Deformation of the O'Grady moduli spaces

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Abstract

In this paper we study moduli spaces of sheaves on an abelian or projective K3 surface. If S is a K3, $v = 2w$ is a Mukai vector on S, where w is primitive and $w^2 = 2$, and H is a $v-$generic polarization on S, then the moduli space M_v of $H-$semistable sheaves on S whose Mukai vector is v admits a symplectic resolution $\widetilde{M}_v$. A particular case is the $10-$dimensional O'Grady example $\widetilde{M}_{10}$ of irreducible symplectic manifold. We show that $\widetilde{M}_v$ is an irreducible symplectic manifold which is deformation equivalent to $\widetilde{M}_{10}$ and that $H^2(M_v, \mathbb{Z})$ is Hodge isometric to the sublattice $v^\perp$ of the Mukai lattice of S. Similar results are shown when S is an abelian surface.

1 Introduction and notations

Moduli spaces of semistable sheaves on abelian or projective K3 surfaces are an important tool to produce examples of irreducible symplectic manifolds. In the following, S will denote an abelian or projective K3 surface.

An element $v \in \widetilde{H}(S, \mathbb{Z}) := H^{2*}(S, \mathbb{Z})$ will be written as $v = (v_0, v_1, v_2)$, where $v_i \in H^{2i}(S, \mathbb{Z})$, and $v_0, v_2 \in \mathbb{Z}$. It will be called *Mukai vector* if $v_0 > 0$ and $v_1 \in NS(S)$, or if $v_0 = 0$ and v_1 is the first Chern class of an effective divisor. Recall that $\widetilde{H}(S, \mathbb{Z})$ has a pure weight-two Hodge structure defined as

$$\widetilde{H}^{2,0}(S) := H^{2,0}(S), \qquad \widetilde{H}^{0,2}(S) := H^{0,2}(S),$$

$$\widetilde{H}^{1,1}(S) := H^0(S, \mathbb{C}) \oplus H^{1,1}(S) \oplus H^4(S, \mathbb{C}),$$

and a lattice structure with respect to the Mukai pairing $(.,.)$. In the following, we let $v^2 := (v, v)$ for every Mukai vector v; moreover, for every Mukai vector v define the sublattice

$$v^\perp := \{\alpha \in \widetilde{H}(S, \mathbb{Z}) \,|\, (\alpha, v) = 0\} \subseteq \widetilde{H}(S, \mathbb{Z}),$$

which inherits a pure weight-two Hodge structure from the one on $\widetilde{H}(S, \mathbb{Z})$.

If $\mathscr{F}$ is a coherent sheaf on S, we define its *Mukai vector* to be

$$v(\mathscr{F}) := ch(\mathscr{F})\sqrt{td(S)} = (rk(\mathscr{F}), c_1(\mathscr{F}), ch_2(\mathscr{F}) + \epsilon rk(\mathscr{F})),$$

where $\epsilon = 1$ if S is K3, and $\epsilon = 0$ if S is abelian. Let H be an ample line bundle on S. For every $n \in \mathbb{Z}$ and every coherent sheaf $\mathscr{F}$, let $\mathscr{F}(nH) := \mathscr{F} \otimes \mathscr{O}_S(nH)$.

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The Hilbert polynomial of $\mathscr{F}$ with respect to H is $P_H(\mathscr{F})(n) := \chi(\mathscr{F}(nH))$, and the reduced Hilbert polynomial of $\mathscr{F}$ with respect to H is

$$p_H(\mathscr{F}) := \frac{P_H(\mathscr{F})}{\alpha_H(\mathscr{F})},$$

where $\alpha_H(\mathscr{F})$ is the coefficient of the term of highest degree in $P_H(\mathscr{F})$.

Definition 1.1. *A coherent sheaf $\mathscr{F}$ is $H-$stable (resp. $H-$semistable) if it is pure and for every proper $\mathscr{E} \subseteq \mathscr{F}$ we have $p_H(\mathscr{E})(n) < p_H(\mathscr{F})(n)$ (resp. $p_H(\mathscr{E})(n) \leq p_H(\mathscr{F})(n)$) for $n \gg 0$.*

Let H be a polarization and v a Mukai vector on S. We write $M_v(S,H)$ (resp. $M_v^s(S,H)$) for the moduli space of $H-$semistable (resp. $H-$stable) sheaves on S with Mukai vector v. If no confusion on S and H is possible, we drop them from the notation.

From now on, we suppose that H is $v-$generic (see section 2.1). We write $v = mw$, where $m \in \mathbb{N}$ and w is a primitive Mukai vector on S. It is known that if $M_v^s \neq \emptyset$, then M_v^s is smooth, quasi-projective, of dimension v^2+2 and carries a symplectic form (see Mukai [13]). If S is abelian, a further construction is necessary: choose $\mathscr{F}_0 \in M_v(S,H)$, and define $a_v : M_v(S,H) \longrightarrow S \times \widehat{S}$ in the following way (see [26]): let $p_{\widehat{S}} : S \times \widehat{S} \longrightarrow \widehat{S}$ be the projection and $\mathscr{P}$ the Poincaré bundle on $S \times \widehat{S}$. For every $\mathscr{F} \in M_v(S,H)$ we let

$$a_v(\mathscr{F}) := (det(p_{\widehat{S}!}((\mathscr{F} - \mathscr{F}_0) \otimes (\mathscr{P} - \mathscr{O}_{S\times\widehat{S}})), det(\mathscr{F}) \otimes det(\mathscr{F}_0)^{-1}).$$

Moreover, we define $K_v(S,H) := a_v^{-1}(0_S, \mathscr{O}_S)$, where 0_S is the zero of S.

If $v = w$ (i. e. $m = 1$) and H is $v-$generic, the moduli space $M_v(S,H)$ (resp. $K_v(S,H)$ if S is abelian) is well understood thanks to the work of several authors (see Mukai [14], Beauville [1], O'Grady [15], Yoshioka [23], [24]). If $v^2 < 2$ (resp. if $v^2 < 6$ if S is abelian), then $M_v(S,H)$ (resp. $K_v(S,H)$) is either empty, a point or a surface. The remaining cases are covered by the following:

Theorem 1.2. ***(Yoshioka)***. *Let S be an abelian or projective K3 surface, v a primitive Mukai vector and H a $v-$generic polarization. Then $M_v(S,H) = M_v^s(S,H)$, and we have the following results:*

1. *if S is K3 and $v^2 \geq 2$, then M_v is an irreducible symplectic variety of dimension $2n = v^2+2$, which is deformation equivalent to $Hilb^n(S)$, the Hilbert scheme of $n-$points on S. Moreover, there is a Hodge isometry between $v^\perp$ and $H^2(M_v,\mathbb{Z})$, where the latter has a lattice structure given by the Beauville form;*

2. *if S is abelian and $v^2 \geq 6$, then $K_v(S,H)$ is an irreducible symplectic variety of dimension $2n = v^2 - 2$, which is deformation equivalent to the generalized Kummer variety $K^n(S)$ on S, i. e. the Albanese fibre of $Hilb^{n+1}(S)$, and there is a Hodge isometry between $v^\perp$ and $H^2(K_v,\mathbb{Z})$.*

If v is not primitive, H is $v-$generic and $w^2 \geq 2$, then M_v is singular: it is then natural to ask if there is a *symplectic resolution* of M_v, i. e. a resolution of the singularities $\pi_v : \widetilde{M}_v \longrightarrow M_v$ such that on $\widetilde{M}_v$ there is a symplectic form extending the one on M_v^s. The first result appearing in the literature is the following:

Theorem 1.3. ***(O'Grady)***.

1. *Let S be a projective K3 surface, $v := (2, 0, -2)$ and H be $v-$generic. Then $M_{10} := M_v(S, H)$ admits a symplectic resolution $\pi : \widetilde{M}_{10} \longrightarrow M_{10}$, and $\widetilde{M}_{10}$ is an irreducible symplectic variety of dimension 10 and second Betti number 24.*

2. *Let S be an abelian surface, $v := (2, 0, -2)$ and H be $v-$generic. Then $K_6 := K_v(S, H)$ admits a symplectic resolution $\pi : \widetilde{K}_6 \longrightarrow K_6$, and $\widetilde{K}_6$ is an irreducible symplectic variety of dimension 6 and second Betti number 8.*

The first example is studied by O'Grady in [16], the computation of its second Betti number is started in [16] and completed in [20]. The second example is contained in [17]. The general case is as follows: if H is $v-$generic and $w^2 < 0$, then $M_v(S, H)$ is either empty or a point, whereas if $w^2 = 0$, then $M_v(S, H)$ is a symmetric product of surfaces. For the remaining cases, we have the following answer about the existence of symplectic resolutions (see [12] and [9]):

Theorem 1.4. *Let S be an abelian or projective K3 surface, $v = mw$ a Mukai vector such that $m \geq 2$ and $w^2 \geq 2$. If $w = (r, \xi, a)$, suppose that if $r = 0$ then $a \neq 0$. Finally, let H be a $v-$generic polarization. Then:*

1. *if $m = 2$ and $w^2 = 2$, then $M_v(S, H)$ admits a symplectic resolution $\pi_v : \widetilde{M}_v = \widetilde{M}_v(S, H) \longrightarrow M_v$, obtained as the blow-up of M_v along the singular locus $\Sigma_v = M_v \setminus M_v^s$ with reduced structure **(Lehn-Sorger)**;*

2. *if $m \geq 3$, or $m = 2$ and $w^2 \geq 4$, then $M_v(S, H)$ does not admit any symplectic resolution and it is locally factorial **(Kaledin-Lehn-Sorger)**.*

In this paper, we deal with the moduli spaces verifying the conditions of point 1 of Theorem 1.4. We resume these conditions in the following:

Definition 1.5. *Let S be an abelian or projective K3 surface, v a Mukai vector, H an ample line bundle on S. We say that (S, v, H) is an OLS-triple if the following conditions are verified:*

1. *the polarization H is primitive and $v-$generic;*

2. *there is a primitive Mukai vector $w \in \widetilde{H}(S, \mathbb{Z})$ such that $v = 2w$ and $w^2 = 2$;*

3. *if $w = (0, \xi, a)$, then $a \neq 0$.*

The name OLS-triple is chosen because they were first studied by O'Grady in [16], [17] and Lehn-Sorger in [12]. If (S, v, H) is an OLS-triple, then $M_v(S, H)$ admits a symplectic resolution $\widetilde{M}_v(S, H)$ obtained by blowing-up $M_v(S, H)$ along its singular locus with reduced structure. If S is abelian, let

$$\widetilde{K}_v = \widetilde{K}_v(S, H) := \pi_v^{-1}(K_v),$$

and we still write $\pi_v : \widetilde{K}_v \longrightarrow K_v$ for the symplectic resolution.

The aim of the present paper it to generalize Theorem 1.2 to OLS-triples. Namely, the first result we prove is the following, about the possible irreducible symplectic manifolds one can produce as symplectic resolution of the moduli spaces $M_v(S, H)$ and $K_v(S, H)$ starting from an OLS-triple (S, v, H):

Theorem 1.6. *Let (S, v, H) be an OLS-triple.*

1. *If S is K3, then $\widetilde{M}_v(S, H)$ is an irreducible symplectic variety which is deformation equivalent to $\widetilde{M}_{10}$.*

2. *If S is abelian, then $\widetilde{K}_v(S, H)$ is an irreducible symplectic variety which is deformation equivalent to $\widetilde{K}_6$.*

The proof of this Theorem is contained in Section 2. The idea is to use deformations of the moduli spaces and of their symplectic resolutions induced by deformations of the underlying surfaces, and isomorphisms between moduli spaces with different Mukai vectors which are induced by Fourier-Mukai transforms (the main ingredient here is given by some results of Yoshioka, see [25]). The second result we prove is about the singular cohomology of the moduli spaces $M_v(S, H)$ and $K_v(S, H)$:

Theorem 1.7. *Let (S, v, H) be an OLS-triple.*

1. *If S is K3, then $\pi_v^* : H^2(M_v, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}_v, \mathbb{Z})$ is injective, and the restrictions to $H^2(M_v, \mathbb{Z})$ of the pure weight-two Hodge structure and of the Beauville form on $H^2(\widetilde{M}_v, \mathbb{Z})$ give a pure weight-two Hodge structure and a compatible lattice structure on $H^2(M_v, \mathbb{Z})$. Moreover, there is a Hodge isometry*
$$\lambda_v : v^\perp \longrightarrow H^2(M_v, \mathbb{Z}).$$

2. *If S is abelian, then $\pi_v^* : H^2(K_v, \mathbb{Z}) \longrightarrow H^2(\widetilde{K}_v, \mathbb{Z})$ is injective, and the restrictions to $H^2(K_v, \mathbb{Z})$ of the pure weight-two Hodge structure and of the Beauville form on $H^2(\widetilde{K}_v, \mathbb{Z})$ give a pure weight-two Hodge structure and a compatible lattice structure on $H^2(K_v, \mathbb{Z})$. Moreover, there is a Hodge isometry*
$$\nu_v : v^\perp \longrightarrow H^2(K_v, \mathbb{Z}).$$

By compatible lattice structure we mean that the classes of type $(2, 0)$ with respect to the weight-two Hodge structure on $H^2(M_v, \mathbb{Z})$ are orthogonal to the classes of type $(1, 1)$ and of type $(2, 0)$. The proof of this is contained in Section 3. The reason why π_v^* is injective is because the singularities of M_v and K_v are rational. The construction of the morphisms λ_v and ν_v is a generalization of that of the Mukai-Donaldson morphism. Using Theorem 1.6 and some commutativity of diagrams we can reduce to the case of M_{10} or K_6: there one finally uses results of [19] and [20] to conclude.

2 Deformations of moduli spaces

In this section we study how moduli spaces and their symplectic resolutions vary under deformation. In section 2.1 we recall the notion of $v-$genericity, $v-$walls and $v-$chambers. In section 2.2 we introduce the main deformation we will look at, i. e. the deformation of a moduli space and of its symplectic resolution induced by a deformation of an OLS-triple along a smooth, connected curve. In section 2.3 we give explicit deformations of OLS-triples whose Mukai vector has positive rank, and in section 2.4 we use these and some results of [25] to prove Theorem 1.6.

2.1 Genericity for polarizations

In this section we recall the notion of $v-$genericity, $v-$walls and $v-$chambers. In the following, S will always denote an abelian or a projective K3 surface, and $v = (v_0, v_1, v_2)$ a Mukai vector on S.

2.1.1 Genericity for $v_0 \geq 2$

Suppose $v_0 \geq 2$. If $\mathscr{F}$ is a coherent sheaf of Mukai vector v, we define the discriminant of $\mathscr{F}$ as $\Delta(\mathscr{F}) := 2v_0c_2 - (v_0 - 1)v_1^2$, where c_2 is the second Chern class of $\mathscr{F}$. We define

$$|v| := \frac{v_0^2}{4}\Delta(\mathscr{F})$$

for some coherent sheaf $\mathscr{F}$ of Mukai vector v: notice that $|v|$ does not depend on the chosen sheaf $\mathscr{F}$, but only on v.

Remark 2.1. Notice that if $M_v \neq \emptyset$, then $|v| \geq 0$: indeed, there is a semistable sheaf $\mathscr{F}$ of Mukai vector v, hence the Bogomolov inequality gives $\Delta(\mathscr{F}) \geq 0$. Moreover, we remark that $|v|$ depends only on (v, v) and v_0: if S is K3, we have $|v| = \frac{v_0^2}{4}(v, v) + \frac{v_0^4}{2}$, and if S is abelian, we have $|v| = \frac{v_0^2}{4}(v, v) + \frac{v_0^2}{2}$. If (S, v, H) is an OLS-triple, then $|v| > 0$.

If $|v| > 0$, we define

$$W_v := \{D \in NS(S) \mid -|v| \leq D^2 < 0\},$$

and we let $W_v := \emptyset$ if $|v| = 0$.

