100\% = 50\%$ CI (dashed segment in Fig. 1, top left
 134 column); $A_{0.8}^\gamma = [1050, 1150]$ ($\subset A_{0.5}^\gamma \subset A_0^\gamma$) is the $(1 - 0.8) \cdot 100\% = 20\%$ CI, and so on. In this
 135 view, the possibility distribution $\pi^\gamma(\gamma)$ can be interpreted as a set of *nested* CIs for parameter γ
 136 [Baudrit and Dubois, 2006; Couso et al., 2001; Dubois et al., 2004].

137 For each possibility (resp., confidence) level α (resp., $1 - \alpha$) in $[0, 1]$, a *family* of PDFs for Y ,
 138 namely $\{p^Y(y|\gamma, \delta)\}_\alpha$, can be generated by letting the epistemically-uncertain parameter γ range
 139 within the corresponding α -cut set A_α^γ , i.e., $\{p^Y(y|\gamma, \delta)\}_\alpha = \{p^Y(y|\gamma, \delta) : \gamma \in A_\alpha^\gamma, \delta = 100\}$. By way
 140 of example, Fig. 1, top right column, shows four PDFs belonging to the family $\{p^Y(y|\gamma, \delta)\}_{\alpha=0}$
 141 (solid lines) and four PDFs belonging to the family $\{p^Y(y|\gamma, \delta)\}_{\alpha=0.5}$ (dashed lines).

142 In the same way, a *bundle* of Cumulative Distribution Functions (CDFs) for Y , namely
 143 $\{F^Y(y|\gamma, \delta)\}_\alpha$, can be constructed by letting γ range within A_α^γ , i.e., $\{F^Y(y|\gamma, \delta)\}_\alpha =$
 144 $\{F^Y(y|\gamma, \delta) : \gamma \in A_\alpha^\gamma, \sigma = 100\}$. This *family* of CDFs (of level α) is bounded above and below by the

145 upper and lower CDFs, $\bar{F}_\alpha^Y(y)$ and $\underline{F}_\alpha^Y(y)$, defined as $\bar{F}_\alpha^Y(y) = \sup_{\gamma \in A_\alpha^Y} \{F^Y(y | \gamma, \delta = 100)\}$ and
 146 $\underline{F}_\alpha^Y(y) = \inf_{\gamma \in A_\alpha^Y} \{F^Y(y | \gamma, \delta = 100)\}$, respectively. Since $\pi^Y(\gamma)$ can be interpreted as a set of nested CIs
 147 for parameter γ (see above), it can be argued that the α -cuts of $\pi^Y(\gamma)$ induce also a *set of nested*
 148 *pairs* of CDFs $\{(F_\alpha^Y(y), \bar{F}_\alpha^Y(y)) : 0 \leq \alpha \leq 1\}$ which bound the “true” CDF $F^Y(y)$ of Y with confidence
 149 larger than or equal to $(1 - \alpha)$, i.e., $P[\underline{F}_\alpha^Y(y) \leq F^Y(y) \leq \bar{F}_\alpha^Y(y)] \geq 1 - \alpha$, with $0 \leq \alpha \leq 1$ [Baudrit et
 150 al., 2007 and 2008]. In passing, notice that the upper and lower CDFs (of level α), $\bar{F}_\alpha^Y(y)$ and
 151 $\underline{F}_\alpha^Y(y)$, can be referred to as the *plausibility* and *belief* functions (of level α) of the set $Z = (-\infty, y]$,
 152 i.e., $\bar{F}_\alpha^Y(y) = Pl_\alpha^Y(Z)$ and $\underline{F}_\alpha^Y(y) = Bel_\alpha^Y(Z)$, respectively. For illustration purposes, Fig. 1, bottom
 153 row, shows the bounding upper and lower CDFs of Y , $\bar{F}_\alpha^Y(y) = Pl_\alpha^Y(Z)$ and $\underline{F}_\alpha^Y(y) = Bel_\alpha^Y(Z)$, built
 154 in correspondence of the α -cuts of level $\alpha = 0$ (solid lines), 0.5 (dashed lines) and 1 (dot-dashed
 155 line) of the possibility distribution $\pi^Y(\gamma)$ of parameter γ (Fig. 1, top left column).
 156 Finally, the *set of nested pairs* of CDFs $\{(F_\alpha^Y(y), \bar{F}_\alpha^Y(y)) : 0 \leq \alpha \leq 1\} = \{(Bel_\alpha^Y(Z), Pl_\alpha^Y(Z)) : 0 \leq \alpha \leq 1\}$,
 157 $Z = (-\infty, y]$, can be synthesized into a *single pair* of plausibility and belief functions as
 158 $Pl^Y(Z) = \int_0^1 Pl_\alpha^Y(Z) d\alpha$ and $Bel^Y(Z) = \int_0^1 Bel_\alpha^Y(Z) d\alpha$, respectively (dotted lines in Fig. 1, bottom
 159 row): in other words, $Pl^Y(Z)$ and $Bel^Y(Z)$ are obtained by *averaging* the different nested
 160 plausibility and belief functions (i.e., $\{(Bel_\alpha^Y(Z), Pl_\alpha^Y(Z)) : 0 \leq \alpha \leq 1\}$) generated at different
 161 possibility levels $\alpha \in [0, 1]$ (i.e., by averaging the different contributions to the plausibility and
 162 belief functions produced by different α -cuts of the epistemic parameter γ). The plausibility and
 163 belief functions $Pl^Y(Z)$ and $Bel^Y(Z)$, $Z = (-\infty, y]$, are shown to represent the “*best bounds*” for the
 164 “true” CDF $F^Y(y)$ of the uncertain variable Y [Ralescu, 2002; Baudrit et al., 2007 and 2008; Couso
 165 et al., 2004; Couso and Dubois, 2009; Couso and Sanchez, 2011].

166 Further details about FRVs are not given here for the sake of brevity: the interested reader is
167 referred to the cited references.

168

169

Fig. 1

170 **Bayesian update of the possibilistic parameters of aleatory probability** 171 **distributions**

172 In this Section, we present the methods employed in this study for updating, in a Bayesian
173 framework, the possibilistic representation of the epistemically-uncertain parameters of (aleatory)
174 probability distributions, as *new* information/evidence (e.g., data) becomes available. In this view,
175 let $\pi^\theta(\theta)$ be the (joint) *prior* possibility distribution for the parameters $\theta = [\theta_1, \theta_2, \dots, \theta_m, \dots, \theta_p]$ of
176 the PDF $p^Y(y|\theta)$ of variable Y (built on the basis of a priori subjective engineering knowledge
177 and/or data). For example, in the risk assessment context of this paper Y may represent the yearly
178 maximal water flow of a river described by a Gumbel distribution: thus, $Y \sim p^Y(y|\theta) = Gum(\theta) =$
179 $Gum(\theta_1, \theta_2) = Gum(\gamma, \delta) = p^Y(y|\gamma, \delta)$ and $\pi^\theta(\theta) = \pi^{\gamma, \delta}(\gamma, \delta)$. Moreover, let
180 $y = [y_1, y_2, \dots, y_k, \dots, y_D]$ be a vector of D *observed* pieces of data representing the *new*
181 information/evidence *available* for the analysis: referring to the example above, y may represent a
182 vector of D values collected over a long period time (e.g., many years) of the yearly maximal water
183 flow of the river under analysis. The objective of the Bayesian analysis is to update the a priori
184 representation $\pi^\theta(\theta) = \pi^{\gamma, \delta}(\gamma, \delta)$ of $\theta = [\gamma, \delta]$ on the basis of the new evidence acquired, i.e., to
185 calculate the *posterior* possibility distribution $\pi^\theta(\theta|y)$ (i.e., $\pi^{\gamma, \delta}(\gamma, \delta|y)$) of θ after y is
186 obtained.

187 In the present paper, two methods are considered to this aim: the purely possibilistic method and the
188 hybrid probabilistic and possibilistic approach.

189 ***Purely possibilistic approach***

190 The purely possibilistic method (hereafter also referred to as ‘Approach A’ for brevity) is based on
 191 a purely possibilistic counterpart of the classical, probabilistic Bayes’ theorem [Dubois and Prade,
 192 1997; Lapointe and Bobée, 2000]:

$$193 \quad \pi^\theta(\theta | y) = \frac{\pi_L^\theta(\theta | y) \cdot \pi^\theta(\theta)}{\sup_{\theta} \{\pi_L^\theta(\theta | y) \cdot \pi^\theta(\theta)\}}, \quad (1)$$

194 where $\pi_L^\theta(\theta | y)$ is the possibilistic likelihood of the parameter vector θ given the newly observed
 195 data y , and quantities $\pi^\theta(\theta | y)$ and $\pi^\theta(\theta)$ are defined above. Notice that $\sup_{\theta} \{\pi_L^\theta(\theta | y) \cdot \pi^\theta(\theta)\}$ is a
 196 normalization factor such that $\sup_{\theta} \{\pi^\theta(\theta | y)\} = 1$, as required by possibility theory [Dubois, 2006].

197 It is worth mentioning that forms of the possibilistic Bayes’ theorem alternative to (1) can be
 198 constructed as a result of other definitions of the operation of ‘conditioning’ with possibility
 199 distributions: the reader is referred to [Dubois and Prade, 1997; Lapointe and Bobée, 2000] for
 200 technical details. In this paper, expression (1) has been chosen because “it satisfies desirable
 201 properties of the revision process and lead to continuous posterior distributions” [Lapointe and
 202 Bobée, 2000].

203 The possibilistic likelihood $\pi_L^\theta(\theta | y)$ is here obtained by *transforming* the classical probabilistic
 204 likelihood function $L^\theta(\theta | y)$ through *normalization*, i.e., $\pi_L^\theta(\theta | y) = \frac{L^\theta(\theta | y)}{\sup_{\theta} \{L^\theta(\theta | y)\}}$ [Anoop et al.,

205 2006] (obviously, $L^\theta(\theta | y) = \prod_{k=1}^D p^Y(y_k | \theta)$ in the case the observations $\{y_k: k = 1, 2, \dots, D\}$ are

206 independent and identically distributed). This choice has been made for the following main reasons:

- 207 i. the transformation is simple and can be straightforwardly applied to any distribution [Anoop
 208 et al., 2006];
- 209 ii. the resulting possibilistic likelihood is *very closely related* to the classical, purely
 210 probabilistic one (which is theoretically well-grounded) by means of the simple and direct

operation of normalization that preserves the “*original structure*” of the experimental evidence;

- iii. it can be easily verified that the resulting possibilistic likelihood keeps the *sequential* nature of the updating procedure typical of the standard Bayes’ theorem;
- iv. the operation of likelihood normalization finds also *theoretical* justifications in some recent works of literature (see the brief discussion below) [Denoeux, 2014; Moral, 2014].

However, two considerations are in order with respect to this choice. First, it has to be admitted that the resulting possibility distributions do not in general adhere to the probability-possibility consistency principle [Dubois and Prade, 1980]. Second, it has to be remembered that the probabilistic likelihood function $L^\theta(\theta|y)$ is *not* a probability distribution: in this view, from a rigorous mathematical viewpoint, speaking of probability/possibility transformation for it would be wrong. On the other side, from the practical engineering viewpoint of interest to the present paper, an operation of normalization can be performed (i.e., $L^\theta(\theta|y) / \int_{\theta} L^\theta(\theta|y) d\theta$) in order to “technically” provide it with the “properties” of a probability distribution function.

It is worth noting that other techniques of transformation of probability density functions into possibility distributions exist, but the corresponding details are not given here for brevity sake: the interested reader is referred to [Dubois et al., 1993, 2004 and 2008; Flage et al., 2010 and 2013] for some proposed techniques, e.g., the principle of maximum specificity [Dubois et al., 1993] and the principle of minimal commitment [Dubois et al., 2008]. Also, it has to be noticed that techniques are also available to construct possibility distributions (and, thus, possibilistic likelihood functions) *directly* from rough experimental *data* (i.e., without resorting to probability-possibility transformations): see [Masson and Denoeux, 2006; Mauris, 2008; Hou and Yang, 2010; Serrurier and Prade, 2011] for more details. Finally, for a thorough *theoretical* justification of a “possibilistic vision” of the likelihood the reader is referred to: e.g., [Dubois et al., 1997], where possibility measures are considered as the *supremum* of a *family* of likelihood functions; [Denoeux, 2014],

236 where the evidence about a parameter (after observing a piece of data) is represented by a
 237 *consonant* “likelihood based” belief function, whose contour function equals the *normalized*
 238 *likelihood function* (see above): in the paper, this is also rigorously derived from three basic
 239 principles, i.e., the likelihood principle [Edwards, 1992], compatibility with Bayes’ rule and the
 240 minimal commitment principle [Smets, 1993]; and finally [Moral, 2014], where the approach by
 241 [Denoeux, 2014] is discussed and the issue of representing likelihood information is taken from the
 242 point of view of imprecise probabilities.

