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THE SYLOW SUBGROUPS OF A FINITE REDUCTIVE GROUP

MICHEL ENGUEHARD AND JEAN MICHEL

Dedicated to professor George Lusztig on the occasion of his 70th birthday

Abstract. We describe the structure of Sylow $\ell$-subgroups of a finite reductive group $G(F_q)$ when $q \not\equiv 0 \pmod{\ell}$ that we find governed by a complex reflection group attached to $G$ and $\ell$, which depends on $\ell$ only through the set of cyclotomic factors of the generic order of $G(F_q)$ whose value at $q$ is divisible by $\ell$. We also tackle the more general case of groups $G^F$ where $F$ is an isogeny some power of which is a Frobenius morphism.

1. Introduction

Definition 1.1. Let $G$ be a connected reductive group over $\overline{\mathbb{F}}_p$, and $F$ an isogeny such that some power of $F$ is a Frobenius endomorphism; then $G^F$ is what we call a finite reductive group. To this situation we attach a positive real number $q$ such that for some integer $n$, the isogeny $F^n$ is the Frobenius endomorphism attached to a $\mathbb{F}_{q^n}$-structure.

The goal of this note is to describe the Sylow $\ell$-subgroups of $G^F$ when $\ell$ is a prime different from $p$ and $G$ is semisimple. The structure of the Sylow $\ell$-subgroups of a Chevalley group was first described by [Gorenstein-Lyons] where they observed that they had a large normal abelian subgroup $(\mathbb{Z}/n)_{\ell}^d$ where $d$ is the multiplicative order of $q$ (mod $\ell$), and they computed a case by case.

In 1992 [Broué-Malle] exhibited subtori of $G^F$ attached to eigenspaces of elements of the Weyl reflection coset of $(G, F)$ whose $F$-stable points are the large abelian groups of [Gorenstein-Lyons]. To these eigenspaces are attached complex reflection groups by Springer’s theory.

We show that the structure of the Sylow $\ell$-subgroups of $G^F$ is determined by these complex reflection groups. The results of this note in the case when $F$ is a Frobenius were obtained by the first author in an unpublished note [Enguehard] of 1992; the second author has found a simpler (containing more casefree steps) proof which is an occasion to publish these results. Some of our results appeared also implicitly in [Malle].

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2. The generic Sylow theorems

Let $G$ be as in 1.1; an $F$-stable maximal torus $T$ of $G$ defines the Weyl group $W = N_G(T)/T$, that we may identify to a reflection subgroup of $GL(X(T))$ where $X(T) := \text{Hom}(T, G_m)$, attached to the root system $\Sigma \subset X(T)$ of $G$ with respect to $T$. The isogeny $F$ induces a $p$-morphism $F^* \in \text{End}(X(T))$ by the formula $F^*(x) = xqF$ for $x \in X(T)$, that is there is a permutation $\sigma$ of $\Sigma$ such that for $\alpha \in \Sigma$ we have $F^*(\alpha) = q_\alpha \sigma(\alpha)$ for some power $q_\alpha$ of $p$; in particular $F^* \in N_{\text{End}(X(T))}(W)$.

If $q, n$ are as in 1.1 then $F^{*n}$ is $q^n$ times an element of $GL(X(T))$ of finite order, thus over $X(T) \otimes \mathbb{Z}[q^{-1}]$ we have $F^* = q\phi$ where $\phi$ is an automorphism of finite order which normalizes $W$. We call $W\phi$ the reflection coset associated to $(G, F)$.

Our setting is more general than that of [Broué-Malle] who considered only the special cases where $F$ is a Frobenius endomorphism, or where $G^F$ is a Ree or Suzuki group. The results of the next subsection allow to extend the definition of Sylow $\Phi$-subtori of [Broué-Malle] to any $(G, F)$ as in 1.1.

**F-indecomposable tori.**

**Definition 2.1.** For $G, F$ as in 1.1, a non-trivial subtorus of $G$ is called $F$-indecomposable if it is $F$-stable and contains no proper non-trivial $F$-stable subtorus.

We say that a group $G$ is an almost direct product of subgroups $G_1$ and $G_2$ if they commute, generate $G$ and have finite intersection, and we define similarly an almost direct product of $k$ subgroups by induction on $k$.

**Proposition 2.2.** For $G, F$ as in 1.1, any $F$-stable subtorus $T$ of $G$ is an almost direct product of $F$-indecomposable tori $S_1, \ldots, S_k$ and $|T^F| = |S_1^F| \cdots |S_k^F|$.

**Proof.** An $F$-stable subtorus $S$ corresponds to a pure $F$-stable sublattice $X' \subset X := X(T)$ (see for example [Borel, III, Proposition 8.12]). Let $d$ be the smallest power of $F$ which is a split Frobenius, thus on $X(T)$ we have $F^{*d} = q^d \text{Id}$. Let $\pi \in \text{End}(X \otimes \mathbb{Q})$ be a projector on $X' \otimes \mathbb{Q}$. Then in $\text{End}(X \otimes \mathbb{Q})$ we can define the $F$-invariant projector $\pi' := d^{-1} \sum_{i=1}^d F^{*i} \pi F^{-i}$ and $\text{Ker} \pi' \cap X$ is another $F$-stable pure sublattice which after tensoring by $\mathbb{Q}$ becomes a complement to $X' \otimes \mathbb{Q}$. This corresponds to an $F$-stable subtorus $S'$ such that $K := S \cap S'$ is finite and $T = SS'$. Iterating, we get the first part of the proposition.

The second part of the proposition results from the next two lemmas. \hfill $\square$

**Lemma 2.3.** For $G, F$ as in 1.1, and $K$ an $F$-stable finite normal subgroup of $G$, then $|(G/K)^F| = |G^F|/|K^F|$.

**Proof.** First, we notice that $K$ is central, thus abelian, since conjugating by $G$ being continuous must be trivial on $K$.

Then, the Galois cohomology long exact sequence: $1 \to K^F \to G^F \to (G/K)^F \to H^1(F, K) \to 1$ shows the result using that $|K^F| = |H^1(F, K)|$. \hfill $\square$

**Lemma 2.4.** Let $G$ as 1.1 be an almost direct product of $F$-stable connected subgroups $G = G_1 \cdots G_k$. Then $|G^F| = |G_1^F| \cdots |G_k^F|$.

**Proof.** It is enough to consider the case $k = 2$ and then iterate. Thus, we assume $G = G_1G_2$ where $K = G_1 \cap G_2$ is finite. We quotient by $K$, which makes the product direct, and apply Lemma 2.3 twice. \hfill $\square$
Lemma 2.5. Let $S$ be an $F$-indecomposable torus, let $q$ be the smallest power such that $q^d \in \mathbb{Z}$, and let $d$ be the smallest power such that $F^{d}q$ is a split Frobenius on $S$. Let $F^* = q^d$ on $X(S)$; then the characteristic polynomial $\Phi$ of $\phi$ is a factor in $\mathbb{Z}[x, q^{-1}]$ of $\Phi_d(x^q)$, where $\Phi_d(x)$ denotes the $d$-th cyclotomic polynomial. Further $q^{-\deg \Phi} \Phi(x/q) \in \mathbb{Z}[x]$ is irreducible and $|S^F| = \Phi(q)$.

Proof. Since $F^{d}q^d$ acts as $q^d\eta$ on $X := X(S)$, the minimal polynomial $P$ of $F^*$ divides $x^{d\eta} - q\eta$.

The polynomial $P$ is irreducible over $\mathbb{Z}$, otherwise a proper nontrivial factor $P_1$ defines an $F^*$-stable pure non-trivial sublattice $\text{Ker}(P_1(F^*))$ of $X$, which contradicts $F$-indecomposability of $S$.

It follows that $X$ is a $\mathbb{Z}[x]/P$-module by making $x$ act by $F^*$, and $X \otimes \mathbb{Q}[x]/P$ is a one-dimensional $\mathbb{Q}[x]/P$-vector space, otherwise a proper nontrivial subspace would define an $F^*$-stable pure sublattice of $X$. It follows that $\dim S = \deg P = \dim X$ and thus $P$ is also the characteristic polynomial of $F^*$.

