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GJMS-TYPE OPERATORS ON A COMPACT RIEMANNIAN MANIFOLD: BEST CONSTANTS AND CORON-TYPE SOLUTIONS

SAIKAT MAZUMDAR

ABSTRACT. In this paper we investigate the existence of solutions to a non-linear elliptic problem involving critical Sobolev exponent for a polyharmonic operator on a Riemannian manifold M . We first show that the best constant of the Sobolev embedding on a manifold can be chosen as close as one wants to the Euclidean one, and as a consequence derive the existence of minimizers when the energy functional goes below a quantified threshold. Next, higher energy solutions are obtained by Coron's topological method, provided that the minimizing solution does not exist. To perform this topological argument, we overcome the difficulty of dealing with polyharmonic operators on a Riemannian manifold and adapting Lions's concentration-compactness lemma. Unlike Coron's original argument for a bounded domain in \mathbb{R}^n , we need to do more than chopping out a small ball from the manifold M . Indeed, our topological assumption that a small sphere on M centred at a point $p \in M$ does not retract to a point in $M \setminus \{p\}$ is necessary, as shown for the case of the canonical sphere where chopping out a small ball is not enough.

1. INTRODUCTION

Let M be a compact manifold of dimension $n \geq 3$ without boundary. Let k be a positive integer such that $2k < n$. Taking inspiration from the construction of the ambient metric of Fefferman-Graham [14] (see [15] for an extended analysis of the ambient metric), Graham-Jenne-Mason-Sparling [18] have defined a family of conformally invariant operators defined for any Riemannian metric. More precisely, for any Riemannian metric g on M , there exists a local differential operator $P_g : C^\infty(M) \rightarrow C^\infty(M)$ such that $P_g = \Delta_g^k + \text{lot}$ where $\Delta_g := -\text{div}_g(\nabla)$, and, given $u \in C^\infty(M)$ and defining $\hat{g} = u^{\frac{4}{n-2k}}g$, we have that

$$(1) \quad P_{\hat{g}}(\varphi) = u^{-\frac{n+2k}{n-2k}} P_g(u\varphi) \text{ for all } \varphi \in C^\infty(M).$$

Moreover, P_g is self-adjoint with respect to the L^2 -scalar product. A scalar invariant is associated to this operator, namely the Q -curvature, denoted as $Q_g \in C^\infty(M)$. When $k = 1$, P_g is the conformal Laplacian and the Q -curvature is the scalar curvature multiplied by a constant. When $k = 2$, P_g is the Paneitz operator introduced in [27]. The Q -curvature was introduced by Branson and Ørsted [9]. The definition of Q_g was then generalized by Branson [7, 8]. In the specific case $n > 2k$, we have that $Q_g := \frac{2}{n-2k} P_g(1)$. Then, taking $\varphi \equiv 1$ in (1), we get that

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$P_g u = \frac{n-2k}{2} Q_{\hat{g}} u^{\frac{n+2k}{n-2k}}$ on M . Therefore, prescribing the Q -curvature in a conformal class amounts to solving a nonlinear elliptic partial differential equation (PDE) of $2k^{th}$ order. Results for the prescription of the Q -curvature problem for the Paneitz operator (namely $k = 2$) are in Djadli-Hebey-Ledoux [12], Robert [29], Esposito-Robert [13]. Recently, Gursky-Malchiodi [19] proved the existence of a metric with constant Q -curvature (still for $k = 2$) provided certain geometric hypotheses on the manifold (M, g) holds. These hypotheses have been simplified by Hang-Yang [20] (see the lecture notes [21]).

