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Abstract

We mainly study a system of two coupled nonlinear Schrödinger equations where one equation includes gain and the other one includes losses. This model constitutes a generalization of the model of pulse propagation in birefringent optical fibers. We answered partially to a question of some authors in [8], that in the Manakov case, the solution stays in $L^2(0,T;H^1)$, that means that the solution can not blow up in finite time. More precisely, the bound that is provided in this paper does not seem to be optimal but different than those that has been given from a previous study [8]. Thanks to the way we treat the a priori estimate, we obtain a sharp bound in $L^2(0,T;H^1)$, which would be difficult to reach from the study of other authors [8]. The result is illustrated by numerical results which have been obtained with a finite element solver well adapted for that purpose.

Keywords: Coupled nonlinear Srchödinger equations, Manakov model, Parity-time symmetry, finite element, solitons
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1. Introduction

Hereafter, a basic model of propagation of weakly dispersive waves is considered by a coupled nonlinear Schrödinger (NLS) equations, which reads

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as follows:

$$\begin{cases} \imath u_t &= -u_{xx} + \kappa v + \imath\gamma u - (g_{11}|u|^2 + g_{12}|v|^2)u, \\ \imath v_t &= -v_{xx} + \kappa u - \imath\gamma v - (g_{12}|u|^2 + g_{22}|v|^2)v, \end{cases} \tag{1}$$

with the coefficients of the nonlinear parts being real. The $\imath$ is the complex such that $\imath^2 = -1$ and the coefficients κ and γ are positive constants that characterize gain and loss in wave components. This model is known for its pertinence for several applications of nonlinear optics and has been studied by several authors [3, 1, 8].

Following the study of [8], let us recall that the solution of the system (1), obeys a parity-time symmetry as soon as there is an equality between coefficients $g_{11} = g_{22}$. Actually, the parity symmetry is defined by the mapping: $\mathcal{P}(u, v) = (v, u)$ and the time reversal operator is defined by the following map:

$$\mathcal{T}(u(t), v(t)) = (\bar{u}(-t), \bar{v}(-t)),$$

where the notation $\bar{u}(t)$ represents the complex conjugate of $u(t)$.

Then, the nonlinearity of the system (1) is such that for any solution $(u(t), v(t))$ defined in a symmetric interval $\mathcal{I} = [-t_0, t_0]$, there exists another one $(u_{\mathcal{PT}}(t), v_{\mathcal{PT}}(t))$ defined in $\mathcal{I}$,

$$(u_{\mathcal{PT}}(t), v_{\mathcal{PT}}(t)) = (\bar{v}(-t), \bar{u}(-t)).$$

2. Main results

According to [8], one knows the existence of a unique global solution $(u(t), v(t)) \in \mathcal{C}(\mathbb{R}, (H^1(\mathbb{R}))^2)$ of the Cauchy problem for the generalized Manakov system (1) with $(u(0), v(0)) = (u_0, v_0) \in (H^1(\mathbb{R}))^2$.

In the Hamiltonian case (for $\gamma = 0$), it is known that for $(u, v) \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R}))$, the two following quantities are conserved, namely the density,

$$Q(t) = \int_{\mathrm{R}} (|u|^2 + |v|^2)dx,$$

and the total energy of the system (1)

$$E(t) = \int_{\mathrm{R}} \Big(|u_x|^2 + |v_x|^2 + \kappa(\bar{u}v + u\bar{v}) - \frac{g_{11}}{2}|u|^4 - \frac{g_{22}}{2}|v|^4 - g_{12}|u|^2|v|^2\Big)dx$$

Our next theorem results in another technique which allows us to find an estimate of the semi-norm in H^1 of the solution of the generalized Manakov system, sharper than that can be obtained from the previous study [8].

Theorem 1. *Assuming that $g_{11} = g_{12} = g_{22}$ then, for the unique solution $(u(t), v(t)) \in \mathcal{C}(\mathbb{R}, (H^1(\mathbb{R}))^2)$ of the Cauchy problem for the generalized Manakov system (1) with $(u(0), v(0)) = (u_0, v_0) \in (H^1(\mathbb{R}))^2$, there exists a positive constant C such that the semi–norm of the solution has the following upper bound, for $t \leq T$*

$$\int_0^T (\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2)dt \leq 4g^2C^2Q(0)^3e^{6\gamma T} + 2Q(0)e^{2\gamma T}(\kappa + \pi\Big(\frac{T}{\omega} + 3\Big)) + 2\pi Q(0) \quad (2)$$

Proof. The proof started similarly to those given in [8], by the fact that the density and the total energy are well defined in the energy space, and that as soon as $\gamma \neq 0$ the integral quantities are not constant in time.