243 ***Hybrid probabilistic and possibilistic approach***

244 The hybrid probabilistic and possibilistic method (hereafter also called ‘Approach B’ for brevity) is
 245 based on the construction of a Fuzzy Probability Distribution Function (FPDF) to be used as a
 246 ‘fictitious’ prior for the epistemically-uncertain parameters θ of the PDF $p^Y(y|\theta)$ of the uncertain
 247 variable Y : in other words, a fictitious (artificial) *probabilistic* function has to be ‘superimposed’
 248 onto the *purely possibilistic* prior $\pi^\theta(\theta)$ that has to be updated. In more detail, let $\tilde{p}^\theta(\theta|\varphi)$ be the
 249 fictitious (prior) FPDF of θ , constructed by the superimposition of an (arbitrarily selected)
 250 fictitious PDF $p^\theta(\theta|\varphi)$ and a vector of parameters φ described by a (properly selected) possibility
 251 distribution $\pi^\varphi(\varphi)$ (it is straightforward to notice that in the case parameters $\{\theta_m: m = 1, 2, \dots, P\}$
 252 are independent, then $p^\theta(\theta|\varphi) = \prod_{m=1}^P p^{\theta_m}(\theta_m|\varphi_m)$, with $\varphi_m = [\phi_{m1}, \phi_{m2}, \dots, \phi_{mN_m}]$, $m = 1, 2, \dots,$
 253 P ; also, $\pi^\varphi(\varphi)$ is expressed as $\pi^\varphi(\varphi) = \{\pi^{\varphi_1}(\varphi_1), \pi^{\varphi_2}(\varphi_2), \dots, \pi^{\varphi_m}(\varphi_m), \dots, \pi^{\varphi_P}(\varphi_P)\}$. In addition,
 254 if also the possibilistic parameters $\phi_{m1}, \phi_{m2}, \dots, \phi_{mN_m}$ of the fictitious PDF $p^{\theta_m}(\theta_m|\varphi_m)$, $m = 1, 2,$
 255 $\dots, P$, are independent, then $\pi^{\varphi_m}(\varphi_m) = \{\pi^{\varphi_{m1}}(\phi_{m1}), \pi^{\varphi_{m2}}(\phi_{m2}), \dots, \pi^{\varphi_{mN_m}}(\phi_{mN_m})\}$). As before, let $Y \sim$
 256 $p^Y(y|\theta) = Gum(\gamma, \delta) = p^Y(y|\gamma, \delta)$, with $\theta = [\theta_1, \theta_2] = [\gamma, \delta]$ epistemically-uncertain. For
 257 simplicity, we consider $\theta_1 = \gamma$ and $\theta_2 = \delta$ independent, such that $\pi^\theta(\theta) = \{\pi^\gamma(\gamma), \pi^\delta(\delta)\}$ and that

258 the (fictitious) PDF $p^\theta(\theta|\varphi)$ can be written as $\prod_{m=1}^P p^{\theta_m}(\theta_m|\varphi_m) = p^{\theta_1}(\theta_1|\varphi_1) \cdot p^{\theta_2}(\theta_2|\varphi_2) =$
 259 $p^\gamma(\gamma|\varphi_\gamma) \cdot p^\delta(\delta|\varphi_\delta)$. By hypothesis, the analyst *arbitrarily* selects Normal distributions to
 260 represent $p^\gamma(\gamma|\varphi_\gamma)$ and $p^\delta(\delta|\varphi_\delta)$, i.e., $p^\gamma(\gamma|\varphi_\gamma) = N(\varphi_\gamma) = N(\mu_\gamma, \sigma_\gamma)$ and $p^\delta(\delta|\varphi_\delta) =$
 261 $N(\varphi_\delta) = N(\mu_\delta, \sigma_\delta)$, respectively (Normal and Uniform distributions are indicated as good choices
 262 for $p^\theta(\theta|\varphi)$ by [Beer, 2009a; Stein and Beer, 2011; Stein et al., 2013]). Then, parameters
 263 $\varphi_\gamma = [\mu_\gamma, \sigma_\gamma]$ and $\varphi_\delta = [\mu_\delta, \sigma_\delta]$ are represented by the possibility distributions
 264 $\pi^{\varphi_\gamma} = \{\pi^{\mu_\gamma}(\mu_\gamma), \pi^{\sigma_\gamma}(\sigma_\gamma)\}$ and $\pi^{\varphi_\delta} = \{\pi^{\mu_\delta}(\mu_\delta), \pi^{\sigma_\delta}(\sigma_\delta)\}$ (by so doing, the fictitious prior FPDF
 265 $\tilde{p}^\theta(\theta|\varphi) = \tilde{p}^{\theta_1}(\theta_1|\varphi_1) \cdot \tilde{p}^{\theta_2}(\theta_2|\varphi_2) = \tilde{p}^\gamma(\gamma|\varphi_\gamma) \cdot \tilde{p}^\delta(\delta|\varphi_\delta)$ is constructed). These possibility
 266 distributions should be *properly* selected by the analyst so as to reflect as closely as possible the
 267 structure of the ‘*real*’ prior possibility distribution $\pi^\theta(\theta) = \{\pi^\gamma(\gamma), \pi^\delta(\delta)\}$ that has to be updated:
 268 for example, π^{φ_γ} and π^{φ_δ} could be identified by ‘imposing’ that the *expected value* of the FPDF
 269 $\tilde{p}^\theta(\theta|\varphi)$ corresponds to the *real* prior possibility distribution $\pi^\theta(\theta)$, i.e., $E_\varphi[\tilde{p}^\theta(\theta|\varphi)] = \pi^\theta(\theta)$
 270 or, in this case, $E_{\varphi_\gamma}[\tilde{p}^\gamma(\gamma|\varphi_\gamma)] = \pi^\gamma(\gamma)$ and $E_{\varphi_\delta}[\tilde{p}^\delta(\delta|\varphi_\delta)] = \pi^\delta(\delta)$.
 271 In extreme synthesis, the method relies on the hybrid combination of (i) Fuzzy Interval Analysis
 272 (FIA) to process the uncertainty described by possibility distributions and (ii) *repeated* Bayesian
 273 updating of the uncertainty represented by probability distributions [Beer, 2009a; Stein and Beer,
 274 2011; Stein et al., 2013]. In more details, the algorithm proceeds as follows:
 275 1. set $\alpha = 0$;
 276 2. select the α -cut A_α^φ of the possibility distribution $\pi^\varphi(\varphi)$ of vector φ of the parameters of the
 277 (fictitious) prior PDF $p^\theta(\theta|\varphi)$;
 278 3. letting the parameter vector φ range within the corresponding α -cut A_α^φ identified at step 2.
 279 above, generate a family of (fictitious) prior PDFs $\{p^\theta(\theta|\varphi)\}_\alpha = \{p^\theta(\theta|\varphi) : \varphi \in A_\alpha^\varphi\}$. This is

empirically done by (i) randomly or deterministically selecting a *finite* number T (e.g., $T = 100$ in this paper) of parameter vectors $\boldsymbol{\varphi}_{l,\alpha}$, $l = 1, 2, \dots, T$, in A_α^φ and (ii) evaluating $p^\theta(\boldsymbol{\theta} | \boldsymbol{\varphi})$ in correspondence of these vectors, i.e., $\{p^\theta(\boldsymbol{\theta} | \boldsymbol{\varphi})\}_\alpha \approx \{p^\theta(\boldsymbol{\theta} | \boldsymbol{\varphi}_{l,\alpha}) : l = 1, 2, \dots, T\}$;

4. apply the classical probabilistic Bayes theorem to *each* (fictitious) prior PDF $p^\theta(\boldsymbol{\theta} | \boldsymbol{\varphi}_{l,\alpha})$ generated at step 3. above to get the corresponding posterior PDF $p_{l,\alpha}^\theta(\boldsymbol{\theta} | \mathbf{y})$:

$$p_{l,\alpha}^\theta(\boldsymbol{\theta} | \mathbf{y}) = \frac{L^\theta(\boldsymbol{\theta} | \mathbf{y}) \cdot p(\boldsymbol{\theta} | \boldsymbol{\varphi}_{l,\alpha})}{\int_{\boldsymbol{\theta}} L^\theta(\boldsymbol{\theta} | \mathbf{y}) \cdot p(\boldsymbol{\theta} | \boldsymbol{\varphi}_{l,\alpha}) d\boldsymbol{\theta}}, \quad l = 1, 2, \dots, T \quad (2)$$

This is equivalent to generating a family $\{p^\theta(\boldsymbol{\theta} | \mathbf{y})\}_\alpha$ of posterior PDFs for $\boldsymbol{\theta}$, i.e., $\{p^\theta(\boldsymbol{\theta} | \mathbf{y})\}_\alpha \approx \{p_{l,\alpha}^\theta(\boldsymbol{\theta} | \mathbf{y}) : l = 1, 2, \dots, T\}$;

5. calculate the *expected value* of *each* posterior PDF $p_{l,\alpha}^\theta(\boldsymbol{\theta} | \mathbf{y})$ generated at step 4. above to obtain a *point estimate* $\boldsymbol{\theta}_{l,\alpha} | \mathbf{y}$ for the epistemically-uncertain parameter vector $\boldsymbol{\theta}$ [Stein et al., 2013]:

$$\boldsymbol{\theta}_{l,\alpha} | \mathbf{y} = E_\theta[p_{l,\alpha}^\theta(\boldsymbol{\theta} | \mathbf{y})], \quad l = 1, 2, \dots, T \quad (3)$$

6. take the hull enveloping the T point estimates $\boldsymbol{\theta}_{l,\alpha} | \mathbf{y}$, $l = 1, 2, \dots, T$, as the (P -dimensional) α -cut $A_\alpha^{\theta|\mathbf{y}}$ of the (joint P -dimensional) posterior possibility distribution $\pi^\theta(\boldsymbol{\theta} | \mathbf{y})$ of $\boldsymbol{\theta}$. By way of example and for illustration purposes, Fig. 2 shows the identification of a (two-dimensional) α -cut $A_\alpha^{\theta|\mathbf{y}} = A_\alpha^{\theta_1, \theta_2 | \mathbf{y}}$ (solid line) as the contour enclosing $T = 20$ point estimates $\boldsymbol{\theta}_{l,\alpha} | \mathbf{y}$, $l = 1, 2, \dots, 20$ (dots), in the simplified case of $P = 2$ parameters θ_1 and θ_2 ;
7. if $\alpha < 1$, then set $\alpha = \alpha + \Delta\alpha$ (e.g., $\Delta\alpha = 0.05$ in this paper) and return to step 2. above; otherwise, stop the algorithm.