We have in $\mathbb{Z}[x]$ the equality $x^{d\eta} - q\eta = \prod_{d|\text{deg} P} (q^{\eta\text{deg} \Phi_d} \Phi_d(x^{\eta/q}))$. Since $P$ is irreducible it divides one of the factors, and since $\text{deg} \Phi$ is minimal such that $F^{d\eta} = q\eta \text{Id}$, that is minimal such that $P$ divides $x^{d\eta} - q\eta$, we have that $P$ divides $q^{\eta\text{deg} \Phi_d} \Phi_d(x^{\eta/q})$; equivalently $\Phi = q^{-\deg P}P(qx)$ divides $\Phi_d(x^q)$.

We have $|S^F| = |\text{Irr}(S^F)| = |X/(F^*-1)X| = \det(F^*-1) = (-1)^{\deg P}P(1) = (-q)^{\deg \Phi}(1/q)$ where the second equality reflects the well known group isomorphism $\text{Irr}(S^F) \simeq X/(F^*-1)X$ and the third is a general property of lattices. Finally, since $\Phi$ is real and divides $\Phi_d(x^q)$, its roots are stable under taking inverses, thus $(-q)^{\deg \Phi}(1/q) = \Phi(q)$.

We call $\Phi$-cyclotomic the polynomials $\Phi$ of Lemma 2.5. In other terms

Definition 2.6. For $q$ as in 1.1, where $q^d$ is the smallest power of $q$ in $\mathbb{Z}$, we call $\Phi$-cyclotomic the monic polynomials $\Phi \in \mathbb{Z}[x, q^{-1}]$ such that $q^{-\deg \Phi}(x/q)$ is a $\mathbb{Z}[x]$-irreducible factor of some $x^{d\eta} - q\eta$.

In the study of semisimple reductive groups we will need the $\Phi$-cyclotomic polynomials of Lemma 2.7. Note that if $d$ is minimal in Definition 2.6, then $\Phi$ is a factor in $\mathbb{Z}[x, q^{-1}]$ of $\Phi_d(x^q)$. We are interested in that number $d$ rather than $\text{deg} \Phi$, and to emphasize this we write $\Phi_{q,d}$ in the following examples.

Lemma 2.7. When $q \in \mathbb{Z}$, the $\Phi$-cyclotomic polynomials are the cyclotomic polynomials.

When $q$ is an odd power of $\sqrt{2}$, the following polynomials are $\Phi$-cyclotomic: $\Phi_{2,1}(x) := \Phi_1(x^2)$, $\Phi_{2,2}(x) := \Phi_2(x^2)$, $\Phi_{2,6}(x) := \Phi_6(x^2)$, the factors $\Phi_{2,4} := x^2 + \sqrt{2}x + 1$ and $\Phi_{2,8} := x^2 - \sqrt{2}x + 1$ of $\Phi_4(x^2)$, and the factors $\Phi_{2,12} := x^4 + x^3\sqrt{2} + x^2 + x\sqrt{2} + 1$ and $\Phi_{2,12} := x^4 - x^3\sqrt{2} + x^2 - x\sqrt{2} + 1$ of $\Phi_{12}(x^2)$.

When $q$ is an odd power of $\sqrt{3}$, the following polynomials are $\Phi$-cyclotomic: $\Phi_{2,1}(x)$, $\Phi_{2,2}(x)$ and the factors $\Phi_{2,6} := x^2 + x\sqrt{3} + 1$ and $\Phi_{2,8} := x^2 - x\sqrt{3} + 1$ of $\Phi_6(x^2)$.

Proof. When $q \in \mathbb{Z}$ the formula $P \mapsto q^{-\deg P}P(qx)$ establishes a bijection between $\mathbb{Z}[x]$-irreducible factors of $x^d - q^d$ and $\mathbb{Z}[x]$-irreducible factors of $x^d - 1$, that is cyclotomic polynomials, which gives the first case of the lemma.

For the other cases, we have to check for each given $\Phi$ that $q^{-\deg \Phi}(x/q)$ is in $\mathbb{Z}[x]$ and irreducible.
Proposition 2.8. Let $S, \eta, d, \Phi$ be as in 2.5 and let $P = q^{\deg \Phi} \Phi(x^n/q^n)$ be the characteristic polynomial of $F^*$. Let $m$ be a divisor of $\Phi(q)$, and assume either that $d \in \{1, 2\}$ and $q \in \mathbb{Z}$ or that $m$ is prime to $dn$; then we have a natural isomorphism $\operatorname{Irr}(S^F)/m \operatorname{Irr}(S^F) \simeq \operatorname{Ker}(F^*-1 \mid X(S)/mX(S))$.

Proof. Proceeding as in the proof of Lemma 2.5 we set $X = X(S)$ and $\bar{X} = X/(F^*-1)X \simeq \operatorname{Irr}(S^F)$. Letting $x$ act as $F^*$ makes $X$ into a $\mathbb{Z}[x]/P$-module, and $X$ a $\mathbb{Z}[x]/(P, x-1)$-module. Since $\mathbb{Z}[x]/(P, x-1) = \mathbb{Z}/P(1) = \mathbb{Z}/\Phi(q)$ we find that the exponent of $X$ divides $\Phi(q)$.

Let $A := \mathbb{Z}[x, q^{-1}]/P$. The extension $\mathbb{Z}[x]/P \hookrightarrow A/P$ is flat thus $\bar{X} \otimes_{\mathbb{Z}[x]/P} A \simeq X'/((F^*-1)X')$ where $X' = X \otimes_{\mathbb{Z}[x]/P} A$; and since the exponent of $X$ divides $\Phi(q)$ which is prime to $q^n$, we have $X \simeq X \otimes_{\mathbb{Z}[x]/P} A$. Under the assumptions of (1) the ring $A$ is Dedekind: if $\eta \neq 1$ then $A$ is integrally closed thus Dedekind; if $\eta = 1$ then $A \simeq \mathbb{Z}[x, q^{-1}]/\Phi_d$ where the isomorphism is given by $x \mapsto x/q$, and is a localization of the Dedekind ring $\mathbb{Z}[x]/\Phi_d$ by $q$. Thus $X'$ identifies to a fractional ideal $\mathfrak{I}$ of $A$ and $\bar{X} \simeq \mathfrak{I}((x-1)\mathfrak{I})$. If $e$ is the exponent of $\bar{X}$ we have thus $c \in (x-1)\mathfrak{I}$, which implies that $x-1$ divides $e$ in $A$. This in turn implies that the norm $(-1)^{\deg P} P(1) = \Phi(q)$ of $(x-1)$ divides $e$ in $\mathbb{Z}$, thus $e = \Phi(q)$ and $\bar{X} \simeq \mathbb{Z}/\Phi(q)$ and the same isomorphism holds for the dual abelian group $S^F$.

(For 2), note that by construction $X/mX$ is the biggest quotient of $X$ on which both $F^*-1$ and the multiplication by $m$ vanish. It is thus equal to the biggest quotient of $X/mX$ on which $F^*-1$ vanishes. Thus the question is that $\operatorname{Ker}(F^*-1)$ has a complement in $X/mX$.

If $q \in \mathbb{Z}$ and $d \in \{1, 2\}$ we have $P = x \pm q$ so $X \simeq \mathbb{Z}$ on which $F^*$ acts by $x \mapsto q$ and $X = X/(q \pm 1)$ of which $X/mX$ is a quotient, so $F^*-1$ vanishes on $X/mX$ which is thus equal to $X/mX$ and there is nothing to prove.