Definition 2.2. *A polarization H is $v-$generic if $H \cdot D \neq 0$ for every $D \in W_v$.*

Notice that if $\rho(S) = 1$, then the ample generator H of the Picard group of S is $v-$generic. If $\rho(S) \geq 2$, we need the following:

Definition 2.3. *Suppose that $\rho(S) \geq 2$. If $D \in W_v$, the $v-$wall associated to D is*

$$W^D := \{\alpha \in Amp(S) \mid D \cdot \alpha = 0\}.$$

The $v-$wall associated to $D \in W_v$ is a hyperplane in $Amp(S)$. Moreover, by Theorem 4.C.2 of [8] the subset $\bigcup_{D \in W_v} W^D \subseteq Amp(S)$ is locally finite.

Definition 2.4. *Suppose that $\rho(S) \geq 2$. A connected component of the open subset $Amp(S) \setminus \bigcup_{D \in W_v} W^D$ of $Amp(S)$ is called $v-$chamber.*

By definition, a polarization is $v-$generic if and only if it lies in a $v-$chamber. The following shows that if we change the polarization inside a chamber, the moduli space does not change (for a proof, see [27]):

Proposition 2.5. *Suppose that $\rho(S) \geq 2$ and that $v = (v_0, v_1, v_2)$ is such that $v_0 \geq 2$. Let $\mathcal{C}$ be a $v-$chamber, and suppose that $H, H' \in \mathcal{C}$. Then a sheaf $\mathscr{E}$ of Mukai vector v is $H-$(semi)stable if and only if it is $H'-$(semi)stable, i. e. there is a natural identification between $M_v(S, H)$ and $M_v(S, H')$.*

We conclude this section with an important property that we will need in the following, which is a particular case of Corollary 4.2:

Lemma 2.6. *Let T be a smooth, connected curve, $f : \mathscr{X} \longrightarrow T$ a smooth, projective family of K3 surfaces (resp. of abelian surfaces) and $\mathscr{H}$ a line bundle on $\mathscr{X}$. For every $t \in T$ write $\mathscr{X}_t := f^{-1}(t)$ and $\mathscr{H}_t := \mathscr{H}_{|\mathscr{X}_t}$, and suppose that for every $t \in T$ the line bundle $\mathscr{H}_t$ is ample. Let $0 \in T$ be such that $\mathscr{X}_0 = S$ and $\mathscr{H}_0 = H$. Moreover, let $v = (v_0, v_1, v_2)$ be a Mukai vector on $\mathscr{X}_0$ such that $v_0 \geq 2$, and write $v_1 = c_1(L)$ for some $L \in Pic(\mathscr{X}_0)$. Let $\mathscr{L} \in Pic(\mathscr{X})$ be such that $\mathscr{L}_0 = L$, and let $v_t := (v_0, c_1(\mathscr{L}_t), v_2)$ be a Mukai vector on $\mathscr{X}_t$. If $\mathscr{H}_0$ is $v-$generic, then the set*

$$T' := \{t \in T \,|\, \mathscr{H}_t \textit{ is not } v_t - \textit{generic}\}$$

is locally given by a finite number of points.

2.1.2 Genericity for $v_0 = 0$

Suppose now $v = (0, v_1, v_2)$, where v_1 is the first Chern class of an effective divisor, and $v_2 \neq 0$.

Definition 2.7. *Let $\mathscr{E}$ be any pure sheaf with Mukai vector v, and let $\mathscr{F} \subseteq \mathscr{E}$ with Mukai vector $u = (0, u_1, u_2)$. The divisor associated to the pair $(\mathscr{E}, \mathscr{F})$ is $D := u_2 v_1 - v_2 u_1$. The set of the non numerically trivial divisors associated to all the possible pairs is denoted W_v, and a polarization H is $v-$generic if $H \cdot D \neq 0$ for every $D \in W_v$.*

As before, if $\rho(S) = 1$, then the ample generator of $Pic(S)$ is $v-$generic. For $\rho(S) \geq 2$ we need again to introduce walls and chambers:

Definition 2.8. *Let $D \in W_v$. The $v-$wall associated to D is*

$$W^D := \{\alpha \in Amp(S) \,|\, \alpha \cdot D = 0\}.$$

As before, the $v-$wall W^D is a hyperplane in $Amp(S)$. As shown in [24], the set of $v-$walls is finite.

Definition 2.9. *Suppose that $\rho(S) \geq 2$. A connected component of the open subset $Amp(S) \setminus \bigcup_{D \in W_v} W^D$ of $Amp(S)$ is called $v-$chamber.*

Again, a polarization is $v-$generic if and only if it lies in a $v-$chamber. As in the previous section, we have the following (see [27]):

Proposition 2.10. *Suppose that $\rho(S) \geq 2$ and that $v = (0, v_1, v_2)$ is such that $v_2 \neq 0$. Let $\mathcal{C}$ be a $v-$chamber, and suppose that $H, H' \in \mathcal{C}$. Then a sheaf $\mathscr{E}$ of Mukai vector v is $H-$(semi)stable if and only if it is $H'-$(semi)stable, i. e. there is a natural identification between $M_v(S, H)$ and $M_v(S, H')$.*

To conclude this section, we state the following lemma about the openness of the $v-$genericity for Mukai vectors v of rank 0, which is a particular case of Lemma 4.4:

Lemma 2.11. *Let T be a smooth, connected curve, $f : \mathscr{X} \longrightarrow T$ a smooth, projective family of K3 surfaces (resp. of abelian surfaces) and $\mathscr{H}$ a line bundle on $\mathscr{X}$. For every $t \in T$ write $\mathscr{X}_t := f^{-1}(t)$ and $\mathscr{H}_t := \mathscr{H}_{|\mathscr{X}_t}$, and suppose that for every $t \in T$ the line bundle $\mathscr{H}_t$ is ample. Let $0 \in T$ be such that $\mathscr{X}_0 = S$ and $\mathscr{H}_0 = H$. Moreover, let $v = (0, v_1, v_2)$ be a Mukai vector on $\mathscr{X}_0$ such that*

$v_2 \neq 0$, and write $v_1 = c_1(L)$ for some $L \in Pic(\mathscr{X}_0)$. Let $\mathscr{L} \in Pic(\mathscr{X})$ be such that $\mathscr{L}_0 = L$, and let $v_t := (0, c_1(\mathscr{L}_t), v_2)$ be a Mukai vector on $\mathscr{X}_t$. If $\mathscr{H}_0$ is $v-$generic, then the set

$$T' := \{t \in T \mid \mathscr{H}_t \text{ is not } v_t - \text{generic}\}$$

is finite.

2.2 Deformations of OLS-triples

We introduce the main construction we use in the following. Let (S, v, H) be an OLS-triple and T a smooth, connected curve, and use the following notation: if $f : Y \longrightarrow T$ is a morphism and $\mathscr{L} \in Pic(Y)$, for every $t \in T$ we let $Y_t := f^{-1}(t)$ and $\mathscr{L}_t := \mathscr{L}_{|Y_t}$.

Definition 2.12. *Let (S, v, H) be an OLS-triple, where $v = 2(r, \xi, a)$ and $\xi = c_1(L)$. A deformation of (S, v, H) along T is a triple $(\mathscr{X}, \mathscr{H}, \mathscr{L})$, where:*

1. *$\mathscr{X}$ is a projective, smooth deformation of S along T, i. e. there is a smooth, projective, surjective map $f : \mathscr{X} \longrightarrow T$ such that $\mathscr{X}_t$ is a projective K3 surface (resp. an abelian surface) for every $t \in T$, and there is $0 \in T$ such that $\mathscr{X}_0 \simeq S$;*

2. *$\mathscr{H}$ is a line bundle on $\mathscr{X}$ such that $\mathscr{H}_t$ is ample for every $t \in T$ and such that $\mathscr{H}_0 \simeq H$;*

3. *$\mathscr{L}$ is a line bundle on $\mathscr{X}$ such that $\mathscr{L}_0 \simeq L$;*

If $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ is a deformation of an OLS-triple (S, v, H) along a smooth, connected curve T, where $v = 2(r, \xi, a)$, then for every $t \in T$ we write $\xi_t := c_1(\mathscr{L}_t)$, $w_t := (r, \xi_t, a)$ and $v_t := 2w_t$.

Remark 2.13. Notice that if (S, v, H) is an OLS-triple and $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ is a deformation of (S, v, H) along a smooth, connected curve T, then $(\mathscr{X}_t, v_t, \mathscr{H}_t)$ is an OLS-triple if and only if $\mathscr{H}_t$ is v_t-generic. Indeed, we have $v_t = 2w_t$, where $w_t = (r, \xi_t, a)$ is primitive and $w_t^2 = 2$. Moreover, if $r = 0$, then ξ_t is effective: we have $\xi_t^2 = 2$, hence either ξ_t or $-\xi_t$ is effective; as ξ is effective, then $-\xi \cdot H < 0$, so that $-\xi_t \cdot \mathscr{H}_t < 0$, hence ξ_t is effective.

Remark 2.14. Consider an OLS-triple (S, v, H) where $v = 2(r, \xi, a)$, $r > 0$ and $\xi = c_1(L)$. Let T be a smooth, connected curve. Moreover, consider a smooth, projective deformation $f : \mathscr{X} \longrightarrow T$ of S such that $\mathscr{X}_0 \simeq S$, and on $\mathscr{X}$ consider two line bundles $\mathscr{H}$ and $\mathscr{L}$ such that $\mathscr{H}_0 \simeq H$ and $\mathscr{L}_0 \simeq L$. In general $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ is not a deformation of the OLS-triple (S, v, H) along T: this is the case if and only if $\mathscr{H}_t$ is ample for every $t \in T$. As the set of $t \in T$ such that $\mathscr{H}_t$ is ample is a Zariski open subset of T, by removing a finite number of points from T we can always assume that $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ is a deformation of the OLS-triple (S, v, H) along T.

The reason why we introduce the notion of deformation of an OLS-triple, is because it allows us to study how the algebraic structure of the corresponding moduli space (and of its symplectic resolution) varies under variations of the algebraic structure of the base surface. The first result we prove is that the relative moduli space $\phi : \mathscr{M} \longrightarrow T$ of semistable sheaves associated to a deformation of an OLS-triple along a smooth, connected curve T, is a flat family over a Zariski open neighborhood of any $t \in T$ such that $(\mathscr{X}_t, v_t, \mathscr{H}_t)$ is an OLS-triple.

Lemma 2.15. *Let (S, v, H) be an OLS-triple, T a smooth, connected curve, and $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ a deformation of (S, v, H) along T. If $t \in T$ is such that $(\mathscr{X}_t, v_t, \mathscr{H}_t)$ is an OLS-triple, then $\phi : \mathscr{M} \longrightarrow T$ is flat over t.*

Proof. Let $t \in T$ be such that $(\mathscr{X}_t, v_t, \mathscr{H}_t)$ is an OLS-triple, $T^0 := T \setminus \{t\}$ and $\mathscr{M}^0 := \phi^{-1}(T^0)$. The morphism ϕ is flat over t if and only if the fiber $\mathscr{M}_t$ is the limit of the fibers $\mathscr{M}_s$ as $s \to t$, by Lemma II-29 of [4]. Now, the limit is the fiber over t of the closure of the family $\mathscr{M}^0$, hence there is an inclusion of the limit in $\mathscr{M}_t$. As $(\mathscr{X}_t, v_t, \mathscr{H}_t)$ is an OLS-triple, we have that $\mathscr{H}_t$ is v_t-generic, hence $\mathscr{M}_t = M_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$ is reduced and irreducible (see [9]), and it has to coincide with the previous limit. □

If (S, v, H) is an OLS-triple, then choosing a non-trivial deformation of it along a smooth, connected curve T, up to removing a finite number of points of T we get a projective, flat deformation $\phi : \mathscr{M} \longrightarrow T$ of $M_v(S, H)$. We now present the main result of this section, which is about local properties of this deformation: it is easy to see that if $t_0 \in T$ is any point and U is any open neighborhood of t_0 in T, then $\phi^{-1}(U)$ is not isomorphic to a product $\mathscr{M}_{t_0} \times U$. However, we show in the following Proposition, that this is true locally around every $t \in T$ such that $(\mathscr{X}_t, v_t, \mathscr{H}_t)$ is an OLS-triple.

Proposition 2.16. *Let (S, v, H) be an OLS-triple, T a smooth, connected curve, and $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ a deformation of (S, v, H) along T. Let $0 \in T$ be such that $(\mathscr{X}_0, v_0, \mathscr{H}_0) = (S, v, H)$. For every $p \in \mathscr{M}_0$ the germ $(\mathscr{M}, p)$ is isomorphic, as germ of complex spaces, to the product $(T, 0) \times (\mathscr{M}_0, p)$.*

Proof. As the statement is analytic, we can suppose from now on that T is a small open disk, and that $\mathscr{H}_t$ is v_t-generic for every $t \in T$ by Lemma 4.1. We need the following definition: let $\phi : \mathscr{M} \longrightarrow T$ (resp. $\phi^s : \mathscr{M}^s \longrightarrow T$) be the relative moduli space of semistable (resp. stable) sheaves associated to the deformation $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ of (S, v, H) along T. Let $\Sigma := \mathscr{M} \setminus \mathscr{M}^s$, which is a closed subset of $\mathscr{M}$. Notice that

$$\Sigma = \bigcup_{t \in T} \Sigma_{v_t},$$

and we use the notation $\Sigma_t := \Sigma \cap \mathscr{M} = \Sigma_{v_t}$. Moreover, for every $t \in T$ let Ω_t be the singular locus of Σ_t, and Ω be the closed subset of $\mathscr{M}$ parameterizing sheaves of the form $\mathscr{E} \oplus \mathscr{E}$, where $\mathscr{E}$ is stable. Notice that

$$\Omega = \bigcup_{t \in T} \Omega_t.$$

As $\mathscr{M}_t = M_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$ for every $t \in T$, the point $p \in \mathscr{M}_0$ is one of the following: p is smooth, i. e. $p \in \mathscr{M}_0^s$; $p \in \Sigma_0 \setminus \Omega_0$, i. e. p is essentially an A_1-singularity: more precisely, we have $(\mathscr{M}_0, p) \simeq (\mathbb{C}^8, 0) \times (\{x^2 + y^2 + z^2 = 0\}, 0)$ (see [12]); the last possibility is $p \in \Omega_0$. If p is smooth, the result is trivial, and there is nothing to prove. Hence, we suppose $p \in \Sigma_0$. We have then two cases:

Case 1: $p \in \Sigma_0 \setminus \Omega_0$. Consider the Zariski tangent space $T_p\mathscr{M}$ of $\mathscr{M}$ at p: we have $dim(T_p\mathscr{M}) = dim(\mathscr{M}) + 1 = 12$. Indeed, as $(\mathscr{M}_0, p) \simeq (\mathbb{C}^8, 0) \times (\{x^2 + y^2 + z^2 = 0\}, 0)$, we have $dim(T_p\mathscr{M}_0) = 11$ and, as $\mathscr{M}_0$ is a Cartier divisor of $\mathscr{M}$, we have $dim(T_p\mathscr{M}) \leq 12$. Now, let $\varphi : \mathscr{N} \longrightarrow T$ be the relative moduli space of semistable sheaves with Mukai vector w, i. e. $\mathscr{N}_t = M_{w_t}(\mathscr{X}_t, \mathscr{H}_t)$ for every $t \in T$: by [12], for every $t \in T$ we have that $\Sigma_t \simeq Sym^2(M_{w_t})$, hence $\phi_{|\Sigma} : \Sigma \longrightarrow T$ is identified with the morphism $\overline{\varphi} : Sym^2_T(\mathscr{N}) \longrightarrow T$ induced by φ. By this identification between $\overline{\varphi}$ and $\phi_{|\Sigma}$, the fact that φ is smooth and projective implies that $\phi_{|\Sigma}$ is submersive at $p \in \Sigma$, i. e. the differential $d_p\phi$ is surjective: hence there is $\tau \in T_p\mathscr{M}$ such that $d_p\phi(\tau) \neq 0$. This means that $\tau \notin T_p\mathscr{M}_0$, so that $dim(T_p\mathscr{M}) = 12$.