First of all, by using the duality between the Sobolev spaces H^{-1} and H^1, we apply the first equation of system (1) to u and the second one to v to form the density.

Then, adding the two previous dual products of left hand side of system (1) and thanks to an integration by parts, we get from the imaginary part that

$$\frac{d}{dt}\int_{\mathrm{R}}(|u|^2 + |v|^2)dx = 2\gamma\int_{\mathrm{R}}(|u|^2 - |v|^2)dx.$$

That gives a control to the evolution of the density

$$\frac{dQ}{dt} = 2\gamma(\|u\|_{L^2}^2 - \|v\|_{L^2}^2) \leq 2\gamma Q(t), \quad (3)$$

and by the Grönwall inequality, the density does not blow up in finite time:

$$\forall t \in [-t_0, t_0], \qquad Q(t) \leq Q(0)\exp(2\gamma|t|). \quad (4)$$

And from the real part, we obtain that

$$\int_{\mathrm{R}}\Big(|u_x|^2 + |v_x|^2\Big)dx + \int_{\mathrm{R}}\mathcal{R}e\{\imath(u_t\bar{u} + v_t\bar{v})\}dx = -\kappa\int_{\mathrm{R}}(v\bar{u} + u\bar{v})dx + \int_{\mathrm{R}} g(|u|^4 + |v|^4 + 2|u|^2|v|^2)dx. \quad (5)$$

Let us focus onto the second integral of the left hand side. As we assume that u can not vanish, we consider that during a time interval its imaginary part is not null (or it can be replaced by its real part). Then, using the same argument for v, we write that

$$\mathcal{R}e\{\imath(u_t\bar{u} + v_t\bar{v})\} = |u|^2\Big(\arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big)_t + |v|^2\Big(\arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big)_t. \quad (6)$$

By integration by parts and thanks to the absolute continuity of their representants,

$$\int_0^T \mathcal{R}e\{\imath(u_t\bar{u} + v_t\bar{v})\}dt = \left[|u|^2 \arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big) + |v|^2 \arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\right]_{t=0}^{t=T}$$
$$- \int_0^T \left\{\arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big(|u|^2\Big)_t + \arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big(|v|^2\Big)_t\right\} dt.$$

From the absolute continuity of the representants of u and v and thanks to the oscillating behaviour of Q ([8]) and relation (3), one can compute explicitly $\|u\|^2_{L^2}$ and $\|v\|^2_{L^2}$. So, one obtains that they are oscillating functions also:

$$\begin{aligned} \|u\|^2_{L^2} &= \frac{1}{2\gamma}\left[\frac{\gamma\kappa C}{\omega^2} + (\gamma A_1 + \omega A_2)\cos 2\omega t + (\gamma A_2 - \omega A_1)\sin 2\omega t\right], \\ \|v\|^2_{L^2} &= \frac{1}{2\gamma}\left[\frac{\gamma\kappa C}{\omega^2} + (\gamma A_1 - \omega A_2)\cos 2\omega t + (\gamma A_2 + \omega A_1)\sin 2\omega t\right]. \end{aligned}$$

Then, easy computations provide that during each time period the time derivative of $\|u\|^2_{L^2}$ and $\|v\|^2_{L^2}$ vanishes at two specific times for each component. We denote the specific instants by $t_u^{(1)}$, $t_u^{(2)} = t_u^{(1)} + \pi/(2\omega)$ and $t_v^{(1)}$, $t_v^{(2)} = t_v^{(1)} + \pi/(2\omega)$. Setting $\alpha = \gamma/\omega$, and $\beta = A_1/A_2$, one obtains

$$t_u^{(1)} = \frac{1}{2\omega}\arctan\Big(\frac{\alpha\beta - 1}{\alpha + \beta}\Big).$$

