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Boundary Element Method Formulation for axisymmetric acoustic problems in a subsonic uniform flow

Bassem Barhoumi

Abstract

This paper presents a new analysis method and numerical development of the direct boundary element method formulation for axisymmetric acoustic radiation and propagation problems in a subsonic uniform flow. It governed by the homogeneous axisymmetric convected Helmholtz equation independent of the flow direction based on the Fourier coefficient, the three-dimensional convected Green’s function in free space according to the azimuth angle represented by the modal kernel arising in the acoustic sources. This axisymmetric integral formula is expressed only in two new operators; the first, concerned the particular normal derivative similar to the temporal derivative and the other, concerned the convected normal derivative reduce the convective terms of normal derivative and the flow direction derivative used in the classical formulations developed in the literature. When, the main source is taken on the generator, the integration modal Green kernel and its convected derivative are performed partly analytically in terms of Laplace coefficients evaluate by a simple recursive formula and partly numerically using a Gauss-Legendre quadrature standard formula to facilitate the analytical resolution and the conventional numerical problems. The boundary element method formulation used to illustrated and tested by applying the infinite axisymmetric cylindrical duct in a subsonic uniform flow.

Keywords:
Convected Helmholtz equation, modal Green’s function, axisymmetric boundary element method, singular integrals, axisymmetric duct.

1. Introduction

Noise nuisance caused by airplanes, satellites and automobiles are becoming a major and growing concern. They are generated by the radiation or the propagation of acoustic waves in a fluid medium in flow or at rest. Thus, several methods have been developed to help understand the physical phenomena involved to reduce this noise such as, the finite element method and boundary element method, when these domains are in revolution, it is necessary to develop new representations for these methods. Firstly, The FEM
in axisymmetric problems (FEMA) is adapted in bounded interior region where [1,2] are determined the acoustic radiation and propagation respectively, for spherically and cylindrically symmetric problems in uniform flow. This simplification by Prandtl-Glauert transformed the flow axisymmetric domain to a no-flow case [3]. However, this method is poorly adapted to resolve the acoustic problems in an unbounded axisymmetric exterior domain. Since, it was been induced reflections or radiation condition at infinity [4,5]. This case is better to use the technique of the axisymmetric boundary element method [6,7] derived from the three-dimensional integral formulation who can be formulated in the transformed acoustic medium based on the Prandtl-Glauert transformation [3] or in the original flow acoustic domain [8,9] and in the no-flow domain [10].

The main disadvantages of conventional axisymmetric integral formula are caused by the complexity of boundary conditions transformation to the no-flow case, the convection effects of the normal derivative and the flow direction derivative [8]. Also, singularity problems of modal Green function and its derivatives can be evaluated by an analytical method [9,11] and by fast Fourier transform techniques [10] containing the additional terms.

In this paper i present a new method to obtain the conventional Boundary integral formulas for axisymmetric acoustic medium (BEMA) based on the axisymmetric convected wave equation and its fundamental solution such as convected modal Greens function. This conventional formula converted into a simple form similar to the no-flow case, dependent on two new operators physically reduce the convective terms and facilitates the analytical resolution of singularity problems. The numerical implementation of BEMA is developed for acoustic propagation in an axisymmetric cylindrical duct and validated by comparison with FEMA method and the analytical solution.

2. Axisymmetric formulation

The axisymmetric physical acoustic medium can be divided into three-regions; the axisymmetric interior and exterior domain or the propagation and radiation problem in a subsonic uniform flow and the interface domain in a subsonic non-uniform low-flow similar to the regions of an aero-engine contains the source region, the duct domain and the Farfield region.

We neglect the non-uniform flow effect, this acoustic domain transform into two regions $\Omega_\epsilon$, with $\epsilon = i$ for the interior domain and $\epsilon = e$ for the exterior domain $\mathbb{R}^2$ in a compressible isentropic fluid, inviscid, of the density $\rho_\infty \epsilon$, of the speed sound $c_\infty \epsilon$ and the flow Mach vector $M_\infty \epsilon = M_\infty e_z$ in the z-direction of a cylindrical coordinate system $(r, \theta, z)$ where $\theta$ is the revolution angle [12] are limited by the fictitious closed generator without needed to impose boundary conditions and unchanged the direction flow (see Figure 1).
where the signs (+, −) designates the acoustic exterior and interior domain in a subsonic uniform flow and \( \partial \Omega = \Gamma_\infty \cup \Gamma \) is the generator limiting the axisymmetric domain \( \Omega \), with \( \Gamma_\infty \) the exterior artificial boundary of the circular generator for sufficiently large radius and \( \Gamma \) is the interior artificial boundary. Which are closed generators of the outward unit normal vector \( n_q(n_{rq}, n_{zq}) \) and of the unit tangent vector \( t_q(t_{rq}, t_{zq}) \) at the boundary point \( q \).