Now, consider an analytic open neighborhood U of p in $T_p\mathscr{M}$, so we can view it as an open neighborhood of 0 in $\mathbb{C}^{12}$. We let $x_1, ..., x_{12}$ be a coordinate system on $\mathbb{C}^{12}$: we can suppose $x_{12} = t$, a local coordinate of T at 0, and the point p corresponds to the point $(0, ..., 0)$. Moreover, $U \cap \mathscr{M}$ is an analytic subvariety of U of codimension 1, hence there is $f \in \mathscr{O}^{hol}(U)$, a holomorphic function on U, such that the equation of $U \cap \mathscr{M}$ is $f(x_1, ..., x_{11}, t) = 0$. Finally, we can choose U so that $U \cap \Omega = \emptyset$. As seen before, we have that $\Sigma \setminus \Omega$ is smooth and submersive on T, so that we can suppose that the equation of $U \cap \Sigma$ is $x_1 = x_2 = x_3 = 0$.

Now, near the point p the fibre $\mathscr{M}_0$ is analytically isomorphic to a product of an A_1-singularity with a smooth polydisc, hence we have

$$f(x_1, ..., x_{11}, 0) = x_1^2 + x_2^2 + x_3^2,$$

so that

$$f(x_1, ..., x_{11}, t) = x_1^2 + x_2^2 + x_3^2 + \sum_j p_j(x_1, ..., x_{11})t^j,$$

where p_j are holomorphic functions on U depending only on $x_1, ..., x_{11}$. Moreover, we have $p_j \in I^2$ for every j, where I is the ideal of $\mathscr{O}^{hol}(U)$ generated by x_1, x_2 and x_3.

Now, let $p : U \longrightarrow T$ defined as $p(x_1, ..., x_{11}, t) := t$, and let $V := U \cap \Sigma$, on which we have coordinates $x_4, ..., x_{11}, t$. Finally, let

$$p' : U \longrightarrow V, \qquad p'(x_1, ..., x_{11}, t) := (x_4, ..., x_{11}, t)$$

and $q : V \longrightarrow T$ be defined as $q(x_4, ..., x_{11}, t) := t$. Notice that $q \circ p' = p$. Moreover, the fibers of p' are all singular, and the fiber over 0 has an A_1-singularity. By the deformation theory of A_1-singularities of [10], there is an open neighborhood $U' \subseteq \mathscr{M}$ of the point p which is a product of an A_1-singularity by a $9-$dimensional polydisc D. As $\phi_{|U'}$ is identified with $p_{|U'}$, then the projection onto D factors ϕ. Hence ϕ is locally trivial at p, and we are done.

Case 2: $p \in \Omega_0$. The strategy is the following: first, we show that for every $n \in \mathbb{N}$, the infinitesimal $n-$th order deformation of $\mathscr{M}_0$ induced by $\mathscr{M}$ is locally trivial at p. Once this is shown, the statement follows in this way: by Corollary 0.2 of [5] there is a maximal subspace $(T', 0) \subseteq (T, 0)$ such that $(\mathscr{M}_{T'}, p)$ is isomorphic, as germ of complex spaces, to the product $(T', 0) \times (\mathscr{M}_0, p)$ (where $\mathscr{M}_{T'} := \mathscr{M} \times_T T'$). Notice that as the infinitesimal $n-$th order deformation of $\mathscr{M}_0$ induced by $\mathscr{M}$ is locally trivial at p for every n, then T' is positive dimensional. As T is a curve, we finally get $(T', 0) = (T, 0)$, and we are done.

We are left to prove that the infinitesimal $n-$th order deformation of $\mathscr{M}_0$ induced by $\mathscr{M}$ is locally trivial at the points of Ω_0 for every n, and we proceed by induction on n. For $n=1$, let

$$T^1 := \mathscr{E}xt^1(\Omega^1_{\mathscr{M}_0}, \mathscr{O}_{\mathscr{M}_0}),$$

where $\Omega^1_{\mathscr{M}_0}$ is the sheaf of holomorphic $1-$forms on $\mathscr{M}_0$: then T^1 is supported on Σ_0, and the local sections of T^1 correspond to local infinitesimal first order deformations of $\mathscr{M}_0$. Moreover, by [12] we know that T^1 is pure.

We show that the infinitesimal first order deformation of $\mathscr{M}_0$ induced by $\mathscr{M}$ is locally trivial at p: consider a Stein open neighborhood U_1 of p in $\mathscr{M}_0$, and let s be the element of T^1 on U_1 induced by $\mathscr{M}$. Let $q \in U_1 \cap (\Sigma_0 \setminus \Omega_0)$: then q is a point of the previous case, hence s is locally trivial at q. This means that there is a Stein open neighborhood $V_q \subseteq U_1$ of q such that $s_{|V_q}$ is trivial. By purity of T^1, s is trivial on U_1, and we are done.

By induction, suppose that the infinitesimal $n-$th order deformation of $\mathscr{M}_0$ induced by $\mathscr{M}$ is locally trivial at p. There are two extensions of it to a local infinitesimal $(n+1)-$th order deformation at p: the trivial one, which we call s_1, and the one induced by $\mathscr{M}$, which we call s_2. By Theorem 2.11 of [22] and Lemma 2.12 of [6] there is a transitive action of T^1 on the space of small extensions, hence there is an element h of T^1 on a Stein open neighborhood U of p such that $h(s_1) = s_2$, where $h(s_1)$ is the action of h on s_1. Let $q \in U \cap (\Sigma_0 \setminus \Omega_0)$: as this is a point of the previous case, the infinitesimal $(n+1)-$th order deformation of $\mathscr{M}_0$ induced by $\mathscr{M}$ is locally trivial at q, hence there is a Stein open neighborhood $V_q \subseteq U$ of q such that $s_{2|V_q} = s_{1|V_q}$. This implies that $h_{|V_q}$ is trivial. Again, by purity of T^1 this implies that h is trivial on U, so that $s_1 = s_2$ on U, and we are done. □

The Proposition we just proved has two important consequences. The first one is that if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two OLS-triples which are related by a deformation of OLS-triples along a smooth, connected curve, then $\widetilde{M}_{v_1}(S_1, H_1)$ and $\widetilde{M}_{v_2}(S_2, H_2)$ are deformation equivalent.

Proposition 2.17. *Let (S, v, H) be an OLS-triple, T a smooth connected curve, and $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ a deformation of (S, v, H) along T.*

1. *If S is a K3 surface, then $\widetilde{M}_v(S, H)$ is irreducible symplectic if and only if $\widetilde{M}_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$ is for some $t \in T$ such that $(\mathscr{X}_t, v_t, \mathscr{H}_t)$ is an OLS-triple, and their deformation classes are equal.*

2. *If S is an abelian surface, Then $\widetilde{K}_v(S, H)$ is irreducible symplectic if and only if $\widetilde{K}_{v_t}(\mathscr{X}_t, \mathscr{H}_t)$ is for some $t \in T$ such that $(\mathscr{X}_t, v_t, \mathscr{H}_t)$ is an OLS-triple, and their deformation classes are equal.*

Proof. Let us suppose that S is K3, and define $\pi : \widetilde{\mathscr{M}} \longrightarrow \mathscr{M}$ to be the blow-up of $\mathscr{M}$ along $\Sigma = \mathscr{M} \setminus \mathscr{M}^s$ with reduced structure. We have a morphism

$$\psi := \phi \circ \pi : \widetilde{\mathscr{M}} \longrightarrow T,$$

which is projective (as ϕ and π are projective) and flat over a Zariski open subset of Tcontaing the subset

$$T_{OLS} := \{t \in T \,|\, (\mathscr{X}_t, v_t, \mathscr{H}_t) \text{ is an OLS} - \text{triple}\}$$

(by Lemma 2.15). Notice that T_{OLS} is open and connected in the classical topology by Lemma 4.1. By [12], for every $t\in T_{OLS}$ the moduli space $M_{v_t}(\mathscr{X}_t,\mathscr{H}_t)$ admits a symplectic resolution $\widetilde{M}_{v_t}(\mathscr{X}_t,\mathscr{H}_t)$ obtained as blow up of $M_{v_t}(\mathscr{X}_t,\mathscr{H}_t)$ along $\Sigma_{v_t,red}$. As an immediate consequence of Proposition 2.16, we have that

$$\widetilde{\mathscr{M}_t}=(Bl_{\Sigma_{red}}\mathscr{M})_t=Bl_{\Sigma_{t,red}}\mathscr{M}_t.$$

Moreover, we have $\mathscr{M}_t=M_{v_t}(\mathscr{X}_t,\mathscr{H}_t)$ and $\Sigma_{t,red}=\Sigma_{v_t,red}$, so that

$$\widetilde{\mathscr{M}_t}=\widetilde{M}_{v_t}(\mathscr{X}_t,\mathscr{H}_t)$$

for every $t\in T_{OLS}$. Hence $\widetilde{\mathscr{M}_t}$ is a smooth, symplectic, projective variety. As T_{OLS} is smooth and connected, the statement follows as in the proof of Corollary 6.2.12 of [8].

If S is an abelian surface, we need one step more: define $\widehat{\mathscr{X}}:=Pic^0(\mathscr{X})$, with the natural map $\widehat{f}:\widehat{\mathscr{X}}\longrightarrow T$, which is again smooth. Consider

$$Z:=\{(0_{\mathscr{X}_t},\mathscr{O}_{\mathscr{X}_t})\in\mathscr{X}_t\times\widehat{\mathscr{X}_t}\,|\,t\in T\}\subseteq\mathscr{X}\times_T\widehat{\mathscr{X}},$$

with the natural morphism $g:Z\longrightarrow T$, which is clearly an isomorphism. Moreover, we define a $T-$morphism

$$a:\mathscr{M}\longrightarrow\mathscr{X}\times_T\widehat{\mathscr{X}},$$

such that $a_{|\mathscr{M}_t}:=a_{v_t}$. Set $\widetilde{a}:=\pi\circ a:\widetilde{\mathscr{M}}\longrightarrow\mathscr{X}\times_T\widehat{\mathscr{X}}$, and let $\widetilde{\mathscr{K}}:=\widetilde{a}^{-1}(Z)$. If $t\in T_{OLS}$, then we have $\widetilde{\mathscr{K}_t}=\widetilde{K}_{v_t}(\mathscr{X}_t,\mathscr{H}_t)$, hence the statement follows again as in the proof of Corollary 6.2.12 of [8]. $\square$

The second consequence of Proposition 2.16 is that the family $\phi:\mathscr{M}\longrightarrow T$ is topologically a product on small open subsets of T parameterizing OLS-triples. More precisely, we have the following:

Corollary 2.18. *Let (S,v,H) be an OLS-triple, T a smooth, connected curve and $(\mathscr{X},\mathscr{H},\mathscr{L})$ a deformation of (S,v,H) along T. Let $\phi:\mathscr{M}\longrightarrow T$ be the relative moduli space induced by $(\mathscr{X},\mathscr{H},\mathscr{L})$. Then for every $t\in T$ such that $(\mathscr{X}_t,v_t,\mathscr{H}_t)$ is an OLS-triple, there is an analytic open neighborhood $U\subseteq T$ of t, and a homeomorphism*

$$h:\phi^{-1}(U)\longrightarrow\mathscr{M}_t\times U,$$

such that $p_U\circ h=\phi$, where $p_U:\mathscr{M}_t\times U\longrightarrow U$ is the projection.

Proof. As the statement is local, by Lemma 4.1 we can suppose that T is a small open disk and that $\mathscr{H}_t$ is v_t-generic for every $t\in T$, i. e. that $(\mathscr{X}_t,v_t,\mathscr{H}_t)$ is an OLS-triple. Let $\mathcal{S}:=\{S_1,S_2,S_3\}$ be the stratification of $\mathscr{M}$ given by $S_1:=\mathscr{M}\setminus\Sigma$, $S_2:=\Sigma\setminus\Omega$ and $S_3:=\Omega$. By Proposition 2.16, we see that $Sing(\mathscr{M})=\bigcup_{t\in T}Sing(\mathscr{M}_t)$. As $Sing(\mathscr{M}_t)=\Sigma_t$, we then get $Sing(\mathscr{M})=\Sigma$. Similarily, we have $Sing(\Sigma)=\Omega$. Moreover, as Ω_t is smooth for every $t\in T$, Proposition 2.16 implies also that Ω is smooth. In conclusion, all the strata of $\mathcal{S}$ are smooth, and if $\overline{S}_i$ is the closure of S_i in $\mathscr{M}$ for every $i=1,2,3$, then we see that S_{i+1} is the singular locus of $\overline{S}_i$. Hence, by Proposition 2.16 we see that every stratum S_i is submersive over T. As $\overline{S}_i$ is proper over T for every

$i = 1, 2, 3$, the statement follows from the Thom First Isotopy Lemma (see Theorem 3.5 in Chapter 1 of [2]) if we prove that $\mathcal{S}$ is a Whitney stratification.

Again, by Proposition 2.16 it is sufficient to prove that for every $t \in T$, the stratification $\mathcal{S}_t = \{S_{1,t}, S_{2,t}, S_{3,t}\}$ of M_{v_t} defined letting $S_{i,t} := S_i \cap \mathscr{M}_t$ for $i = 1, 2, 3$ (i. e. $S_{1,t} = M^s_{v_t}$, $S_{2,t} = \Sigma_{v_t} \setminus \Omega_{v_t}$ and $S_{3,t} = \Omega_{v_t}$), is Whitney. To do so, we need to show that $S_{i,t}$ is Whitney regular over $S_{j,t}$ for every $j > i$ (see Definition 1.7 of [2]). We have two cases:

Case 1: $S_{1,t}$ is Whitney regular over $S_{2,t}$. Let $p \in S_{2,t} = \Sigma_t \setminus \Omega_t$: then there is an open neighborhood $U \subseteq M_{v_t}$ of p, which is a product of a type A_1-singularity by an $8-$dimensional polydisc. As the stratification of the singularities of the type A_1-singularity is Whitney, this implies the Whitney regularity of S_1 over S_2.

Case 2: $S_{1,t}$ and $S_{2,t}$ are Whitney regular over $S_{3,t}$. Let $q \in S_{3,t} = \Omega_t$: by [12] there is open neighborhood of q in $\mathscr{M}_t$ which is of the form $Z \times V$, where V is a smooth polydisk of dimension 4, and Z is a $6-$dimensional singular variety whose singular locus Z' has dimension 4, and such that $Z'' := Sing(Z')$ is $0-$dimensional. In $Z \times V$ the stratification $\mathcal{S}_t$ is $\{(Z \setminus Z') \times V, (Z' \setminus Z'') \times V, Z'' \times V\}$. Now, the strata $S_{1,t}$ and $S_{2,t}$ are Whitney regular over $S_{3,t}$ if and only if $(Z \setminus Z') \times V$ and $(Z' \setminus Z'') \times V$ are Whitney regular over $Z'' \times V$. But this is true by Lemma 1.10 in Chapter 1 of [2], as Z'' is $0-$dimensional, and we are done. $\square$

2.3 Deformations and Mukai vectors of positive rank

In this section we consider OLS-triples with Mukai vector v of positive rank, and we show that the deformation classes of $\widetilde{M}_v$ and $\widetilde{K}_v$ depend only on the rank of v. To do so, we follow closely the arguments used by O'Grady in [15]. As a first step, we remark that the tensorization via a line bundle does not change the moduli spaces. Let S be an abelian or projective K3 surface.

