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An extension of the Tusnady inequality for general distributions
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Under mild assumptions an analog of Tusnady’s inequality for conditional
distributions of partial sums of independent nonidentically distributed r.v.’s is
obtained. These type of inequalities provide very sharp bounds for couplings of
random variables via quantile transformations, which are main tools for proving
various strong invariance principles for sums of independent random variables.
We refer to Komlós, Major and Tusnády (1975, 1976), Sakhanenko (1984),
and to references therein for details. Recently quantile transformations have
been applied for establishing a constructive asymptotic equivalence of statistical
experiments, see Brown, Carter, Low and Zhang (2004).

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\lambda\) be a real number satisfying
\(0 < \lambda < \infty\). Denote by \(\mathcal{D}(\lambda)\) the subset of all real valued random variables \(S\) on the
probability space \((\Omega, \mathcal{F}, P)\), which admits for some \(n \geq 1\), a representation of
the form \(S = X_1 + \ldots + X_n\), where \(X_1, \ldots, X_n\) are independent r.v.’s, satisfying
assumptions

\begin{itemize}
  \item[A1] The r.v.’s \(X_1, \ldots, X_n\) are of means zero and finite variances:
        \[ EX_i = 0, \quad 0 < EX_i^2 < \infty, \quad i = 1, \ldots, n. \]
  \item[A2] The following Sakhanenko’s condition holds true:
        \[ \lambda E |X_i|^3 \exp \{ \lambda |X_i| \} \leq EX_i^2, \quad i = 1, \ldots, n. \]
\end{itemize}

Let \(i = \sqrt{-1}\) and \(\varepsilon, \nu\) be real numbers, such that \(0 < \varepsilon \leq 1/96\) and \(0 < \nu < \infty\). Denote by \(\mathcal{D}_1(\lambda)\) the subset of all r.v.’s \(S \in \mathcal{D}(\lambda)\) which additionally satisfy
the smoothness assumption

\begin{itemize}
  \item[S1] The r.v. \(S\) is such that,
        \[ \sup_{|h| \leq \lambda} \int_{|t| > \varepsilon \lambda} \left| \frac{E \exp \{(it + h)S\}}{E \exp \{hS\}} \right| dt \leq \frac{\nu}{\lambda ES^2}. \]
\end{itemize}

Denote by \(\mathcal{D}_2(\lambda)\) the subset of all r.v.’s \(S \in \mathcal{D}(\lambda)\) which take discrete values \(a + \Delta k, \quad k \in \{0, \pm 1, \pm 2, \ldots\}\) with maximal span \(\Delta\), where \(0 \leq a < \Delta\) and \(\lambda \Delta \leq \pi/\varepsilon\), and which additionally satisfy the assumption (S2)

\begin{itemize}
  \item[S2] The r.v. \(S\) is such that,
        \[ \sup_{|h| \leq \lambda, \varepsilon \lambda < |t| \leq \pi/\Delta} \left| \frac{E \exp \{(it + h)S\}}{E \exp \{hS\}} \right| dt \leq \frac{\nu}{\lambda ES^2}. \]
\end{itemize}
Let us agree that $X$ is the range of a r.v. $S$ in the class $\mathcal{D}_1(\lambda)$ or $\mathcal{D}_2(\lambda)$, i.e. $X = R$, if $S \in \mathcal{D}_1(\lambda)$ and $X = \{a + \Delta k : k = 0, \pm 1, \pm 2, \ldots\}$, if $S \in \mathcal{D}_2(\lambda)$.