If $\alpha \neq \beta$, one has the expression for $t_v^{(1)}$,

$$t_v^{(1)} = \frac{1}{2\omega}\arctan\Big(\frac{\alpha\beta + 1}{\alpha - \beta}\Big),$$

and if $\alpha = \beta$ one gets:

$$t_v^{(1)} = \frac{\pi}{4\omega}.$$

An interesting reader can sort the zeros of the time derivative of $\|u\|^2_{L^2}$ and $\|v\|^2_{L^2}$ according to the values of parameters. But our concern is to derive an upper bound which is valid for each case, if the sign of $\big(\|u\|^2_{L^2}\big)_t$ and $\big(\|v\|^2_{L^2}\big)_t$ are the same or not. Let us give details in the case of the sign of each component is different and that their sign change from an interval of the subdivision to the next one. For instance, $(|u|^2)_t \leq 0$ and $(|v|^2)_t \geq 0$ in $[t_1, t_2]$ whereas $(|u|^2)_t \geq 0$ and $(|v|^2)_t \leq 0$ in $[t_2, t_3]$.

Let us denote by I, the integral that follows:

$$\begin{aligned}
&\int_0^T \int_{\mathrm{R}} \Big\{ \arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big(|u|^2\Big)_t + \arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big(|v|^2\Big)_t \Big\}\, dx\, dt \\
&\quad \leq \int_{t_0}^{t_1} \int_{\mathrm{R}} \Big\{ \arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big(|u|^2\Big)_t + \arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big(|v|^2\Big)_t \Big\}\, dx\, dt + \\
&\sum_{j=0}^{N-1} \Big[\int_{t_1}^{t_2} \int_{\mathrm{R}} \Big\{ \arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big(|u|^2\Big)_t + \arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big(|v|^2\Big)_t \Big\}\, dx\, dt \\
&\quad + \int_{t_2}^{t_3} \int_{\mathrm{R}} \Big\{ \arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big(|u|^2\Big)_t + \arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big(|v|^2\Big)_t \Big\}\, dx\, dt \Big].
\end{aligned}$$

Thus, by identyfying the sign of each term, and denoting the next left hand side by I_j

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{\mathrm{R}} \Big\{ \arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big(|u|^2\Big)_t + \arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big(|v|^2\Big)_t \Big\}\, dx\, dt \\
&\quad + \int_{t_2}^{t_3} \int_{\mathrm{R}} \Big\{ \arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big(|u|^2\Big)_t + \arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big(|v|^2\Big)_t \Big\}\, dx\, dt \\
&\quad \leq \int_{t_1}^{t_2} \int_{\mathrm{R}} \Big\{ \Big|\arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big| \Big(|v|^2\Big)_t - \Big|\arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big| \Big(|u|^2\Big)_t \Big\}\, dx\, dt \\
&\quad + \int_{t_2}^{t_3} \int_{\mathrm{R}} \Big\{ \Big|\arctan\Big(\frac{\mathcal{R}e\ u}{\mathcal{I}m\ u}\Big)\Big| \Big(|u|^2\Big)_t - \Big|\arctan\Big(\frac{\mathcal{R}e\ v}{\mathcal{I}m\ v}\Big)\Big| \Big(|v|^2\Big)_t \Big\}\, dx\, dt
\end{aligned}$$

The arctan function being bounded,

$$\begin{aligned}
I_j &\leq \frac{\pi}{2} \int_{\mathrm{R}} \Big\{ \int_{t_1}^{t_2} \Big[\Big(|v|^2\Big)_t - \Big(|u|^2\Big)_t\Big] + \int_{t_2}^{t_3} \Big[\Big(|u|^2\Big)_t - \Big(|v|^2\Big)_t\Big] \Big\} \\
&\leq \frac{\pi}{2} \int_{\mathrm{R}} \Big(|v(t_2)|^2 - |u(t_2)|^2 - |v(t_1)|^2 + |u(t_1)|^2 \\
&\qquad + |u(t_3)|^2 - |v(t_3)|^2 - |u(t_2)|^2 + |v(t_2)|^2 \Big) dx \\
&\leq \frac{\pi}{2}(Q(t_1) + 2Q(t_2) + Q(t_3)).
\end{aligned}$$