299 The (joint P -dimensional) posterior possibility distribution $\pi^\theta(\boldsymbol{\theta} | \mathbf{y})$ for the epistemically-uncertain
 300 parameter vector $\boldsymbol{\theta}$ is empirically constructed as the (discrete) collection of the α -cuts $A_\alpha^{\theta|y}$, $\alpha = 0,$
 301 $0.05, \dots, 0.95, 1$, found at step 6. above.
 302
 303 It is worth noting that the application of both approaches A and B always produces a *joint* P -
 304 dimensional posterior possibility distribution $\pi^\theta(\boldsymbol{\theta} | \mathbf{y})$ (*whatever* the state of dependence between
 305 the priors), characterized by P -dimensional α -cuts $A_\alpha^{\theta|y}$, with $0 < \alpha < 1$: as a consequence, there is
 306 an *interactive dependence* between the values that parameters $\{\theta_m: m = 1, 2, \dots, P\}$ can take when
 307 ranging within a given α -cut $A_\alpha^{\theta|y}$: for example, in Fig. 2 it is impossible that parameter θ_1 takes on
 308 low values *and* parameter θ_2 takes on high values at the same time.
 309 From $\pi^\theta(\boldsymbol{\theta} | \mathbf{y})$ it is straightforward to obtain the *marginal* posterior possibility distribution
 310 $\pi^{\theta_m}(\theta_m | \mathbf{y})$ for each parameter θ_m as $\pi^{\theta_m}(\theta_m | \mathbf{y}) = \max_{\theta_j \in \mathfrak{R}, j \neq m} \{\pi^\theta(\boldsymbol{\theta} | \mathbf{y})\}$, $\forall \theta_m \in \mathfrak{R}$, $m = 1, 2, \dots, P$
 311 [Baudrit et al., 2006]: $\pi^{\theta_m}(\theta_m | \mathbf{y})$ is the projection of $\pi^\theta(\boldsymbol{\theta} | \mathbf{y})$ onto the m -th axis. The (one-
 312 dimensional) α -cut $A_\alpha^{\theta_m|y} = [\underline{\theta}_{m,\alpha} | y, \bar{\theta}_{m,\alpha} | y]$ of the marginal possibility distribution $\pi^{\theta_m}(\theta_m | \mathbf{y})$ is
 313 then related to the (P -dimensional) α -cut $A_\alpha^{\theta|y}$ of the joint possibility distribution $\pi^\theta(\boldsymbol{\theta} | \mathbf{y})$ by the
 314 following straightforward relation, i.e., $A_\alpha^{\theta_m|y} = [\underline{\theta}_{m,\alpha} | y, \bar{\theta}_{m,\alpha} | y] = [\min_{\theta \in A_\alpha^{\theta|y}} \{\theta_m\}, \max_{\theta \in A_\alpha^{\theta|y}} \{\theta_m\}]$. In this
 315 view, notice that the use of the P -dimensional α -cut $A_{\alpha, Cart}^{\theta|y}$ constructed by the Cartesian product of
 316 the (one-dimensional) α -cuts $A_\alpha^{\theta_m|y}$ of the marginal distributions, $m = 1, 2, \dots, P$ (i.e., $A_{\alpha, Cart}^{\theta|y} = A_\alpha^{\theta_1|y}$
 317 $\times A_\alpha^{\theta_2|y} \times \dots \times A_\alpha^{\theta_m|y} \times \dots \times A_\alpha^{\theta_P|y}$) would (incorrectly) imply *independence* between the posterior
 318 estimates of the parameters $\{\theta_m: m = 1, 2, \dots, P\}$; however, since $A_{\alpha, Cart}^{\theta|y}$ completely *contains* $A_\alpha^{\theta|y}$
 319 (i.e., by definition $A_\alpha^{\theta|y} \subset A_{\alpha, Cart}^{\theta|y}$), then conservatism would be still guaranteed [Stein et al., 2013].
 320 For illustration purposes and with reference to the example above, Fig. 2 shows also the two-

321 dimensional α -cut $A_{\alpha, Cart}^{\theta_1, \theta_2 | y}$ (dashed line) generated by the Cartesian product of the (one-dimensional)
 322 α -cuts $A_{\alpha}^{\theta_1 | y}$ and $A_{\alpha}^{\theta_2 | y}$, i.e., $A_{\alpha, Cart}^{\theta_1, \theta_2 | y} = A_{\alpha}^{\theta_1 | y} \times A_{\alpha}^{\theta_2 | y}$.

323

324

325

Fig. 2

326 The characteristics of the two approaches are summarized in Table 1. Notice that both methods
 327 relies on *arbitrary assumptions* about either the prior or the likelihood functions: in the purely
 328 possibilistic approach (A) the “original” possibilistic prior is employed, but a possibilistic
 329 likelihood function has to be constructed (e.g., by probability-possibility transformations, directly
 330 from rough experimental data and/or by resorting to the guidelines provided by [Dubois et al., 1997;
 331 Denoeux, 2014]); instead, in the hybrid method (B) the original probabilistic likelihood function is
 332 used, but a “fictitious” prior Fuzzy Probability Distribution Function needs to be identified by
 333 superimposing an *arbitrarily selected* probabilistic PDF onto the “original” possibilistic prior that
 334 has to be updated.

335

336

Table 1.

337 A final consideration is in order with respect to the two approaches here outlined. In the hybrid
 338 probabilistic-possibilistic framework of interest to the present paper, the knowledge a priori
 339 available on the parameters $\theta = [\theta_1, \theta_2, \dots, \theta_m, \dots, \theta_p]$ of a given (aleatory) probability model
 340 $p^Y(y|\theta)$ is described by the *prior* possibility distribution function $\pi^\theta(\theta)$. As detailed in the
 341 previous Section, the possibilistic approach is particularly suitable to address those situations where
 342 the information a priori available on θ is scarce and imprecise, i.e., *not sufficient* for assigning a
 343 *single specific* probability distribution to (describe the epistemic uncertainty in) θ . Actually, the
 344 *possibilistic* function $\pi^\theta(\theta)$ is in practice “equivalent” to the *family* of *all* those *probability*
 345 *distributions* (of possibly *different shapes*) that are *coherent* with the scarce information available
 346 on θ . On the other hand, it is worth mentioning that in a classical *purely probabilistic* framework,
 347 imprecision in prior information about θ can be also accounted for by means of the so-called

348 *Hierarchical Bayes* approach. Hierarchical Bayes is so-named because it utilizes *hierarchical* or
 349 *multistage* prior distributions [Gelman, 2006; Gelman et al., 2008; Congdon, 2010; Kelly and
 350 Smith, 2011; Chung et al., 2015; Shirley and Gelman, 2015]. To develop a hierarchical model for
 351 θ , we need to specify a *first-stage prior* (say, $p^\theta(\theta|\varphi)$), which is often of a *particular functional*
 352 *form*, often a conjugate prior. However, analysts find it difficult to express their incertitude
 353 numerically at all, much less as particular probability distributions. Thus, a higher-dimensional
 354 model is defined to represent such (epistemic) uncertainty: in particular, we need to specify an
 355 “additional” prior distribution (say, $p^\varphi(\varphi|\omega)$) on the first-stage parameters φ . Distribution
 356 $p^\varphi(\varphi|\omega)$ is called the *second-stage prior*, or *hyper-prior*. This way of proceeding amounts to
 357 generating a ‘parametric’ family $\{p^\theta(\theta|\varphi)\}$ of (first-stage prior) probability distributions for θ , all
 358 obtained in correspondence of different possible values of the first-stage parameters φ (described by
 359 hyper-prior $p^\varphi(\varphi|\omega)$). This method has been investigated in the field of social and behavioral
 360 sciences with the *main* aim of treating hierarchical data with different levels of variables in the same
 361 statistical model. For example, the hierarchical data for sociological survey analysis include
 362 measurements from individuals with different historical, geographic, or economic variables. To this
 363 end, the hierarchical modeling was proposed to account for the different grouping or times at which
 364 data are measured [Gill, 2002]. In addition, a common application of hierarchical Bayes analysis in
 365 the Probabilistic Risk Assessment (PRA) of nuclear power plants has been as a model of variability
 366 among data sources, for example variability in Emergency Diesel Generator (EDG) performance
 367 across different plants, or across time [Siu and Kelly, 1998; Atwood et al., 2003; Kelly and Smith,
 368 2009]. Finally, similar analogy can be made for the collected measurements from a structure under
 369 different ambient and environmental conditions. This framework has been recently implemented for
 370 uncertainty quantification applications in structural dynamics [Behmanesh et al., 2015; Ballesteros
 371 et al., 2014]. Although hierarchical Bayes can address the issue of imprecise prior information by
 372 means of multi-level models, the following conceptual and practical considerations should be made

about its applicability: (i) in principle, one could define even higher dimensional models to represent the uncertainty. Some analysts have attempted three- and even four-level models [Jaworska and Aldenberg, 2000], but this hardly seems a workable solution when the complexity of the analysis is the primary problem; (ii) in addition, “if concern about the uncertainty in parameters can in principle drive the analysis to a higher level, one could fall into an insoluble infinite cascade” [Ferson, 2005]; (iii) finally, both first- and second-stage priors, $p^\theta(\theta|\varphi)$ and $p^\varphi(\varphi|\omega)$, respectively, are often of a *particular* functional form, which forces the analyst to make “overly-optimistic” and excessively precise statements about parameter uncertainties, even when the information and data available are scarce and/or vague, i.e., *not sufficient* for assigning a *single specific* probability distribution. In such cases, non-probabilistic methods (e.g., intervals, possibility distributions or Dempster-Shafer structures from evidence theory) have been shown to provide reliable and robust results [Ferson et al., 2003; Beer et al., 2013a]. On the basis of considerations (i)-(iii) above and given that the objective of the paper is the comparison of methods for the update of *possibilistic* parameters (of aleatory probability distributions), Hierarchical Bayes approaches are not considered in the present work.

Case study: flood protection risk-based design

The case study deals with the design of a protection dike in a residential area closely located to a river with potential risk of floods. Two issues of concern are: (i) high construction and annual maintenance costs of the dike; (ii) uncertainty in the natural phenomenon of flooding. Then, the different design options must be evaluated within a flooding risk analysis framework accounting for uncertainty.

The model

The maximal water level of the river (i.e., the output variable of the model, Z_c) is given as a function of several (and some uncertain) parameters (i.e., the inputs to the model) [Limbourg and de Rocquigny, 2010]:

$$Z_c = f(Q, Z_m, Z_v, K_s, B, L) = f(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6) \quad (4)$$

where: $Y_1 = Q$ is the yearly maximal water discharge [m^3/s]; $Y_2 = Z_m$ and $Y_3 = Z_v$ are the riverbed levels [m asl] at the upstream and downstream parts of the river under investigation, respectively; $Y_4 = K_s$ is the Strickler friction coefficient; $Y_5 = B$ and $Y_6 = L$ are the width and length of the river part [m], respectively. Quantities $Y_5 = B$ ($= 300\text{m}$) and $Y_6 = L$ ($= 5000\text{m}$) are *constant* parameters, whereas quantities $Y_1 = Q$, $Y_2 = Z_m$, $Y_3 = Z_v$, $Y_4 = K_s$ are *uncertain* variables.

The input variables: physical description and representation of the associated uncertainty

The $n = 4$ input variables Y_i , $i = 1, 2, 3, 4$, are affected by aleatory and epistemic uncertainties. The aleatory part of the uncertainty is described by probability distributions of defined shape. The parameters of the probability distributions describing the aleatory uncertainty are themselves affected by epistemic uncertainty and represented in terms of possibility distributions.

The yearly maximal water flow, $Y_1 = Q$

The aleatory uncertainty in the yearly maximal water flow $Y_1 = Q$ is well described by a Gumbel probability distribution $p^Q(q|\gamma, \delta) = \text{Gum}(\gamma, \delta) = \frac{1}{\delta} \exp\left[-\exp\left(\frac{\gamma-q}{\delta}\right)\right] \exp\left[\frac{\gamma-q}{\delta}\right]$ [Limbourg and de Rocquigny, 2010]. The extreme physical bounds on variable Q are [Limbourg and de Rocquigny, 2010]:

- $Q_{\min} = 10 \text{ m}^3/\text{s}$, which is a typical drought flow level (irrelevant within a flood study);
- $Q_{\max} = 10000 \text{ m}^3/\text{s}$, which is three times larger than the maximal flood ever occurred.

When Approach A is used, the prior possibility distributions $\pi^\gamma(\gamma)$ and $\pi^\delta(\delta)$ for the epistemically-uncertain parameters γ and δ are subjectively chosen as triangular functions $TR(a_\gamma, c_\gamma, b_\gamma)$ and $TR(a_\delta, c_\delta, b_\delta)$, respectively, with cores (i.e., preferred or most likely values) $c_\gamma = 955 \text{ m}^3/\text{s}$ and $c_\delta = 600 \text{ m}^3/\text{s}$, and supports $[a_\gamma, b_\gamma] = [869, 1157] \text{ m}^3/\text{s}$ and $[a_\delta, b_\delta] = [455, 660] \text{ m}^3/\text{s}$,

421 respectively. When Approach B is employed, the fictitious prior FPDFs $\tilde{p}^\gamma(\gamma)$ and $\tilde{p}^\delta(\delta)$ are
 422 subjectively chosen as normal probability distributions $p^\gamma(\gamma) = N(\mu_\gamma, \sigma_\gamma)$ and $p^\delta(\delta) =$
 423 $N(\mu_\delta, \sigma_\delta)$, respectively, with possibilistic parameters $\mu_\gamma, \sigma_\gamma, \mu_\delta$ and σ_δ : the characteristics of the
 424 corresponding triangular possibility distributions $\pi^{\mu_\gamma}(\mu_\gamma)$, $\pi^{\sigma_\gamma}(\sigma_\gamma)$, $\pi^{\mu_\delta}(\mu_\delta)$ and $\pi^{\sigma_\delta}(\sigma_\delta)$ are
 425 summarized in Table 2 for brevity. The Bayesian update of these uncertainty representations (based
 426 on prior subjective knowledge) is realized with the aid of a vector $\mathbf{y}_1 = [y_{11}, y_{12}, \dots, y_{1k}, \dots, y_{1D_1}]$ of
 427 $D_1 = 149$ (independent and identically distributed – iid) values of the annual maximal flow of the
 428 river, i.e., $\mathbf{y}_1 = \mathbf{q} = [q_1, q_2, \dots, q_k, \dots, q_{149}]$. The point estimates for γ and δ obtained by the
 429 classical, purely probabilistic Maximum Likelihood Estimation (MLE) method are $\hat{\gamma}^{MLE} = 1013.21$
 430 m^3/s and $\hat{\delta}^{MLE} = 558.21 \text{ m}^3/\text{s}$, respectively.