### 2.1. Axisymmetric wave equation

A linear combination of fluid mechanics equations based on the mass conservation, the linear momentum and the adiabatic state to produce the standard form of the wave equation, to describe the three-dimensional homogeneous convected Helmholtz equation for the harmonic acoustic pressure perturbation is expressed in the form \( P = p \exp(+i \omega \epsilon t) \) of angular frequency \( \omega \epsilon \) and \( p \) represents the complex amplitude, \( i \) is the imaginary unit, which can be written as

\[
H_{3D}(p) = \Delta_{3D}(p) + k_{\epsilon}^2 p - (M_{\infty} \cdot \nabla_{3D}) \left[ (2i k_{\epsilon} + (M_{\infty} \cdot \nabla_{3D})) (p) \right] = 0, \text{ in } V_{\epsilon}
\]  

where \( V_{\epsilon} \) is the three-dimensional acoustics problems, \( H_{3D} \) signifies the three-dimensional convected Helmholtz operator, \( \Delta_{3D} \) and \( \nabla_{3D} \) are the operators of Laplace and of Gradient in three-dimensional, \( k_{\epsilon} = \omega_{\epsilon} / c_{\infty} \) is the original wavenumber. The axisymmetric acoustics problems are handled by expanding the physical quantity into a complex Fourier series according to the revolution angle

\[
p(r, \theta, z) = \sum_{n=-\infty}^{n=+\infty} p_n(r, z) e^{in\theta}
\]

where \( p_n \) is the modal acoustic pressure or the Fourier coefficient of the three-dimensional acoustic pressure of azimuthal order \( n \) independent on the revolution angle but still dependent on \( r \) and \( z \) can be represented in the form

\[
p_n(r, z) = \frac{1}{2\pi} \int_0^{2\pi} p(r, \theta, z) e^{-in\theta} d\theta
\]
By expanding all variables of the wave equation Eq. (1) in complex Fourier series Eq. (2) and integrate the resulting equation multiply by \( \exp(-im\theta)/2\pi \) over the revolution interval \([0, 2\pi]\) with \( m \in \mathbb{Z} \), the three-dimensional convected Helmholtz operator \( H_{3D} \) transformed to the axisymmetric convected operator \( H_{axi} \) for modal acoustic pressure decompose into no-flow part and a convection part
\[
H_{axi}(p_n) = \left[ \Delta + k^2_c - \left( \frac{n^2}{r^2} \right) \right] (p_n) - (M_{\infty} \cdot \nabla) \left[ (2ik_c + (M_{\infty} \cdot \nabla)) (p_n) \right] = 0 \tag{4}
\]
where \( \Delta \) and \( \nabla \) are the operators of Laplace and of Gradient in axisymmetric problems \( r-z \). The fundamental solution of this axisymmetric wave equation for an acoustic source in free space is the convected modal Green’s function \( G_k(m, q) \), the Fourier coefficient of the three-dimensional convected Green function \( G_k(M, Q) \) in a unbounded medium at the locations observer-source points are respectively denoted by \( M(r_M, \theta_M, z_M) \) and \( Q(r_Q, \theta_Q, z_Q) \) identified in axisymmetric problems by this points \( m(r_m = r_M, z_m = z_M) \) and \( q(r_q = r_Q, z_q = z_Q) \) (Figure 2).

![Figure 2: Axisymmetric body.](image)

The convected modal Greens function of the azimuth angle \( \beta = \theta_M - \theta_Q \) and the order \( n \) defined by Eq. (3) satisfies the inhomogeneous axisymmetric convected Helmholtz equation take the form
\[
H_{axi}(G^k_n(m, q)) = \left[ \Delta_q + k^2_c - \left( \frac{n^2}{r_q^2} \right) \right] G^k_n(m, q)
+ (M_{\infty} \cdot \nabla_q) \left[ (2ik_c + (M_{\infty} \cdot \nabla_q)) G^k_n(m, q) \right] = \frac{1}{2\pi r_q} \delta(q - m) \tag{5}
\]
where \( \nabla_q \) is the axisymmetric gradient operator at point \( q \) and \( \delta \) is the Dirac delta function. The intrinsic form of three-dimensional convected Green function in free space \( G_k(M, Q) \) for \( M \neq Q \) independent of the flow direction, which is represented by
\[
G_k(M, Q) = \exp \left[ -ik^* (MQ, M^*_\infty + R^*) \right] / (4\pi c R^*) \tag{6}
\]
where \( c \) is the Prandtl-Glauert factor, \( k^* = k_c/c \) is the convected wavenumber, \( M^*_\infty = M_{\infty}/c \) is the convected Mach vector and \( R^* \) is the three-dimensional convected radius,
given by

\[ R^* = \sqrt{R^2 + (MQ \cdot M^*)^2} \] (7)

where \( R = MQ \) is the physical distance between source-observer in three-dimensional. Using the cylindrical coordinate system of the points \( M \) and \( Q \), the convected radius \( R^* \) can be rewritten as a function of the azimuth angle

\[ R^* = \tau^* \sqrt{1 - \tau^2 \cos^2(\theta)} \] (8)

where the angle \( \theta = \beta / 2 \) and the parameters \( \tau \) and \( \tau^* \) given by

\[ \tau^2 = \frac{4r_q r_m}{\tau^*}, \quad \tau^* = \sqrt{r^* + 4r_q r_m} \] (9)

The convected distance \( r^* = \sqrt{r^2 + (mq \cdot M^*)^2} \) is the axisymmetric convected physical radius such as \( r = mq \) is the axisymmetric physical distance between axisymmetric observer-source points \( m \) and \( q \).