Assume now $m$ prime to $dn$. There exists $R \in \mathbb{Z}[x]$ such that in $\mathbb{Z}[x]$ we have $P = (x-1)R + P(1)$. Taking derivatives, we get $P' = (x-1)R' + R$, whence $P(1) = P'(1)$. Let $\delta$ be the discriminant of $P$; we can find polynomials $M, N \in \mathbb{Z}[x]$ such that $MP + NP' = \delta$, which evaluating at 1 gives $M(1)P(1) + N(1)P'(1) = \delta$. Since $q$ is prime to $P(1)$, thus to $m$, and $\delta$ is a divisor of the discriminant of $X^{dn} - q^{dn}$, equal to $q^{dn(dn-1)(dn)^{\delta/dn}}$, thus prime to $m$, we find that $P'(1)$ is prime to $m$. In $(\mathbb{Z}/m)[x]$ we have $P = (x-1)R$, thus applied to $F^*$ we get that on $X/mX$ we have $0 = P(F^*) = (F^*-1)R(F^*)$, whence $\operatorname{Ker}(F^*-1) \cap \operatorname{Ker}(R(F^*)) = X/mX$. Since $R(1)$ is prime to $m$, we can write $1 = Q(x-1) + aR$ in $(\mathbb{Z}/m)[x]$ for some $Q \in (\mathbb{Z}/m)[x]$ and the inverse $(\mod m)$ of $R(1)$. This proves that $\operatorname{Ker}(F^*-1) \cap \operatorname{Ker}(R(F^*)) = 0$ thus $X/mX$ is the direct sum of $\operatorname{Ker}(F^*-1)$ and $\operatorname{Ker}(R(F^*))$. q.e.d.

Complex reflection cosets. (1) to (3) below are classical results of Springer and Lehrer.

Proposition 2.9. Let $V$ be a finite dimensional vector space over a subfield $k$ of $\mathbb{C}$, let $W \subset \operatorname{GL}(V)$ be a finite complex reflection group and let $\phi \in N_{\operatorname{GL}(V)}(W)$, so that $W \phi$ is a reflection coset; let $(d_1, \varepsilon_1), \ldots, (d_n, \varepsilon_n)$ be its generalized degrees (see for instance [Broué, 4.2]). For $\varsigma$ a root of unity define $a(\varsigma)$ as the multiset of the $d_i$ such that $\varepsilon_i = \varepsilon_i$. Then:
(1) For any root of unity $\zeta$, the maximum dimension when $w\phi$ runs over $W\phi$ of a $\zeta$-eigenspace of $\psi$ on $V \otimes_k k[\zeta]$ is $|a(\zeta)|$.

(2) For $w\phi \in W\phi$ denote $V_{w,\zeta} \subset V \otimes_k k[\zeta]$ its $\zeta$-eigenspace. Assume $\dim V_{w,\zeta} = |a(\zeta)|$ and let $C = C_W(V_{w,\zeta})$ and $N = N_W(V_{w,\zeta})$. Then $N/C$ is a complex reflection group acting on $V_{w,\zeta}$, with reflection degrees $a(\zeta)$.

(3) Any two subspaces $V_{w,\zeta}$ and $V_{w',\zeta}$ of dimension $|a(\zeta)|$ are $W$-conjugate.

(4) For $\psi$ as in (2) the natural actions of $w\phi$ on $N$ and $C$ induce the trivial action on $N/C$.

(5) Let $a \in \mathbb{Z}$ be such that $(W\phi)^a = W\phi$ and such that $\zeta$ and $\zeta^a$ are conjugate by $\Gal(k(\zeta)/k)$. Then for $w\phi$ as in (2) there exists $v \in N_W(N) \cap N_W(C)$ which conjugates $w\phi C$ to $(w\phi)^a C$.

Proof. For (1) see for instance [Broué, 5.2], for (2) see [Broué, 5.6(3) and (4)] and for (3) see [Broué, 5.6 (1)]. (4) results from the observation that if $n \in N$ and $v \in V_{w,\zeta}$ then $(n^{-1} \cdot \psi n v) = (n^{-1} \cdot w\psi n (w\phi)^{-1} v) = (n^{-1} \cdot w\psi n (\zeta^{-1} v)) = (n^{-1} \cdot w\psi n (\zeta^{-1} n v)) = (n^{-1} \cdot n v) = v$ thus $n^{-1} \cdot w\psi n \in C$.

For (5), $\Gal(k(\zeta)/k)$ acts naturally on $V \otimes_k k[\zeta]$, commuting with $GL(V)$, in particular with $W$ and $\phi$. If $\sigma \in \Gal(k(\zeta)/k)$ is such that $\sigma(\zeta) = \zeta^a$, let $\zeta^a = \sigma^{-1}(\zeta)$. Then $\sigma^{-1}(V_{w,\zeta}) = V_{w,\zeta^a}$. It follows that $N = N_W(V_{w,\zeta^a})$ and $C = C_W(V_{w,\zeta^a})$.

Now since $a'$ is the inverse of a modulo the order of $\zeta$ the space $V_{w,\zeta^a}$ is the $\zeta$-eigenspace of $(w\phi)^a$. By assumption we have $(\psi a) \in W\phi$. Since two maximal $\zeta$-eigenspaces of elements of $W\phi$ are conjugate by (3) there exists $v \in W$ which conjugates $V_{w,\zeta}$ to $V_{w,\zeta^a}$, and $v \in N_W(N) \cap N_W(C)$ since $N = N_W(V_{w,\zeta^a})$ and $C = C_W(V_{w,\zeta^a})$. The element $v$ thus conjugates the set $w\phi C$ of elements which have $V_{w,\zeta}$ as $\zeta$-eigenspace to the set $(w\phi)^a C$ of elements which have $V_{w,\zeta^a}$ as $\zeta$-eigenspace.

Generic Sylow subgroups. We define the Sylow $\Phi$-subtori of $(G, F)$, first in the case when $G$ is quasi-simple, then in the case of descent of scalars.

From now on we assume $G$ semisimple. Then, if $(d_1, \epsilon_1), \ldots, (d_n, \epsilon_n)$ are the generalized degrees of the reflection coset $W\phi$, we have (see [Steinberg, 11.16])

\begin{equation}
|G^F| = q^{\Sigma_k(d_k-1)} \prod_i (q^{d_i} - \epsilon_i).
\end{equation}

Proposition 2.11. Let $G$ be as in 1.1 and quasi-simple. Then we can rewrite the order formula 2.10 for $|G^F|$ as

\begin{equation}
|G^F| = q^{\Sigma_k(d_k-1)} \prod_{\Phi \in \mathcal{P}} \Phi(q)^{n(\Phi)}
\end{equation}

where $\mathcal{P}$ is a set of $q$-cyclotomic polynomials, and where $0 \neq n(\Phi) = |a(\zeta)|$ (see 2.9) for any root $\zeta$ of $\Phi$. For each $\Phi \in \mathcal{P}$ there exists a non-trivial $F$-stable subtorus $S_\Phi$ of $G$ such that $|S_\Phi^F| = \Phi(q)^{n(\Phi)}$.

We note that if $G^F$ is a Ree or Suzuki group, the $\eta$ of Definition 2.6 is 2. Otherwise $\eta = 1$ and the $q$-cyclotomic polynomials are cyclotomic polynomials.

We call any $F$-stable torus $S$ such that $|S^F|$ is a power of $\Phi(q)$ a $\Phi$-torus, and tori $S_\Phi$ as above are called Sylow $\Phi$-subtori of $(G, F)$ — we abuse notation and call
them Sylow $\Phi$-subtori of $G$ when $F$ is clear from the context; they are the most direct product of $n_\phi$ $F$-indecomposable $\Phi$-tori.

Proof. Proposition 2.11 is essentially in [Broué-Malle] but let us reprove it.

First, we note that assuming $|G^F|$ has a decomposition of the form 2.12, the value of $n_\phi$ results from 2.10: let $\zeta$ be any root of $\Phi(x)$. Then $(x-\zeta)$ divides $\Phi(x)$ with multiplicity one, and does not divide any other $\Phi'(x)$ for $\Phi' \in \mathcal{P}$ since the $\Phi(x/q)$ are distinct irreducible polynomials in $\mathbb{Q}[x]$. Thus $n_\phi$ is the number of pairs $(d_i, \varepsilon_i)$ such that $x-\zeta$ divides $x^{d_i} - \varepsilon_i$.