Assume that we are given three r.v.'s $S_0$, $S_1$ and $S_2$ with mean 0 and finite variance, so that $S_1$ and $S_2$ are independent and $S_0 = S_1 + S_2$, $S_i \in \mathcal{D}(\lambda)$. Set $B_k^2 = E S_k^2$, $k = 0, 1, 2$ and $B = B_1 B_2 / B_0$. Let $\Psi_k(h)$ be the cumulant generating function of the r.v. $S_k$, i.e.

$$
\Psi_k(h) = \log E \exp\{h S_k\}, \quad |h| \leq \lambda, \quad k = 0, 1, 2.
$$

It is easy to see that $\Psi_0'(h)$ is strictly increasing, for $|h| \leq \lambda$. Let $H_0(x) = h$ be the solution of the equation $\Psi_0'(h) = x$, for $x$ in the range of $\Psi_0'(h)$. Set $B = B_1 B_2 / B_0$,

$$
\mu(y) = \Psi_1'(H_0(y)) = y - \Psi_2'(H_0(y)),
$$

and

$$
\sigma^2(y) = \Psi_1''(H_0(y)) \Psi_2''(H_0(y)) / \Psi_0''(H_0(y)).
$$

Define the conditional distribution function

$$
F(x | y) = P(S_1 \leq x | S_0 = y).
$$

In the sequel $c_i$, $i = 0, 1, \ldots$ denote positive absolute constants and $\theta_i$, $i = 1, 2, \ldots$ real numbers satisfying $|\theta_i| \leq 1$.

**Theorem 1.** Assume that $S_0, S_1, S_2 \in \mathcal{D}_1(\lambda)$ or $S_0, S_1, S_2 \in \mathcal{D}_2(\lambda)$, for some $0 < \lambda < \infty$. Then, for any $x, y \in X$ satisfying $|x| \leq c_0 \lambda B^2$ and $|y| \leq c_0 \lambda B^2$, we have

$$
F(x + \mu(y) | y) = \Phi\left(\frac{x}{\sigma(y)} + \theta_1 \frac{c_1}{\lambda \sigma(y)} \left(1 + \frac{x^2}{\sigma^2(y)}\right)\right),
$$

provided $\lambda B \geq c_2$. Moreover, for $y \in X$ satisfying $|y| \leq c_0 \lambda B^2$, we have

$$
\mu(y) = \beta_1 y + \theta_2 c_3 \frac{y^2}{\lambda B^2}, \quad \frac{1}{\sigma^2(y)} = \frac{1}{B^2} + \theta_3 c_4 \frac{|y|}{\lambda B^2},
$$

where $\beta_1 = B_1^2 / B_0^2$.

In the case case of sums of i.i.d. r.v.'s we get the following result. Let $\xi_1, \ldots, \xi_n, n \geq 1$ be a sequence of i.i.d. r.v.'s of means 0 and finite variances $\sigma^2 = E \xi^2 < \infty$, obeying one of the two conditions (G1) or (G2) below:

**G1** For some $0 < \lambda < \infty$ and $q \geq 1$,

$$
\sup_{|h| \leq \lambda} \int_{-\infty}^{\infty} |E \exp((it + h) \xi)|^q dt < \infty.
$$

**G2** The r.v. $\xi$ takes values on the lattice $a + \Delta k$, $k \in \mathbb{Z}$ with maximal span $\Delta$ and for some $0 < \lambda < \infty$,

$$
E \exp(\lambda |\xi_i|) < \infty.
$$

The common cumulant generating function of $\xi_i$-th is denoted by $\psi(t) = \log E \exp(t \xi_i)$, for $|t| \leq \lambda$. Let $h(x), |x| \leq \lambda / 8$ be the inverse of the monotone
function $\psi'(t)$, $|t| \leq \lambda/4$. Assume that $n_1, n_2$ are positive integers such that $n_1 + n_2 = n$. Set $S_1 = \xi_1 + \ldots + \xi_n$, $S_0 = \xi_1 + \ldots + \xi_n$ and $F(x|y) = P(S_1 \leq x|S_0 = y)$.

**Theorem 2.** For any $|x| \leq c_0\lambda\sigma^2$ and $|y| \leq c_0\lambda\sigma^2$, we have

$$F(x + \mu(y)|y) = \Phi\left(\frac{x}{\sigma(y)} + \theta_1 \frac{c_1}{\lambda\sigma(y)} \left(1 + \frac{x^2}{\sigma^2(y)}\right)\right),$$

where

$$\mu(y) = \frac{n_1}{n} y, \quad \sigma^2(y) = \frac{n_1 n_2}{n} \psi'' \left(\frac{y}{n}\right).$$

Proofs of Theorems 1 and 2 rely on Lemmas 1 and 2 below, which are of independent interest.