2.2. Boundary conditions

For the physical acoustic interior and exterior problems it is necessary to introduce the boundary conditions of the fictitious closed generator and the exterior boundary at infinity represented by the duct conditions and the Sommerfeld radiation condition in a subsonic uniform flow. Firstly, the duct conditions can be represented by the rigid duct or vibration duct without the need to apply. Secondly, the Sommerfeld radiation condition represents the outgoing radiation waves, can be rewritten the radiation condition in a subsonic uniform flow \([3,5,8]\) take the advanced form, given by

\[ \lim_{R \to \infty} \left\{ R \left( \frac{dp}{dR} + ik_{\infty} p \right) \right\} = 0 \] (10)

where \( d/dR \) is the particular temporal derivative in harmonic-time, \( k_{\infty} = \tilde{k}_t - kM_{\infty} \) is the non-standard convected wavenumber depended the mean flow with \( k_t \) is the tangential convected wavenumber transformed by Prandtl-Glauert of the tangential Mach number \( M_{\infty t} = M_{\infty} \cdot t_q \), and \( M_{\infty n} = M_{\infty} \cdot n_q \) is the normal Mach number, given by

\[ \tilde{k}_t = \frac{k_t}{\sqrt{1 - M_{\infty t}^2}} \] (11)

This advanced form of the Sommerfeld radiation similar to the radiation condition for a fluid at rest without transforming the domain and the non-standard convected wavenumber \( k_{\infty} \) becomes the original wavenumber \( k_\epsilon \), the particular temporal derivative \( d/dR \) becomes the normal derivative \( \partial / \partial R \)

\[ \lim_{R \to \infty} \left\{ R \left( \frac{\partial p}{\partial R} + ik_\epsilon p \right) \right\} = 0 \] (12)

These two Sommerfeld radiations conditions are verified by the convected Greens function in free space.
3. Boundary element method for axisymmetric problems

The Boundary element method for axisymmetric acoustics problems is obtained by multiplying the homogeneous axisymmetric convected Helmholtz equation Eq. (4) for the modal acoustic pressure \( H_{n}(p_{n}(q)) \) at an point \( q \) by the convected modal Green’s function \( G^{k}_{n}(m,q) \) and multiplying the inhomogeneous convected Helmholtz equation Eq. (5) for the convected modal Green’s function \( H_{n}^{k}(G^{k}_{n}(m,q)) \) by the modal acoustic pressure \( p_{n}(q) \), subtracting these two terms followed by integration over the axisymmetric domain \( \Omega \), with respect to \( q \neq m \) can be written as

\[
\int_{\Omega} \frac{p_{n}(q)}{2\pi r_{q}} \delta(q - m) d\Omega_{eq} = \int_{\Omega} \left[ \Delta_{q} G^{k}_{n}(m,q) p_{n}(q) - \Delta_{q} p_{n}(q) G^{k}_{n}(m,q) \right] d\Omega_{eq}
- 2i\kappa_{eq} \int_{\Omega} [G^{k}_{n}(m,q)(M_{\infty} \cdot \nabla_{q}) p_{n}(q) + p_{n}(q)(M_{\infty} \cdot \nabla_{q}) G^{k}_{n}(m,q)] d\Omega_{eq}
- \int_{\Omega} \left[ c^{k}_{n}(m,q) M_{\infty} \cdot \nabla_{q} \left( (M_{\infty} \cdot \nabla_{q}) p_{n}(q) \right) - p_{n}(q) M_{\infty} \cdot \nabla_{q} \left( (M_{\infty} \cdot \nabla_{q}) G^{k}_{n}(m,q) \right) \right] d\Omega_{eq}
\]