Definition 2.19. *Let $v, v' \in \widetilde{H}(S, \mathbb{Z})$ be two Mukai vectors, $v = (v_0, v_1, v_2)$, $v' = (v'_0, v'_1, v'_2)$ and $v_0, v'_0 > 0$. We say that v and v' are equivalent if there is a line bundle L on S such that $v' = v \cdot ch(L)$.*

Notice that if v is equivalent to v', then $v_0 = v'_0$ and $v^2 = (v')^2$, so that $W_v = W_{v'}$ (by Remark 2.1), and H is $v-$generic if and only if it is $v'-$generic. Moreover, (S, v, H) is an OLS-triple if and only if (S, v', H) is an OLS-triple. If (S, v, H) and (S, v', H) are two OLS-triples such that $v' = v \cdot ch(L)$ for some line bundle $L \in Pic(S)$, then the tensorization with L defines an isomorphism between $M_v(S, H)$ and $M_{v'}(S, H)$. This is due to the following, which is Lemma 1.1 of [24]:

Lemma 2.20. *If v is a Mukai vector of positive rank, H is $v-$generic and $L \in Pic(S)$, then the tensorization with L gives an isomorphism between $M_v(S, H)$ and $M_{v'}(S, H)$.*

Remark 2.21. This Lemma is originally stated only for stable sheaves, but the argument goes through for semistable sheaves.

In order to give explicit deformations of an OLS-triple (S, v, H) where $v = 2(r, \xi, a)$ and $r > 0$, we use the irreducibility of the moduli space of polarized K3 or abelian surfaces. Hence, it is useful to suppose $\xi = c_1(H)$, which is always possible by the following:

Lemma 2.22. *Let (S, v, H) be an OLS-triple where $v = 2(r, \xi, a)$ is such that $r > 0$. Suppose that $\rho(S) \geq 2$, and let $\mathcal{C}$ be the $v-$chamber such that $H \in \mathcal{C}$. Then there exists a Mukai vector $v' = 2(r, \xi', a')$ such that*

1. *v' is equivalent to v;*
2. *ξ' is a primitive ample class lying in $\mathcal{C}$.*

Moreover, we can choose v' so that $(\xi')^2 \gg 0$.

Proof. The proof is essentially the one of Lemma II.6 of [15]: there one requires ξ to be primitive, but Yoshioka noticed that the same argument goes through in the more general case of r and ξ prime to each other (see [25]). This last condition is always verified: write $w = (r, \xi, a)$, which is primitive and $w^2 = 2$, i. e. $\xi^2 = 2ra + 2$. Suppose that $s \in \mathbb{N}$ is such that $r = sr'$ and $\xi = s\xi'$. Then $s^2(\xi')^2 = 2sar' + 2$. As S is abelian or K3, we have $(\xi')^2 = 2l$ for some $l \in \mathbb{Z}$, so that $s(sl - ar') = 1$. As $s \in \mathbb{N}$ this implies $s = 1$, and we are done. □

An important class of OLS-triples is given by those on elliptic K3 or abelian surfaces, as in this case we have a priviledged class of polarizations. In order to prove that the deformation class of $\widetilde{M}_v(S, H)$ depends only on the rank of v, the strategy will be to deform the OLS-triple (S, v, H) to an OLS-triple on an elliptic K3 or abelian surface with a polarization in this priviledged class. Let then Y be an elliptic K3 or abelian surface such that $NS(Y) = \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \sigma$, where f is the class of a fiber and σ is the class of a section. Let v be a Mukai vector on Y, and recall the following definition (see [15]):

Definition 2.23. *A polarization H on Y is called $v-$suitable if H is in the unique $v-$chamber whose closure contains f.*

We have an easy numerical criterion to guarantee that a polarization on Y is $v-$suitable:

Lemma 2.24. *Let Y be an elliptic K3 or abelian surface such that $NS(Y) = \mathbb{Z} \cdot \sigma \oplus \mathbb{Z} \cdot f$, where σ is a section and f is a fibre, and let $v = (v_0, v_1, v_2)$ be a Mukai vector on Y such that $v_0 > 0$. Let H be a polarization such that $c_1(H) = \sigma + lf$ for some $l \in \mathbb{Z}$.*

1. *If S is K3, then H is $v-$suitable if $l \geq |v| + 1$.*
2. *If S is abelian, then H is $v-$suitable if $l \geq |v|$.*

Proof. If S is a K3 surface, this is Lemma I.0.3 of [15]. For the abelian case the proof is similar: H is $v-$suitable if and only if $D \cdot H$ and $D \cdot f$ have the same sign for every $D \in W_v$. Notice that $D = a\sigma + bf$ for some $a, b \in \mathbb{Z}$, so that $D \cdot f = a$. Suppose $D \cdot f > 0$, i. e. $a > 0$. We have $D \cdot H = la + b$ and $D^2 = 2ab$. As $D^2 \geq -|v|$, we then get $b \geq -l/2a$. If $l \geq |v|$, we then get

$$D \cdot H = la + b \geq |v|a - (|v|/2a) > 0,$$

and we are done. □

The main result of this section is the following:

Proposition 2.25. *Let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two OLS-triples verifying the two following conditions:*

1. *S_1 and S_2 are two projective K3 surfaces or two abelian surfaces;*
2. *if $v_i = 2(r_i, \xi_i, a_i)$, then $r_1 = r_2 > 0$.*

Then $\widetilde{M}_{v_1}(S_1, H_1)$ is deformation equivalent to $\widetilde{M}_{v_2}(S_2, H_2)$. In particular, Theorem 1.6 is true for (S_1, v_1, H_1) if and only if it is true for (S_2, v_2, H_2).

Proof. The argument we present here was first used by O'Grady in [15], then extended by Yoshioka in [23]. For $i = 1, 2$, we let $L_i \in Pic(S_i)$ be such that $\xi_i = c_1(L_i)$.

First of all, we can always assume $\rho(S_i) > 1$. Indeed, consider a non-trivial deformation $(\mathscr{X}_i, \mathscr{H}_i, \mathscr{L}_i)$ of the OLS-triple (S_i, v_i, H_i) along a smooth, connected curve T, and let $0 \in T$ be such that $(\mathscr{X}_{i,0}, v_{i,0}, \mathscr{H}_{i,0}) = (S_i, v_i, H_i)$. By Lemma 2.6, there is a small open neighborhood U of 0 in T such that the triple $(\mathscr{X}_{i,t}, v_{i,t}, \mathscr{H}_{i,t})$ is an OLS-triple for all $t \in U$. By the Main Theorem of [18], we know that the locus of $t \in U$ such that $\rho(\mathscr{X}_{i,t}) > 1$ is dense in the classical topology of U: by Proposition 2.17 we can then suppose $\rho(S_i) > 1$.

By Lemma 2.22 and Proposition 2.5 we may also suppose (S_i, v_i, H_i) to be such that $v_i = 2(r, c_1(H_i), a_i)$ and $H_i^2 = 2d_i$, where $d_i \gg 0$. Now, let Y be a K3 (resp. abelian) surface admitting an elliptic fibration and such that

$$NS(Y) = \mathbb{Z} \cdot \sigma \oplus \mathbb{Z} \cdot f,$$

where f is the class of a fiber, and σ is the class of a section. For $i = 1, 2$, there is a smooth, connected curve T_i and a deformation $(\mathscr{X}'_i, \mathscr{H}'_i, \mathscr{L}'_i)$ over T_i of the OLS-triple (S_i, v_i, H_i) such that there is $t \in T_i$ with the property $(\mathscr{X}'_{i,t}, v'_{i,t}, \mathscr{H}'_{i,t}) = (Y, v'_i, H'_i)$, where

1. $c_1(H'_i) = \sigma + l_i f$, where $l_i \gg 0$.
2. $v'_i = 2(r, c_1(H'_i), a_i)$.

By Proposition 2.17, we have that $\widetilde{M}_{v_i}(S_i, H_i)$ is deformation equivalent to $\widetilde{M}_{v'_i}(Y, H'_i)$, so we just need to show the statement for $(S_i, v_i, H_i) = (Y, v'_i, H'_i)$, for $i = 1, 2$. Let $\xi'_i := c_1(H'_i)$. Notice that $(v'_1)^2 = (v'_2)^2$ and they have the same rank: hence $|v'_1| = |v'_2|$, so that by Lemma 2.24 a polarization is v'_1-suitable if and only if it is v'_2-suitable. Again by Lemma 2.24, we have that H'_i is v'_i-suitable for $i = 1, 2$, as $l_i \gg 0$. Then H'_1 and H'_2 lie in the same chamber $\mathcal{C}$ (which is clearly a v'_i-chamber for $i = 1, 2$). By Proposition 2.5 we then change to a common generic polarization $H \in \mathcal{C}$, which is v'_i-generic for $i = 1, 2$.

As $(v'_1)^2 = (v'_2)^2$, we have $(\xi'_1)^2 - 2ra_1 = (\xi'_2)^2 - 2ra_2$, and as

$$(\xi'_i)^2 = (\sigma + l_i f)^2 = 2l_i - 2,$$

(in the abelian case we have $\xi_i^2 = 2l_i$), we then get the equation

$$l_1 = l_2 + r(a_1 - a_2). \tag{1}$$

Notice that v'_1 and v'_2 are then equivalent: indeed, we have

$$v'_2 \cdot ch(\mathscr{O}_Y((a_1 - a_2)f)) = 2(r, \sigma + l_2 f, a_2) \cdot (1, (a_1 - a_2)f, 0) =$$

$$= 2(r, \sigma + l_1 f, a_1) = v_1',$$

where the second equality follows from equation (1). By Lemma 2.20 we are then done. $\square$

Remark 2.26. We observe that in order to relate $\widetilde{M}_{v_1}(S_1, H_1)$ and $\widetilde{M}_{v_2}(S_2, H_2)$ in the previous proof, we only used deformations of the symplectic resolutions induced by deformations of OLS-triples along a smooth, connected curve, and isomorphisms between moduli spaces given by tensorization with a line bundle.

2.4 Proof of Theorem 1.6

In this section we finally prove Theorem 1.6: the crucial facts are two lemmas due to Yoshioka [25]. If S is an abelian or projective K3 surface, write Δ for the diagonal of $S \times S$, $\mathscr{I}_\Delta$ for the ideal sheaf of Δ and, if S is abelian, let $\mathscr{P}$ be the Poincaré bundle on $S \times \widehat{S}$ and $\widehat{H}$ the dual polarization on $\widehat{S}$.

Lemma 2.27. ***(Yoshioka)****. Let (S, v, H) be an OLS-triple where $v = 2(0, \xi, a)$ is such that $a \gg 0$.*

1. *If S is K3, let $\widehat{v} := 2(a, \xi, 0)$, and suppose that H is $\widehat{v}-$generic. Then the Fourier-Mukai transform $\mathcal{F} : D^b(S) \longrightarrow D^b(S)$ with kernel $\mathscr{I}_\Delta$ sends any $H-$(semi)stable sheaf with Mukai vector v to an $H-$(semi)stable sheaf with Mukai vector $\widehat{v}$. In particular, it defines an isomorphism between $\widetilde{M}_v(S, H)$ and $\widetilde{M}_{\widehat{v}}(S, H)$.*

2. *If S is abelian, let $\widehat{v} := 2(a, \widehat{\xi}, 0)$, where $\widehat{\xi}$ is the dual of ξ. Then the Fourier-Mukai transform $\mathcal{F} : D^b(S) \longrightarrow D^b(\widehat{S})$ with kernel $\mathscr{P}$ sends any $H-$(semi)stable sheaf with Mukai vector v to an $\widehat{H}-$(semi)stable sheaf with Mukai vector $\widehat{v}$. In particular, it defines an isomorphism between $\widetilde{K}_v(S, H)$ and $\widetilde{K}_{\widehat{v}}(\widehat{S}, \widehat{H})$.*

Proof. We prove the statement for K3 surfaces, the case of abelian surfaces is analogue. Let $w := (0, \xi, a)$ and $\widehat{w} := (a, \xi, 0)$, and notice that as H is $v-$generic (resp. $\widehat{v}-$generic), then it is $w-$generic (resp. $\widehat{w}-$generic). By Proposition 3.14 of [25], as $a \gg 0$ the Fourier-Mukai functor of the statement sends an $H-$stable sheaf with Mukai vector v (resp. w) to an $H-$stable sheaf of Mukai vector $\widehat{v}$ (resp. $\widehat{w}$). As $M_v = M_v^s \cup Sym^2 M_w$ and $M_w = M_w^s$ (as w is primitive and H is $w-$generic), then $\mathcal{F}$ induces an open embedding $f : M_v(S, H) \longrightarrow M_{\widehat{v}}(S, H)$: as H is $\widehat{v}-$generic and $M_v(S, H)$ is projective, this implies that f is an isomorphism, which induces an isomorphism between $\widetilde{M}_v(S, H)$ and $\widetilde{M}_{\widehat{v}}(S, H)$. $\square$

The following lemma is Theorem 3.18 of [25]:

Lemma 2.28. ***(Yoshioka)****. Let (S, v, H) be an OLS-triple such that $NS(S) = \mathbb{Z} \cdot h$, where $h = c_1(H)$ is ample and $h^2 = 2$. Write $v = 2(r, nh, a)$, and suppose $n \gg 0$.*

1. *If S is K3, then the Fourier-Mukai transform $\mathcal{F} : D^b(S) \longrightarrow D^b(S)$ with kernel $\mathscr{I}_\Delta$ sends $H-$(semi)stable sheaves with Mukai vector $2(r, nh, a)$ to $H-$(semi)stable sheaves with Mukai vector $2(a, nh, r)$. In particular, it defines an isomorphism between $\widetilde{M}_{2(r,nh,a)}(S, H)$ and $\widetilde{M}_{2(a,nh,r)}(S, H)$.*

2. *If S is abelian, then the Fourier-Mukai transform $\mathcal{F} : D^b(S) \longrightarrow D^b(\widehat{S})$ with kernel $\mathscr{P}$ sends $H-$(semi)stable sheaves with Mukai vector $2(r, nh, a)$ to $\widehat{H}-$(semi)stable sheaves with Mukai vector $2(a, n\widehat{h}, r)$. In particular, it defines an isomorphism between $\widetilde{K}_{2(r,nh,a)}(S, H)$ and $\widetilde{K}_{2(a,n\widehat{h},r)}(\widehat{S}, \widehat{H})$.*

Remark 2.29. Suppose that S is a K3 or abelian surface such that $NS(S) = \mathbb{Z}\cdot h$, where $h = c_1(H)$ is ample and $h^2 = 2$, and let $v = 2w$ be a Mukai vector on S such that w is primitive and $w^2 = 2$. It is then easy to see that $w = (r, nh, a)$ for some $r, n, a \in \mathbb{Z}$ such that $gcd(r, n, a) = 1$ and $n^2 = ra + 1$.

We now proceed with the proof of Theorem 1.6:

Theorem 1.6. *Let (S, v, H) be an OLS-triple.*

1. *If S is K3, then $\widetilde{M}_v(S, H)$ is irreducible symplectic and deformation equivalent to $\widetilde{M}_{10}$.*

2. *If S is abelian, then $\widetilde{K}_v(S, H)$ is irreducible symplectic and deformation equivalent to $\widetilde{K}_6$.*

Proof. Let (S, v, H) be an OLS-triple where S is a projective K3 surface (the proof in the abelian case is analogue), and write $v = 2(r, \xi, a)$. We show that $\widetilde{M}_v(S, H)$ is deformation equivalent to $\widetilde{M}_{2(0,h,2)}(X, \overline{H})$, where X is a surface such that $NS(X) = \mathbb{Z} \cdot h$, $h = c_1(\overline{H})$ is ample and $\overline{H}^2 = 2$. The equivalence is obtained using deformations of the simplectic resolutions induced by deformations along smooth, connected curves of the corresponding OLS-triple, and isomorphism between moduli spaces. As a particular case is $\widetilde{M}_{10}$, we are done.

Step 1: suppose that $S = X$ and $v = 2(0, h, a)$, where $a = 2k$ for some $k \in \mathbb{Z}$. Then $\widetilde{M}_v(X, \overline{H}) \simeq \widetilde{M}_{2(0,h,2)}(X, \overline{H})$: indeed $v = 2(0, h, 2) \cdot ch(\mathscr{O}_X((k-1)\overline{H}))$, and as tensoring with a multiple of $\overline{H}$ does not change $\overline{H}-$(semi)stability, we get an isomorphism

$$M_{2(0,h,2)}(X, \overline{H}) \longrightarrow M_{2(0,h,a)}(X, \overline{H}), \qquad E \mapsto E \otimes \mathscr{O}_X((k-1)\overline{H}),$$

and we are done.