There is a decomposition of the form 2.12: if $\eta = 1$ we get such a decomposition of $|G^F|$ by decomposing each term of 2.10 into a product of cyclotomic polynomials. Otherwise $G^F$ is a Ree or Suzuki group, $\eta = 2$ and $q$ is an odd power of $\sqrt{2}$ or $\sqrt{3}$, and the set $\mathcal{P}$ and the decomposition of the form 2.12 is given by what follows:

| $(G, F)$ | $|G^F|$ | generalized degrees of $W\phi$ |
|---|---|---|
| $2B_2(q^r)$ | $q^r(\Phi_1^2, \Phi_2^2, \Phi_3^2)(q)$ | $\{(2, 1), (4, -1)\}$ |
| $2F_4(q^r)$ | $q^{24}(\Phi_1^2, \Phi_2^2, \Phi_3^2, \Phi_4^2, \Phi_5^2, \Phi_6^2, \Phi_7^2, \Phi_8^2, \Phi_9^2, \Phi_{10}^2, \Phi_{11}^2, \Phi_{12}^2)(q)$ | $\{(2, 1), (6, -1), (8, 1), (12, -1)\}$ |
| $2G_2(q^r)$ | $q^6(\Phi_1^2, \Phi_2^2, \Phi_3^2, \Phi_4^2, \Phi_5^2, \Phi_6^2)(q)$ | $\{(2, 1), (6, -1)\}$ |

Note that for $\eta = 2$ our “$q$-cyclotomic polynomials” are the “($tp$)-cyclotomic polynomials” defined in [Broué-Malle, 3.14].

To construct the torus $S_\Phi$ for $\Phi \in \mathcal{P}$, let us choose $\zeta$ a root of $\Phi$ and $w$ as in (2) of Proposition 2.9. Then if $T_w$ is a maximal torus of type $w$ with respect to $T$, so that $(T_w, F) \simeq (T, wF)$, the characteristic polynomial of $w\phi$ on $X(T)$ has $\Phi(x)^{n_\phi}$ as a factor; the kernel of $\Phi(w\phi)$ on $X(T)$ is a pure sublattice corresponding to a subtorus $S_\Phi$ of $T_w$ such that $|S_\Phi^F| = \Phi(q)^{n_\phi}$.

Proposition 2.13. Let $(G, F)$ be as in 1.1, semisimple and such that the Dynkin diagram of $G$ has $n$ connected components permuted transitively by $F$. Then there exists a reductive group $G_1$ with isogeny $F_1$ such that up to isomorphism $G$ is a “descent of scalars” $G = G_1^F$ with $F(g_1, \ldots, g_n) = (g_2, \ldots, g_n, F_1(g_1))$.

Then $G^F \simeq G_1^{F_1}$, and if the scalar associated to $(G, F)$ is $q$ that associated to $(G_1, F_1)$ is $q_1 := q^{n_1}$. Thus we have $|G^F| = q^{n \sum d_i - 1} \prod_{\Phi \in \mathcal{P}} \Phi(q)^{n_\phi}$ where $d_i, \mathcal{P}, n_\phi$ are as given by 2.11 for $(G_1, F_1, q_1)$.

Here again, for $\Phi \in \mathcal{P}$ there exists a Sylow $\Phi$-subtorus of $G$, that is an $F$-stable subtorus $S_\Phi$ such that $|S_\Phi^F| = \Phi(q)^{n_\phi}$.

Proof. The proposition is obvious apart perhaps for the statement about the existence of $S_\Phi$. This results in particular from the following lemma that we need for future reference.

Lemma 2.14. In the situation of Proposition 2.13, let $(T, wF)$ where $T = T_1^F$ be a maximal torus of type $w = (1, \ldots, 1, w_1)$ of $G$ and define $\phi$ on $V = X(T) \otimes \mathbb{C}$ (resp. $\phi_1$ on $V_1 = X(T_1) \otimes \mathbb{C}$) by $F^* = q \phi$ (resp. $F_1^* = q_1 \phi_1$). Then if the characteristic polynomial of $w_1\phi_1$ is $P(x)$, that of $w\phi$ is $P(x^n)$. Let $\Phi$ be a $q_1$-cyclotomic factor of $P$ (corresponding to a $\mathbb{Z}[x]$-irreducible factor of the characteristic polynomial of $w_1\phi_1$) and let $\zeta$ be a root of $\Phi(x^n)$. Denote by $V_{\zeta}$ the $\zeta$-eigenspace of $w\phi$ (resp. by $V_1_{\zeta}$ the $\zeta^n$-eigenspace of $w_1\phi_1$).

Let $S_1$ be the Sylow $\Phi$-subtorus of $(G_1, F_1)$ determined by $\text{Ker}(\Phi(w_1\phi_1))$, and $S$ be the $wF$-stable subtorus of $T$ determined by $\text{Ker}(\Phi((w\phi)^n))$. Then $S$ is a Sylow
composes in several cyclotomic polynomials according to the formula
\[ \Phi \]
where \( \Sigma \) is the map \( (x) \rightarrow w \). By the analysis above, \( \phi \) is an isomorphism. Extended to a Coxeter torus of a semisimple group \( Z \), we have an isomorphism \( N(w) \rightarrow C(W) \). Let \( \Phi \) be defined in terms of \( w \). We have the following commutative diagram
\[ \begin{array}{ccc}
X & \xrightarrow{(w)^{-1}} & X \\
\downarrow{\Sigma} & & \downarrow{\Sigma} \\
X_1 & \xrightarrow{(w)^{-1}} & X_1
\end{array} \]
where \( \Sigma \) is the map \( (x_1, \ldots, x_n) \rightarrow x_1 + \cdots + x_n \). Since we have \( \Sigma \circ (w)^n = w_1 \circ \Sigma \), for any polynomial \( Q \) the morphism \( \Sigma \) induces a surjective morphism \( \text{Ker}(Q((wF)^n)) \rightarrow \text{Ker}(Q(wF)) \). Whence for \( Q = P \) a surjection \( \text{Irr}(S^{wF}) \rightarrow \text{Irr}(S^{wF}) \), since \( |S^{wF}| \) is prime to \( |T^{wF}/S^{wF}| \) this surjection must be an isomorphism. Extended to \( V = X \otimes \mathbb{C} \), the map \( \Sigma \) sends \( V \) to \( V_{1, \zeta} \) and sends the action of \( N_{W}(V_{1, \zeta})/C_{W}(V_{1, \zeta}) \) to that of \( N_{W}(V_{1, \zeta})/C_{W}(V_{1, \zeta}) \), whence the last statement of the lemma.

Note that any element of \( W \phi \) is conjugate to an element of the form \( (1, \ldots, 1, w_1) \phi_1 \) so the form of \( w \) in the statement of Lemma 2.14 covers all the types of maximal tori.

**Remark 2.15.** If the generalized degrees of \( W_1 \phi_1 \) are \( (d_i, \epsilon_i) \), those of \( W \phi \) are \( (d_i, n_j) \) where \( n_j \) runs over the \( n \)-th roots of \( \epsilon_i \). It follows that \( n_\Phi \) can be defined in terms of \( W \phi \) as it is also the number of \( (d_i, n_j) \) such that \( \zeta^{d_i} = n_j \), where \( \zeta \) is any root of \( \Phi(x^n) \).

**Remark 2.16.** For \( \Phi \in \mathcal{P}(G) \), a Sylow \( \Phi \)-subtorus of \( G \) is a “power” of a subtorus \( S_0 \) such that \( S_0^\Phi = \Phi(q) \). If \( G \) is quasi-simple, such a subtorus \( S_0 \) is \( F \)-indecomposable (since then the polynomial \( \Phi \) is \( q \)-cyclic). But this is no longer true for a descent of scalars. First, a cyclotomic polynomial in \( x^n \) decomposes in several cyclotomic polynomials according to the formula \( \Phi_d(x^n) = \prod_{\{\mu(n, d) \text{ prime to } d\}} \Phi_{\mu(d)}(x) \) (see [Broué-Malle, Appendice 2]). But there could be further decompositions: for instance, the characteristic polynomial of \( F^* \) on a Coxeter torus of a semisimple group \( G \) of type \( B_2 \) over \( \mathbb{F}_2 \) is \( x^2 + 4 \), which is \( \mathbb{Z} \)-irreducible. But on a descent of scalars \( G \times G \), the characteristic polynomial...
of \( F^* \) on a lift of scalars of this torus is \( x^4 + 4 \) which is no longer \( \mathbb{Z} \)-irreducible: 
\[ x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2), \]
so the torus seen inside the descent of scalars is no longer \( F \)-indecomposable.

We could have decomposed \( |G^F| \) into a product of \( q \)-cyclotomic polynomials corresponding to \( F \)-indecomposable tori, but in the case of descent of scalars it was convenient to use larger tori.