In the present paper, we are interested in a generalization of the prescription of the Q -curvature problem. Namely, given $f \in C^\infty(M)$, we investigate the existence of $u \in C^\infty(M)$, $u > 0$, such that

$$(2) \quad Pu = f u^{2_k^\sharp - 1} \text{ in } M,$$

where $2_k^\sharp := \frac{2n}{n-2k}$ and $P : C^\infty(M) \rightarrow C^\infty(M)$ is a smooth self-adjoint $2k^{th}$ order partial differential operator defined by

$$(3) \quad Pu = \Delta_g^k u + \sum_{l=0}^{k-1} (-1)^l \nabla^{j_l \dots j_1} (A_l(g)_{i_1 \dots i_l, j_1 \dots j_l} \nabla^{i_1 \dots i_l} u)$$

where the indices are raised via the musical isomorphism and for all $l \in \{0, \dots, k-1\}$, $A_l(g)$ is a smooth symmetric T_{2l}^0 -tensor field on M (that is: $A_l(g)(X, Y) = A_l(g)(Y, X)$ for all T_0^l -tensors X, Y on M). When $P := P_g$, then (2) is equivalent to say that $Q_{\hat{g}} = \frac{2}{n-2k} f$ with $\hat{g} = u^{\frac{4}{n-2k}} g$.

The conformal invariance (1) of the geometric operator P_g yields obstruction to the existence of solutions to (2). The historical reference here is Kazdan-Warner [23]; for the general GJMS operators, we refer to Delanoë-Robert [11]. In particular, it follows from [11] that on the canonical sphere $(\mathbb{S}^n, \text{can})$, there is no positive solution $u \in C^\infty(\mathbb{S}^n)$ to $P_{\text{can}} u = (1 + \epsilon \varphi) u^{2_k^\sharp - 1}$ for all $\epsilon \neq 0$ and all first spherical harmonic φ . For the conformal Laplacian (that is $k = 1$), Aubin [3] proved that the existence of solutions is guaranteed if a functional goes below a specific threshold. We generalize this result for any $k \geq 1$ in Theorem 3. In the case of a smooth bounded domain, Coron [10] introduced a variational method based on topological arguments, provided the minimizing solution does not exist. Our main theorem is in this spirit:

Theorem 1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. We let P be a coercive operator as in (3). Let $\iota_g > 0$ be the injectivity radius of the manifold M . Suppose that the manifold M contains a point x_0 such that the embedded $(n-1)$ -dimensional sphere $\mathbb{S}_{x_0}(\iota_g/2) := \{x \in M / d_g(x, x_0) = \iota_g/2\}$ is not contractible in $M \setminus \{x_0\}$. Then there exists $\epsilon_0 \in (0, \frac{\iota_g}{2})$ such that the equation*

$$(4) \quad \begin{cases} Pu = |u|^{2_k^\sharp - 2} u & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1 \end{cases}$$

has a non-trivial $C^{2k}(\Omega_M)$ solution for $\Omega_M := M \setminus \overline{B}_{x_0}(\epsilon_0)$. Moreover, if the Green's Kernel of P on Ω_M is positive, then we can choose $u > 0$.

In the original result of Coron [10] (see also Weth and al. [5] for the case $k = 2$), the authors work with a smooth domain of \mathbb{R}^n and assume that it has a small ‘‘hole’’.

In the context of a compact manifold, this assumption is not enough: indeed, the entire compact manifold minus a small hole might retract on a point. We discuss the example of the canonical sphere in Section 7, where the existence of a hole is not sufficient to get solutions to (2).

Concerning higher-order problems, we refer to Bartsch-Weth-Willem [5], Pucci-Serrin [28], Ge-Wei-Zhou [17], the general monograph Gazzola-Grunau-Sweers [16] and the references therein.

Among other tools, the proof of Theorem 1 uses a Lions-type Concentration Compactness Lemma adapted to the context of a Riemannian manifold: this will be the object of Theorem 4.

Equation (2) has a variational structure. Since P is self-adjoint in L^2 , we have that for all $u, v \in C^\infty(M)$.