Step 2: suppose that (S, v, H) is an OLS-triple such that $r > 0$. By Proposition 2.25 we know that $\widetilde{M}_v(S, H)$ is deformation equivalent to $\widetilde{M}_{2(r,nh,a)}(X, \overline{H})$, for some $n \in \mathbb{Z}$ and $a = (n^2 - 1)/r$ (by Remark 2.29). Choose $n \gg 0$ such that the corresponding a is even. As $n \gg 0$, point 1 of Lemma 2.28 gives an isomorphism between $M_{2(r,nh,a)}(X, \overline{H})$ and $M_{2(a,nh,r)}(X, \overline{H})$, which is deformation equivalent to $M_{2(a,h,0)}(X, \overline{H})$ by Proposition 2.25. Moreover, as $n \gg 0$ we have $a \gg 0$, hence by point 1 of Lemma 2.27 we have an isomorphism between $M_{2(a,h,0)}(X, \overline{H})$ and $M_{2(0,h,a)}(X, \overline{H})$. As a is even, we have $M_{2(0,h,a)}(X, \overline{H}) \simeq M_{2(0,h,2)}(X, \overline{H})$ by Step 1, and we are done.

Step 3: suppose that (S, v, H) is any OLS-triple such that $r = 0$. Let $v' := v \cdot ch(\mathscr{O}_S(dH))$ for some $d \in \mathbb{N}$ such that $d \gg 0$ and $v'_2 \neq 0$. A straightforward computation shows that H is $v'-$generic, and that the tensorization with dH does not change the $H-$(semi)stability. Hence we have an isomorphism

$$M_v(S, H) \longrightarrow M_{v'}(S, H), \qquad E \mapsto E \otimes \mathscr{O}_S(dH).$$

Notice that

$$v' = 2(0, \xi, a) \cdot (1, dH, d^2H^2/2) = 2(0, \xi, a + dH \cdot \xi).$$

As H is ample and ξ is effective, we have $a + d\xi \cdot H \gg 0$ as $d \gg 0$, hence we assume $a \gg 0$. Let now $\widehat{v} := 2(a, \xi, 0)$. If H is not $\widehat{v}-$generic, by Proposition 2.10 we may replace H with a $v-$generic polarization H' lying in the same $v-$chamber of H which is $\widehat{v}-$generic. By point 1 of Lemma 2.27 we have then an isomorphism between $M_v(S, H)$ and $M_{\widehat{v}}(S, H)$: we are now in the situation of Step 2, hence we are done. $\square$

3 The second integral cohomology of the moduli spaces

Let (S, v, H) be an OLS-triple. In this section we define a morphism

$$\lambda_v : v^\perp \longrightarrow H^2(M_v, \mathbb{Z}).$$

For primitive Mukai vectors v with $v^2 = 0$, this was defined (using semi-universal families) first by Mukai [14], who showed that it gives a Hodge isometry between $v^\perp/\mathbb{Z} \cdot v$ and $H^2(M_v, \mathbb{Z})$ (in this case M_v is a K3 surface). For v primitive and $v^2 \geq 2$, this morphism was constructed by Mukai [14], O'Grady [15] and Yoshioka [23], who showed that λ_v gives a Hodge isometry between $v^\perp$ and $H^2(M_v, \mathbb{Z})$ (the latter being a lattice with respect to the Beauville form, as it is an irreducible symplectic manifold).

In the present section we define λ_v for any OLS-triple (S, v, H): as in the case of primitive Mukai vectors, using a semi-universal family on $S \times M_v^s$ one defines a morphism λ_v^s which, a priori, takes values only in $H^2(M_v^s, \mathbb{Q})$. As (S, v, H) is an OLS-triple, M_v^s is an open subset which is strictly contained in M_v, and we show that $H^2(M_v^s, \mathbb{Q}) \simeq H^2(M_v, \mathbb{Q})$, so that we finally define a morphism $\lambda_v : v^\perp \longrightarrow H^2(M_v, \mathbb{Q})$.

The main result of the section is to show that the morphism λ_v takes values in $H^2(M_v, \mathbb{Z})$, and moreover that it is a Hodge isometry between $v^\perp$ and $H^2(M_v, \mathbb{Z})$ (or $H^2(K_v, \mathbb{Z})$ if S is abelian). Before doing this, we need to show that on $H^2(M_v, \mathbb{Z})$ there are a pure weight-two Hodge structure and an integral bilinear form: as shown in section 3.1, these are induced by the pure weight-two Hodge structure on $H^2(\widetilde{M}_v, \mathbb{Z})$ and by the Beauville form of $\widetilde{M}_v$ (which is now known to be an irreducible symplectic manifold), as a consequence of the fact that the singularities of M_v are rational. In section 3.3 we show that λ_v takes values in $H^2(M_v, \mathbb{Z})$ and that it is a Hodge isometry: by [19], this is the case for the O'Grady examples; by following the steps of the proof of Theorem 1.6, we show that this is then always verified.

3.1 Hodge structure and integral bilinear form

In this section we show that on $H^2(M_v, \mathbb{Z})$ and $H^2(K_v, \mathbb{Z})$ there are a pure weight-two Hodge structure and an integral bilinear form for every OLS-triple (S, v, H). As a first step we show that they are free $\mathbb{Z}-$modules.

Lemma 3.1. *Let X be a normal, irreducible projective variety with rational singularities, and let $f : \widetilde{X} \longrightarrow X$ be a resolution of the singularities. The morphism $f^* : H^2(X,\mathbb{Z}) \longrightarrow H^2(\widetilde{X},\mathbb{Z})$ is injective.*

Proof. As X is a normal, irreducible projective variety having rational singularities and $f : \widetilde{X} \longrightarrow X$ is a resolution of singularities, then $R^i f_* \mathscr{O}_{\widetilde{X}} = 0$ for every $i > 0$. Moreover, by the Zariski Main Theorem the natural morphism $\mathscr{O}_X \longrightarrow f_* \mathscr{O}_{\widetilde{X}}$ is an isomorphism, and $f_* \mathscr{O}^*_{\widetilde{X}} \simeq \mathscr{O}^*_X$. Applying the functor Rf_* to the exponential sequence of $\widetilde{X}$ we then find $R^1 f_* \mathbb{Z} = 0$. Consider the Leray spectral sequence

$$E_2^{p,q} := H^p(X, R^q f_* \mathbb{Z}) \Longrightarrow H^{p+q} := H^{p+q}(\widetilde{X},\mathbb{Z}).$$

As $E_2^{p,1} = 0$ for every $p \in \mathbb{Z}$, the map $E_2^{2,0} \longrightarrow H^2$ is injective. But this is the map $f^* : H^2(X,\mathbb{Z}) \longrightarrow H^2(\widetilde{X},\mathbb{Z})$, and we are done. □

Corollary 3.2. *Let (S, v, H) be an OLS-triple.*

1. *If S is K3, then $H^2(M_v,\mathbb{Z})$ is free.*
2. *If S is abelian, then $H^2(K_v,\mathbb{Z})$ is free.*

Proof. If S is a K3 surface, then M_v has rational singularities: indeed, it admits a symplectic resolution, therefore the singularities are canonical, hence rational by Elkik [3]. By Lemma 3.1,

$$\pi_v^* : H^2(M_v,\mathbb{Z}) \longrightarrow H^2(\widetilde{M}_v,\mathbb{Z})$$

is injective. Finally, by Theorem 1.6 we know that $\widetilde{M}_v$ is an irreducible symplectic manifold, hence it is simply connected. This implies that $H^2(\widetilde{M}_v,\mathbb{Z})$ is free, so $H^2(M_v,\mathbb{Z})$ is free. The case of abelian surfaces is analogue. □

Remark 3.3. By Lemma 3.1 and the proof of Corollary 3.2, the pull-back morphism $\pi_v^* : H^2(M_v,\mathbb{Z}) \longrightarrow H^2(\widetilde{M}_v,\mathbb{Z})$ is an injection of mixed Hodge structures. By strict compatibility of the weight filtrations, the mixed Hodge structure on $H^2(M_v,\mathbb{Z})$ is then pure of weight two. Explicitely, the pure weight-two Hodge structure on $H^2(M_v,\mathbb{Z})$ is defined as follows:

Definition 3.4. *Let (S, v, H) be an OLS-triple where S is a K3 surface. The pure weight-two Hodge structure on $H^2(M_v,\mathbb{Z})$ is defined as follows:*

$$H^{2,0}(M_v) := \pi_v^*(H^2(M_v,\mathbb{C})) \cap H^{2,0}(\widetilde{M}_v),$$

$$H^{1,1}(M_v) := \pi_v^*(H^2(M_v,\mathbb{C})) \cap H^{1,1}(\widetilde{M}_v),$$

$$H^{0,2}(M_v) := \pi_v^*(H^2(M_v,\mathbb{C})) \cap H^{0,2}(\widetilde{M}_v).$$

Similarily, we define the pure weight-two Hodge structure on $H^2(K_v,\mathbb{Z})$.

We now deal with the quadratic form. Recall that if (S, v, H) is an OLS-triple, then $\widetilde{M}_v(S,H)$ and $\widetilde{K}_v(S,H)$ are irreducible symplectic manifolds by Theorem 1.6. This implies that on $H^2(\widetilde{M}_v,\mathbb{Z})$ and $H^2(\widetilde{K}_v,\mathbb{Z})$ we have a lattice structure with respect to the Beauville form. As π_v^* is injective, this induces an integral bilinear form on $H^2(M_v,\mathbb{Z})$ and $H^2(K_v,\mathbb{Z})$ which is compatible with the Hodge structure we just defined. More explicitely, we have the following:

Definition 3.5. *Let (S, v, H) be an OLS-triple where S is a K3 surface. We define an integral bilinear form on $H^2(M_v, \mathbb{Z})$:*

$$q_v : H^2(M_v, \mathbb{Z}) \times H^2(M_v, \mathbb{Z}) \longrightarrow \mathbb{Z}, \qquad q_v(\alpha, \beta) := \widetilde{q}_v(\pi_v^* \alpha, \pi_v^* \beta),$$

where $\widetilde{q}_v$ is the Beauville form of $\widetilde{M}_v(S, H)$. Similarily, we define an integral bilinear form on $H^2(K_v, \mathbb{Z})$ for every OLS-triple (S, v, H) where S is an abelian surface.

Remark 3.6. The integral bilinear form on $H^2(M_v, \mathbb{Z})$ will be shown to be non-degenerate for every OLS-triple (S, v, H) (as a corollary of Theorem 1.7), hence it defines a lattice structure on $H^2(M_v, \mathbb{Z})$ which is compatible with the pure weight-two Hodge structure.

3.2 Mukai-Donaldson-Le Potier morphism

In this section we define a morphism

$$\lambda_v : v^\perp \longrightarrow H^2(M_v, \mathbb{Z})$$

for every OLS-triple (S, v, H). The strategy is the following: consider a semi-universal family $\mathscr{F}$ on $S \times M_v^s$ of similitude ρ. Then define

$$\lambda_{v,\mathscr{F}}^s : \widetilde{H}(S, \mathbb{Z}) \longrightarrow H^2(M_v^s, \mathbb{Q}), \quad \lambda_{v,\mathscr{F}}^s(\alpha) := \frac{1}{\rho}[p_{M*}(p_S^*(\alpha^\vee \cdot \sqrt{tdS}) \cdot ch(\mathscr{F}))]_1.$$

Here, if $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, we define $\alpha^\vee := (\alpha_0, -\alpha_1, \alpha_2)$, and p_M and p_S are the two projections of $S \times M_v^s$ to M_v^s and S respectively. If S is abelian, composing with the inclusion morphism $j_v^s : K_v^s \longrightarrow M_v^s$ we then get a morphism $\nu_{v,\mathscr{F}}^s := j_v^* \circ \lambda_{v,\mathscr{F}}^s$.

Now, if $\alpha \in v^\perp$ and $\mathscr{F}, \mathscr{F}'$ are two semi-universal families, then $\lambda_{v,\mathscr{F}}^s(\alpha) = \lambda_{v,\mathscr{F}'}^s(\alpha)$ (resp. $\nu_{v,\mathscr{F}}^s(\alpha) = \nu_{v,\mathscr{F}'}^s(\alpha)$). We have then a map

$$\lambda_v^s : v^\perp \longrightarrow H^2(M_v^s, \mathbb{Q}), \qquad (\text{resp. } \nu_v^s : v^\perp \longrightarrow H^2(K_v^s, \mathbb{Z}))$$

which does not depend on the chosen semi-universal family. The problem is to extend λ_v^s to a morphism

$$\lambda_v : v^\perp \longrightarrow H^2(M_v, \mathbb{Z}),$$

i. e. such that if $i_v : M_v^s \longrightarrow M_v$ is the inclusion, we have $\lambda_v^s = i_v^* \circ \lambda_v$. If S is abelian, and $j_v : K_v \longrightarrow M_v$ is the inclusion, we then get a morphism

$$\nu_v := j_v^* \circ \lambda_v : v^\perp \longrightarrow H^2(K_v, \mathbb{Z}).$$

In order to do this, we need to study the relation between $H^2(M_v)$ and $H^2(M_v^s)$. We have the following:

Lemma 3.7. *Let (S, v, H) be an OLS-triple, and let $i_v : M_v^s \longrightarrow M_v$ (resp. $i_v : K_v^s \longrightarrow K_v$) be the inclusion. Then*

$$i_v^* : H^2(M_v, \mathbb{Z}) \longrightarrow H^2(M_v^s, \mathbb{Z})$$

(resp. $i_v^* : H^2(K_v, \mathbb{Z}) \longrightarrow H^2(K_v^s, \mathbb{Z})$) is injective.

Proof. We have a commutative diagram, every row of which is exact:

$$(2) \qquad \begin{array}{ccccc} H^2(M_v, M_v^s) & \xrightarrow{c} & H^2(M_v,\mathbb{Z}) & \xrightarrow{i_v^*} & H^2(M_v^s,\mathbb{Z}) \\ f\downarrow & & \pi_v^*\downarrow & & \pi_v^*\downarrow \\ H^2(\widetilde{M}_v, \pi_v^{-1}(M_v^s)) & \xrightarrow{\widetilde{c}} & H^2(\widetilde{M}_v,\mathbb{Z}) & \xrightarrow{\widetilde{i}_v^*} & H^2(\pi_v^{-1}(M_v^s),\mathbb{Z}) \end{array}$$

where $\widetilde{i}_v : \pi_v^{-1}(M_v^s) \longrightarrow \widetilde{M}_v$ is the inclusion. As $\widetilde{M}_v \setminus \pi_v^{-1}(M_v^s) = \widetilde{\Sigma}_v$, the exceptional divisor of π_v, which is irreducible, we have $H^2(\widetilde{M}_v, \pi_v^{-1}(M_v^s)) \simeq \mathbb{Z}$, and $\widetilde{c}(1) = c_1(\widetilde{\Sigma}_v)$. Let $\alpha \in H^2(M_v,\mathbb{Z})$ be such that $i_v^*(\alpha) = 0$, so that $\widetilde{i}_v^* \circ \pi_v^*(\alpha) = 0$. As the second row of the diagram (2) is exact, there is $n \in \mathbb{Z}$ such that $\pi_v^*(\alpha) = \widetilde{c}(n) = nc_1(\widetilde{\Sigma}_v)$.