Using the Dirac property for the left term, the axisymmetric Greens identity formula for the right first term, applying the Gradient property on the right second term and the divergence theorem for the last term. The surface integral for the right terms in Eq. (13) transformed into a boundary integral for the modal acoustic pressure can be expressed as

\[
\frac{p_{n}(m)}{2\pi} = \pm \int_{\partial \Omega} \left[ p_{n}(q) \partial_{n_{q}}(G^{k}_{n}(m,q)) - G^{k}_{n}(m,q) \partial_{n_{q}}(p_{n}(q)) \right] r_{q} d\Gamma_{q}
+ \pm \int_{\partial \Omega} 2i\kappa_{eq} M_{\infty} p_{n}(q) G^{k}_{n}(m,q) r_{q} d\Gamma_{q}
+ \pm \int_{\partial \Omega} M_{\infty} \left[ G^{k}_{n}(m,q) \left( (M_{\infty} \cdot \nabla_{q}) p_{n}(q) \right) - p_{n}(q) \left( (M_{\infty} \cdot \nabla_{q}) G^{k}_{n}(m,q) \right) \right] r_{q} d\Gamma_{q}
\]

where \( \nabla_{q} = \sigma_{n} n_{q} + \sigma_{t} t_{q} \) is the axisymmetric Gradient operator such as the couple \( (\sigma_{n} = \partial_{n_{q}}, \sigma_{t} = \partial_{t_{q}}) \) designates the normal and tangential derivative with respect to the direction indicated by the vectors \( n_{q} \) and \( t_{q} \), respectively.

In a no-flow case, the axisymmetric boundary integral equation represented by the right first term of Eq. (14) depending on the symmetric modal Green function of \( G^{k} = \exp(i\kappa R)/4\pi c R \) and its antisymmetric modal normal derivative \( \partial_{n_{q}}(G^{k}) \), but in a uniform flow case of the integral formula Eq. (14) they becomes more complicated because their contains of convection terms on the right-hand side of the normal derivative \( \partial_{n_{q}} \) and the flow direction derivative \( (M_{\infty} \cdot \nabla_{q}) \) for the convected modal Green’s function. These convection operators can be converted into a simple particular normal derivative \( \partial_{n_{q}} \) take into account of the flow direction similar to the particular temporal derivative adapted to the theoretical developments and numerical implementation, defined by

\[
d_{n_{q}}(\cdot) = \partial_{n_{q}}(\cdot) - M_{\infty} (M_{\infty} \cdot \nabla_{q})(\cdot)
\]

The particular normal derivative of the convected modal Green function \( d_{n_{q}}(G^{k}) \) is the Fourier coefficient of \( d_{n_{q}}(G^{k}) = \partial_{n_{q}}(G^{k}) - M_{\infty} (M_{\infty} \cdot \nabla_{q})(G^{k}) \) with respect to the direction indicated by the unit outward normal vector \( n_{Q} \) at the boundary point \( Q \), of order
n according to the azimuth angle $\beta$. Using the normal derivative of the convected Greens function $\frac{\partial n}{\partial n} (G^k)$ and the flow direction derivative $(M_{\infty} \cdot \nabla Q)(G^k)$ given by [7], one obtains

$$\frac{\partial n}{\partial n} (G^k(M, Q)) = - \left[ (1 + ik^* R^*) \frac{R_{n\beta}}{R^3} + ik^* M_{\infty} \right] G^k(M, Q)$$

where $r_{n\beta} = MQ_{\text{axi}} \cdot n_q$ is the normal distance depended on the azimuth angle, the vector $MQ_{\text{axi}}$ is coordinate $(r_Q - r_M \cos(\beta), z_Q - z_M)$. The particular normal derivative of the convected Greens function Eq. (16) decomposed into two terms, the first way, it generalizes the normal derivative expression in no-flow case and the second way it explains by the convection influence. Substitution of the particular normal derivative Eqs. (15) and (16) into the axisymmetric integral formula Eq. (10), yields a relatively simple form of the modal acoustic pressure

$$\frac{p_n(m)}{2\pi} = \pm \int_{\partial \Omega} \left[ G_n^k(m, q) \frac{\partial n}{\partial n} (p_n(q)) - p_n(q) \frac{\partial n}{\partial n} (G_n^k(m, q)) \right] n_q d\Gamma_q$$

These modified axisymmetric integral formula Eq. (17) form an explicit expression for the modal acoustic pressure and its particular normal derivative becomes as a no-flow integral formula given by the right first term of Eq. (14) over the generator used to determine the modal acoustic pressure in the exterior and interior domain. From the particular normal derivative operator Eq. (16), we obtain the convected normal derivative of the convected modal Green function $\frac{\partial n}{\partial n} (G_n^k) = \frac{\partial n}{\partial n} (G_n^k) + 2iM_{\infty} G_n^k$ as the Fourier coefficient of $\frac{\partial n}{\partial n} (G_n^k)$ of order $n$ and the azimuth angle $\beta$, given by

$$\frac{\partial n}{\partial n} (G_n^k(M, Q)) = - \left[ (1 + ik^* R^*) \frac{R_{n\beta}}{R^3} - ik^* M_{\infty} \right] G_n^k(M, Q)$$

The axisymmetric integral formula expressed by Eq. (17) over the large circular generator $\Gamma_{\infty}$ represent the contribution wave on the interior due of the source placed at infinity such as $q \to \infty$ implies that the integration is properly convergent and becoming zero value. This integral term leads two convected boundary conditions verified by the modal acoustic pressure and the modal Greens function. The first indicates that the pressure must decrease, as we go to infinity (non-reflection condition) and the second generalize the usual radiation condition in a subsonic uniform flow, ensures that all radiated waves are outgoing justified by the choice of these new operators Eqs. (15) and (18). When the point $m \to q$, the modal kernel function $G_n^k(m, q)$ and their convected normal derivative contain singular part localized as well as regular part more global.