**Remark 2.17.** An arbitrary semisimple reductive group is of the form \( G = G_1 \ldots G_k \), an almost direct product of descents of quasi-simple groups \( G_i \), corresponding to the orbits of \( F \) on the connected components of the Dynkin diagram of \( G \). Then we have \( |G^F| = |G_1^F| \ldots |G_k^F| \) by Lemma 2.4, and similarly, if \( S \) is an \( F \)-stable torus of \( G \), and \( S_i = S \cap G_i \), then \( |S^F| = |S_1^F| \ldots |S_k^F| \). This can be used to give a global decomposition of \( |G^F| \), but the polynomials \( P \) in one factor could divide those in another. For instance we could have \( \Phi_{2,4} \) for a factor of \( G \) of type \( 2B_2 \) and \( \Phi_8 \) for another factor of type \( B_2 \). Because of this it is cumbersome to give a global statement.

From now on we fix \((G, F)\) as in 2.13, which determines \( q, n \), and \( \eta \) minimal such that \( q^{\eta n} \in \mathbb{Z} \). This allows in the next definition to omit the mention of \( G \) and \( F \) from the notation \( d(\ell) \).

**Definition 2.18.** Let \( \ell \) be a prime number different from \( p \). In the context of 2.13 we define \( d(\ell) \) as the order of \( q^{\eta n} \) \( \pmod{\ell} \) \( \pmod{4} \) if \( \ell = 2 \).

In particular \( \ell | \Phi_\delta(\ell)(q^{\eta n}) \).

The next proposition extends some of the Sylow theorems of [Broué-Malle], and introduces a complex reflection group \( W_\Phi \) attached to each \( \Phi \) in the set \( \mathcal{P} \) of 2.11.

**Proposition 2.19.** Under the assumptions of 2.13, let \( T \) be an \( F \)-stable maximal torus of \( G \) in an \( F \)-stable Borel subgroup, and let \( W_\Phi \subset \text{GL}(X(T)) \) be the reflection coset associated to \((G, F)\). Then for each \( \Phi \in \mathcal{P} \):

1. \( \zeta \) is a root of \( \Phi(x^n) \) and \( w \) is as in 2.9(2), a maximal torus of \( G \) of type \( w \) with respect to \( T \) contains a unique Sylow \( \Phi \)-subtorus \( S \).

For \( \zeta, w \) as in (1) let \( W_\Phi = N_W(V_\zeta)/C_W(V_\zeta) \) where \( V_\zeta \) is the \( \zeta \)-eigenspace of \( w_\Phi \) on \( V = X(T) \otimes \mathbb{C} \).

2. For \( S \) as in (1) we have \( N_{G^F}(S)/C_{G^F}(S) = N_G(S)/C_G(S) \cong W_\Phi \), and \( W_\Phi \) can be identified to a subgroup of \( \text{GL}(X(S)) \).

3. The Sylow \( \Phi \)-tori of \( G \) are \( G^F \)-conjugate.

4. Let \( \ell \neq p \) be a prime number, and assume that \( \Phi \) divides \( \Phi_{d(\ell)} \) (see Definition 2.18). Then unless \( \ell = 2 \) and \((G_1, F_1)\) is of type \( 2G_2 \), any Sylow \( \ell \)-subgroup of \( W_\Phi \) acts faithfully on the subgroup of \( \ell \)-elements \( S_\ell^F \) of \( S^F \).

**Proof.** For (1) we consider a torus \((T, wF)\) of type \( w \). Then a \( wF \)-stable subtorus corresponds to the span of a subset of eigenspaces of \( w_\Phi \) on \( V \). Since the polynomials \( \Phi \) are prime to each other the polynomials \( \Phi(x^n) \) are also, thus \( q^\zeta \) is root of no other factor of the characteristic polynomial of \( w_\Phi \) than \( \Phi(x^n) \). Thus the \( S \) defined in Lemma 2.14, which we will denote \( S_0 \), is unique.

Let us show (2). Let \((T_w, F, S)\) be conjugate to \((T, wF, S_0)\). Let \( L = C_G(S) \), which, as the centralizer of a torus, is a Levi subgroup. Then we note that \( N_G(S) \subset N_G(L) \). It follows that we can find representatives of \( N_G(S) \) modulo \( L \) in \( N_G(T_w) \) since for \( n \in N_G(S) \) the torus \( ^nT_w \) is another maximal torus of \( L \) which
is thus $L$-conjugate to $T_w$. We thus get that $N_G(S)/L = N_G(S, T_w)/(N_G(T_w) \cap L)$; transferring this to $T$ and then to $W$ we get $N_G(S, T_w)/(N_G(T_w) \cap L) \simeq N_W(S_0)/C_W(S_0)$ where $S_0$ is the subtorus of $T$ determined by $\text{Ker}(P(wF^*))$ where $P = \Phi(x^n/q^n)$. The action of $F$ is transferred to the action of $w\phi$ on this quotient.

That $N_W(S_0) = N_W(V_\zeta)$ and $C_W(S_0) = C_W(V_\zeta)$ was given in 2.14.

By 2.9(4) we see that the action of $w\phi$ on $N_W(S_0)/C_W(S_0)$ is trivial, thus also that of $F$ on $N_G(S)/C_G(S)$, thus $N_G(S)/C_G(S) = (N_G(S)/C_G(S))^F = N_G(S)^F/C_G(S)^F = N_G(S)/C_G(S)$, the second equality since $L = C_G(S)$ is connected. Finally, the last part of (2) results from the fact that the representation of $W_\Phi$ on $X(S_0)$, extended to $X(S_0) \otimes \mathbb{C}$ has as summand the representation of $W_\Phi$ on $V_\zeta$, which is the reflection representation, thus faithful.

(3) is a direct translation of 2.9(3): when brought to subtori of $T$ corresponding to eigenspaces of $w\phi$ (resp. $w'\phi$) the $G^F$-conjugacy of two Sylow $\Phi$-subtori corresponds to the $W$-conjugacy of the corresponding eigenspaces.

For (4) we first remark that we can reduce to the case where $G$ is quasi-simple, using 2.14. Thus either $q \in \mathbb{Z}$ or $G^F$ is a Ree or a Suzuki group. Let $\delta$ be the order of the coset $W\phi$, that is the smallest integer such that $(W\phi)^\delta = W$. We have $\delta \in \{1, 2, 3\}$. We first show the

**Lemma 2.20.** If $G$ is quasi-simple and we are in one of the cases:

1. $q \in \mathbb{Z}$ and $\delta \in \{1, 2\}$.
2. $q \in \mathbb{Z}$, $\delta = 3$ and $d$ is prime to $3$.
3. $q$ is an odd power of $\sqrt{2}$ and $d = 3$.

then $W_\Phi$ acts faithfully on $S_0^F$.

**Proof.** On $X(T) \otimes \mathbb{Q}(q^{-1})$ we have $wF^* = qw\phi$. The characteristic polynomial $Q$ of $wF^*$ on $X(S)$ is $q^{n_q \deg \Phi} \Phi(x/q)^{n_q}$; as $wF^*$ is semisimple, the minimal polynomial of $wF^*$ is $P = q^{\deg \Phi(x/q)}$. We can identify $X(S)$ with $\text{Ker}(P(qw\phi))$ on $X(T)$. As in the proof of Proposition 2.8, if $X = X(S)$ we can make $X' = X \otimes \mathbb{Z}[q^{-1}]$ an $A$-module where $A = \mathbb{Z}[x, q^{-n}]$. Under the assumptions of the lemma $A$ is a Dedekind ring. This results from the proof of 2.8(1) when $q \in \mathbb{Z}$. In the remaining case (3) of Lemma 2.20, $\eta = 2$ and the order of $q^2 \pmod{3}$ is 2, thus $\Phi = x^2 + 1$ and $P = x^2 + q^2$; we have $A = \mathbb{Z}[x, q^{-2}]/P \simeq \mathbb{Z}[1/2, \sqrt{-2}]$ which is integrally closed (thus Dedekind) since localized of $\mathbb{Z}[\sqrt{-2}]$ which is integrally closed. As an $A$-module of rank $n_{\Phi}$, the module $X'$ is a sum of projective rank 1 submodules thus $S$ is a product of $n_{\Phi}$ copies of a $wF$-indecomposable torus. By Proposition 2.19(2) we can identify $W_\Phi$ to a subgroup of $\text{GL}(X)$. With the notations of 2.8, since the assumption of 2.8(1) is satisfied, $X := X/(wF^* - 1).X \simeq \text{Irr}(S_0^F)$ is isomorphic to $(Z/\Phi(q))^{n_q}$. The representation of $W_\Phi$ on $X$ reduces to $X$. We will show it is faithful on $X/\ell X$ (or $X/4X$ when $\ell = 2$).