Now, we introduce the following notation: let $\Sigma_v^0 \subseteq \Sigma_v$ be the smooth locus of Σ_v. Following [16] we know that $\pi_v : \pi_v^{-1}(\Sigma_v^0) \longrightarrow \Sigma_v^0$ is a $\mathbb{P}^1-$bundle, whose generic fiber is then a rational curve δ. As δ is contracted by π_v, we have $\pi_v^*(\alpha)\cdot\delta = 0$. On the other hand, by adjunction the normal bundle to $\widetilde{\Sigma}_v$ is the canonical bundle of $\widetilde{\Sigma}_v$, hence it has degree -2 on δ. In conclusion, we have

$$0 = \pi_v^*(\alpha)\cdot\delta = nc_1(\widetilde{\Sigma}_v)\cdot\delta = -2n,$$

so that $n = 0$. Hence $\pi_v^*(\alpha) = 0$, but as π_v^* is injective by Lemma 3.1, we then have $\alpha = 0$, so that i_v^* is injective, and we are done for the K3 surface case. The proof of the abelian case is similar. □

If $\alpha \in v^\perp$ and $\mu_1(\alpha), \mu_2(\alpha) \in H^2(M_v,\mathbb{Z})$ are such that $i_v^*(\mu_1(\alpha)) = i_v^*(\mu_2(\alpha))$, then by Lemma 3.7 we have that $\mu_1(\alpha) = \mu_2(\alpha)$. Hence, if there is an extension of $\lambda_v^s(\alpha)$ to an element of $H^2(M_v,\mathbb{Z})$, then this extension is unique, and we call it $\lambda_v(\alpha)$. In conclusion, the problem is only to find an extension.

In order to do so, we recall a construction due to Le Potier. Let $K_{hol}(S)$ be the holomorphic $K-$theory of S, and let

$$vect^\vee : K_{hol}(S) \longrightarrow \widetilde{H}(S,\mathbb{Z}), \qquad vect^\vee([E]) := (v(E))^\vee,$$

where $[E]$ is the class in $K_{hol}(S)$ of a sheaf E on S. Notice that $vect^\vee$ gives an isomorphism between $K_{hol}(S)$ and $\widetilde{H}^{1,1}(S) \cap \widetilde{H}(S,\mathbb{Z})$. Let

$$(.,.) : K_{hol}(S) \times K_{hol}(S) \longrightarrow \mathbb{Z}, \qquad ([E],[F]) := \chi(E \otimes F),$$

and it is easy to see that $([E],[F]) = -(v(E), v(F))$ for every sheaves E, F on S. Let $\mathscr{E}$ be any sheaf parameterized by $M_v(S,H)$, $e_v := [\mathscr{E}]$ and $e_v^\perp \subseteq K_{hol}(S)$ the orthogonal of e_v with respect to $(.,.)$. Finally, let $R_v \subseteq Q_v$ be the open subset of $H-$semistable quotients in a Quot-scheme Q_v, such that $M_v = R_v/GL(N)$ for some $N \in \mathbb{Z}$. Let q_R and q_S be the projections of $S \times R_v$ onto R_v ad S respectively, and let $\mathscr{F}$ be a universal family on $S \times R_v$. Then define

$$L_{v,\mathscr{F}}^R : K_{hol}(S) \longrightarrow Pic(R_v), \quad L_{v,\mathscr{F}}^R([E]) := det(p_{R!}(p_S^*[E]\cdot[\mathscr{F}])).$$

If $\mathscr{F}$ and $\mathscr{F}'$ are two universal families on $S \times R_v$, then $L_{v,\mathscr{F}}^R([E]) = L_{v,\mathscr{F}'}^R([E])$ for every $[E] \in e_v^\perp$ (see Lemma 1.2 of [11]), so we get a morphism

$$L_v^R : e_v^\perp \longrightarrow Pic(R_v).$$

Lemma 3.8. *Let (S, v, H) be an OLS-triple. Then for every $[E] \in e_v^\perp$ the line bundle $L_v^R([E])$ descends to a line bundle $L_v([E]) \in Pic(M_v)$.*

Proof. The line bundle $L_v^R([E])$ has a natural $GL(N)-$linearization inherited from the one we have on $\mathscr{F}$. Let $[P] \in R_v$ be a point with closed $GL(N)-$orbit corresponding to a sheaf $F \in M_v$. Let $\pi : R_v \longrightarrow M_v$ be the quotient morphism, so that $\pi([P]) = F$. We need to show that the action of the stabilizer $Stab([P])$ is trivial on the fiber $L_v^R([E])_{[P]}$. We know that this is the case if F is $H-$stable by [11], hence we suppose $F = (F_1 \otimes V_1) \oplus (F_2 \otimes V_2)$, where F_1, F_2 are $H-$stable and V_1, V_2 are vector spaces. We know that

$$Stab([P]) \simeq Aut(F) \simeq GL(V_1) \times GL(V_2).$$

Moreover, we have

$$L_v^R([E])_{[P]} \simeq \bigotimes_{i=1}^{2} \left(det(H^\bullet(F_i \otimes E))^{dim(V_i)} \otimes (det(V_i))^{\chi(F_i \otimes E)} \right),$$

and the action of an element $(M_1, M_2) \in Stab([P])$ is simply the multiplication by $det(M_1)^{\chi(F_1 \otimes E)} det(M_2)^{\chi(F_2 \otimes E)}$. As the polarization H is $v-$generic, then $v(F_1) = v(F_2) = v/2$: hence, as $[E] \in e_v^\perp$, we get $\chi(F_1 \otimes E) = \chi(F_2 \otimes E) = 0$. In conclusion, the action of any element of the stabilizer is trivial, so that $L_v^R([E])$ descends to a line bundle $L_v([E]) \in Pic(M_v)$. $\square$

We have, in conclusion, a morphism $L_v : e_v^\perp \longrightarrow Pic(M_v)$. The main result of the section is the following:

Proposition 3.9. *Let (S, v, H) be any OLS-triple. Then there is a morphism*

$$\lambda_v : v^\perp \longrightarrow H^2(M_v, \mathbb{Z})$$

such that $i_v^* \circ \lambda_v = \lambda_v^s$.

Proof. By Lemma 3.8 we have a morphism $L_v : e_v^\perp \longrightarrow Pic(M_v)$. In the following we use the notation $(v^\perp)^{1,1} := v^\perp \cap \widetilde{H}^{1,1}(S)$. An application of the Grothendieck-Riemann-Roch Theorem shows that if $\alpha \in (v^\perp)^{1,1}$ and $[E] \in e_v^\perp$ is such that $vect^\vee([E]) = \alpha$, then

$$\lambda_v^s(\alpha) = i_v^*(c_1(L_v([E])))$$

(for a similar computation, see [19]). Hence, we define $\lambda_v(\alpha) := c_1(L_v([E]))$, so that we finally get a morphism

$$\lambda_v : (v^\perp)^{1,1} \longrightarrow H^2(M_v, \mathbb{Z}).$$

It remains to show that we can define λ_v on the whole $v^\perp$. To do so, we use a deformation argument. Let T be a smooth, connected curve and $(\mathscr{X}, \mathscr{H}, \mathscr{L})$ be a deformation of the OLS-triple (S, v, H) along T, and write $f : \mathscr{X} \longrightarrow T$ for the associated map, which is smooth and projective. Write $v = 2(r, c_1(L), a)$, and let $0 \in T$ be such that $(\mathscr{X}_0, v_0, \mathscr{H}_0) = (S, v, H)$. Finally, assume that the Kodaira-Spencer map of the given family $f : \mathscr{X} \longrightarrow T$ is injective at 0, and let $\phi : \mathscr{M} \longrightarrow T$ be the relative moduli space of semistable sheaves associated with the deformation $(\mathscr{X}, \mathscr{H}, \mathscr{L})$.

By Corollary 2.18 and Lemmas 2.6 and 2.11 there is a small analytic open neighborhood U of 0 in T parameterizing OLS triples, such that $f^{-1}(U)$ is homeomorphic to the product $S \times U$ over U, and $\phi^{-1}(U)$ is homeomorphic to the product $M_v \times U$ over U. Up to shrinking U, we can even suppose that the local systems $R^{2i}f_*\mathbb{Z}$, $R^2\phi_*\mathbb{Z}$ and $R^2\phi^s_*\mathbb{Z}$ are constant (hence even those with coefficients in $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are constant). Notice that this means that we can identify $\widetilde{H}(\mathscr{X}_t, \mathbb{Z})$ with $\widetilde{H}(S, \mathbb{Z})$ (as lattices), $H^2(M_{v_t}, \mathbb{Z})$ with $H^2(M_v, \mathbb{Z})$ and $H^2(M^s_{v_t}, \mathbb{Q})$ with $H^2(M^s_v, \mathbb{Q})$.

As v is constant over U, we can then consider $\mathcal{V} \subseteq \oplus_{i=0}^2 R^{2i}f_*\mathbb{Z}$ to be a local system such that for every $t \in T$ we have $\mathcal{V}_t = v_t^\perp$. As we have relative semi-universal families, then we define a morphism

$$\lambda : \mathcal{V} \longrightarrow R^2\phi^s_*\mathbb{Q},$$

such that for every $t \in U$ we have $\lambda_t = \lambda_{v_t}$.

Notice that we just need to show that there is a finite set of generators $\{\alpha_1, ..., \alpha_n\}$ of $v^\perp$ such that for every $i = 1, ..., n$ there is $t_i \in U$ such that $\alpha_i \in v^\perp \cap \widetilde{H}^{1,1}(\mathscr{X}_{t_i})$. Indeed, by Lemma 3.8 we have that $\lambda^s_{v_{t_i}}(\alpha_i)$ extends to an element $\lambda_{v_{t_i}}(\alpha_i) \in H^2(M_{v_{t_i}}, \mathbb{Z})$. Hence even $\lambda^s_v(\alpha_i)$ extends to $\lambda_v(\alpha_i) \in H^2(M_v, \mathbb{Z})$. Now, let $\alpha \in v^\perp$: then there are $\mu_1, ..., \mu_n \in \mathbb{Z}$ such that

$$\alpha = \sum_{i=1}^n \mu_i \alpha_i.$$

But this implies that $\lambda^s_v(\alpha)$ extends to the element

$$\lambda_v(\alpha) := \sum_{i=1}^n \mu_i \lambda_v(\alpha_i) \in H^2(M_v, \mathbb{Z}).$$

Since by Lemma 3.7 the extension is unique, the previous equality gives us the desired morphism of $\mathbb{Z}-$modules $\lambda_v : v^\perp \longrightarrow H^2(M_v, \mathbb{Z})$.

We then prove that there is a finite set of generators $\{\alpha_1, ..., \alpha_n\}$ of $v^\perp$ such that for every $i = 1, ..., n$ there is $t_i \in U$ such that $\alpha_i \in v^\perp \cap \widetilde{H}^{1,1}(\mathscr{X}_{t_i})$. To do so, define

$$V := \{\alpha \in \oplus_{i=0}^2 H^{2i}(S, \mathbb{C}) \,|\, (\alpha, v)_{\mathbb{C}} = 0\},$$

where $(.,.)_{\mathbb{C}}$ is the extension of the Mukai pairing to $\oplus_{i=0}^2 H^{2i}(S, \mathbb{C})$. Notice that $V = v^\perp \otimes \mathbb{C}$, and it is a $23-$dimensional complex vector space. Finally, let $\mathbb{P} := \mathbb{P}(V)$.

Let $\Omega \subseteq \mathbb{P}(H^2(S, \mathbb{C}))$ be the period domain, and let $P : U \longrightarrow \Omega$ be the period map sending $t \in U$ to $H^{2,0}(\mathscr{X}_t)$. Recall that we assumed that the Kodaira-Spencer map of the family $f : \mathscr{X} \longrightarrow T$ is injective at 0: then, the injectivity of the period map P implies that up to shrinking U we can identify U with its image $P(U) \subseteq \mathbb{P}(H^2(S, \mathbb{C}))$. In this way we may identify t with the period of $\mathscr{X}_t$. Now, for every $t \in U$ we have that $v_t \in \widetilde{H}^{1,1}(\mathscr{X}_t)$, then $(t, v)_{\mathbb{C}} = (t, v_t)_{\mathbb{C}} = 0$, so that $t \in \mathbb{P}$. In conclusion we have $U \subseteq \mathbb{P}$.

Consider the incidence variety $I \subseteq U \times \mathbb{P}$, i. e.

$$I := \{(t, [w]) \in U \times \mathbb{P} \,|\, (t, w)_{\mathbb{C}} = 0\}.$$

First of all, we notice that I is a smooth, projective variety such that $dim(I) = dim(\mathbb{P})$. Indeed, if $g : I \longrightarrow U$ is the projection, then for every $t \in U$ we have

$$g^{-1}(t) \simeq \mathbb{P}(V \cap (\widetilde{H}^{2,0}(\mathscr{X}_t) \oplus \widetilde{H}^{1,1}(\mathscr{X}_t))) \subseteq \mathbb{P},$$

which is a $21-$dimensional projective space.

Let now $h : I \longrightarrow \mathbb{P}$ be the projection to the second factor. Notice that if $\alpha \in v^\perp$ and $[\alpha] \in im(h)$, then there is $t \in U$ such that $\alpha \in V \cap (\widetilde{H}^{2,0}(\mathscr{X}_t) \oplus \widetilde{H}^{1,1}(\mathscr{X}_t))$. Since α is integral we have finally $\alpha \in V \cap \widetilde{H}^{1,1}(\mathscr{X}_t)$, so that the proposition follows if we prove that there is a finite set $\{\alpha_1, ..., \alpha_n\}$ of generators of $v^\perp$ over $\mathbb{Z}$ such that $[\alpha_i] \in im(h)$ for $i = 1, ..., n$.

We claim that there is $w \in V \cap \oplus_{i=0}^2 H^{2i}(S, \mathbb{R})$ such that $[w] \in im(h)$ and $[w]$ admits an open analytic neighborhood in $\mathbb{P}$ which is contained in $im(h)$. This will be enough to conclude: indeed, as a consequence $im(h)$ contains a non-empty open subset of $\mathbb{P}(V \cap \oplus_{i=0}^2 H^{2i}(S, \mathbb{R})) = \mathbb{P}(v^\perp \otimes \mathbb{R})$. As for any non-empty open subset $\mathcal{U}$ of $\mathbb{P}(v^\perp \otimes \mathbb{R})$ there is a finite set $\{\alpha_1, ..., \alpha_n\}$ of generators of $v^\perp$ over $\mathbb{Z}$ such that $[\alpha_i] \in \mathcal{U}$ for $i = 1, ..., n$, we are done.

In conclusion, we just need to prove our claim. This is an immediate consequence of the following:

Lemma 3.10. *There is $w \in V \cap \widetilde{H}^{1,1}(S) \cap (\oplus_{i=0}^2 H^{2i}(S, \mathbb{R}))$ such that the map*

$$dh_{(0,[w])} : T_{(0,[w])} I \longrightarrow T_{[w]}\mathbb{P}$$

is an isomorphism.

Proof. Let $V = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ be the pure weight-two Hodge decomposition of $v^\perp$ induced by the Hodge decomposition of $\widetilde{H}(S, \mathbb{Z})$, and write $(.,.)_V$ for the $\mathbb{C}-$bilinear form on V induced by the Mukai pairing $(.,.)_\mathbb{C}$. More explicitely, we have $V^{i,j} = V \cap \widetilde{H}^{i,j}(S)$, and $0 \in \mathbb{P}$ is just the line $V^{2,0}$. Moreover, let $\Theta \subseteq \mathbb{P}$ be the projective tangent line at U in 0, and let $W \supseteq V^{2,0}$ be a $2-$dimensional linear subspace of V such that $\mathbb{P}(W) = \Theta$.