431 **The upstream riverbed level, $Y_2 = Z_m$**

432 The aleatory part of the uncertainty in the upstream riverbed level $Y_2 = Z_m$ is represented by a
 433 normal distribution, i.e., $Z_m \sim p^{Z_m}(z_m | \mu_{Z_m}, \sigma_{Z_m}) = N(\mu_{Z_m}, \sigma_{Z_m})$ [Limbourg and de Rocquigny,
 434 2010]. This distribution is truncated at the minimum and maximum physical bounds on Z_m , i.e.,
 435 $Z_{m,\min} = 53.5 \text{ m}$ (given by plausible lower geomorphologic limits to erosion) and $Z_{m,\max} = 57 \text{ m}$
 436 (given by plausible upper geomorphologic limits to sedimentation), respectively. In Approach A,
 437 the prior possibility distributions $\pi^{\mu_{Z_m}}(\mu_{Z_m})$ and $\pi^{\sigma_{Z_m}}(\sigma_{Z_m})$ for μ_{Z_m} and σ_{Z_m} are chosen as
 438 triangular functions (Table 2). In Approach B, the fictitious prior FPDFs $\tilde{p}^{\mu_{Z_m}}(\mu_{Z_m})$ and $\tilde{p}^{\sigma_{Z_m}}(\sigma_{Z_m})$
 439 are chosen as normal PDFs $p^{\mu_{Z_m}}(\mu_{Z_m}) = N(\mu_{\mu_{Z_m}}, \sigma_{\mu_{Z_m}})$ and $p^{\sigma_{Z_m}}(\sigma_{Z_m}) = N(\mu_{\sigma_{Z_m}}, \sigma_{\sigma_{Z_m}})$ with
 440 parameters described by the triangular possibilistic functions $\pi^{\mu_{\mu_{Z_m}}}(\mu_{\mu_{Z_m}})$, $\pi^{\sigma_{\mu_{Z_m}}}(\sigma_{\mu_{Z_m}})$,
 441 $\pi^{\mu_{\sigma_{Z_m}}}(\mu_{\sigma_{Z_m}})$ and $\pi^{\sigma_{\sigma_{Z_m}}}(\sigma_{\sigma_{Z_m}})$ reported in Table 2. The Bayesian update of these uncertainty
 442 representations is carried out using a vector $\mathbf{y}_2 = [y_{21}, y_{22}, \dots, y_{2k}, \dots, y_{2D_2}]$ of $D_2 = 29$ (iid) values

of the upstream riverbed level, i.e., $y_2 = z_m = [z_{m,1}, z_{m,2}, \dots, z_{m,k}, \dots, z_{m,29}]$. The point estimates $\hat{\mu}_{Z_m}^{MLE}$ and $\hat{\sigma}_{Z_m}^{MLE}$ for μ_{Z_m} and σ_{Z_m} obtained by the MLE method are $\hat{\mu}_{Z_m}^{MLE} = 50.19$ m and $\hat{\sigma}_{Z_m}^{MLE} = 0.38$ m, respectively.

446 **The downstream riverbed level, $Y_3 = Z_v$**

447 As for $Y_2 = Z_m$, the aleatory part of the uncertainty in the downstream riverbed level $Y_3 = Z_v$ is
 448 represented by a normal distribution, i.e., $Z_v \sim p^{Z_v}(z_v | \mu_{Z_v}, \sigma_{Z_v}) = N(\mu_{Z_v}, \sigma_{Z_v})$, truncated at
 449 $Z_{v,\min} = 48$ m and $Z_{v,\max} = 51$ m. As before, the prior possibility distributions $\pi^{\mu_{Z_v}}(\mu_{Z_v})$ and
 450 $\pi^{\sigma_{Z_v}}(\sigma_{Z_v})$ used in Approach A are triangular functions (Table 2) and the FPDFs $\tilde{p}^{\mu_{Z_v}}(\mu_{Z_v})$ and
 451 $\tilde{p}^{\sigma_{Z_v}}(\sigma_{Z_v})$ employed in Approach B are normal PDFs $p^{\mu_{Z_v}}(\mu_{Z_v}) = N(\mu_{\mu_{Z_m}}, \sigma_{\mu_{Z_m}})$ and $p^{\sigma_{Z_v}}(\sigma_{Z_v}) =$
 452 $N(\mu_{\sigma_{Z_v}}, \sigma_{\sigma_{Z_v}})$ with parameters described by the triangular possibility distributions $\pi^{\mu_{Z_v}}(\mu_{Z_v})$,
 453 $\pi^{\sigma_{Z_v}}(\sigma_{Z_v})$, $\pi^{\mu_{\sigma_{Z_v}}}(\mu_{\sigma_{Z_v}})$ and $\pi^{\sigma_{\sigma_{Z_v}}}(\sigma_{\sigma_{Z_v}})$ of Table 2. These representations are updated by means of
 454 a vector $y_3 = [y_{31}, y_{32}, \dots, y_{3k}, \dots, y_{3D_3}]$ of $D_3 = 29$ (iid) values of the downstream riverbed level,
 455 i.e., $y_3 = z_v = [z_{v,1}, z_{v,2}, \dots, z_{v,k}, \dots, z_{v,29}]$. The MLE estimates of the parameters are $\hat{\mu}_{Z_v}^{MLE} = 55.03$ m
 456 and $\hat{\sigma}_{Z_v}^{MLE} = 0.45$ m, respectively.

457 **The Strickler friction coefficient, $Y_4 = K_s$**

458 The Strickler friction coefficient $Y_4 = K_s$ is the most critical source of uncertainty because it is
 459 usually a simplification of a complex hydraulic model. The absolute physical limits of K_s are 5 and
 460 60, respectively [Limbourg and de Rocquigny, 2010]:

- 461 - $K_s < 5$ corresponds to an “extremely sinuous shape of the canal, with large dents and strong
 462 vegetation”;
- 463 - $K_s = 60$ corresponds to a “canal with smoothest earth surface, rectilinear, without any
 464 vegetation”.

465 The friction coefficient K_s is affected by random events modifying the river status (e.g., erosion,
 466 sedimentation, ...): the corresponding variability is typically described by a normal distribution, i.e.,
 467 $K_s \sim p^{K_s}(k_s | \mu_{K_s}, \sigma_{K_s}) = N(\mu_{K_s}, \sigma_{K_s})$ [Limbourg and de Rocquigny, 2010]. However, the
 468 parameters of this normal distribution are difficult to estimate because data can only be obtained
 469 through “indirect calibration characterized by significant uncertainty” [Limbourg and de
 470 Rocquigny, 2010]: the uncertainty in these parameters is described by possibility distributions.
 471 Details about the possibilistic functions $\pi^{\mu_{K_s}}(\mu_{K_s})$ and $\pi^{\sigma_{K_s}}(\sigma_{K_s})$ and the FPDFs $\tilde{p}^{\mu_{K_s}}(\mu_{K_s})$ and
 472 $\tilde{p}^{\sigma_{K_s}}(\sigma_{K_s})$ used to represent the a priori knowledge on μ_{K_s} and σ_{K_s} in Approaches A and B,
 473 respectively, are reported in Table 2. The Bayesian revision of these a priori representations is
 474 performed by means of a vector $\mathbf{y}_4 = [y_{41}, y_{42}, \dots, y_{4k}, \dots, y_{4D_4}]$ of $D_4 = 5$ (iid) values of the
 475 Strickler friction coefficient, i.e., $\mathbf{y}_4 = \mathbf{k}_s = [k_{s,1}, k_{s,2}, \dots, k_{s,k}, \dots, k_{s,5}]$. The MLE estimates of the
 476 parameters are $\hat{\mu}_{K_s}^{MLE} = 27.8$ and $\hat{\sigma}_{K_s}^{MLE} = 5.26$, respectively.

477
 478 **Table 2.**
 479

480 Results

481 In order to simplify the notation, in what follows let θ be one of the uncertain parameters of the
 482 PDFs of $Y_1 = Q$, $Y_2 = Z_m$, $Y_3 = Z_v$ and $Y_4 = K_s$, i.e., $\theta = \gamma, \delta, \mu_{Z_m}, \sigma_{Z_m}, \mu_{Z_v}, \sigma_{Z_v}, \mu_{K_s}$ or σ_{K_s} . Fig.
 483 3 illustrates the possibility distributions of the epistemically-uncertain parameters of the aleatory
 484 PDFs $p^Q(q | \gamma, \delta)$ (top row), $p^{Z_m}(z_m | \mu_{Z_m}, \sigma_{Z_m})$ (middle-top row), $p^{Z_v}(z_v | \mu_{Z_v}, \sigma_{Z_v})$ (middle-bottom
 485 row) and $p^{K_s}(k_s | \mu_{K_s}, \sigma_{K_s})$ (bottom row) of the uncertain input variables $Y_1 = Q$, $Y_2 = Z_m$, $Y_3 = Z_v$
 486 and $Y_4 = K_s$, respectively, of the model of the previous Section: in particular, the prior possibility
 487 distributions $\pi^\theta(\theta)$ ($= \pi^\gamma(\gamma), \pi^\delta(\delta), \pi^{\mu_{Z_m}}(\mu_{Z_m}), \pi^{\sigma_{Z_m}}(\sigma_{Z_m}), \pi^{\mu_{Z_v}}(\mu_{Z_v}), \pi^{\sigma_{Z_v}}(\sigma_{Z_v}), \pi^{\mu_{K_s}}(\mu_{K_s})$
 488 and $\pi^{\sigma_{K_s}}(\sigma_{K_s})$) are shown as solid lines, whereas the *marginal* posterior possibility distributions

489 $\pi^\theta(\theta | \mathbf{y})$ ($= \pi^\gamma(\gamma | \mathbf{q}), \pi^\delta(\delta | \mathbf{q}), \pi^{\mu_{Zm}}(\mu_{Zm} | \mathbf{z}_m), \pi^{\sigma_{Zm}}(\sigma_{Zm} | \mathbf{z}_m), \pi^{\mu_{Zv}}(\mu_{Zv} | \mathbf{z}_v), \pi^{\sigma_{Zv}}(\sigma_{Zv} | \mathbf{z}_v),$
490 $\pi^{\mu_{K_s}}(\mu_{K_s} | \mathbf{k}_s)$ and $\pi^{\sigma_{K_s}}(\sigma_{K_s} | \mathbf{k}_s)$) obtained by Approaches A and B using $D_1 = 149, D_2 = 29, D_3 =$
491 29 and $D_4 = 5$ pieces of data are shown in dashed and dot-dashed lines, respectively; the point
492 estimates $\hat{\theta}^{MLE}$ ($= \hat{\gamma}^{MLE}, \hat{\delta}^{MLE}, \hat{\mu}_{Zm}^{MLE}, \hat{\sigma}_{Zm}^{MLE}, \hat{\mu}_{Zv}^{MLE}, \hat{\sigma}_{Zv}^{MLE}, \hat{\mu}_{K_s}^{MLE}$ and $\hat{\sigma}_{K_s}^{MLE}$) produced by the
493 classical MLE method are also shown for comparison (dots).