If $q \in \mathbb{Z}$ and $\ell = 2$ then $d \in \{1, 2\}$ and we can apply Proposition 2.8(2) taking $m = 4$. We get that $X/4X \simeq Ker(wF^* - 1 \mid X/4X)$. We have as observed in the proof of Proposition 2.8 that $\text{Ker}(wF^* - 1) = X/4X$ and the representation of $W_\Phi$ on $X/4X$, which is a quotient of $\text{Irr}(S_0^F)$, is faithful by Lemma 4.3.

If $q \in \mathbb{Z}$ and $\ell \neq 2$ then $d$ is prime to $\ell$; and in case (3) of Lemma 2.20 $\eta = 2$, $\ell = 3$ thus $d = 2$ and $\ell$ is prime to $d\eta$. In both cases we can apply Proposition 2.8(2) with $m = \ell$ to get that $X/\ell X \simeq Ker(wF^* - 1 \mid X/\ell X)$. We know by Lemma 4.3 that the representation of $W_\Phi$ on $X/\ell X$ is faithful and we would like to conclude that it is faithful on the submodule $\text{Ker}(wF^* - 1)$. We use the element $v$ given
by Proposition 2.9(5): it preserves the kernel of \( \Phi(w\phi) \) thus induces an element of \( \text{GL}(X) \) which defines an automorphism \( \sigma \) of \( W_\ell \) which sends \( w\phi \) to \((w\phi)^a\), so it remains true after reduction \((\mod \ell)\) that \( \sigma \) sends \( w\phi \) to \((w\phi)^a\), thus permutes the eigenspaces of \( wF^* \) on \( X/\ell X \): since \( d \) is the order of \( q \) \((\mod \ell)\), all the primitive \( d \)-th roots of unity live in \( \mathbb{F}_\ell \) and the eigenvalues of \( wF^* \) are the product of one primitive \( d \)-th root of unity, which is \( q \), by the other primitive \( d \)-th roots of unity so are of the form \( q^{1-a} \) where a runs over \((\mathbb{Z}/d)^\times\). And under the assumption \((W\phi)^a = W\phi \) of 2.9(5) we can find \( v \) thus \( \sigma \) which sends the \( q^{1-a} \)-eigenspace of \( wF^* \) to the \( q^{1-1} = 1 \)-eigenspace.

If every \( a \) prime to \( d \) has a representative in \( 1 + \delta \mathbb{Z} \) we can satisfy \((W\phi)^a = W\phi \) for such \( a \) thus every eigenspace is isomorphic as a \( W_\ell \)-module to \( \text{Ker}(wF^* - 1) \). Then \( W_\ell \) is faithful on the whole \( X/\ell X \) if and only if it is faithful on \( \text{Ker}(wF^* - 1) \), thus we conclude. If \( a \equiv 1 \pmod{\gcd(d, \delta)} \) then by Bezout’s theorem there exist integers \( \alpha, \beta \) such that \( a = 1 + ad + \beta \delta \), and then \( a - ad \in 1 + \delta \mathbb{Z} \) is a representative of \( a \).

If \( \delta = 1 \) or \( \delta = 2 \) then every \( a \) prime to \( d \) is \( \equiv 1 \pmod{\gcd(d, \delta)} \) and we conclude. We conclude similarly if \( \delta = 3 \) and \( d \) is prime to 3, or in case (3) of Lemma 2.20 since in this case \( d = 2 \).

When \( q \in \mathbb{Z} \) the only case not covered by the lemma is \( ^3D_4 \) and \( d \) divisible by 3, that is \( d \in \{3, 6, 12\} \). But in this case \( \ell > 3 \), since \( d \) is the order of \( q \) \((\mod \ell)\), thus \(|W|\) is prime to \( \ell \) and a fortiori the Sylow \( \ell \)-subgroup of \( W_\ell \) is trivial.

For the Ree and Suzuki groups we do not have to consider \( ^2B_2 \) since \( W \) is a 2-group and \( \ell \neq p \), and the groups \( ^2G_2 \) since only the prime \( \ell = 2 \) divides \(|W|\) and is different from \( p \), and this case is excluded in the proposition.

For the groups \( ^2F_4 \) the only prime \( \ell \neq p \) such that \( \ell ||W| \) is \( \ell = 3 \) and we are in case (3) of the lemma.

The Ree group \( ^2G_2 \) with \( \ell = 2 \) is a genuine counterexample since the Sylow \( 2 \)-subgroups of \( ^2G_2(q) \) are isomorphic to \((\mathbb{Z}/2)^3\).

3. The structure of the Sylow \( \ell \)-subgroups

**Definition 3.1.** Let \( G, F, G_1, \mathcal{P} \) and \( n \) be as in 2.13 and let \( \ell \neq p \) be a prime number. We define \( D(\ell) \) as the set of integers \( d \) such that for some \( \Phi \in \mathcal{P} \) dividing \( \Phi_d(x^n) \) we have \( \ell | \Phi(q^n) \), where \( n \) is as in Definition 2.18.

The following proposition is [Enguehard, Théorème 1] when \( n = 1 \); we give here a shorter proof. Since [Enguehard] was written, Malle ([Malle, 5.14 and 5.19]) has published a proof of (2) below — thus implicitly of (1) also — when \( n = 1 \) (giving more, see Theorem 3.3).

**Theorem 3.2.** Assume in the situation of 3.1 that \( D(\ell) \neq \emptyset \), or equivalently that \( \ell | G^F \). Then

1. \( d(\ell) \in D(\ell) \).
2. There exists a unique \( \Phi \in \mathcal{P} \) such that \( \ell | \Phi(q^n) \) and \( \Phi \) divides \( \Phi_{d(\ell)}(x^n) \). If \( S \) is a Sylow \( \Phi \)-torus then \( N_G(S) \) contains a Sylow \( \ell \)-subgroup of \( G^F \) which is an extension of \((Z^0C_G(S))_\ell^F \) by a Sylow \( \ell \)-subgroup of \( W_\ell \).
3. The Sylow \( \ell \)-subgroups of \( G^F \) are abelian if and only if \(|D(\ell)| = 1 \) (which is equivalent to \( W_\ell \) being an \( \ell' \)-group), apart from the exception where
(G_1, F_1) is of type 2G_2 and ℓ = 2 in which case |D(ℓ)| = 2 and |W_Φ| = 6 but the 2-Sylow is abelian, isomorphic to (Z/2)^3.

Further, if S is as in (2), then (Z^0C_G(S))^F = S^F except if:

- ℓ = 3 and G_1 of type 3D_4,
- ℓ = 2, d = 1 and for some odd degree ε_i = -1. Equivalently G_1 is non-split and has an odd reflection degree, that is, is one of 2A_n, 2D_{2n+1} or 2E_6,
- ℓ = 2, d = 2 and for some odd degree ε_i = 1; equivalently G_1 is split and has an odd reflection degree, that is, is one of A_n(n > 1), D_{2n+1} or E_6.

In the above exceptions, Z^0C_G(S) = C_G(S) is a maximal torus of G.

Proof. Let us note that to prove (2) when we are not in an exception, that is the stronger statement that a Sylow ℓ-subgroup is in an extension of S^F by a Sylow ℓ-subgroup of W_Φ, it is enough to prove that

\[ v_ℓ(|G^F|) = v_ℓ(|S^F|) + v_ℓ(|W_Φ|) \]  

(*)

where v_ℓ denotes the ℓ-adic valuation, and in the exceptions, if we have proved that Z^0C_G(S) = C_G(S) it is enough to show

\[ v_ℓ(|G^F|) = v_ℓ(|C_G(S)^F|) + v_ℓ(|W_Φ|) \]  

(**)

Note also that by the definition of d(ℓ) and D(ℓ) in Proposition 2.13, assertion (1) as well as formulae (*) and (**) are equivalent in G and G_1, that is we may assume G quasi-simple to prove them which we do now. Also, in view of (2) and Proposition 2.19(4), (3) reduces to proving:

(3') |D(ℓ)| = 1 is equivalent to W_Φ being an ℓ'-group.