As a first step, we show that there is $w \in V \cap \widetilde{H}^{1,1}(S) \cap (\oplus_{i=0}^2 H^{2i}(S, \mathbb{R}))$ such that $w \notin W^\perp$, where $W^\perp$ is the orthogonal to W with respect to $(.,.)_V$. As every $t \in U$ is represented by a class of type $(2,0)$ of $\mathscr{X}_t$, U is included in the smooth quadric Q defined by $(.,.)_V$. Hence Θ is included in the projective tangent space at Q in 0, which is just $\mathbb{P}(V^{2,0} \oplus V^{1,1})$ as $V^{2,0} \oplus V^{1,1}$ is the orthogonal of $V^{2,0}$ with respect to $(.,.)_V$. Hence $V^{2,0} \subseteq W \subseteq V^{2,0} \oplus V^{1,1} \subseteq V$. As $dim(W) = 2$, we have $W \cap V^{1,1} \neq 0$. Since $(.,.)_V$ is non-degenerate on $V^{1,1}$, there must be $w \in V^{1,1}$ such that $w \notin W^\perp$. As $V^{1,1}$ is defined over $\mathbb{R}$, we can finally assume that $w \in V \cap \widetilde{H}^{1,1}(S) \cap (\oplus_{i=0}^2 H^{2i}(S, \mathbb{R}))$.

It remains to show that $dh_{(0,[w])}$ is an isomorphism. Let $M \subseteq V$ be the line such that $\mathbb{P}(M) = [w]$, so that M is not contained in $W^\perp$. Recall that

$$T_{[w]}\mathbb{P} \simeq Hom(M, V/M), \qquad T_0 U = Hom(V^{2,0}, W/V^{2,0}),$$

$$T_{(0,[w])}(U \times \mathbb{P}) \simeq Hom(V^{2,0}, W/V^{2,0}) \times Hom(M, V/M).$$

By definition of I, we then have

$$T_{(0,[w])} I = \{(\phi, \psi) \in T_{(0,[w])}(U \times \mathbb{P}) \mid (\phi(l), m)_V + (l, \psi(m))_V = 0\},$$

where the equation is true for every $l \in V^{2,0}$ and every $m \in M$ (for a similar computation, see [7], Example 16.20). Moreover, we have

$$dh_{(0,[w])} : T_{(0,[w])}I \longrightarrow T_{[w]}\mathbb{P}, \qquad dh_{(0,[w])}(\phi,\psi) = \psi.$$

As I is smooth and $dim(I) = dim(\mathbb{P})$, in order to show that $dh_{(0,[w])}$ is an isomorphism we just need to show that it is surjective. Consider then $\psi \in Hom(M, V/M)$: as M is not contained in $W^{\perp}$, for every $l \in V^{2,0}$ there is an element $\phi(l) \in W$ such that

$$(\phi(l), m)_V = -(l, \psi(m))_V$$

for every $m \in M$. But this defines an element $\phi \in Hom(V^{2,0}, W/V^{2,0})$ such that $(\phi,\psi) \in T_{(0,[w])}I$ and $dh_{(0,[w])}(\phi,\psi) = \psi$, and we are done. $\square$

Now, if w is as in the statement of Lemma 3.10, then it admits an open analytic neighborhood $\mathcal{U}$ contained in $im(h)$ as desidered, and we are done. $\square$

3.3 Proof of Theorem 1.7

The aim of this section is to prove the following:

Theorem 1.7. *Let (S, v, H) be an OLS-triple.*

1. *If S is K3, then $\lambda_v : v^{\perp} \longrightarrow H^2(M_v, \mathbb{Z})$ is a Hodge isometry.*
2. *If S is abelian, then $\nu_v : v^{\perp} \longrightarrow H^2(K_v, \mathbb{Z})$ is a Hodge isometry.*

Proof. Let (S, v, H) be an OLS-triple. We need to show the three following properties:

1. λ_v (resp. ν_v) is an isomorphism of $\mathbb{Z}-$modules;
2. λ_v (resp. ν_v) is an isometry;
3. λ_v (resp. ν_v) is a Hodge morphism.

We introduce the following notations:

$$\widetilde{\lambda}_v := \pi_v^* \circ \lambda_v : v^{\perp} \longrightarrow H^2(\widetilde{M}_v, \mathbb{Z}), \qquad \widetilde{\nu}_v := \pi_v^* \circ \nu_v : v^{\perp} \longrightarrow H^2(\widetilde{K}_v, \mathbb{Z}).$$

Step 1. If S is an abelian or projective K3 surface such that $NS(S) = \mathbb{Z} \cdot h$, where $h = c_1(H)$ is ample and $h^2 = 2$, and $v = (2, 0, -2)$, then λ_v and ν_v are Hodge isometries: this is proved in [19].

Step 2. Let (S, v, H) be an OLS-triple. We show that λ_v is an isomorphism of $\mathbb{Z}-$modules. Following the proof of Theorem 1.6, we reduce to the case of Step 1: the only transformations we use are deformations of the moduli spaces induced by deformations of the corresponding OLS-triple along a smooth, connected curve, and isomorphisms between moduli spaces induced by some Fourier-Mukai transforms.

Deforming an OLS-triple along a smooth, connected curve T, by Corollary 2.18 the $\mathbb{Z}-$module structures of $v^{\perp}$ and of $H^2(M_v, \mathbb{Z})$ remain constant along the locus of T parameterizing OLS-triples; for the isomorphism induced by the Fourier-Mukai transform we have the following:

Lemma 3.11. *Let (S, v, H) be an OLS-triple.*

1. *If S is K3, let $\mathscr{P} \in D^b(S \times S)$ and $\mathfrak{F}_{\mathscr{P}} : D^b(S) \longrightarrow D^b(S)$ the Fourier-Mukai transform with kernel $\mathscr{P}$. Moreover, let $\phi_{\mathscr{P}}$ be the morphism induced by $\mathfrak{F}_{\mathscr{P}}$ in cohomology, and let $v' := \phi_{\mathscr{P}}(v)$. If $\mathfrak{F}_{\mathscr{P}}$ is an equivalence and it induces an isomorphism $f_{\mathscr{P}} : M_{v'}(S, H) \longrightarrow M_v(S, H)$, then λ_v is an isomorphism if and only if $\lambda_{v'}$ is an isomorphism.*

2. *If S is abelian, let $\mathscr{P} \in D^b(S \times \widehat{S})$ and $\mathfrak{F}_{\mathscr{P}} : D^b(S) \longrightarrow D^b(\widehat{S})$ the Fourier-Mukai transform with kernel $\mathscr{P}$. Moreover, let $\phi_{\mathscr{P}}$ be the morphism induced by $\mathfrak{F}_{\mathscr{P}}$ in cohomology, and let $v' := \phi_{\mathscr{P}}(v)$. If $\mathfrak{F}_{\mathscr{P}}$ is an equivalence and it induces an isomorphism $f_{\mathscr{P}} : K_{v'}(\widehat{S}, \widehat{H}) \longrightarrow K_v(S, H)$, then ν_v is an isomorphism if and only if $\nu_{v'}$ is an isomorphism.*

Proof. We show the first point, as the second is similar. We show that the diagram

$$\begin{array}{ccc} v^{\perp} & \xrightarrow{\phi_{\mathscr{P}}} & (v')^{\perp} \\ {\scriptstyle \lambda_v}\downarrow & & \downarrow{\scriptstyle \lambda_{v'}} \\ H^2(M_v, \mathbb{Z}) & \xrightarrow{f^*_{\mathscr{P}}} & H^2(M_{v'}, \mathbb{Z}) \end{array} \tag{3}$$

is commutative. By the construction of λ_v and $\lambda_{v'}$, this is true if the following diagram

$$\begin{array}{ccc} v^{\perp} & \xrightarrow{\phi_{\mathscr{P}}} & (v')^{\perp} \\ {\scriptstyle \lambda^s_v}\downarrow & & \downarrow{\scriptstyle \lambda^s_{v'}} \\ H^2(M^s_v, \mathbb{Q}) & \xrightarrow{f^*_{\mathscr{P}}} & H^2(M^s_{v'}, \mathbb{Q}) \end{array} \tag{4}$$

is commutative, and this is shown to be true by standard computations (see for example Proposition 2.4 of [24]). As $\phi_{\mathscr{P}}$ and $f^*_{\mathscr{P}}$ are isomorphisms, then λ_v is an isomorphism if and only if $\lambda_{v'}$ is, and we are done. □

In conclusion, we reduce to the case of Step 1, so that λ_v is an isomorphism of $\mathbb{Z}-$modules for every OLS-triple (S, v, H).

Step 3. We prove now that λ_v is an isometry between $v^{\perp}$ (which has a lattice structure given by the Mukai pairing) and $H^2(M_v, \mathbb{Z})$ (on which we have an integral bilinear form, as seen in section 3.1). Again, we reduce to the case of Step 1 following the proof of Theorem 1.6: as in the previous step, the only transformations we use are deformations of the moduli spaces induced by deformations of the corresponding OLS-triple along a smooth, connected curve, and isomorphisms between moduli spaces induced by some Fourier-Mukai transforms.

Deforming an OLS-triple along a smooth, connected curve T, by Corollary 2.18 the integral bilinear forms on $v^{\perp}$ and on $H^2(M_v, \mathbb{Z})$ remain constant along the locus of T parameterizing OLS-triples; we then just need to analyze the isomorphisms induced by Fourier-Mukai transforms. We have the following:

Lemma 3.12. *Let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two OLS-triples.*

1. *If S_1 and S_2 are K3 and there is an isomorphism $f : M_{v_1} \longrightarrow M_{v_2}$, then the morphism $f^* : H^2(M_{v_2}, \mathbb{Z}) \longrightarrow H^2(M_{v_1}, \mathbb{Z})$ is an isometry.*

2. If S_1 and S_2 are abelian and there is an isomorphism $f : K_{v_1} \longrightarrow K_{v_2}$, then the morphism $f^* : H^2(K_{v_2}, \mathbb{Z}) \longrightarrow H^2(K_{v_1}, \mathbb{Z})$ is an isometry.

Proof. We prove the first point, as the second is similar. We have a commutative diagram

$$\begin{array}{ccc} H^2(M_{v_2}, \mathbb{Z}) & \xrightarrow{f^*} & H^2(M_{v_1}, \mathbb{Z}) \\ {\scriptstyle \pi^*_{v_2}}\downarrow & & \downarrow{\scriptstyle \pi^*_{v_1}} \\ H^2(\widetilde{M}_{v_2}, \mathbb{Z}) & \xrightarrow{\widetilde{f}^*} & H^2(\widetilde{M}_{v_1}, \mathbb{Z}). \end{array}$$

By hypothesis, we have that f is an isomorphism. Moreover, $\pi^*_{v_1}$ and $\pi^*_{v_2}$ are isometries onto their images, and $\widetilde{f}^*$ is an isometry by [15]. Hence f^* is an isometry, and we are done. □

In conclusion, we reduce to the case of Step 1, so that λ_v is an isometry for every OLS-triple (S, v, H).

Step 4. We show that λ_v is a Hodge morphism. Suppose in the following that S is a K3 surface (the proof for S abelian is analogue). To show that λ_v is a Hodge morphism is equivalent to show that $\widetilde{\lambda}_v$ is a Hodge morphism. Notice that as λ_v is an isometry by Step 3, then $\widetilde{\lambda}_v$ is an isometry onto its image. Recall that λ_v is defined as an extension of the morphism

$$\lambda^s_v : v^\perp \longrightarrow H^2(M^s_v, \mathbb{Q}), \qquad \lambda^s_v(\alpha) = \frac{1}{\rho}[p_*(q^*(\alpha^\vee \cdot \sqrt{td(S)}) \cdot ch(\mathscr{F})]_1,$$

where ρ is the similitude of $\mathscr{F}$ and p, q are the two projections of $S \times M^s_v$ onto M^s_v and S respectively.

As $ch(\mathscr{F}) \in H^{2*}(S \times M^s_v, \mathbb{Q})$, and as M^s_v is (up to isomorphism) an open subset of $\widetilde{M}_v$, taking the closure of the cycles $ch_i(\mathscr{F})$ we get a class $c \in H^{2*}(S \times \widetilde{M}_v, \mathbb{Q})$, whose component $c_i \in H^{2i}(S \times \widetilde{M}_v, \mathbb{Q})$ represents a $(i,i)-$class. Let $\widetilde{p}$ and $\widetilde{q}$ be the projections of $S \times \widetilde{M}_v$ onto $\widetilde{M}_v$ and S respectively, and consider the morphism

$$\widetilde{\mu}_v : v^\perp \longrightarrow H^2(\widetilde{M}_v, \mathbb{Q}), \qquad \widetilde{\mu}_v(\alpha) := \frac{1}{\rho}[\widetilde{p}_*(\widetilde{q}^*(\alpha^\vee \cdot \sqrt{td(S)}) \cdot c)]_1.$$

On $v^\perp$ and $H^2(\widetilde{M}_v, \mathbb{Q})$ we have pure weight-two Hodge structures, and $\widetilde{\mu}_v$ is a Hodge morphism. Now, a priori the morphisms $\widetilde{\lambda}_v$ and $\widetilde{\mu}_v$ are not equal, but we have the following:

Lemma 3.13. *For every $\omega \in H^{2,0}(S)$ we have $\widetilde{\lambda}_v(\omega) = \widetilde{\mu}_v(\omega)$.*

Proof. Let $\widetilde{i}_v : \pi_v^{-1}(M^s_v) \longrightarrow \widetilde{M}_v$ be the inclusion. By the very definition of $\widetilde{\lambda}_v$ and $\widetilde{\mu}_v$, for every $\omega \in H^{2,0}(S)$ we have $\widetilde{i}^*_v(\widetilde{\lambda}_v(\omega)) = \widetilde{i}^*_v(\widetilde{\mu}_v(\omega))$. Moreover, the kernel of the morphism $\widetilde{i}^*_v : H^2(\widetilde{M}_v, \mathbb{C}) \longrightarrow H^2(\pi_v^{-1}(M^s_v), \mathbb{C})$ is $\mathbb{C} \cdot c_1(\widetilde{\Sigma}_v)$ (see the proof of Lemma 3.7), so that $\widetilde{\lambda}_v(\omega) - \widetilde{\mu}_v(\omega) = lc_1(\widetilde{\Sigma}_v)$ for some $l \in \mathbb{C}$. But

$$-2l = lc_1(\widetilde{\Sigma}_v) \cdot \delta = (\widetilde{\lambda}_v(\omega) - \widetilde{\mu}_v(\omega)) \cdot \delta = 0$$

(see the proof of Lemma 3.7 for the definition of δ), so that $l = 0$, and $\widetilde{\lambda}_v(\omega) = \widetilde{\mu}_v(\omega)$ for every $\omega \in H^{2,0}(S)$. □

As $\widetilde{\mu}_v$ is a Hodge morphism, by Lemma 3.13 we have that $\widetilde{\lambda}_v(\omega) \in H^{2,0}(\widetilde{M}_v)$ for every $\omega \in H^{2,0}(S)$. Hence $\widetilde{\lambda}_v$ sends the $(2,0)-$part of the Hodge structure of $v^\perp$ to the $(2,0)-$part of the Hodge structure of $H^2(\widetilde{M}_v, \mathbb{Z})$. Now, consider $\alpha \in v^\perp \otimes \mathbb{C} \cap (\widetilde{H}^{2,0}(S) \oplus \widetilde{H}^{1,1}(S))$. Then, for every $\omega \in H^{2,0}(S)$ we have $(\alpha, \omega) = 0$, where $(.,.)$ is the Mukai pairing on $v^\perp \otimes \mathbb{C}$. As $\widetilde{\lambda}_v$ is an isometry on its image by assumption, we have then

$$\widetilde{q}_v(\widetilde{\lambda}_v(\alpha), \widetilde{\lambda}_v(\omega)) = 0,$$

where $\widetilde{q}_v$ is the Beauville form of the irreducible symplectic manifold $\widetilde{M}_v$. But this implies that $\widetilde{\lambda}_v(\alpha) \in H^{2,0}(\widetilde{M}_v) \oplus H^{1,1}(\widetilde{M}_v)$ for every $\alpha \in v^\perp \otimes \mathbb{C} \cap (\widetilde{H}^{2,0}(S) \oplus \widetilde{H}^{1,1}(S))$. In conclusion, we see that $\widetilde{\lambda}_v$ respects the Hodge filtrations, hence it is a Hodge morphism, and we are done. $\square$

Remark 3.14. Theorem 1.7 tells us that the integral bilinear form q_v on $H^2(M_v, \mathbb{Z})$ is indeed non-degenerate, hence it defines a lattice structure on $H^2(M_v, \mathbb{Z})$.