Obviously, the way that has been generated the upper bound for this sign configuration can be extended to any other configuration. Hence, combining and adding the upper bound for each term, one gets that,

$$I \leq \frac{\pi}{2}\left[Q(t_0) + Q(t_1) + \sum_{j=0}^{p-1} (Q(t_1) + 2Q(t_2) + Q(t_3)) \right].$$

Thanks to the density estimate (4), one obtains

$$|I| \leq \frac{\pi}{2}(Q(0) + (4p+5)Q(0)e^{2\gamma T}).$$

Returning to the integral of the second term of equation (6) left hand side, the upper bound is as follows.

$$\begin{aligned}\int_0^T \int_{\mathrm{R}} \mathcal{R}e\{\imath(u_t\bar{u} + v_t\bar{v})\}dx\,dt &\leq \tfrac{\pi}{2}Q(0)(1+e^{2\gamma T}) + \tfrac{\pi}{2}Q(0)(1+(4p+5)e^{2\gamma T}) \\ &\leq \pi Q(0)(1+\Big(\frac{T}{w}+3\Big)e^{2\gamma T}).\end{aligned}$$

Setting $D(T) = \int_0^T (\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2)dt$, and thanks to the Gagliardo–Nirenberg inequality which is valid for function of H^1 Sobolev space [7], we then obtain that,

$$\begin{aligned}D(T) + \int_0^T \int_{\mathrm{R}} \mathcal{R}e\{\imath(u_t\bar{u} + v_t\bar{v})\}dx\,dt \leq \kappa \int_0^T Q(t)\,dt \\ + 2gC\int_0^T (\|u\|_{L^2}^3\|u_x\|_{L^2} + \|v\|_{L^2}^3\|v_x\|_{L^2})\,dt. \quad (7)\end{aligned}$$

Using the relation $a^3b + c^3d \leq (a^2+b^2)^{3/2}(b^2+d^2)^{1/2}$, one obtains from the inequality (7), Jensen inequality (concavity of the square root function) and relation (4) that,

$$D(T) - 2gCQ(0)^{3/2}e^{3\gamma T}D(T)^{1/2} - \kappa Q(0)e^{2\gamma T} - \pi Q(0)(1+\Big(\frac{T}{w}+3\Big)e^{2\gamma T}) \leq 0,$$

that can be considered as an univariable quadratic function of unknown $D(T)^{1/2}$. The discriminant is computed as

$$\Delta = 4g^2C^2Q(0)^3e^{6\gamma T} + 4Q(0)e^{2\gamma T}(\kappa + \pi\Big(\frac{T}{w}+3\Big)) + 4\pi Q(0),$$

which is undoubtedly positive as the density. Thus, computing the solutions, the variable $D(T)^{1/2}$ being positive, it must stay clustered as follows:

$$D(T)^{1/2} \leq gCQ(0)^{3/2}e^{3\gamma T} + \sqrt{g^2C^2Q(0)^3e^{6\gamma T} + Q(0)e^{2\gamma T}(\kappa + \pi\Big(\frac{T}{\omega}+3\Big)) + \pi Q(0)}.$$

Taking the square, we finally get the upper bound for $t \leq T$ that completes the proof.

$$D(T) \leq 4g^2C^2Q(0)^3e^{6\gamma T} + 2Q(0)e^{2\gamma T}\left\{\kappa + \pi\Big(\frac{T}{\omega}+3\Big)\right\} + 2\pi Q(0). \quad (8)$$

$\square$

Remark 1. *The previous theorem provides an upper bound of the velocity norm in* $L^2(0,T;H^1)$ *as roughly a constant times* $\exp(6\gamma T)$*. One can not compare straightly with the results that have been obtained by other studies. But, if we wish to derive from the previous study [8], an approximation in* $L^2(0,T;H^1)$ *norm, one has to combine their inequality (21) with their equality (20) and the behaviour of the mass (11). And it likely seems that this combination provides an upper bound as a constant times* $\exp(10\gamma T)$ *which is less sharp than the bound that we have obtained.*

Theorem 2. *Let us assume that* $g_{11} = g_{22} = g_{12}$.