494
495 **Fig. 3.**
496

497 From a mere visual and qualitative inspection of Fig. 3 it can be seen that both approaches are
498 suitable for revising the prior possibility distributions (based on a priori purely subjective
499 knowledge) by means of empirical data. In particular, it is evident that: (i) the most likely (i.e.,
500 preferred) values c_θ of the epistemically-uncertain parameters (i.e., those values in correspondence
501 of which the possibility function equals 1) are moved towards the MLE estimates $\hat{\theta}^{MLE}$ in all the
502 cases considered; (ii) the area S_θ underlying the corresponding possibility distributions is
503 significantly reduced: noting that this area is related to the imprecision in the knowledge of the
504 possibilistic parameter (i.e., the larger the area, the higher the imprecision), it can be concluded that
505 both approaches succeed in reducing the epistemic uncertainty. With respect to that, Table 3 reports
506 the most likely values c_θ and the areas S_θ underlying the (marginal) possibility distributions of the
507 uncertain parameter θ ($= \gamma, \delta, \mu_{Zm}, \sigma_{Zm}, \mu_{Zv}, \sigma_{Zv}, \mu_{K_s}$ and σ_{K_s}) before and after the Bayesian
508 update performed by Approaches A and B; the point estimates $\hat{\theta}^{MLE}$ obtained by the classical MLE
509 method are also reported for completeness. In addition, in order to perform a quantitative
510 comparison between the methods, two indicators are defined:

- 511 i. the relative absolute distance d_θ^{MLE} between the (posterior) most likely value c_θ of parameter
512 θ and the corresponding MLE estimate $\hat{\theta}^{MLE}$, i.e.:

$$d_{\theta}^{MLE} = \frac{|c_{\theta} - \hat{\theta}^{MLE}|}{\hat{\theta}^{MLE}}. \quad (5)$$

Obviously, the lower is d_{θ}^{MLE} the closer is the most likely value c_{θ} to the MLE estimate $\hat{\theta}^{MLE}$, i.e., the higher is the *strength* of the approach in updating the prior possibilistic distribution on the basis of newly available experimental evidence;

- ii. the percentage relative difference R_{θ} between the areas underlying the possibility distribution of parameter θ before and after the Bayesian update, namely S_{θ}^{Prior} and $S_{\theta}^{Posterior}$, respectively:

$$R_{\theta} = \frac{S_{\theta}^{Prior} - S_{\theta}^{Posterior}}{S_{\theta}^{Prior}} \cdot 100. \quad (6)$$

In this case, the higher is R_{θ} , the higher is the reduction in the area (i.e., in the epistemic uncertainty) and, thus, the higher is the “updating strength” of the approach.

Table 3.

It is evident that the *strength* of Approach A in moving the most likely values c_{θ} towards the corresponding MLE estimates $\hat{\theta}^{MLE}$ is *always higher* than that of Approach B. Actually, the values of d_{θ}^{MLE} (5) produced by Approach A are 1.58-5.24 times *lower* than those generated by Approach B for all the parameters.

From the analysis of quantitative indicator R_{θ} (6) it can be seen that both methods succeed in reducing the area underlying the possibility distributions of the uncertain parameters: in particular, the percentage reduction R_{θ} ranges between 8.33% and 30.49% for approach A and between 3.89% and 33.01% for Approach B. In addition, as expected, the strength of *both* approaches in reducing epistemic uncertainty *decreases* with the *size* of the data set used to perform the Bayesian update. For example, the area S_{γ} underlying the possibility distribution of γ (Fig. 3, top left column) is reduced by 30.49–33.01% with the aid of a *large* data set of size $D_1 = 149$; on the contrary, the area

537 $S_{\sigma_{K_s}}$ underlying the possibility distribution of σ_{K_s} (Fig. 3, bottom right column) is reduced only by
 538 3.89–8.76% by means of $D_4 = 5$ pieces of data. Moreover, it is interesting to note that the strength
 539 of Approach B in reducing epistemic uncertainty is *slightly higher* than that of Approach A *only*
 540 when the amount of available data is quite *large* (i.e., in the revision of the possibility distributions
 541 of parameters γ and δ of the PDF of $Y_1 = Q$, by means of $D_1 = 149$ pieces of data): actually, the
 542 values of R_γ and R_δ range within 25.56-30.49% and 28.74-33.01% for Approaches A and B,
 543 respectively. In all the other cases, the power of Approach A in reducing epistemic uncertainty is
 544 *higher* than that of Approach B and this difference becomes more and *more evident* as the size of
 545 the data set *decreases*: in particular, for medium-sized data sets (i.e., in the revision of the
 546 possibility distributions of the parameters μ_{Z_m} , σ_{Z_m} , μ_{Z_v} , σ_{Z_v} of the PDFs of $Y_2 = Z_m$ and $Y_3 = Z_v$
 547 with $D_2 = D_3 = 29$ pieces of data) the values of R_θ (6) produced by Approaches A and B range
 548 within 8.33-24.38% and 4.17-22.00%, respectively; instead, for small-sized data sets (i.e., in the
 549 revision of the possibility distributions of the parameters μ_{K_s} and σ_{K_s} of the PDF of $Y_4 = K_s$ with
 550 $D_4 = 5$ pieces of data) the values of R_θ produced by Approaches A and B range within 8.76-16.28%
 551 and 3.89-10.08%, respectively. This is particularly evident in the estimation of the standard
 552 deviation σ_{K_s} of K_s (Fig. 3, bottom right column): on one side, the posterior distribution produced by
 553 the hybrid approach (B) seems not to be influenced by the revision process (actually, the most likely
 554 value of the parameter, $c_{\sigma_{K_s}} = 6.72$, and the area underlying the corresponding posterior possibility
 555 distribution, $S_{\sigma_{K_s}} = 3.95$, are quite close to those of the prior, i.e., 6.89 and 4.11, respectively); on
 556 the other side, the posterior distribution generated by the purely possibilistic approach (A) is almost
 557 centered on the point estimates obtained by the MLE method and the corresponding area is reduced
 558 by about 9%.

559

560 Finally, it has to be remarked that for the sake of simplicity the quantitative analyses above have
 561 been performed only on the *marginal* posterior possibility distributions of the uncertain parameters

562 $(\pi^\gamma(\gamma|\mathbf{q}), \pi^\delta(\delta|\mathbf{q}), \pi^{\mu_{Z_m}}(\mu_{Z_m}|\mathbf{z}_m), \pi^{\sigma_{Z_m}}(\sigma_{Z_m}|\mathbf{z}_m), \pi^{\mu_{Z_v}}(\mu_{Z_v}|\mathbf{z}_v), \pi^{\sigma_{Z_v}}(\sigma_{Z_v}|\mathbf{z}_v), \pi^{\mu_{K_s}}(\mu_{K_s}|\mathbf{k}_s)$
 563 and $\pi^{\sigma_{K_s}}(\sigma_{K_s}|\mathbf{k}_s))$; in fact, as highlighted before, the posterior possibility distributions of the
 564 uncertain parameters are *multi-dimensional* functions (and the posterior estimates of the parameters
 565 are *dependent*). Only for illustration purposes, Fig. 4 shows the α -cuts of level $\alpha = 0.05, 0.50$ and
 566 0.95 of the joint posterior possibility distributions $\pi^{\gamma,\delta}(\gamma,\delta|\mathbf{q}), \pi^{\mu_{Z_m},\sigma_{Z_m}}(\mu_{Z_m},\sigma_{Z_m}|\mathbf{z}_m),$
 567 $\pi^{\mu_{Z_v},\sigma_{Z_v}}(\mu_{Z_v},\sigma_{Z_v}|\mathbf{z}_v)$ and $\pi^{\mu_{K_s},\sigma_{K_s}}(\mu_{K_s},\sigma_{K_s}|\mathbf{k}_s)$ of the parameters of the PDFs of $Y_1 = Q$ (top right
 568 column), $Y_2 = Z_m$ (top left column), $Y_3 = Z_v$ (bottom left column) and $Y_4 = K_s$ (bottom right column),
 569 respectively, produced by Approaches A (solid lines) and B (dashed lines).

570
571 **Fig. 4.**
572

573 The visual inspection of Fig. 4 confirms the results obtained by the quantitative analysis carried out
 574 on the marginal distributions. For example, it can be seen that the areas of the α -cuts of level $\alpha =$
 575 0.95 of $\pi^{\mu_{Z_v},\sigma_{Z_v}}(\mu_{Z_v},\sigma_{Z_v}|\mathbf{z}_v)$ and $\pi^{\mu_{K_s},\sigma_{K_s}}(\mu_{K_s},\sigma_{K_s}|\mathbf{k}_s)$ produced by Approach A are consistently
 576 smaller than those generated by Approach B; the α -cuts of $\pi^{\mu_{Z_m},\sigma_{Z_m}}(\mu_{Z_m},\sigma_{Z_m}|\mathbf{z}_m)$ produced by
 577 Approaches A and B are comparable in size, whereas the area of the α -cut of $\pi^{\gamma,\delta}(\gamma,\delta|\mathbf{q})$ obtained
 578 by Approach B is slightly lower than the one produced by Approach A.

579
 580 In order to show the effect that the reduction of the epistemic uncertainty in the distribution
 581 parameters has on the uncertain input variables $Y_1 = Q, Y_2 = Z_m, Y_3 = Z_v$ and $Y_4 = K_s$, Fig. 5 reports
 582 the upper and lower CDFs, $\bar{F}^{Y_j}(y_j)$ and $\underline{F}^{Y_j}(y_j)$ (i.e., the plausibility and belief functions,
 583 $Pl^{Y_j}((-\infty, y_j])$ and $Bel^{Y_j}((-\infty, y_j])$, respectively), $j = 1, 2, 3, 4$, of $Y_1 = Q$ (top left column), $Y_2 = Z_m$
 584 (top right column), $Y_3 = Z_v$ (bottom left column) and $Y_4 = K_s$ (bottom right column) before (solid
 585 lines) and after the Bayesian update performed by Approaches A (dashed lines) and B (dot-dashed

lines). The calculation of CDFs is of great importance in risk assessment since they summarize the uncertainty “contained” in the variables of interest.

Fig. 5

Obviously, the gap between the plausibility and belief functions $Pl^{Y_j}((-\infty, y_j]) = \bar{F}^{Y_j}(y_j)$ and $Bel^{Y_j}((-\infty, y_j]) = \underline{F}^{Y_j}(y_j)$, $j = 1, 2, 3, 4$, is larger before the Bayesian update in all the cases considered: in particular, the ‘prior’ CDFs (solid lines) completely envelop the ‘posterior’ ones (dashed and dot-dashed lines). This larger gap is explained by the larger area contained under the possibility distribution functions of the corresponding epistemically-uncertain parameters (actually, as highlighted before, the larger the area, the higher the imprecision in the knowledge of the possibilistic parameters).

Then, in order to provide a fair and quantitative comparison between the two approaches adopted, the intervals $\left[\left[\bar{F}^{Y_j} \right]^{-1}(\beta), \left[\underline{F}^{Y_j} \right]^{-1}(\beta) \right]$ for the $\beta \cdot 100$ -th percentiles Y_j^β , $\beta = 0.05, 0.50$ and 0.95 , of the variables Y_j , $j = 1, 2, 3, 4$, are computed (Table 4). For example, analyzing Q^β , $\beta = 0.05, 0.50$ and 0.95 , it can be seen that the width of the intervals is reduced by 30.54–40.34% and 29.88–43.29% by Approaches A and B, respectively: coherently with the results reported in Table 3, when a *large* data set is available (i.e., $D_1 = 149$ in this case) the strength of Approach B in reducing the epistemic uncertainty is *slightly* (i.e., 2-3%) *higher* than that of Approach A. On the contrary, analyzing K_s^β , $\beta = 0.05, 0.50$ and 0.95 , it is evident that the width of the intervals is reduced of 25.30-30.80% and 5.74-19.45% by Approaches A and B, respectively: as highlighted before (Table 3), when a *small* data set is available (i.e., $D_4 = 5$ in this case) the power of Approach A in reducing epistemic uncertainty is *consistently* (i.e., 15-20%) *higher* than that of Approach B. This is confirmed *also* by the analysis of the quantiles of $Y_2 = Z_m$ and $Y_3 = Z_v$ (see Table 4).

Table 4.