### 3.1. Singular behavior

A classical procedure to separate these singular and regular parts obtained by a Taylor expansion, decomposing the convected Greens function $G^k = G^0 + g^k$ into a static singular part $G^0$ contain a singularity at $M \to Q$ and regular part $g^k$ depending on the wavenumber

$$G^0(M, Q) = \frac{1}{4\pi c R^*}$$

$$g^k(M, Q) = \frac{\exp(-ik^* R) - 1}{4\pi c R^*}$$
where $\tilde{R} = MQ.M_{\infty}^* + R^*$ is the special radius. The convected normal derivative from this function to the same form $T_k^s = T_k^s + T_k^r$ where the singular function $T_k^s = d_{nk}^s(G^0)$ is convected normal derivative of the static Green function given by Eq. (6) and the function $T_k^r$ is the regular part, given by

$$T_k^s(M, Q) = -\frac{r_{n\beta}}{4\pi c R^*} + ik_c M_{\infty} G^0(M, Q) \tag{21}$$

$$T_k^r(M, Q) = \left(1 - (1 + jk^* R^*) \exp(-ik^* R) \right) \frac{r_{n\beta}}{4\pi c R^*} + ik_c M_{\infty} G^0_k(M, Q) \tag{22}$$

In the same way, the Fourier coefficients of the convected Greens function and its convected normal derivative in the integral equation Eq. (17) decomposed as follows

$$G^0_k(m, q) = G^0_0(m, q) + g^k_0(m, q) \tag{23}$$

$$T^k(m, q) = T^s_{kn}(m, q) + T^r_{kn}(m, q) \tag{24}$$

where $T^k$ is the Fourier coefficients of the convected normal derivative $T^k$. Became different between the singular and regular behavior. It appears to be efficient to treat the two parts in a different fashion. The modal functions $(g_k^0, T_k^r)$ is the Fourier coefficients of the regular functions $(g^k, T_k^r)$ are evaluated numerically with standard Gauss-Legendre quadrature, and the singular terms of the static modal Green function $G^0_0$ and its modal convected normal derivative $T^s_{kn}$ will be evaluated analytically.

Substituting the normal distance in Eq. (21), and the using of convected radius Eq. (8) of the $2\pi$-periodic even function, the modal functions $(g_k^0, T_k^r)$ is the Fourier coefficients of the static modal Green function $G^0_0$ and its modal convected normal derivative $T^s_{kn}$ will be evaluated analytically.

In the same way, the Fourier coefficients of the convected Greens function and its convected normal derivative in the integral equation Eq. (17) decomposed as follows

$$G^0_k(m, q) = 1 + \frac{\alpha^3}{8\pi c R^*} \frac{b_{1/2}^n(\alpha)}{r_{n\beta}} \tag{25}$$

$$T^s_{kn}(m, q) = -r_{n/2} t^0_{n, m} \frac{n_q}{2} \frac{t^0_{n+1}(m, q) + t^0_{n-1}(m, q)}{t^0_{n+1}(m, q) + t^0_{n-1}(m, q)} + ik_c M_{\infty} G^0_0(m, q) \tag{26}$$

where the modal function $t^0_{n}$ is the Fourier coefficient of the function $t^0 = 1/4cR^3$ and of the azimuth angle $\beta$ satisfies the following relation

$$t^0_{n}(m, q) = \left(1 + \frac{\alpha^3}{8\pi c R^*} \frac{b_{1/2}^n(\alpha)}{r_{n\beta}} \right) \tag{27}$$

The function $b_{1/2}^n(\alpha)$ define the Laplace coefficients of argument $\alpha$ and of order $n$, the symbol $s$ means a half odd number. These Fourier coefficients can be defined by [13]

$$b_{1/2}^n(\alpha) = \frac{4}{\pi} \int_0^{\pi/2} \frac{\cos(2n\theta)}{(1 - 2\alpha \cos(2\theta + \alpha^2)^2) d\theta} \tag{28}$$

The argument $\alpha$ of Laplace coefficients in the integrand of the static modal Green function and the modal function $t^0_{n}$, can be written as [13]

$$\alpha = \frac{\tau^2}{(1 + \sqrt{1 - \tau^2})^2} \tag{29}$$
The coefficients and their derivatives with respect to $\alpha$ for $s = (1/2, 3/2)$, are all linear functions of the complete elliptic integrals with argument $\alpha$ explained by a particular of the first and second coefficients [14]

$$b_0^{1/2}(\alpha) = \frac{4}{\pi} K(\alpha) \text{ and } b_1^{1/2}(\alpha) = b_{-1}^{1/2}(\alpha) = \frac{4}{\pi} \frac{K(\alpha) - E(\alpha)}{\alpha}$$