(5)

$$\int_M uP(v) dv_g = \int_M vP(u) dv_g = \int_M \Delta_g^{k/2} u \Delta_g^{k/2} v dv_g + \sum_{l=0}^{k-1} \int_M A_l(g)(\nabla^l u, \nabla^l v) dv_g$$

where

$$\Delta_g^{l/2} u := \begin{cases} \Delta_g^m u & \text{if } l = 2m \text{ is even} \\ \nabla \Delta_g^m u & \text{if } l = 2m + 1 \text{ is odd} \end{cases}$$

and, when $l = 2m + 1$ is odd, $\Delta_g^{k/2} u \Delta_g^{k/2} v = (\nabla \Delta_g^m u, \nabla \Delta_g^m v)_g$. If P is coercive and $f > 0$, then, up to multiplying by a constant, any solution $u \in C^\infty(M)$ to (2) is a critical point of the functional

$$(6) \quad u \mapsto J_P(u) := \frac{\int_M uP(u) dv_g}{\left(\int_M f|u|^{2_k^\#} dv_g \right)^{2/2_k^\#}}.$$

It follows from (5) that J_P makes sense in the Sobolev spaces $H_k^2(M)$, where for $1 \leq l \leq k$, $H_l^2(M)$ which is the completion of $C^\infty(M)$ with respect to the $u \mapsto \sum_{\alpha=0}^l \|\nabla^\alpha u\|_2$. Equivalently (see Robert [30]), $H_l^2(M)$ is also the completion of the space $C^\infty(M)$ with respect to the norm

$$(7) \quad \|u\|_{H_l^2}^2 := \sum_{\alpha=0}^l \int_M (\Delta_g^{\alpha/2} u)^2 dv_g.$$

By the Sobolev embedding theorem we get a continuous but not compact embedding of $H_k^2(M)$ into $L^{2_k^\#}(M)$. The continuity of the embedding $H_k^2(M) \hookrightarrow L^{2_k^\#}(M)$ yields a pair of real numbers A, B such that for all $u \in H_k^2(M)$

$$(8) \quad \|u\|_{L^{2_k^\#}}^2 \leq A \int_M (\Delta_g^{k/2} u)^2 dv_g + B \|u\|_{H_{k-1}^2}^2.$$

See for example Aubin [4] or Hebey [22]. Following the terminology introduced by Hebey, we then define

$$(9) \quad \mathcal{A}(M) := \inf\{A \in \mathbb{R} : \exists B \in \mathbb{R} \text{ with the property that inequality (8) holds}\}.$$

As for the classical case $k = 1$ (see Aubin [4]), the value of $\mathcal{A}(M)$ depends only on k and the dimension n . More precisely, we let $\mathcal{D}^{k,2}(\mathbb{R}^n)$ be the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $u \mapsto \|\Delta^{k/2}u\|_2$, and we define $K_0(n, k) > 0$

$$(10) \quad \frac{1}{K_0(n, k)} := \inf_{u \in \mathcal{D}^{k,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} (\Delta^{k/2}u)^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{2_k^\#} dx \right)^{\frac{2}{2_k^\#}}}.$$

as the best constant in the Sobolev's continuous embedding $\mathcal{D}^{k,2}(\mathbb{R}^n) \hookrightarrow L^{2_k^\#}(\mathbb{R}^n)$. Our second result is the following:

Theorem 2. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Then $\mathcal{A}(M) = K_0(n, k) > 0$. In particular, for any $\epsilon > 0$, there exists $B_\epsilon \in \mathbb{R}$ such that for all $u \in H_k^2(M)$ one has*

$$(11) \quad \left(\int_M |u|^{2_k^\#} dv_g \right)^{\frac{2}{2_k^\#}} \leq (K_0(n, k) + \epsilon) \int_M (\Delta_g^{k/2}u)^2 dv_g + B_\epsilon \|u\|_{H_{k-1}^2}^2.$$

As a consequence of this result, we will be able to prove the existence of solutions to (2) when the functional J_P goes below a quantified threshold, see Theorem 3.

This paper is organized as follows. In Section 2, we study the best-constant problem and prove Theorem 2. In Section 3, we prove Theorem 3 by classical minimizing method. In Section 4, we prove a Concentration-Compactness Lemma in the spirit of Lions. Section 5 is devoted to test-functions estimates and the proof of the existence of solutions to (4) via a Coron-type topological method. Section 6 deals with positive solutions, and Section 7 with the necessity of the topological assumption of Theorem 1. The appendices concern regularity and a general comparison between geometric norms.