4 Appendix: openness of $v-$genericity

This section is dedicated to prove some properties of $v-$genericity we used in the paper which are related with the behaviour of $v-$genericity in families. The basic tool is the following, which is the main technical tool we needed for the proof of Lemma 2.6:

Lemma 4.1. *Let*

$$\begin{array}{ccc} S & \xrightarrow{i} & \mathcal{S} \\ \downarrow & & \downarrow\phi \\ 0 & \longrightarrow & B \end{array} \tag{5}$$

be family of smooth projective surfaces over a smooth base. Let $\mathscr{H} \in Pic(\mathcal{S})$, and for every $b \in B$ write $S_b := \phi^{-1}(b)$ and $H_b := \mathscr{H}_{|S_b}$. Suppose that $\mathscr{H}_b$ is ample for every $b \in B$. Then, for any $n \in \mathbb{N}$ the set B_n of points $b \in B$ such that there exists $\alpha \in NS(S_b) \cap H_b^\perp$ with $-n \leq \alpha^2 < 0$ is locally a finite union of analytic subvarieties of B.

Proof. Since the statement is local, we may suppose that B is a small polydisk. In this case, the family $\mathcal{S}$ is topologically trivial over B and we have a natural identification $H^2(S, \mathbb{Z}) = H^2(S_b, \mathbb{Z})$. By this identification, the class H_b does not depend on b, and we write $H = H_b$. Denote by b_2 the second Betti number of S and set $h^{2,0} := \dim(H^{2,0}(S))$. Let $\mathbb{G}r(b_2 - h^{2,0}, H^2(S, \mathbb{C}))$ be the complex Grassmannian parametrizing $\mathbb{C}-$vector subspaces of $H^2(S, \mathbb{C})$ of dimension $b_2 - h^{2,0}$. As well known, the period map

$$P : B \longrightarrow \mathbb{G}r(b_2 - h^{2,0}, H^2(S, \mathbb{C})), \qquad P(b) := H^{2,0}(S_b) \oplus H^{1,1}(S_b) \subset H^2(S, \mathbb{C})$$

is holomorphic. For every $\alpha \in H^2(S, \mathbb{Z})$ write

$$G_\alpha := \{W \in \mathbb{G}r(b_2 - h^{2,0}, H^2(S, \mathbb{C})) \,|\, \alpha \in W\}.$$

Since $\alpha \in H^2(S,\mathbb{Z})$, we have that $\alpha \in NS(S_b)$ if and only if $P(b) \in G_\alpha$. It follows that

$$B_n = \bigcup_{\substack{\alpha \in H^\perp \cap H^2(S,\mathbb{Z}) \\ -n \leq \alpha^2 < 0}} P^{-1}(G_\alpha).$$

As G_α is an analytic subvariety of $\mathbb{G}r(b_2 - h^{2,0}, H^2(S,\mathbb{C}))$, we then see that B_n is a countable union of analytic subvarieties of B. It is then enough to show that, up to shrinking B, there are only finitely many $\alpha \in H^\perp \cap H^2(S,\mathbb{Z})$ satisfying $-n \leq \alpha^2 < 0$ and such that G_α intersects the image of P.

To do so, let $V \subset H^2(S,\mathbb{R})$ be the $\mathbb{R}-$vector space consisting of real cohomology classes orthogonal to H with respect to the intersection form on S, and let $\mathbb{G}r(b_2 - 2h^{2,0} - 1, V)^-$ be the real Grassmannian parameterizing negative definite $\mathbb{R}-$vector subspaces of V of dimension $b_2 - 2h^{2,0} - 1$. Since the map P is holomorphic, the map

$$\widehat{P} : B \longrightarrow \mathbb{G}r(b_2 - 2h^{2,0} - 1, V)^-, \qquad \widehat{P}(b) := V \cap H^{1,1}(S_b)$$

is continuous. For $\alpha \in H^2(S,\mathbb{Z})$, we let

$$\Gamma_\alpha := \{W \in \mathbb{G}r(b_2 - 2h^{2,0} - 1, V)^- \mid \alpha \in W\}.$$

For every $\alpha \in H^\perp \cap H^2(S,\mathbb{Z})$, we have that $\alpha \in H^{2,0}(S_b) \oplus H^{1,1}(S_b)$ if and only if $\alpha \in V \cap H^{1,1}(S_b)$, hence $G_\alpha \cap im(P) \neq \emptyset$ if and only if $\Gamma_\alpha \cap im(\widehat{P}) \neq \emptyset$. As $\widehat{P}$ is a continuous map, it will then be sufficient to show that there exist an open neighborhood U of $V \cap H^{1,1}(S)$ in $\mathbb{G}r(b_2 - 2h^{2,0} - 1, V))^-$ such that $U \cap \Gamma_\alpha \neq \emptyset$ only for finitely many $\alpha \in H^\perp \cap H^2(S,\mathbb{Z})$ satisfying $-n \leq \alpha^2 < 0$.

Set $W_1 := V \cap H^{1,1}(S)$ and $W_2 := V \cap (H^{2,0}(S) \oplus H^{0,2}(S))$. Notice that $V = W_1 \oplus W_2$; moreover, by the Hodge-Reimann bilinear relations we have that W_1 is negative definite, W_2 is positive definite and the direct sum is orthogonal with respect to the intersection form on $H^2(S)$. For $i = 1, 2$ denote by $\pi_i : V \longrightarrow W_i$ the projection associated with the given decomposition. We define a norm on V by

$$||.|| : V \longrightarrow \mathbb{R}, \quad ||v|| := \sqrt{-\pi_1(v)^2 + \pi_2(v)^2},$$

where $\pi_i(v)^2$ is the self-intersection of $\pi_i(v)$ with respect to the cup product on $H^2(S,\mathbb{R})$, for $i = 1, 2$.

As the integral classes form a discrete subset of V, it is enough to show that there is an open neighborhood U of W_1 in $\mathbb{G}r(b_2 - 2h^{2,0} - 1, V)^-$ such that the set

$$A := \bigcup_{W \in U} \{\alpha \in H^2(S,\mathbb{R}) \mid -n \leq \alpha^2 < 0,\ \alpha \in W\}$$

is bounded, i. e. there is a constant K such that $||\alpha|| < K$ for every $\alpha \in A$.

To do so, notice that as W_1 is transverse to W_2, then there exists a neighborhood $\widetilde{U}$ of W_1 in $\mathbb{G}r(b_2 - 2h^{2,0} - 1, V)^-$ consisting of negative definite $\mathbb{R}-$vector subspaces of V which are transverse to W_2. Then $\widetilde{U}$ can be identified with a neighborhood of the trivial morphism in the vector space $Hom(W_1, W_2)$ of linear morphisms: the identification sends $W \in \widetilde{U}$ to

$$L_W : W_1 \to W_2, \quad L_W(x) := y,$$

where y is the unique vector of W_2 such that $x+y \in W$. As for $i=1,2$ the restriction of $||.||$ to W_i is still a norm, we get a norm on $Hom(W_1, W_2)$ by setting

$$N(L_W) := max\left\{\frac{||L_W(x)||}{||x||}\right\}_{x\in W_1\setminus\{0\}} = max\left\{\sqrt{-\frac{\pi_2(w)^2}{\pi_1(w)^2}}\right\}_{w\in W\setminus\{0\}}.$$

Consider U to be the open neighborhood of W_1 defined as

$$U := \{W \in \widetilde{U} \,|\, N(L_W) < 1/\sqrt{2}\}.$$

Now, consider $\alpha \in A$. Recall that $-n \leq \alpha^2 = \pi_1(\alpha)^2 + \pi_2(\alpha)^2$. As $\alpha \in W$ for some $W \in U$ we have $\pi_2(\alpha)^2 < -N(L_W)^2\pi_1(\alpha)^2 = -(1/2)\pi_1(\alpha)^2$. Hence, we have the inequality

$$-\pi_1(\alpha)^2 \leq n + \pi_2(\alpha)^2 < n - \frac{1}{2}\pi_1(\alpha)^2.$$

It follows that $-\pi_1(\alpha)^2 < 2n$ and $\pi_2(\alpha)^2 < n$. Therefore

$$||\alpha|| = \sqrt{-\pi_1(\alpha)^2 + \pi_2(\alpha)^2} < \sqrt{3n}$$

and we are done. □

As a corollary of Lemma 4.1 we have a more general version of Lemma 2.6. The definition of the discriminant of a sheaf we gave in section 2.1 is completely general: if S is any smooth, projective surface and $\mathscr{F}$ is a coherent sheaf of rank r and Chern classes c_1 and c_2, then we define the discriminant to be $\Delta := 2rc_2 - (r-1)c_1^2$. If $\mathscr{F}$ is semistable with respect to some polarization, then $\Delta(\mathscr{F}) \geq 0$. A polarization H is $(r, c_1, c_2)-$*generic* if $H \cdot D \neq 0$ for every divisor D$\in NS(S)$ such that $-\frac{r^2}{4}\Delta \leq D^2 < 0$. We have then the following, which is an immediate corollary of Lemma 4.1, and of which Lemma 2.6 is a particular case:

Corollary 4.2. *Let B be a smooth, connected scheme, $f : \mathscr{X} \longrightarrow B$ a smooth, projective family of surfaces and $\mathscr{H}, \mathscr{L} \in Pic(\mathscr{X})$. For every $b \in B$ write $\mathscr{X}_b := f^{-1}(b)$, $\mathscr{H}_b := \mathscr{H}_{|\mathscr{X}_b}$ and $\mathscr{L}_b := \mathscr{L}_{|\mathscr{X}_b}$. Suppose that for every $b \in B$ the line bundle $\mathscr{H}_b$ is ample, and let $r, c_2 \in \mathbb{Z}$, $r \geq 2$. The set*

$$B' := \{b \in B \,|\, \mathscr{H}_b \textit{ is not } (r, c_1(\mathscr{L}_b), c_2) - \textit{generic}\}$$

is locally a finite union of analytic subvarieties of B.

We now deal with the openness of genericity for rank 0. As in the previous case, we study the problem over an arbitrary surface S. If $v_1 \in H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$ is the first Chern class of an effective divisor and $\chi \in \mathbb{Z} \setminus \{0\}$, we say that a polarization H on S is $(0, v_1, \chi)-$*generic* if $H \cdot D \neq 0$ for every non numerically trivial divisor $D = \chi c_1(\mathscr{F}) - \chi(\mathscr{F})v_1$, where $\mathscr{F}$ is a subsheaf of a sheaf $\mathscr{E}$ with $c_1(\mathscr{E}) = v_1$ and $\chi(\mathscr{E}) = \chi$ (see [24]). Notice that if S is K3 or abelian and $\mathscr{E}$ is a sheaf with $v(\mathscr{E}) = (0, v_1, v_2)$, then $c_1(\mathscr{E}) = v_1$ and $\chi(\mathscr{E}) = v_2$.

Remark 4.3. By definition, if a polarization H is not $(0, v_1, \chi)-$generic, then there is effective curve C on S such that $[C] \notin \mathbb{Q} \cdot v_1$, $\chi \cdot (C \cdot H)/(v_1 \cdot H) \in \mathbb{Z}$ and $h^0(L(-C)) > 0$, where $L \in Pic(S)$ is such that $v_1 = c_1(L)$: here C is the support with multiplicity of a subsheaf $\mathscr{F}$ of a sheaf $\mathscr{E}$ of Mukai vector v, and the condition $h^0(L(-C)) > 0$ comes from the fact that $\mathscr{E}/\mathscr{F}$ is supported on the zero-scheme of a section of $L(-C)$. Conversely, if such a curve exists, then H is not $v-$generic.

We now prove that the $(0, v_1, \chi)-$genericity is an open property. Namely, we prove the following, which is an analogue (but stronger) version of Corollary 4.2, and which is a more general version of Lemma 2.11:

Lemma 4.4. *Let B be a smooth, connected scheme, $f : \mathscr{X} \longrightarrow B$ a smooth, projective family of surfaces and $\mathscr{H}$ a line bundle on $\mathscr{X}$. For every $b \in B$ write $\mathscr{X}_b := f^{-1}(b)$ and $\mathscr{H}_b := \mathscr{H}_{|\mathscr{X}_b}$, and suppose that for every $b \in B$ the line bundle $\mathscr{H}_b$ is ample. Let $0 \in B$, $v_1 = c_1(L)$ for some effective $L \in Pic(\mathscr{X}_0)$ and $\chi \in \mathbb{Z} \setminus \{0\}$. Let $\mathscr{L} \in Pic(\mathscr{X})$ be such that $\mathscr{L}_0 = L$. Then the set*

$$B' := \{b \in B \,|\, \mathscr{H}_b \textit{ is not } (0, c_1(\mathscr{L}_b), \chi) - \textit{generic}\}$$

is a Zariski closed subset of B.

Proof. Let d be the degree of v_1 with respect to $\mathscr{H}$. For every $d', p_a \in \mathbb{Z}$ let $\mathscr{H}ilb_{d',p_a}(\mathscr{X}) \longrightarrow B$ be the relative Hilbert scheme of curves of degree d' and arithmetic genus p_a. Moreover, let $\mathscr{C}_d$ be the set of the curves in the fibres of f of degree at most d with respect to $\mathscr{H}$. We first show that the set A of the arithmetic genera of the curves of $\mathscr{C}_d$ is finite. As a consequence, the union

$$\mathscr{H}ilb_d := \bigcup_{0<d'\leq d,\ p_a\in A} \mathscr{H}ilb_{d',p_a}(\mathscr{X})$$

has only a finite number of irreducible components. Moreover, it has a natural morphism $\phi : \mathscr{H}ilb_d \longrightarrow B$, which is projective.

To show that A is finite, let $p \in \mathbb{N}$ be such that $p\mathscr{H}_b$ is very ample for every $b \in B$. If $C \in \mathscr{C}_d$ is irreducible, then $0 \leq p_a(C) \leq (pd-1)(pd-2)/2$, so that the set A' of the arithmetic genera of the irreducible curves of $\mathscr{C}_d$ is finite. Therefore

$$\mathscr{F} := \bigcup_{0<d'\leq d,\ p_a\in A'} \mathscr{H}ilb_{d',p_a}(\mathscr{X})$$

has a finite number of irreducible components. Moreover, if $C', C'' \subseteq \mathscr{X}_b$ are two curves in $\mathscr{F}$, then $C' \cdot C''$ and $C' \cdot K_{\mathscr{X}_b}$ depend only on the connected component of $\mathscr{F}$ where C' and C'' lie, hence there are only a finite number of possibilities for $C' \cdot C''$ and $C' \cdot K_{\mathscr{X}_b}$. Now, if $C \subseteq \mathscr{X}_b$ is any curve in $\mathscr{C}_d$, then $p_a(C) = 1 + (C^2 + C \cdot K_{\mathscr{X}_b})/2$: as the number of irreducible components of C is at most d, then $p_a(C)$ takes a finite number of values, i. e. the set A is finite.

Let Y be the union of all the connected components of $\mathscr{H}ilb_d$ parameterizing curves C such that $[C] \notin \mathbb{Q} \cdot c_1(\mathscr{L}_b)$ and $\chi \cdot (C \cdot \mathscr{H}_b)/d \in \mathbb{Z}$, for $b \in B$ such that $C \subseteq \mathscr{X}_b$. Let $Z \subseteq Y$ be the locus parameterizing curves C such that $h^0(L'(-C)) > 0$ for some $L' \in Pic(\mathscr{X}_b)$ with $c_1(L') = c_1(\mathscr{L}_b)$: by upper semicontinuity Z is closed in Y, and by Remark 4.3 we have $B' = \phi(Z)$. As ϕ is projective, we have that B' is a Zariski closed subset of B. $\square$

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