- *if* $\gamma < \kappa$*, for* $(u_0, v_0) \in (H^1(\mathbb{R}))^2$ *a local solution at* $t = 0$ *of the system (1), there exists a constant* $Q_{max} > 0$ *such that the solution of the (1) verifies:*
$$\sup_{t\in\mathrm{R}} \left(||u(t)||^2_{L^2} + ||v(t)||^2_{L^2}\right) \leq Q_{max}. \tag{9}$$
- *if* $\gamma \geq \kappa$*, there exists a global solution of the system (1) such that,*
$$\lim_{t\to\infty} \left(||u(t)||^2_{L^2} + ||v(t)||^2_{L^2}\right) = \infty.$$

The proof is given in [8].

Theorem 3. *Let us assume that* $g_{11} = g_{22} = g_{12}$.

- *if* $\gamma < \kappa$*, for* $(u_0, v_0) \in (H^1(\mathbb{R}))^2$ *a local solution at* $t = 0$ *of the system (1), there exists a constant* $E_{max} > 0$ *such that the solution of the (1) verifies:*
$$\int_0^T (||u(t)||^2_{H^1} + ||v(t)||^2_{H^1})dt \leq E_{max}, \tag{10}$$
- *if* $\gamma \geq \kappa$*, there exists a global solution of the system (1) such that,*
$$\lim_{t\to\infty} \left(||u(t)||^2_{H^1} + ||v(t)||^2_{H^1}\right) = \infty. \tag{11}$$

Proof. As $g_{11} = g_{22} = g_{12}$ and $\gamma < \kappa$ and according to the conclusions of theorem 2, the density $Q(t)$ est bounded, *i.e.* there exists a positive constant Q_{max} such that
$$\sup_{t\in\mathrm{R}} \left(||u(t)||^2_{L^2} + ||v(t)||^2_{L^2}\right) \leq Q_{max}.$$
Moreover the estimate (2) shows that $D(T)$ exists. Consequently, the solution of the generalized Manakov system lies in $L^2(0,T;H^1)$, *i.e.* there exists a positive constant E_{max} such that
$$\int_0^T (||u(t)||^2_{H^1} + ||v(t)||^2_{H^1})dt \leq E_{max}.$$

The case $\gamma \geq \kappa$ is a direct consequence of conclusion of theorem 2, that ends the proof. $\square$

3. Numerical results

The results of theorem 3 indicate that for $\gamma < \kappa$, the global solutions of Parity-Time symmetric system (1) remain bounded in $L^2(0,T;H^1)$. In order to confirm this result and to investigate the behaviour of the solutions in other cases, numerical solutions have been computed.

As we started from an existing solver that we developped already for one NLS equation, to solve the system (1), we developped an algorithm that substitutes the solution of the first equation into the second equation. The numerical approximation is based on Crank-Nicholson scheme for the temporal discretization and Lagrange P1-Galerkin finite element method for spatial discretization.

By conformal-finite element approximation in $H_0^1(I)$, for $I =]-l,l[$ an open interval, N a positive integer, we then consider $h = 1/(N+1)$ a mesh space and the following approximation space:

$$V_h = \{v_h \in \mathcal{C}^0(\bar{I}),\, v_h(\pm l) = 0,\, v_{h|]-l+(i-1)h,-l+ih[} \in \mathbb{C}[X],\, i = 1,\ldots,N+1\}.$$

Obviously, the approximation space is spanned by classical hat functions denoted by $(\Psi_i)_{1\leq i\leq N}$ and the nonlinearity is approached in a common way by:

$$\int_{-l+(i-1)h}^{-l+ih} |u|^2 u \Psi_k dx \approx \sum_{j=1}^{N} |u_j|^2 u_j \int_{-l+(i-1)h}^{-l+ih} \Psi_j(x)\Psi_k(x)dx.$$

Its treatment has been done by a fixed-point iterative strategy which was not so coslty. Let us consider δt the time step, and mention by an upper index n the approximate value of the component of the solution at time $t_n = n\,\delta t$, then

$$u(t_n,.) \in V_h, \text{ that means that } \exists u_j^n,\, u(t_n,x) = \sum_{j=1}^{N} u_j^n \Psi_j(x),\ x \in I.$$