We first look at a case of a Ree or Suzuki group, where η = 2.

Let us prove (1) first. By Lemma 4.2 if ℓ divides |G^F| then there is an element of D(ℓ) of the form d(ℓ)^b with b ≥ 0. By inspecting the order formula for |G^F| given in the proof of 2.11 the elements of D(ℓ) have all their prime factors in {2, 3}, so b > 0 implies ℓ ∈ {2, 3} thus d(ℓ) ∈ {1, 2}; inspecting again the formula, we see that then d(ℓ) in D(ℓ) and that |D(ℓ)| = 1 unless ℓ ∈ {2, 3}.

To prove (2) for ℓ ∉ {2, 3}, we observe there is a single Φ ∈ P such that Φ(q) since the two numbers Φ_2(q), Φ_2(q) are prime to each other, and the same observation applies to Φ_2(q), Φ_2(q) and Φ_2(q), Φ_2(q). Thus for ℓ ∉ {2, 3} assertions (3') and (*) are obvious since |G^F| = |S^F| and ℓ /|W|.

Let us prove (*) for ℓ ∈ {2, 3}; since ℓ ≠ p and the elements of D(ℓ) have only 2 as prime factor in the case 2B_2, we have just to consider:

- ℓ = 3 for 2F_4: we have d(3) = 2, W_{Φ_{2,2}} = G_{12} of order 48; the only factor Φ(q) with a value divisible by 3 apart from |S^F| = Φ_{2,2}(q)^2 is Φ_{2,6}(q) and \( v_3(Φ_{2,6}(q)) = 1 = v_3([G_{12}]) \) which proves this case.
- ℓ = 2 for 2G_2: we have d(2) = 2 and |W_{Φ_{2,2}}| = 6; the only factor Φ(q) with an even value apart from |S^F| = Φ_{2,2}(q) is Φ_{2,1}(q) and \( v_2(Φ_{2,1}(q)) = 1 = v_2([W_Φ]) \) which proves this case.

We have seen (3') along the way.

Now we look at the other quasi-simple groups thus η = 1. We notice generally that, assuming we have proved (1) then if |D(ℓ)| = 1 assertion (2) is trivial since a Sylow ℓ-subgroup is then in S, and (3') reduces to checking that W_Φ is an ℓ'-group.

We consider separately 3D_4 where \( |3D_4(q)| = q^{12}(Φ_3(q))^3Φ_2(q)^{(1q_1)}(q) \). Again, since the only prime factors of elements of D(ℓ) are {2, 3}, we see that d(ℓ) ∈ D(ℓ)
except possibly if \( \ell \in \{2, 3\} \); but in that case \( d(\ell) \in \{1, 2\} \) and there is a factor \( \Phi_{d(\ell)}(q) \), whence (1). Since \( |W| = 3 \cdot 2^k \) assertion \((3')\) is proved when \( \ell(\ell) = 1 \).

It remains to prove (2) when \( \ell \in \{2, 3\} \). In both cases \( W_{\Phi_{d(\ell)}} = W(2) \) and by Lemma 4.2 \( v_\ell(|G^F/|S^F|) = 2 \). If \( \ell = 2 \) then \( 2 = v_\ell(|W(2)|) \) which proves \((*)\). If \( \ell = 3 \) a Sylow \( \Phi \)-subtorus \( S \) is in a torus \( T_w = C_G(S) \) where \( w = 1 \) if \( d + 1 \) (resp. \( w = w_0 \) if \( d = 2 \)). We have \( |T_w^F| = \Phi_4(q^2) \Phi_4(q) \) (resp. \( |T_w^{W_0}| = \Phi_2(q^2) \Phi_4(q) \)) which has same 3-valuation as \( |G^F|/|W_\Phi| \) which proves \((**)*\).

In the remaining cases \( \varepsilon_i = \pm 1 \) for all \( i \). Let us set \( \zeta_d = e^{2i\pi /d} \). We have \( \Phi = \Phi_{d(\ell)} \) and \( v_\ell(|S^F|) = |a(\zeta_d(\ell))|v_\ell(\Phi_{d(\ell)}(q)) \).

We first treat the case \( \ell \) odd. We have \( a(\zeta_d) = \{ d_i \mid \varepsilon_i = 1 \} \) and \( |W_\Phi| = \prod_{d_i \in a(\zeta_d(\ell))} d_i \). By Lemma 4.2, a factor \( \Phi_e(q) \) of \( |G^F| \) can contribute to the \( \ell \)-valuation only if \( e \) is of the form \( d(\ell) \ell^b \) for some \( b \geq 0 \). Further such a factor appears if and only if \( a(\zeta_e) \neq \emptyset \), that is for some \( i \) we have \( \zeta_d^{d_i(\ell)} = \varepsilon_i \). Thus \( \ell \) is odd raising this equality to the power \( \ell^b \) gives \( \zeta_d^{d_i(\ell)\ell^b} = \varepsilon_i \) thus \( d_i \in a(\zeta_d(\ell)) \) and in particular \( d(\ell) \in D(\ell) \). And \( \zeta_d^{d_i(\ell)\ell^b} = \varepsilon_i \) implies that \( \ell^b \) divides \( d_i \). Thus only the \( d_i \) in \( a(\zeta_d(\ell)) \) contribute to \( v_\ell(|G^F|) \) and each of them contributes \( v_\ell(\Phi_{d(\ell)}(q)) \) to the \( \ell \)-valuation of \( a(\zeta_d(\ell)) \). By Lemma 4.2 this is \( v_\ell(\Phi_{d(\ell)}(q)) + v_\ell(d_i) \). Summing over \( d_i \in a(\zeta_d(\ell)) \) proves \((*)\).

It remains the case \( \ell = 2 \) where we proceed similarly. We have \( d(\ell) \in \{1, 2\} \). If \( d(2) = 1 \) then \( a(1) = \{ d_i \mid \varepsilon_i = 1 \} \). Thus the condition \( \zeta_d^{d_i} = \varepsilon_i \) is still equivalent to \( 2^k|d_i| \); but there could be some more solutions of this equation than elements of \( a(1) \) when \( b = 1 \): any odd \( d_i \) such that \( \varepsilon_i = -1 \) brings an additional factor \( 1 = v_2(\Phi_2(q)) \). If \( d(2) = 2 \) then \( a(-1) = \{ d_i \mid \varepsilon_i = (-1)^{d_i} \} \). The contribution of the even \( d_i \) can be worked out as before; but this time the odd \( d_i \) where \( \varepsilon_i = 1 \) bring additional factors \( v_2(\Phi_4(q)) \). In the exceptions in each case \( C_G(S) \) is a maximal torus of type 1 or \( w_0 \); looking at the orders of these tori, they contain enough extra \( \Phi_1 \) or \( \Phi_2 \) factors (which correspond to the eigenvalues 1 or \( -1 \) of \( \phi \) or \( w_0 \phi \)) to compensate the discrepancy.

Let us show now \((3')\), which reduces to proving that \( |D(\ell)| > 1 \) implies \( v_\ell(|W_\Phi|) > 0 \). Thus we assume \( |D(\ell)| > 1 \). We first do the case \( \ell = 2 \); then \( d(\ell) \in \{1, 2\} \) from which it follows, since the 1 and \(-1\)-eigenspaces are defined over the reals, that \( W_\Phi \) is a Coxeter group, whose order is always even. We consider finally \( \ell \) odd; then \( D(\ell) \supseteq d(\ell) \ell^a \) for some \( a > 0 \). But we have seen above that there exists a factor \( \Phi_{d(\ell)\ell^a}(q) \) only if \( \ell^a|d_i \) for some \( d_i \in a(\zeta_d(\ell)) \).