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2. THE BEST CONSTANT

It follows from Lions [24] and Swanson [32] that the extremal functions for the Sobolev inequality (10) exist and are exactly multiples of the functions

$$(12) \quad U_{a,\lambda} = \alpha_{n,k} \left(\frac{\lambda}{1 + \lambda^2|x-a|^2} \right)^{\frac{n-2k}{2}} \quad a \in \mathbb{R}^n, \lambda > 0$$

where the choice of $\alpha_{n,k}$'s are such that for all λ , $\|U_{a,\lambda}\|_{2_k^\#} = 1$ and $\|U_{a,\lambda}\|_{\mathcal{D}^{k,2}}^2 = \frac{1}{K_0(n,k)}$. They satisfies the equation $\Delta^k u = \frac{1}{K_0(n,k)} |u|^{2_k^\#-2} u$ in \mathbb{R}^n

Next we consider the case of a compact Riemmanian manifold. The first result we have in this direction is the following.

Lemma 2.1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Any constant A in inequality (8) has to be greater than or equal to $K_0(n, k)$, whatever the constant B be.*

Proof of Lemma 2.1: We fix $\epsilon > 0$ small. It follows from Lemma 9.1 that there exists, $\delta_0 \in (0, \iota_g)$ depending only on $(M, g), \epsilon$, where ι_g is the injectivity radius of M , such that for any point $p \in M$, any $0 < \delta < \delta_0$, $l \leq k$ and $u \in C_c^\infty(B_0(\delta))$

$$(13) \quad \int_M (\Delta_g^{l/2}(u \circ \exp_p^{-1}))_g^2 dv_g \leq (1 + \epsilon) \int_{\mathbb{R}^n} (\Delta^{l/2} u)^2 dx$$

and

$$(14) \quad (1 - \epsilon) \left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp} dx \right)^{2/2_k^\sharp} \leq \left(\int_M |u \circ \exp_p^{-1}|^{2_k^\sharp} dv_g \right)^{2/2_k^\sharp}$$

Then plugging the above inequalities into (8) we obtain that any $u \in C_c^\infty(B_0(\delta))$ satisfies

$$(15) \quad \left(\int_{\mathbb{R}^n} |u|^{2_k^\sharp} dx \right)^{2/2_k^\sharp} \leq \frac{1 + \epsilon}{1 - \epsilon} A \int_{\mathbb{R}^n} (\Delta^{k/2} u)^2 dx + C_\epsilon \sum_{l=0}^{k-1} \int_{\mathbb{R}^n} |\nabla^l u|^2 dx.$$

Let $v \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp}(v) \subset B_0(R_0)$. For $\lambda > 1$ let $v_\lambda = v(\lambda x)$. Then for λ large, $\text{supp}(v_\lambda) \subset B_0(\delta)$. Taking $u \equiv v_\lambda$ in (15), a change of variable yields

$$(16) \quad \frac{1}{\lambda^{n-2k}} \left(\int_{\mathbb{R}^n} |v|^{2_k^\sharp} dx \right)^{2/2_k^\sharp} \leq \frac{1 + \epsilon}{1 - \epsilon} \cdot \frac{A}{\lambda^{n-2k}} \int_{\mathbb{R}^n} (\Delta^{k/2} v)^2 dx + C_\epsilon \sum_{l=0}^{k-1} \frac{1}{\lambda^{n-2l}} \int_{\mathbb{R}^n} |\nabla^l v|^2 dx.$$

Multiplying by λ^{n-2k} and letting $\lambda \rightarrow +\infty$, we get that for all $v \in \mathcal{D}^{k,2}(\mathbb{R}^n)$, we have

$$(17) \quad \left(\int_{\mathbb{R}^n} |v|^{2_k^\sharp} dx \right)^{2/2_k^\sharp} \leq \frac{1 + \epsilon}{1 - \epsilon} A \int_{\mathbb{R}^n} (\Delta^{k/2} v)^2 dx.$$

Therefore $\frac{1+\epsilon}{1-\epsilon} A \geq K_0(n, k)$ for all $\epsilon > 0$, and letting $\epsilon \rightarrow 0$ yields $A \geq K_0(n, k)$. This ends the proof of Lemma 2.1. \square

We now prove (11) to get Theorem 2.