(30)

$$b_0^{3/2}(\alpha) = \frac{4}{\pi} \frac{(\alpha^2 - 1)K(\alpha) + 2E(\alpha)}{(1 - \alpha^2)^2} \text{ and } b_1^{3/2}(\alpha) = b_{-1}^{3/2}(\alpha) = \frac{4}{\pi} \frac{(\alpha^2 - 1)K(\alpha) + (\alpha^2 + 1)E(\alpha)}{\alpha(1 - \alpha^2)^2}$$

(31)

where $K(\alpha)$ and $E(\alpha)$ are the complete elliptic integrals of the first and second kinds in which the upper limit is $\pi/2$ will be respectively denoted by [15]

$$K(\alpha) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \alpha^2 \cos^2(\theta)}} d\theta \text{ and } E(\alpha) = \int_0^{\pi/2} \sqrt{1 - \alpha^2 \cos^2(\theta)} d\theta$$

(32)

The difficulty of the analytic evaluation of the static modal Green function and its derivative for largest value of order $n$ is the necessity to develop a recursive relationship between these Fourier coefficients. Using the relations of Laplace coefficients and their derivatives developed in Celestial mechanics calculations [14], the recursive formulas take the form

$$b_{n+1}^s(\alpha) = \frac{n(1 + \alpha^2)}{(n + 1 - s)\alpha} b_n^s(\alpha) - \frac{(n - 1 + s)}{(n + 1 - s)\alpha} b_{n-1}^s(\alpha)$$

(33)

The static modal functions $G_n^0$ and $t_n^0$ given by Eqs. (25) and (27) depending directly of the Laplace coefficients. Then, they are satisfied the recursive formulas Eq. (33), which are used the first and the second Laplace coefficients Eqs. (30) and (31) to calculate the first terms of these functions. We observe that the static modal convected function $T_{kn}^s$ is expressed by an extremely simple formula written in convection term depending of the static modal Green function $G_n^0$ and the modal function term $t_n^0$ evaluated by a simple analytical recursive formula. For the axisymmetric modal case $(n=0)$, the axisymmetric static convected Greens function $G_n^0$ and its associated convected derivative $T_{k0}^s$ are evaluated by the complete elliptic integrals of the first and second kinds, respectively, $K(\alpha)$ and $E(\alpha)$ of the first and second Laplace coefficients Eqs. (30) and (31), can be used to evaluate and simplifies the convection terms [2,8].

This method determined the static modal functions $(G_n^0, T_{kn}^s)$ reduced the complexity of the conventional approximate solution for the static Greens function and its derivatives, the normal derivative and the flow direction derivative and not contain the supplementary terms of the method binomial expansion [16], the simplification of the fast Fourier transform techniques [10]. When the source point $m$ is taken on the generator with $m \rightarrow q (\tau \rightarrow 1)$ corresponding of the limit argument $\alpha \rightarrow \tau^2$, this has reduced the relative error between the analytical methods used the parameter $\tau$ and this analytical method of the Laplace coefficients depending of the argument $\alpha$, another way between the complete elliptic integrals of the parameter $\tau$ and of argument $\alpha$.

3.2. Analytical evaluation of the singular integral terms

Using the convected radiation conditions and the classical procedure to isolate the singularity parts of modal Green function and its convected normal derivative Eqs. (23)
and (24), we get the new original form of the integral representation in a uniform flow for a point \( m \) of the acoustic domain

\[
\frac{c^\pm(m)}{2\pi} p_n(m) = \pm \int_\Gamma p_n(q) T_n^k(m, q) r_q d\Gamma_q \mp \int_\Gamma G_n^k(m, q) d_n(p_n(q)) r_q d\Gamma_q
\]

(34)

where \( c^\pm(m) \) is the convected angle related to singular integrals derived from the acoustic exterior and anterior problem has been determined by evaluation of the following simple integral, can be rewritten as

\[
c^\pm(m) = 1 \pm 2\pi \int_\Gamma T_{00}^l(m, q) r_q d\Gamma_q
\]

(35)

The Integral term of convected angle Eq. (35) multiply by the revolution angle cylindrical \( 2\pi \) interpreted as the convected angles at the center \( m \). For the point \( m \) in an acoustic exterior domain \( c^+(m) = 1 \) and in the acoustic interior domain \( c^-(m) = 1 \). When \( m \) a regular point on the generator \( c^\pm(m) = 1/2 \). In order to neglect the axisymmetric acoustic pressure contribution at infinity and used the exclusion procedure of the singularity, we find the same form of the axisymmetric BEM developed for Poisson equation in static case [17] and Laplace equation in no-flow case [10,11] which reduces the computational burden of conventional methods in a subsonic uniform flow and then becomes execution times of no-flow case.