We remark that if \( \ell \) divides only one \( \Phi_{d(\ell)}(q) \), a Sylow \( \ell \)-subgroup \( S \) lies in a single Sylow \( \Phi \)-torus \( S \) (the intersection of two tori has lower dimension so cannot have same order polynomial). It follows that \( N_{G^F}(S) = N_{G^F}(S) \) and \( C_{G^F}(S) = C_{G^F}(S) \). This observation is a start for describing the \( \ell \)-Frobenius category of \( G^F \) in terms of the category of \( \Phi \)-eigenspaces of \( W_{G^F} \).

In general, one can deduce the following unicity theorem from the work of Cabanes, Enguehard and Malle.

**Theorem 3.3.** Consider \( G, F, n, G_1, q \) as in 2.13 with \( q^n \in \mathbb{Z} \) and let \( \Phi \) as defined in Theorem 3.2, (2). Assume that we are not in one of the following cases:

- \( \ell = 3 \), \( G_1 \) simply connected of type \( A_2, 2A_2 \) or \( G_2 \).
- \( \ell = 2 \), \( G_1 \) simply connected of type \( C_n, n \geq 1 \).
Let $Q$ be a Sylow $\ell$-subgroup of $G^F$. There is a unique Sylow $\Phi$-subtorus $S$ of $G$ such that $Q \subseteq N_G(S)$.

**Proof.** In the context of Theorem 3.2(2), let $Q$ be a Sylow $\ell$-subgroup of $G^F$ contained in $N_G(S)$; then according to [Cabanes], $S^F_{\ell}$ is often characteristic in $Q$ (for example when $\ell \geq 5$), thus in these cases $N_{G^F}(Q) \subseteq N_G(S)$. Using inductively that property and inspecting small cases, G. Malle has proved the inclusion

$$N_{G^F}(Q) \subseteq N_G(S)$$

for all quasi-simple groups $G$ short of the cases excluded in Theorem 3.3, see [Malle, Theorems 5.14 and 5.19]. Here $S$ is a Sylow $\Phi_{\ell(t)}$-subtorus of $(G, F)$ as defined in Definition 2.18 with $\eta = 1$ (note that $N_{G^F}(Q) \subseteq N_G(S)$ implies $Q \subseteq N_G(S)$).

We first verify that the last inclusion holds more generally in a "descent of scalars". With hypotheses and notations of Proposition 2.13 and Lemma 2.14 assume $q^n \in \mathbb{Z}$. If $e = d(\ell)$ is the order of $q^n$ modulo $\ell$, take $\Phi = \Phi_e \in \mathcal{P}$, defining $S = S_e$ and $S_1$. There is a morphism from $G$ onto $G_1$, sending $S$ to $S_1$, with restriction an isomorphism from $G^F$ to $G_1^F$. Then a Sylow $\ell$-subgroup $Q_1$ of $G^F$ contained in $N_{G_1}(S_1)$ is the isomorphic image of a Sylow $\ell$-subgroup $Q$ of $G^F$ contained in $N_G(S)$. The inclusion 3.4 written with $(G_1, F_1, Q, S_1)$ instead of $(G, F, Q, S)$ implies 3.4 in $(G, F)$.

From 3.4 the unicity of $S$, given $Q$, follows:

**Lemma 3.5.** Let $\Phi \in \mathcal{P}$, let $S$ be a Sylow $\Phi$-subtorus of $(G, F)$ and $Q$ a Sylow $\ell$-subgroup of $G^F$. If $N_{G^F}(Q) \subseteq N_G(S)$, then $S$ is the unique Sylow $\Phi$-torus of $(G, F)$ such that $Q \subseteq N_G(S)$.

**Proof.** Assume $Q \subseteq N_G(S')$ for some Sylow $\Phi$-torus $S'$ of $(G, F)$. By Proposition 2.19 there exists $g \in G^F$ such that $S = (S')^g$, hence $Q^g \subseteq N_G(S)$. By Sylow’s theorem in $N_G(S)^F$, $Q = Q^{gh}$ for some $h \in N_G(S)^F$ hence $gh \in N_G(S)$ by our hypothesis. \[\square\]

4. Appendix

We gather here arithmetical lemmas used above.

**Lemma 4.1.** Let $x, f, \ell \in \mathbb{N}$ where $\ell$ is prime, and assume $x \equiv 1 \pmod{\ell}$ (resp. (mod 4) if $\ell = 2$). Then $\nu_{\ell}(\frac{x^{f^2} - 1}{x - 1}) = \nu_{\ell}(f)$.

**Proof.** From $\frac{x^{f^2} - 1}{x - 1} = \frac{x^{f^2} - 1 - x^f - 1}{x - 1} = \frac{x^f - 1}{x - 1}^f$ we see that it is enough to show the lemma when $f$ is prime. We have $\frac{x^f - 1}{x - 1} = f + \sum_{i=2}^{f-1} (x - 1)^{i-1}(f)$. Let $S$ be this last sum; we have $S \equiv f \pmod{\ell}$, since $x - 1 \equiv 0 \pmod{\ell}$, thus $S$ is prime to $\ell$ when $f \neq \ell$ which shows the lemma in this case. When $f = \ell$ then all the terms of $S$ but the first one and possibly the last one are divisible by $\ell^2$ since $(f)$ is divisible by $\ell$ when $2 \leq i < \ell$; the last term is divisible by $\ell^2$ when $\ell - 1 \geq 2$ which fails only for $f = \ell = 2$; but when $\ell = 2$ we have arranged that $\nu_{\ell}(x - 1) \geq 2$ and this time $2(f - 1) \geq 1$; thus $S \equiv f \pmod{\ell^2}$, whence the lemma. \[\square\]

The following lemma is in [Malle, 5.2]; a short elementary proof results immediately from Lemma 4.1.
Lemma 4.2. Let \( q, \ell \in \mathbb{N} \) where \( \ell \) is prime. Let \( d \) be the order of \( q \) (mod \( \ell \)) (or (mod 4) if \( \ell = 2 \)). Then \( \ell \) divides \( \Phi_e(q) \) if and only if \( e \) is of the form \( dp^b \) with \( b \in \mathbb{N} \) (or additionally \( b = -1 \) when \( \ell = 2 \)), and \( v_e(\Phi_{dp^b}(q)) = 1 \) if \( b \neq 0 \).

The following lemma is in [Minkowski]; we give the proof since it is very short and the original German proof may be less accessible.

Lemma 4.3. Let \( m \in \mathbb{N}, m > 2 \). Then the kernel of the reduction map \( \text{GL}(\mathbb{Z}^n) \to \text{GL}(\mathbb{Z}/m^\mathbb{N}) \) is torsion-free.

Note that the bound \( m > 2 \) is sharp since \(-\text{Id} \equiv \text{Id} \) (mod 2).

Proof. Let \( w \in \text{GL}(\mathbb{Z}^n) \) be of finite order, \( w \neq \text{Id} \) and assume its reduction \( v \equiv \text{Id} \). We will derive a contradiction.

Possibly replacing \( w \) by a power, we may assume that \( w \) is of prime order \( p \).

Also \( \text{GL}(\mathbb{Z}^n/m) = \prod_i \text{GL}(\mathbb{Z}^n/p_i) \) where \( m = \prod_i p_i \) is the decomposition of \( m \) into prime powers, thus we may assume that \( m \) is a prime power.

Since \( w \) is of order \( p \), the polynomial \( \Phi_p(x) \) is a factor of the characteristic polynomial of \( w \). The characteristic polynomial of \( v \) is the reduction (mod \( m \)) of that of \( w \), thus we must have \( \Phi_p(x) \equiv (x-1)^p \pmod{m} \), in particular \( (p-1) \equiv -1 \) (mod \( m \)) thus \( m \mid p \) which implies \( m = p \).

Write now \( w = \text{Id} + px^a \) where \( x \equiv 1 \pmod{m} \) and \( a \in \mathbb{N} \). Then the equation \( w^m = \text{Id} \) gives \( \sum_{i=1}^m \binom{m}{i} x^i a^i = 0 \), which after dividing by \( m^{a+1} \) becomes \( x = -\sum_{i=2}^m \binom{m}{i} x^i m^{i(i-1)-1} \) where all coefficients on the right-hand side are divisible by \( m \) (since \( m \geq 3 \)), which contradicts \( x \equiv 1 \pmod{m} \). \( \square \)

References


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