612

613 Comparable conclusions can be drawn by the analysis of the upper and lower CDFs $\bar{F}^{Z_c}(z_c)$ and
 614 $\underline{F}^{Z_c}(z_c)$ (i.e., the plausibility and belief functions, $Pl^{Z_c}((-\infty, z_c])$ and $Bel^{Z_c}((-\infty, z_c])$,
 615 respectively), of the maximal water level of the river Z_c (i.e., the model output) obtained before
 616 (Fig. 6, solid lines) and after the Bayesian update performed by Approaches A (Fig. 6, dashed lines)
 617 and B (Fig. 6, dot-dashed lines). Notice that the distributions for Z_c have been obtained by
 618 propagating the two-level mixed probabilistic and possibilistic uncertainty through the
 619 mathematical model by means of a hybrid Monte Carlo (MC) and Fuzzy Interval Analysis (FIA)
 620 approach. This method combines the MC technique [Kalos and Withlock, 1986] with the extension
 621 principle of fuzzy set theory [Guyonnet et al., 2003; Kentel and Aral, 2004 and 2007] in two
 622 hierarchical, repeated steps [Baudrit et al., 2008; Kentel and Aral, 2005; Moller and Beer, 2004 and
 623 2008; Moller et al., 2003 and 2006; Pedroni and Zio, 2012; Pedroni et al., 2013]: the reader is
 624 referred to the cited references for details. Again, in order to provide a fair comparison between the
 625 two approaches employed, proper quantitative indicators are computed. The final goal of the case
 626 study presented in the previous Section is to determine (i) the dike level necessary to guarantee a
 627 given flood return period or (ii) the flood risk for a given dike level. With respect to issue (i) above,
 628 the quantity of interest that is most relevant to the decision maker is the $\beta \cdot 100\%$ -th quantile of Z_c
 629 (i.e., Z_c^β): this corresponds to the yearly maximal water level with a $\beta \cdot 100$ -year return period. With
 630 respect to issue (ii) above, the quantity of interest that is most relevant to the decision maker is the
 631 probability that the maximal water level of the river Z_c exceeds a given threshold z^* , i.e.,
 632 $P[Z_c \geq z^*]$: in the present paper, $z_c^* = 55.5$ m (Table 5). Analyzing the intervals $[\bar{F}^{Z_c}]^{-1}(\beta)$,
 633 $[\underline{F}^{Z_c}]^{-1}(\beta)$ for the percentiles Z_c^β , $\beta = 0.05, 0.50$ and 0.95 , and the intervals
 634 $[1 - \bar{F}^{Z_c}(z_c^*), 1 - \underline{F}^{Z_c}(z_c^*)]$ for the exceedance probability $P[Z_c > z_c^*]$, it can be seen that their width is
 635 reduced of 28.14–38.63% and 16.88–29.23% by Approaches A and B, respectively (i.e., the

636 strength of Approach A in reducing epistemic uncertainty is about 4-11% *higher* than that of
637 Approach B).

638

639 **Fig. 6.**

640

641 **Table 5.**

642

643 Several considerations are in order with respect to the results obtained. When the Bayesian update is
644 performed based on a data set of *large* size (e.g., > 100 in this case), the difference in the behavior
645 of the two approaches is quite *low*. This demonstrates that although the two methods are
646 conceptually and algorithmically quite different, in presence of a “*strong*” experimental evidence
647 they produce “*coherent*” results (i.e., posterior possibility distributions that bear the *same* overall
648 “uncertainty content”): this is a fair outcome since the results provided by the two methods are
649 expected to be more and more *similar* (i.e., more and more coherent with the experimental
650 evidence) as the size of the data set *increases* (experts and practitioners may find a similarity and
651 parallelism between these results and those obtained in purely probabilistic, graphical Bayesian
652 models [Gelman et al., 2004]; in particular, see references concerning approaches used to “borrow
653 strength” in (Hierarchical) Bayesian analyses [Atwood et al., 2003; Kelly and Smith, 2009 and
654 2011]). Instead, when the Bayesian update is performed on a data set of *small-medium* size (e.g., $\approx$
655 5-30 in this case), the *strength* of Approach A in reducing epistemic uncertainty is *significantly*
656 *higher* than that of Approach B. This can be explained as follows. In Approach A the *purely*
657 *possibilistic* likelihood (i.e., the function “containing” the experimental evidence available) has a
658 *direct* and strong influence on the *purely possibilistic* prior (actually, they are directly multiplied in
659 (1)); on the contrary, in Approach B the *purely probabilistic* likelihood has a direct influence *only*
660 on the (fictitious) *probabilistic* distributions that are superimposed onto the prior possibilistic
661 parameters: this “artificial” procedure may in practice “soften” (i.e., reduce) the effect of the newly
662 available information (i.e., of the data) in the revision of the possibilistic priors (obviously, this

663 effect is expected to be more evident if the amount of data, i.e., the *strength* of the experimental
664 evidence, is *small*). In such cases, embracing one method instead of the other may significantly
665 change the outcome of a decision making process in a risk assessment problem involving
666 uncertainties: this is of paramount importance in systems that are critical from the safety view point,
667 e.g., in the nuclear, aerospace, chemical and environmental fields.

668 Finally, it is absolutely important to acknowledge that even if the strength of one method (i.e., A in
669 this case) in reducing epistemic uncertainty is higher than that of the other one (i.e., B in this case),
670 this does *not necessarily* imply that one method is “*better*” or “*more effective*” than the other
671 *overall*. Actually, if on one side a consistent reduction in the epistemic uncertainty is in general
672 *desirable* in decision making processes related to risk assessment problems (since it significantly
673 *increases* the analyst’s *confidence* in the decisions), on the other side this reduction must be
674 *coherent* with the amount of *information available*. In this view, an objection may arise in the
675 present case: is the *remarkable* strength of Approach A in reducing epistemic uncertainty (with very
676 few pieces of data) *fully justified* by such a small amount of data? In other words, is this
677 considerable reduction of epistemic uncertainty coherent with the strength of the experimental
678 evidence or is it too optimistic? These issues will be thoroughly discussed in the following
679 dedicated Section.

680
681 Finally, in addition to the strength of the approaches in revising the (prior) possibilistic description
682 of the uncertain parameters of aleatory variables, also the *computational time* associated to the
683 methods has to be taken into account. Table 6 reports the computational time t_{comp} required by the
684 Bayesian update of all the parameters of the PDFs of $Y_1 = Q$, $Y_2 = Z_m$, $Y_3 = Z_v$ and $Y_4 = K_s$ performed
685 by Approach A and by Approach B (with $T = 100$ repetitions of the purely probabilistic Bayes’
686 theorem for each of the $N_\alpha = 21$ α -cuts analyzed). Obviously, the computational time required by
687 Approach B is approximately $T \cdot N_\alpha$ (i.e., $T \cdot N_\alpha = 100 \cdot 21 = 2100$ in this case) times larger than that of
688 Approach A. On the other hand, notice that since Approach B is based on several *repetitions* of the

689 purely probabilistic Bayes' theorem, if possible, *parallelization* could be in principle employed to
690 reduce the associated computational cost.

691
692
693

Table 6.

694 **Discussion of the results and comparison of the approaches**

695 The results of the comparisons performed in the previous Sections can be summarized as follows:

- 696 • both methods succeed in updating the possibilistic description of the epistemically-uncertain
697 parameters of (aleatory) probability distributions by means of data. This is highlighted by
698 the fact that in most cases the posterior possibility distributions produced by the two
699 approaches are significantly different from the corresponding priors. In particular:
 - 700 ▪ the most likely values of the parameters (i.e., those values in correspondence of
701 which the possibility function equals 1) are moved towards the point estimates of the
702 parameters obtained by the classical, purely probabilistic MLE method;
 - 703 ▪ the area underlying the posterior possibility distributions is consistently lower than
704 that of the priors: since this area is related to the imprecision in the knowledge of the
705 possibilistic parameter (i.e., the larger the area, the higher the imprecision), it can be
706 concluded that both approaches succeed in reducing the epistemic uncertainty in the
707 possibilistic parameters of the aleatory probability distributions. This is also
708 confirmed by the reduction of the gap between the upper and lower CDFs (i.e., the
709 plausibility and belief functions) of the corresponding aleatory variables;
- 710 • when the Bayesian update is performed using a data set of *large* size (e.g., > 100), the
711 *strength* of the two approaches in reducing the epistemic uncertainty is quite *similar*. By
712 way of example, in the case study considered Approaches A and B reduce the areas
713 underlying the possibility distributions of the uncertain parameters by 25.56-30.49% and
714 28.74-33.01%, respectively. This demonstrates that although the two methods are

conceptually and algorithmically quite different, in presence of a “strong” experimental evidence they produce “coherent” results (i.e., possibility distributions that bear almost the *same* overall “uncertainty content”): this is a fair outcome since the it is “desired” that both methods provide results that become more and more similar (i.e., that become more and more coherent with the experimental evidence) as the size of the data set increases;

- the strength of the purely possibilistic approach (A) in reducing epistemic uncertainty is *consistently higher* than that of the hybrid one (B) in presence of *medium-* and *small-*sized data sets (e.g., ≈ 5 -30) (which is often the case in the risk analysis of complex safety-critical systems). For example, the width of the intervals for the quantiles of the variables of interest is reduced by 25.30-30.80% and 5.74-19.45% by Approaches A and B, respectively. This significantly different behavior is explained by the fact that in Approach A the (purely possibilistic) likelihood has an immediate and strong influence on the (purely possibilistic) prior (i.e., they are directly multiplied); on the contrary, in Approach B the (purely probabilistic) likelihood has a direct influence *only* on the (fictitious) probabilistic function that is superimposed onto the possibilistic parameter (subject to the Bayesian update): this “artificial” procedure could in practice weaken the effect of the newly available information (i.e., the data) in the revision of the possibilistic prior (this effect is expected to be more evident if the amount of data, i.e., the *strength* of the experimental evidence, is small).

As highlighted above, the fact that the power of Approach A in reducing epistemic uncertainty is higher than that of Approach B does *not necessarily* imply that one method is “better” or “more effective” than the other *overall*. For example, by hypothesis the remarkable strength of Approach A in reducing epistemic uncertainty by means of very few pieces of data (e.g., 5-30 in this case) might *not* be coherent with the “real” strength of such a scarce experimental evidence and could be accidentally due to some bias in the procedure. With respect to that, it has to be admitted that the uncertainty reduction power of the purely possibilistic approach (A) is strongly dependent on the shape of a *constructed* possibilistic

likelihood that could in principle *bias* the analysis. However: (i) in the present paper, this possibilistic function is *very closely related* to the classical, purely probabilistic one (which is theoretically well-grounded) by a simple and direct operation of *normalization* that preserves the “original structure” of the experimental evidence; (ii) the operation of normalization of the probabilistic likelihood finds also some *theoretical* justification in the work by [Denoeux, 2014]; (iii) in general, a probability-to-possibility transformation (properly performed according to the rules of possibility theory) *always* introduces *additional artificial* epistemic uncertainty into the analysis, i.e., it does not artificially reduce it (because it replaces a *single* probabilistic distribution by a *family* of distributions) [Dubois et al., 1993, 2004 and 2008; Flage et al., 2010 and 2013]. On the basis of considerations (i)-(iii) above, it seems unlikely that the purely possibilistic approach (A) may produce results that are dangerously over-optimistic with respect to those of the hybrid one (B). On the other hand, future research should be devoted to the study and development of rigorous, generalized methods for Bayesian model comparison and validation in a *purely possibilistic* framework, in order to complement and strengthen the conclusions drawn by means of the metrics originally introduced in the present manuscript: in this light, techniques from the classical, purely probabilistic field may serve as inspiring references [Gelman et al., 1996; Bayarri and Berger, 1999 and 2000; Johnson Valen, 2004];

- the computational time required by the hybrid approach (B) is *consistently higher* than that associated to the purely possibilistic one (A): this is explained by the necessity of repeatedly applying many times the purely probabilistic Bayes’ theorem for *each* α -cut analyzed. More precisely, the application of method A just requires *one single* evaluation of the purely possibilistic Bayes’ formula; on the other hand, approach B entails repeating $T \cdot N_\alpha$ times the classical probabilistic Bayes’ theorem. In this respect, notice that N_α (i.e., the number of α -cuts processed) is typically of the order of 10 [Baudrit et al., 2006], whereas T (i.e., the number of parameter values selected to explore *each* α -cut) cannot be in principle prescribed

a priori: however, it needs to be large enough to thoroughly explore *each dimension* of the epistemic parameter space (*at least* 10 values should be selected for each parameter);

- both methods relies on *assumptions* about either the prior or the likelihood functions: in the purely possibilistic approach (A) the “original” possibilistic prior is employed, but a possibilistic likelihood function has to be constructed (e.g., by probability-possibility transformations or directly from rough experimental data); instead, in the hybrid method (B) the original probabilistic likelihood function is used, but a “fictitious” prior Fuzzy Probability Distribution Function needs to be identified by superimposing an *arbitrarily selected* probabilistic PDF onto the “original” possibilistic prior that has to be updated.