4. Axisymmetric cylindrical duct

Consider an infinite axisymmetric cylindrical duct of truncated domain \( \Omega_i \) of radius \( R = 0.1 \text{m} \), of length \( L = 0.5 \text{m} \), of symmetry axis \((oz)\) and the revolution angle \( \theta \), of flow Mach vector \( M_{\infty} = M_{\infty} e_z \) limited by the rigid wall \( \Gamma_R \) of normal vector \( n_{\Gamma_R} \) and two generators, the left \( \Gamma_- \) in \( z = 0 \) and the right \( \Gamma_+ \) en \( z = L \) of normal vector respectively \( n_{\Gamma_-} \) and \( n_{\Gamma_+} \) (Figure 3).

![Figure 3: Axisymmetric cylindrical duct.](image-url)
4.1. Analytical solution

The axisymmetric acoustic propagation represent by a modal acoustic pressure \((n,n')\) of the azimuthal order \(n\) and the radial order \(n'\), in the axisymmetric duct checking the axisymmetric wave equation into the harmonic regime Eq. (4) is given by \[18\]

\[
p_{n,n'}^{\pm} = A_{n,n'} J_n(k_{r,n,n'} r)\exp(ik_{z,n,n'} z) \tag{36}
\]

where \(A_{n,n'}\) is the axisymmetric normalization amplitude, \(k_{z,n,n'}^{\pm}\) and \(k_{r,n,n'}\) are the axial and radial wavenumber (\(\pm\) indicates the propagation mode in the increased and decreased \(z\)-direction), \(J_n\) is the Bessel function of the first kind of order \(n\).

Since, the term depending of tangential derivative function \(\sigma_t\) over the axisymmetric generator duct is zero. Thus, the new condition of radiation or Neumann condition given by the particular derivative of the acoustic pressure Eq. (15) where right generator \(\Gamma_+\), resulted on

\[
\sigma_+ = ic^2k_{z,n,n'}^{\pm} P \tag{37}
\]

We note \(k_{n,n'}^{\pm} = c^2k_{z,n,n'}^{\pm}\) the new module wavenumber lower to that of \(k_{z,n,n'}^{\pm}\). Indeed, when the increasing flow, the axial wavenumber module \(|k_{z,n,n'}^{\pm}|\) is increased but the new axial wavenumber module \(|k_{n,n'}^{\pm}|\) will shrink to a zero value. This explains the efficacy of the new radiation condition for cylindrical duct in a subsonic uniform flow that is tend before the classical radiation condition \(\sigma_c = ik_{z,n,n'}^{\pm} P\), within the no-Flow case, this condition returns to the classical case and replace the particular normal derivative by the normal derivative (see Figures 4 and 5).

Figure 4: \(|k_{z,n,n'}^{\pm}(\mu_{n,n'})|\) and \(|k_{n,n'}^{\pm}(\mu_{n,n'})|\) for azimuthal order \(n=0\) and the radial order \(0 \leq n' \leq 20\) when the wavenumber \(k_i = 5\), for different values of Mach number.
Figure 5: $|k_{z,n,n'}^\pm(\mu_{n,n'})|$ and $|k_{n,n'}^\pm(\mu_{n,n'})|$ for azimuthal order $n=0$ and the radial order $0 \leq n' \leq 20$ when the wavenumber $k_i = 100$, for different values of Mach number.

where $\mu_{n,n'}$ is the $n$th root of the Bessel function derivative. For the azimuthal order $n \neq 0$, the axial wavenumber module $|k_{z,n\neq0,n'}^\pm|$ and the new axial wavenumber module $|k_{n\neq0,n'}^\pm|$ as a the function of $\mu_{n\neq0,n'}$ represent the same form of axisymmetric case because it has always increasing values.

4.2. Numerical implementation

The numerical implementation of the Axisymmetric Boundary element method Eq. (34) requires the discretization of the interior axisymmetric duct surface $\Omega_i$ by triangular elements structured and regular with 3 nodes isoparametric of two straight perpendiculars. Therefore, the correspondante generator $\Gamma = \Gamma_- \cup \Gamma_R \cup \Gamma_+$ is divided in $N$ nodes and $N-1$ elements of the convergence criterion in a subsonic uniform flow explain by the relation $n_e^* h_e^* \leq \lambda$ where $n_e^*$ is the convected number of finite elements for a radius and axis wavelength $\lambda$ and $h_e^*$ is the convected diameter of the finite element, into taking account convected angle in point $m = \Gamma_- \cup \Gamma_R$ and in point $m = \Gamma_R \cup \Gamma_+$ of the axisymmetric duct $c^-(m) = 3/4$ (Figure 6).

Figure 6: A discretization of the axisymmetric duct surface.