Assume that we are given a real valued r.v. $X$, which has a density $p(x)$ w.r.t. Lebesque measure on the real line. We shall impose conditions (C1-C2) below, where $\gamma_0, \ldots, \gamma_4 > 0$ and $\sigma > 1$ denote real quantities not depending on $x$.

**C1** For any $|x| \leq \gamma_0\sigma$,

$$p(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2} + \theta_1 \frac{\gamma_1}{2} \frac{x^3}{\sigma}\right) \left(1 + \theta_2 \gamma_2 \frac{1 + |x|}{\sigma}\right),$$

where $|\theta_i| \leq 1$, $i = 1, 2$.

**C2** For any $|x| \leq \gamma_0\sigma$ and any real $z$ satisfying $zx \geq 0$,

$$p(x+z) \leq \gamma_3 \exp\left(-\frac{x^2}{2} + \frac{\gamma_1}{2} \frac{x^3}{\sigma} - \gamma_4 z x\right).$$

**Lemma 1.** Assume that the density $p(x)$ satisfies conditions (C1-C2). Then, for any $x$ satisfying $|x| \leq c_0\sigma$, we have

$$\Phi\left(x - c_1 \frac{1 + x^2}{\sigma}\right) \leq P(X \leq x) \leq \Phi\left(x + c_1 \frac{1 + x^2}{\sigma}\right),$$

provided $\sigma \geq c_2$, where $c_0, c_1$ and $c_2$ are constants depending only on $\gamma_0, \ldots, \gamma_4$.

Let $\Delta_0 > 0$ and $\Delta \in (0, \Delta_0]$. Assume that we are given a r.v. $X$, which takes discrete values $a_k = a + k\Delta$, $k \in \{0, \pm 1, \pm 2, \ldots\}$, with maximal span $\Delta$ and with $a \in [0, \Delta)$. In the sequel we shall impose conditions (D1-D2) below, where we assume that $\gamma_0, \ldots, \gamma_4 > 0$ and $\sigma > 1$ denote real quantities not depending on values $a_k$, $k \in \{0, \pm 1, \pm 2, \ldots\}$. For the sake of brevity we set $x_k = a_k/\sigma$.

**D1** For any integer $k$ satisfying $|x_k| \leq \gamma_0\sigma$,

$$P(X = a_k) = \frac{\Delta}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x_k^2}{2} + \theta_1 \frac{\gamma_1 x_k^3}{\sigma}\right) \left(1 + \theta_2 \gamma_2 \frac{1 + |x_k|}{\sigma}\right).$$

**D2** For any integers $k, j$ satisfying $|x_k| \leq \gamma_0\sigma$ and $(x_{k+j} - x_k)x_k \geq 0$,

$$P(X = a_{k+j}) \leq \gamma_3 \frac{\Delta}{\sigma} \exp\left(-\frac{x_k^2}{2} + \frac{\gamma_1}{2} \frac{x_k^3}{\sigma} - \gamma_4 (x_{k+j} - x_k)x_k\right).$$
Lemma 2. Assume that conditions (D1-D2) are satisfied. Then, for any \( x_k \) satisfying \( |x_k| \leq c_0 \sigma \),

\[
\Phi \left( x_k - c_1 \frac{1 + x_k^2}{\sigma} \right) \leq P(X \leq a_k) \leq \Phi \left( x_k + c_1 \frac{1 + x_k^2}{\sigma} \right),
\]

provided \( \sigma \geq c_2 \), where \( c_0, c_1 \) and \( c_2 \) are constants depending only on \( \gamma_0, \ldots, \gamma_4 \) and \( \Delta_0 \).

Note that condition \( \Delta \in (0, \Delta_0] \) is used to assure the uniformity of the assertion of the lemma in \( \Delta \).