Based on these findings, the advantages and drawbacks of the two approaches are summarized in the following Table 7.

Table 7.

Conclusions

In this paper, we have considered two methods for the Bayesian update of the possibilistic parameters of aleatory probability distributions, with exemplification on a case study concerning the risk-based design of a flood protection dike. The first method considered is based on a purely possibilistic counterpart of the classical probabilistic Bayes’ theorem; the second is a hybrid (probabilistic and possibilistic) method combining Fuzzy Interval Analysis and the classical probabilistic Bayes’ theorem.

The findings of the work show that in general adopting different methods may generate different results and possibly different decisions in risk problems involving uncertainties: this is of paramount importance in systems that are critical from the safety viewpoint, e.g., in the nuclear, aerospace, chemical and environmental fields.

793 In particular, on the basis of the results obtained, it seems advisable to suggest the use of the purely
794 possibilistic approach (instead of the hybrid one) for the following reasons: (i) its strength in
795 reducing epistemic uncertainty is significantly higher, in particular when the amount of available
796 data is small: this is important in decision making processes since reducing epistemic uncertainty
797 significantly increases the analyst confidence in the decisions; (ii) the computational time required
798 is consistently lower.

799 However, it has to be remarked that the construction of a possibilistic likelihood required by the
800 purely possibilistic method, although recently tackled in the literature, still represents an issue to be
801 *further* investigated from both the theoretical and practical viewpoint in order to avoid introducing
802 biases in the analysis and to suggest the application of the approach for real risk assessment
803 problems: with respect to that, future research will be devoted to the investigation of additional
804 methods developed to this aim. Also, future studies will be aimed at developing generalized
805 methods for Bayesian model comparison and validation in a *purely possibilistic* framework, in order
806 to complement and strengthen the conclusions drawn by means of the metrics originally introduced
807 in the present manuscript.

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Figure captions list

Fig. 1. Top left: triangular possibility distribution $\pi^\gamma(\gamma)$ of the epistemically-uncertain parameter γ of the Gumbel probability distribution of Y ; in evidence the α -cuts of level $\alpha = 0, 0.5$ and 1 . **Top right:** four PDFs belonging to the family $\{p^Y(y | \gamma, \delta)\}_{\alpha=0}$ and four PDFs belonging to the family $\{p^Y(y | \gamma, \delta)\}_{\alpha=0.5}$. **Bottom:** bounding upper and lower CDFs of Y , $\bar{F}_\alpha^Y(y) = Pl_\alpha^Y(Z)$ and $\underline{F}_\alpha^Y(y) = Bel_\alpha^Y(Z)$, $Z = (-\infty, y]$, built in correspondence of the α -cuts of level $\alpha = 0, 0.5$ and 1 of $\pi^\gamma(\gamma)$; the plausibility and belief functions $Pl^Y(Z)$ and $Bel^Y(Z)$, $Z = (-\infty, y]$, are also shown

Fig. 2. Identification of the P -dimensional α -cut $A_\alpha^{\theta|y}$ of the posterior possibility distribution of θ as the hull enveloping $T = 20$ point estimates $\theta_{l,\alpha} | y$, $l = 1, 2, \dots, 20$ (dots), generated by the repeated application of the Bayes' theorem in the hybrid probabilistic-possibilistic approach (B), in the particular case of $P = 2$ parameters θ_1 and θ_2 . The corresponding α -cut $A_{\alpha, Cart}^{\theta_1, \theta_2 | y}$ generated by the Cartesian product of the (one-dimensional) α -cuts $A_\alpha^{\theta_1 | y}$ and $A_\alpha^{\theta_2 | y}$ of the marginal possibility distributions is also shown

Fig. 3. Prior and posterior possibility distributions of the epistemically-uncertain parameters of the aleatory PDFs of $Y_1 = Q$ (top row, left and right column), $Y_2 = Z_m$ (middle-top row, left and right column), $Y_3 = Z_v$ (middle-bottom row, left and right column) and $Y_4 = K_s$ (bottom row, left and right column). The point estimates of the parameters obtained by the classical MLE method are also shown for comparison. Adapted and extended from [Pedroni et al., 2014]

1150 **Fig. 4.** Exemplary α -cuts of level $\alpha = 0.05, 0.50$ and 0.95 of the joint posterior possibility
1151 distributions of the parameters of the PDFs of $Y_1 = Q$ (**top left**), $Y_2 = Z_m$ (**top right**), $Y_3 = Z_v$
1152 (**bottom left**) and $Y_4 = K_s$ (**bottom right**), produced by Approaches A (solid lines) and B
1153 (dashed lines)

1154

1155 **Fig. 5.** Upper and lower CDFs, $\bar{F}^{y_j}(y_j)$ and $\underline{F}^{y_j}(y_j)$, $j = 1, 2, 3, 4$, of the uncertain input variables
1156 $Y_1 = Q$ (**top left**), $Y_2 = Z_m$ (**top right**), $Y_3 = Z_v$ (**bottom left**) and $Y_4 = K_s$ (**bottom right**) before
1157 and after the Bayesian update performed by Approaches A and B

1158

1159 **Fig. 6.** Plausibility and belief functions, $Pl^{Z_c}((-\infty, z_c])$ and $Bel^{Z_c}((-\infty, z_c])$, of the maximal water
1160 level of the river Z_c before and after the Bayesian update performed by Approaches A and B.
1161 Adapted and extended from [Pedroni et al., 2014]

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Figures and corresponding associated captions

Fig. 1. Top left: triangular possibility distribution $\pi^\gamma(\gamma)$ of the epistemically-uncertain parameter γ of the Gumbel probability distribution of Y ; in evidence the α -cuts of level $\alpha = 0, 0.5$ and 1 . **Top right:** four PDFs belonging to the family $\{p^Y(y|\gamma, \delta)\}_{\alpha=0}$ and four PDFs belonging to the family $\{p^Y(y|\gamma, \delta)\}_{\alpha=0.5}$. **Bottom:** bounding upper and lower CDFs of Y , $\bar{F}_\alpha^Y(y) = Pl_\alpha^Y(Z)$ and $\underline{F}_\alpha^Y(y) = Bel_\alpha^Y(Z)$, $Z = (-\infty, y]$, built in correspondence of the α -cuts of level $\alpha = 0, 0.5$ and 1 of $\pi^\gamma(\gamma)$; the plausibility and belief functions $Pl^Y(Z)$ and $Bel^Y(Z)$, $Z = (-\infty, y]$, are also shown

Fig. 2. Identification of the P -dimensional α -cut $A_{\alpha}^{\theta|y}$ of the posterior possibility distribution of θ as the hull enveloping $T = 20$ point estimates $\theta_{l,\alpha} | y$, $l = 1, 2, \dots, 20$ (dots), generated by the repeated application of the Bayes' theorem in the hybrid probabilistic-possibilistic approach (B), in the particular case of $P = 2$ parameters θ_1 and θ_2 . The corresponding α -cut $A_{\alpha, Cart}^{\theta_1, \theta_2|y}$ generated by the Cartesian product of the (one-dimensional) α -cuts $A_{\alpha}^{\theta_1|y}$ and $A_{\alpha}^{\theta_2|y}$ of the marginal possibility distributions is also shown

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1189 **Fig. 3.** Prior and posterior possibility distributions of the epistemically-uncertain parameters of the
1190 aleatory PDFs of $Y_1 = Q$ (top row, left and right column), $Y_2 = Z_m$ (middle-top row, left and
1191 right column), $Y_3 = Z_v$ (middle-bottom row, left and right column) and $Y_4 = K_s$ (bottom row,
1192 left and right column). The point estimates of the parameters obtained by the classical MLE
1193 method are also shown for comparison. Adapted and extended from [Pedroni et al., 2014]
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Fig. 4. Exemplary α -cuts of level $\alpha = 0.05, 0.50$ and 0.95 of the joint posterior possibility

distributions of the parameters of the PDFs of $Y_1 = Q$ (**top left**), $Y_2 = Z_m$ (**top right**), $Y_3 = Z_v$

(**bottom left**) and $Y_4 = K_s$ (**bottom right**), produced by Approaches A (solid lines) and B

(dashed lines)

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Fig. 5. Upper and lower CDFs, $\bar{F}^{y_j}(y_j)$ and $\underline{F}^{y_j}(y_j)$, $j = 1, 2, 3, 4$, of the uncertain input variables $Y_1 = Q$ (top left), $Y_2 = Z_m$ (top right), $Y_3 = Z_v$ (bottom left) and $Y_4 = K_s$ (bottom right) before and after the Bayesian update performed by Approaches A and B

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1211 **Fig. 6.** Plausibility and belief functions, $Pl^{Z_c}((-\infty, z_c])$ and $Bel^{Z_c}((-\infty, z_c])$, of the maximal water
 1212 level of the river Z_c before and after the Bayesian update performed by Approaches A and B.

1213 Adapted and extended from [Pedroni et al., 2014]

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Tables

	Purely possibilistic method (A)	Hybrid probabilistic-possibilistic method (B)
Prior	Possibility distribution	Fuzzy Probability Distribution Function
Likelihood	Possibility distribution: - Probability-possibility transformation in this paper (e.g., normalization: see also [Denoeux, 2014])	Probabilistic function
Posterior	Possibility distribution	Fuzzy Probability Distribution Function: - Expected value: possibility distribution

1217 *Table 1. Characteristics of the purely possibilistic (A) and the hybrid probabilistic and possibilistic*
1218 *(B) approaches*