To solve the first equation, the numerical procedure consists of giving an initial guest $u_j^{n+1,0}$ as u_j^n and solving the following systems as soon as the threshold is not reached, $\|u_j^{n+1,l+1} - u_j^{n+1,l}\| > \varepsilon$:

$$\sum_{j=1}^{N} u_j^{n+1,l+1}\langle\Psi_j,\Psi_k\rangle + \frac{\imath\delta t}{2}\sum_{j=1}^{N} u_j^{n+1,l+1}\langle\nabla\Psi_j,\nabla\Psi_k\rangle - \frac{\gamma\delta t}{2}\sum_{j=1}^{N} u_j^{n+1,l+1}\langle\Psi_j,\Psi_k\rangle$$
$$- \imath\delta t\sum_{j=1}^{N}(g_{11}|u_j^{n+1,l}|^2 + g_{12}|v_j^n|^2)u_j^{n+1,l+1}\langle\Psi_j,\Psi_k\rangle = \frac{\gamma\delta t}{2}\sum_{j=1}^{N} u_j^n\langle\Psi_j,\Psi_k\rangle$$
$$- \frac{\imath\delta t}{2}\sum_{j=1}^{N} u_j^n\langle\nabla\Psi_j,\nabla\Psi_k\rangle + \sum_{j=1}^{N} u_j^n\langle\Psi_j,\Psi_k\rangle - \frac{\imath\kappa\delta t}{2}\sum_{j=1}^{N} v_j^n\langle\nabla\Psi_j,\nabla\Psi_k\rangle.$$

At convergence, one obtains $\left(u_j^{n+1}\right)_{1\le j\le N}$ whose values are substituting into the second equation.

A computer program has been developped in Python interfacing with Fortran that solves the generalized Manakov system (1) in a finite domain $]-l,l[$ endowed by homogeneous Dirichlet boundary continions, with initial conditions (at $t=0$) that have been given by Gaussian beams:

$$u_0(x) = \frac{A}{\pi^{1/4}a^{1/2}}\exp\left\{-\frac{x^2}{2a^2}\right\}, \quad v_0(x) = \frac{B}{\pi^{1/4}b^{1/2}}\exp\left\{-\frac{x^2}{2b^2}\right\}.$$

In a first test series, we report the results for the set of initial conditions and parameters that have considered by other authors with some more details ([8]).

The computational domain has been taken sufficiently large that the numerical value of the initial conditions nearly verifies the homogeneous Dirichlet boundary conditions at the boundary locations. The first tests serie stands for different initial conditions (different values of (A,a,B,b)) with different γ and κ (such that $\gamma < \kappa$) for the case of the Manakov nonlinearity *i.e.* $g_{11} = g_{22} = g_{12} = 1$ (see Fig.1).
In the case of Manakov nonlinearity, the results are in agreement of our theoretical computations. The third pattern of Fig. 1 indicates that the density oscillates in time and that the semi-norm of the solution stays bounded. Also, from the second pattern, one recovers numerically that the norm of each component of the solution oscillates but not in phase.

Let us recall that one property already known, is that

$$S_1 = \int_{\mathrm{R}}(\bar{u}v + u\bar{v})dx,$$

the first Stokes integral of the solution of the Cauchy problem related to the Manakov system must stay constant in time [8]. The third pattern of Fig. 1 also numerically proves that the solution that is obtained has a Stokes integral S_1 which stays constant in time.