The elementary integrals related to the modal Greens function and to the convected normal derivative function of this approximate boundary integral equation in a collocation
point m are evaluated analytically by the recursive formula and numerically by Gauss-Legendre quadrature standard method which uses the Acoustics group codes for numerical resolution.

4.2.1. Axisymmetric mode

For the axisymmetric mode of n=0 azimuthal order, we apply the analytical solution (S.Anal) associated to the axisymmetric acoustic pressure without and with flow propagated in the increased z-direction $P_0^+ = P_{0,0}^+$, BEMA with these new operators, and the FEMA [19,20] as a function of the new non-reflection condition Eq. (37) where the elementary integrals evaluated analytically because the integrands are regular polynomials except that the second member will be evaluated by Gauss quadrature standard method since it depends on the Bessel function of the first kind. Figures 7 and 8 represents the real parts of the axisymmetric acoustic pressure ($P_n^+/A_n^+$) for different values of Mach vector by these three methods.

Figure 7: Real part of the axisymmetric acoustic pressure ($P_0^+/A_0$) by S.Anal (top), FEMA (middle) and BEMA (bottom) for $M_{\infty} = 0$ (left) and $M_{\infty} = 0.4e_2$ (right) when $k_i = 30$. 13
Figures 8 and 7 present a real parts of the axisymmetric acoustic pressure \((P^+_n/A_n')\) by FEMA and BEMA which are in very good concord with analytical solution of relative error given by the \(L_2\) norm. For the axisymmetric finite element method, the relative error is less than 0.5% and for the axisymmetric boundary integral method, the error is less than 0.8%. This error difference between the two methods explained by evaluating the integrals terms of BEMA in the origin point and in the other point of \((r=0,z=L)\). We observed the initial mode \((0,0)\) for the axisymmetric acoustic pressure is uniform along radial and axial wave. The wavelength increases in the case where the increases uniform flow. These explained of the offset wave in the increased z-direction (see Figure 7). In the other mode of axisymmetric acoustic pressure \((0,2)\) from high wave number is non-uniform in a presence and in the absence of flow, along the radial axis and his symmetry axis. Indeed, in the upstream region, the acoustic pressure field is radially amplified to be greater than that obtained in the downstream region. This explains of the Sommerfeld radiation condition Eq. (10) and (12), and we notice also that the uniform
flow is proportional to the wavelength (see Figure 8).

4.2.2. Non-Axisymmetric mode

The second test case concerns a non-axisymmetric mode or a non-axisymmetric excitation represented by the \((n \neq 0, n')\) order propagated in the increased \(z\)-direction of condition wave \(ck_{r,n,n'} < k_i\) with the cutoff parameter of these mode \(\alpha_{r,n,n'} = \sqrt{k_i^2 - ck_{r,n,n'}^2}\). The figures 9 and 10 shows the modal acoustic pressure obtained by application of the present method, such as S.anal and S.BEMA from low wavenumber \(k_i = 40\) and a high wave number \(k_i = 80\) without flow of Mach vector \(\mathbf{M}_\infty = 0\) and with a mean flow of Mach vector \(\mathbf{M}_\infty = 0.4e_z\).

![Figure 9: Real part of the axisymmetric acoustic pressure \((P_{1,1}^+ / A_{1,1})\) by S.Anal (top) and BEMA (bottom) for \(M_\infty = 0\) (left) and \(M_\infty = 0.4e_z\) (right) when \(k_i = 40\).](image)

![Figure 10: Real part of the axisymmetric acoustic pressure \((P_{2,2}^+ / A_{2,2})\) by S.Anal (top) and BEMA (bottom) for \(M_\infty = 0\) (left) and \(M_\infty = 0.4e_z\) (right) when \(k_i = 80\).](image)

In figures 9 and 10 are shown the comparison between the results for the real part of non-axisymmetric mode obtained by the boundary element method and those from analytical method for the \((1,1)\) mode and \((2,2)\) mode. The figures showed a very good agreement. In both cases the maximum relative error of about 1%.
The increase of the subsonic uniform flow from the same mode in the z-direction generates a decrease of the wavelength, but for the increase azimuthal order and radial order generates a decrease of the wavelength.

5. Conclusion

A simple and effective new boundary element method formula is proposed for the axisymmetric acoustic exterior domain and the axisymmetric acoustic interior domain, with or without an arbitrary axisymmetric mean flow, as a function of the two new operators such as particular and convected terms substantially reduce the convective effect of mean flow in the classical axisymmetric boundary integral formulations for original acoustic medium. The main advantage of this method is that the singularity behavior avoids the supplementary evaluation of the modal Green’s function and its derivatives like the normal derivative and the flow direction derivative, and requires only the use the simple recursive formula for the static modal Green’s function and its convected derivative based on the Laplace coefficients, which reduces the analysis time and the implementation numerical time compared to the conventional formulations and the finite element method.

References


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