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Purely possibilistic method (Approach A)		
Aleatory uncertainty		Epistemic uncertainty (Purely possibilistic priors)
Q	$Gum(\gamma, \delta)$	$\pi^\gamma(\gamma) = TR(a_\gamma, c_\gamma, b_\gamma) = TR(869, 955, 1157)$
		$\pi^\delta(\delta) = TR(a_\delta, c_\delta, b_\delta) = TR(455, 600, 660)$
Z_m	$N(\mu_{Z_m}, \sigma_{Z_m})$	$\pi^{\mu_{Z_m}}(\mu_{Z_m}) = TR(a_{\mu_{Z_m}}, c_{\mu_{Z_m}}, b_{\mu_{Z_m}}) = TR(54.78, 54.93, 55.28)$
		$\pi^{\sigma_{Z_m}}(\sigma_{Z_m}) = TR(a_{\sigma_{Z_m}}, c_{\sigma_{Z_m}}, b_{\sigma_{Z_m}}) = TR(0.33, 0.51, 0.58)$
Z_v	$N(\mu_{Z_v}, \sigma_{Z_v})$	$\pi^{\mu_{Z_v}}(\mu_{Z_v}) = TR(a_{\mu_{Z_v}}, c_{\mu_{Z_v}}, b_{\mu_{Z_v}}) = TR(49.98, 50.11, 50.40)$
		$\pi^{\sigma_{Z_v}}(\sigma_{Z_v}) = TR(a_{\sigma_{Z_v}}, c_{\sigma_{Z_v}}, b_{\sigma_{Z_v}}) = TR(0.23, 0.45, 0.54)$
K_s	$N(\mu_{K_s}, \sigma_{K_s})$	$\pi^{\mu_{K_s}}(\mu_{K_s}) = TR(a_{\mu_{K_s}}, c_{\mu_{K_s}}, b_{\mu_{K_s}}) = TR(21.37, 25.23, 34.23)$
		$\pi^{\sigma_{K_s}}(\sigma_{K_s}) = TR(a_{\sigma_{K_s}}, c_{\sigma_{K_s}}, b_{\sigma_{K_s}}) = TR(1.16, 6.91, 9.37)$
Hybrid probabilistic-possibilistic method (Approach B)		
Aleatory uncertainty	Epistemic uncertainty (Fuzzy Probability Density Functions - FPDFs)	
	Probabilistic part	Possibilistic part
Q	$Gum(\gamma, \delta)$	$p^\gamma(\gamma) = N(\mu_\gamma, \sigma_\gamma)$
		$p^\delta(\delta) = N(\mu_\delta, \sigma_\delta)$
		$\pi^{\mu_\gamma}(\mu_\gamma) = TR(a_{\mu_\gamma}, c_{\mu_\gamma}, b_{\mu_\gamma}) = TR(869, 955, 1157),$
		$\pi^{\sigma_\gamma}(\sigma_\gamma) = TR(a_{\sigma_\gamma}, c_{\sigma_\gamma}, b_{\sigma_\gamma}) = TR(39, 52, 57)$
Z_m	$N(\mu_{Z_m}, \sigma_{Z_m})$	$\pi^{\mu_\delta}(\mu_\delta) = TR(a_{\mu_\delta}, c_{\mu_\delta}, b_{\mu_\delta}) = TR(455, 600, 660)$
		$\pi^{\sigma_\delta}(\sigma_\delta) = TR(a_{\sigma_\delta}, c_{\sigma_\delta}, b_{\sigma_\delta}) = TR(25, 31, 43)$
		$p^{\mu_{Z_m}}(\mu_{Z_m}) = N(\mu_{\mu_{Z_m}}, \sigma_{\mu_{Z_m}})$
		$p^{\sigma_{Z_m}}(\sigma_{Z_m}) = N(\mu_{\sigma_{Z_m}}, \sigma_{\sigma_{Z_m}})$
Z_v	$N(\mu_{Z_v}, \sigma_{Z_v})$	$\pi^{\mu_{Z_m}}(\mu_{Z_m}) = TR(a_{\mu_{Z_m}}, c_{\mu_{Z_m}}, b_{\mu_{Z_m}}) = TR(54.78, 54.93, 55.28)$
		$\pi^{\sigma_{Z_m}}(\sigma_{Z_m}) = TR(a_{\sigma_{Z_m}}, c_{\sigma_{Z_m}}, b_{\sigma_{Z_m}}) = TR(0.06, 0.09, 0.11)$
		$\pi^{\mu_{Z_v}}(\mu_{Z_v}) = TR(a_{\mu_{Z_v}}, c_{\mu_{Z_v}}, b_{\mu_{Z_v}}) = TR(49.98, 50.11, 50.40)$
		$\pi^{\sigma_{Z_v}}(\sigma_{Z_v}) = TR(a_{\sigma_{Z_v}}, c_{\sigma_{Z_v}}, b_{\sigma_{Z_v}}) = TR(0.04, 0.08, 0.10),$
K_s	$N(\mu_{K_s}, \sigma_{K_s})$	$\pi^{\mu_{Z_v}}(\mu_{Z_v}) = TR(a_{\mu_{Z_v}}, c_{\mu_{Z_v}}, b_{\mu_{Z_v}}) = TR(0.23, 0.45, 0.54)$
		$\pi^{\sigma_{Z_v}}(\sigma_{Z_v}) = TR(a_{\sigma_{Z_v}}, c_{\sigma_{Z_v}}, b_{\sigma_{Z_v}}) = TR(0.02, 0.04, 0.08)$
		$p^{\mu_{K_s}}(\mu_{K_s}) = N(\mu_{\mu_{K_s}}, \sigma_{\mu_{K_s}})$
		$p^{\sigma_{K_s}}(\sigma_{K_s}) = N(\mu_{\sigma_{K_s}}, \sigma_{\sigma_{K_s}})$
		$\pi^{\mu_{K_s}}(\mu_{K_s}) = TR(a_{\mu_{K_s}}, c_{\mu_{K_s}}, b_{\mu_{K_s}}) = TR(21.37, 25.23, 34.23)$
		$\pi^{\sigma_{K_s}}(\sigma_{K_s}) = TR(a_{\sigma_{K_s}}, c_{\sigma_{K_s}}, b_{\sigma_{K_s}}) = TR(0.50, 2.80, 3.79)$
		$\pi^{\mu_{K_s}}(\mu_{K_s}) = TR(a_{\mu_{K_s}}, c_{\mu_{K_s}}, b_{\mu_{K_s}}) = TR(1.16, 6.91, 9.37)$
		$\pi^{\sigma_{K_s}}(\sigma_{K_s}) = TR(a_{\sigma_{K_s}}, c_{\sigma_{K_s}}, b_{\sigma_{K_s}}) = TR(0.10, 0.86, 2.64)$

Table 2. Characteristics and parameters of the prior distributions of the uncertain variables $Y_1 = Q$, $Y_2 = Z_m$, $Y_3 = Z_v$ and $Y_4 = K_s$ used in Approaches A and B, respectively

Marginal possibility distributions update								
Variable	θ	MLE $\hat{\theta}^{MLE}$	Most likely value, $c_\theta (d_\theta^{MLE})$			Area, $S_\theta (R_\theta)$		
			Prior	Posterior A	Posterior B	Prior	Posterior A	Posterior B
Q [m ³ /s]	γ	1013.21	955.55	1002.70 (0.010)	990.42 (0.023)	144.65	100.55 (30.49)	96.90 (33.01)
	δ	558.48	599.15	566.35 (0.014)	581.40 (0.041)	103.35	76.94 (25.56)	73.65 (28.74)
Z_m [m]	μ_{Z_m}	55.03	54.93	55.00 (5.45e-4)	54.98 (9.09e-4)	0.25	0.190 (24.00)	0.195 (22.00)
	σ_{Z_m}	0.45	0.51	0.47 (0.044)	0.50 (0.111)	0.12	0.110 (8.33)	0.115 (4.17)
Z_v [m]	μ_{Z_v}	50.19	50.11	50.17 (3.98e-4)	50.15 (7.97e-4)	0.21	0.165 (21.43)	0.170 (19.05)
	σ_{Z_v}	0.38	0.45	0.39 (0.026)	0.43 (0.132)	0.16	0.121 (24.38)	0.125 (21.88)
K_s [l]	μ_{K_s}	27.80	25.24	26.95 (0.031)	26.43 (0.049)	6.45	5.40 (16.28)	5.80 (10.08)
	σ_{K_s}	5.26	6.89	5.54 (0.053)	6.72 (0.278)	4.11	3.75 (8.76)	3.95 (3.89)

Table 3. Most likely values c_θ of the parameters $\theta = \gamma, \delta, \mu_{Z_m}, \sigma_{Z_m}, \mu_{Z_v}, \sigma_{Z_v}, \mu_{K_s}$ and σ_{K_s} of the aleatory PDFs of the input variables $Y_1 = Q, Y_2 = Z_m, Y_3 = Z_v$ and $Y_4 = K_s$ and areas S_θ underlying the corresponding (marginal) possibility distributions before and after the Bayesian update performed by Approaches A and B. The point estimates $\hat{\theta}^{MLE}$ obtained by the classical MLE method are also shown for comparison together with the values of the quantitative indicators d_θ^{MLE} (5) and R_θ (6) (in parentheses). Adapted and extended from [Pedroni et al., 2014]

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INPUT VARIABLES				
Variable	Indicator	Prior	Purely possibilistic approach (A) (% width reduction)	Hybrid approach (B) (% width reduction)
$Y_1 = Q$	$Q^{0.05}$	[284.8, 480.9]	[329.3, 446.3] (0.4033)	[322.9, 434.1] (0.4329)
	$Q^{0.50}$	[1110.5, 1295.5]	[1143.6, 1269.1] (0.3216)	[1145.5, 1263.7] (0.3610)
	$Q^{0.95}$	[2478.2, 2917.3]	[2553.4, 2857.9] (0.3065)	[2571.1, 2879.0] (0.2987)
$Y_2 = Z_m$	$Z_m^{0.05}$	[53.98, 54.36]	[54.09, 54.37] (0.2631)	[54.06, 54.36] (0.2105)
	$Z_m^{0.50}$	[54.85, 55.12]	[54.92, 55.09] (0.4074)	[54.93, 55.12] (0.2592)
	$Z_m^{0.95}$	[55.56, 56.02]	[55.66, 55.96] (0.3478)	[55.67, 55.99] (0.3043)
$Y_3 = Z_v$	$Z_v^{0.05}$	[49.22, 49.64]	[49.35, 49.64] (0.3095)	[49.29, 49.62] (0.2142)
	$Z_v^{0.50}$	[50.04, 50.29]	[50.10, 50.25] (0.4000)	[50.10, 50.27] (0.3199)
	$Z_v^{0.95}$	[50.63, 51.09]	[50.72, 51.00] (0.3913)	[50.72, 51.06] (0.2608)
$Y_4 = K_s$	$K_s^{0.05}$	[10.95, 22.33]	[12.75, 21.25] (0.2530)	[11.15, 20.38] (0.1889)
	$K_s^{0.50}$	[23.49, 31.15]	[24.43, 29.73] (0.3080)	[24.05, 30.22] (0.1945)
	$K_s^{0.95}$	[32.66, 44.14]	[33.01, 41.40] (0.2691)	[33.32, 44.14] (0.0574)

1235 **Table 4.** Intervals $\left[\left[\bar{F}^{Y_j}\right]^{-1}(\beta), \left[F^{Y_j}\right]^{-1}(\beta)\right]$ for the β -100-th percentiles Y_j^β , $\beta = 0.05, 0.50$ and 0.95 ,
1236 of the input variables Y_j , $j = 1, 2, 3, 4$, before and after the Bayesian update performed by
1237 Approaches A and B. The percentage reduction in the width of the intervals produced by
1238 Approaches A and B is also shown in parentheses for comparison purposes

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OUTPUT VARIABLE Z_c				
Variable	Indicator	Prior	Purely possibilistic approach (A) (% width reduction)	Hybrid approach (B) (% width reduction)
Z_c	$Z_c^{0.05}$	[50.70, 51.67]	[50.90, 51.56] (0.3196)	[50.84, 51.53] (0.2887)
	$Z_c^{0.50}$	[52.16, 53.46]	[52.38, 53.23] (0.3462)	[52.36, 53.28] (0.2923)
	$Z_c^{0.95}$	[54.13, 56.44]	[54.21, 55.87] (0.2814)	[54.28, 56.20] (0.1688)
	$P[Z_c > z_c^*]$	[0.8908, 0.9946]	[0.9284, 0.9921] (0.3863)	[0.9103, 0.9946] (0.1879)

Table 5. Intervals $\left[\left[Pl^{Z_c} \right]^{-1}(\beta), \left[Bel^{Z_c} \right]^{-1}(\beta) \right]$ for the $\beta \cdot 100$ -th percentiles Z_c^β , $\beta = 0.05, 0.50$ and 0.95 , of the maximal water level of the river Z_c (i.e., the model output) and intervals $[1 - \overline{F}^{Z_c}(z_c^*), 1 - \underline{F}^{Z_c}(z_c^*)]$ for the exceedance probability $P[Z_c > z_c^* = 55.5m]$, before and after the Bayesian update performed by Approaches A and B. The percentage reduction in the width of the intervals produced by Approaches A and B is also shown in parentheses for comparison purposes. Adapted and extended from [Pedroni et al., 2014]

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	Approach A	Approach B ($T = 100, N_\alpha = 21$)
Computational time t_{comp} [s] on a Intel® Core™2 Duo CPU E7600 @ 3.06 and 3.07 GHz	0.73	1536.65

Table 6. Computational time t_{comp} required by the Bayesian update of all the parameters of the PDFs of $Y_1 = Q$, $Y_2 = Z_m$, $Y_3 = Z_v$ and $Y_4 = K_s$ performed by Approach A and by Approach B with $T = 100$ repetitions of the purely probabilistic Bayes' theorem for each of the $N_\alpha = 21$ α -cuts analyzed

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	Purely possibilistic approach (A)	Hybrid probabilistic-possibilistic approach (B)
Advantages	<ul style="list-style-type: none">- High strength in updating the possibilistic parameters independently on the size of the data set available- Computationally cheap	<ul style="list-style-type: none">- Well-established and rigorous theoretical framework (i.e., probabilistic Bayes' theorem)
Drawbacks	<ul style="list-style-type: none">- Necessity to “build” a possibilistic likelihood (e.g., by probability-possibility transformations, normalization or directly from rough experimental data)	<ul style="list-style-type: none">- Necessity to “build” a (fictitious) Fuzzy Probability Distribution Functions as a prior- High strength in updating the possibilistic parameters only when the size of the data set available is large (e.g., > 100)- Computationally burdensome

1256 *Table 7. Advantages and drawbacks of the purely possibilistic (A) and the hybrid probabilistic and*
1257 *possibilistic (B) approaches*

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