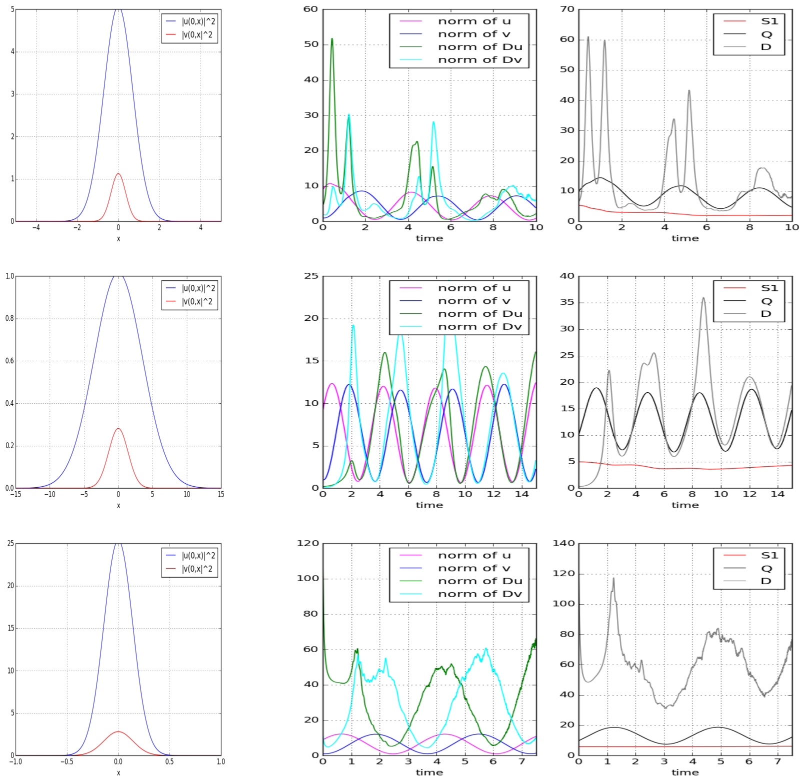

Figure 1: Numerical results of the Cauchy problem for the PT-symmetric system (1) in Manakov case ($g_{11} = g_{22} = g_{12} = 1$). For each line the first plot stands for the initial condition, the second one traces the L^2-norm of u, v and its gradient. The last graph represents the behavior of the density Q, the semi-norm D of the solution and one Stokes integral S_1.

The second tests serie also tends to illustrate that the H^1-norm of the solution of the Cauchy problem for the PT-symmetric system (1) can blow up in finite time.

4. Conclusions

In this work we confirm the result that has been found previously by other authors. Despite the fact that we were not able to clarify if the H^1-norm of the solution of the generalized Manakov system is globally bounded, we improved the estimate of the solution. Numerical results are also given to illustrate the theoretical result with a suitable numerical procedure.

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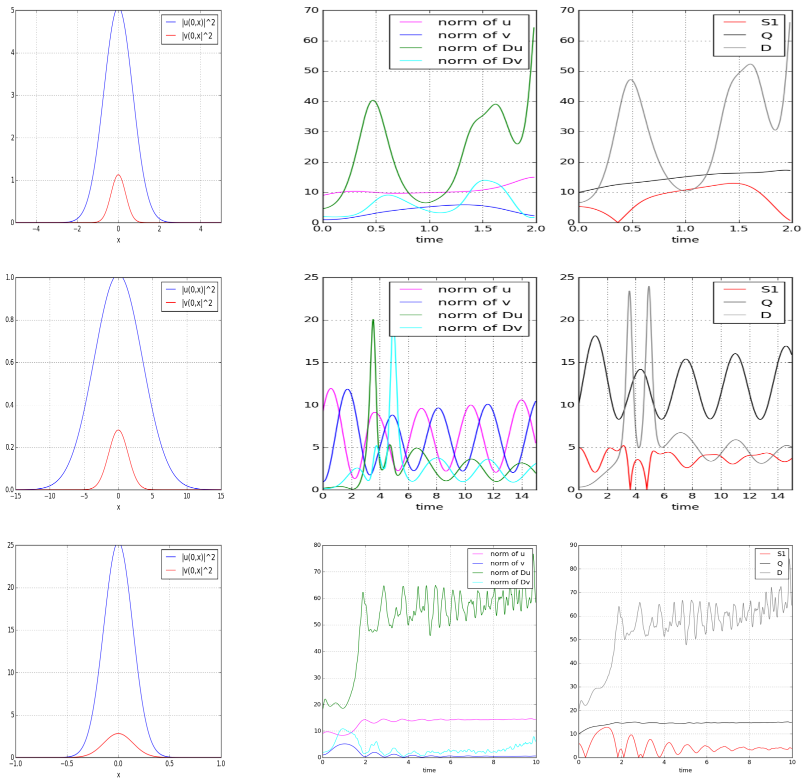

Figure 2: Numerical results of the Cauchy problem for the PT-symmetric system (1) in non-Manakov case ($g_{11} = g_{22} = 1$ and $g_{12} = 1/2$). For each line the first plot stands for the initial condition, the second one traces the L^2-norm of u, v and its gradient. The last graph represents the behavior of the density Q, the semi-norm D of the solution and one Stokes integral S_1.