Step 1: A local inequality. From a result of Anderson (*Main lemma 2.2* of [2]) it follows that for any point $p \in M$ there exists a harmonic coordinate chart φ around p . Then from Lemma 9.1, for any $0 < \epsilon < 1$, there exists $\tau > 0$ small enough such that for any point $p \in M$ and for any $u \in C_c^\infty(B_p(\tau))$, one has

$$(18) \quad \int_{\mathbb{R}^n} (\Delta^{k/2}(u \circ \varphi^{-1}))^2 dx \leq \left(1 + \frac{\epsilon}{3K_0(n, k)} \right) \int_M (\Delta_g^{k/2} u)^2 dv_g$$

and

$$(19) \quad \left(\int_M |u|^{2_k^\sharp} dv_g \right)^{2/2_k^\sharp} \leq \left(1 + \frac{\epsilon}{3K_0(n, k)} \right) \left(\int_{\mathbb{R}^n} |u \circ \varphi^{-1}|^{2_k^\sharp} dx \right)^{2/2_k^\sharp}.$$

The expression for the Laplacian Δ_g in the harmonic coordinates is $\Delta_g u = -g_{ij}\partial_{ij}u$. Then (10) implies that for any $u \in C_c^\infty(B_p(\tau))$

$$(20) \quad \left(\int_M |u|^{2_k^\#} dv_g \right)^{2/2_k^\#} \leq (K_0(n, k) + \epsilon) \int_M (\Delta_g^{k/2} u)^2 dv_g.$$

Step 2: Finite covering and proof of the global inequality. Since M is compact, it can be covered by a finite number of balls $B_{p_i}(\tau/2)$, $i = 1, \dots, N$. Let $\alpha_i \in C_c^\infty(B_{p_i}(\tau))$ be such that $0 \leq \alpha_i \leq 1$ and $\alpha_i = 1$ in $B_{p_i}(\tau/2)$. We set

$$(21) \quad \eta_i = \frac{\alpha_i^2}{\sum_{j=1}^N \alpha_j^2}.$$

Then $(\eta_i)_{i=1, \dots, N}$ is a partition of unity subordinate to the cover $(B_{p_i}(\tau))_{i=1, \dots, N}$ such that $\sqrt{\eta_i}$'s are smooth and $\sum_{i=1}^N \eta_i = 1$. In the sequel, C denote any positive constant depending on k, n , the metric g on M and the functions $(\eta_i)_{i=1, \dots, N}$. Now for any $u \in C^\infty(M)$, we have

$$(22) \quad \|u\|_{2_k^\#}^2 = \|u^2\|_{2_k^\#/2} = \left\| \sum_{i=1}^N \eta_i u^2 \right\|_{2_k^\#/2} \leq \sum_{i=1}^N \|\eta_i u^2\|_{2_k^\#/2} = \sum_{i=1}^N \|\sqrt{\eta_i} u\|_{2_k^\#}^2.$$

So for any $u \in C^\infty(M)$, using inequality (20) we obtain that

$$(23) \quad \left(\int_M |u|^{2_k^\#} dv_g \right)^{2/2_k^\#} \leq (K_0(n, k) + \epsilon) \sum_{i=1}^N \int_M (\Delta_g^{k/2} (\sqrt{\eta_i} u))_g^2 dv_g.$$

Next we claim that there exists $C > 0$ such that

$$(24) \quad \sum_{i=1}^N \int_M (\Delta_g^{k/2} (\sqrt{\eta_i} u))^2 dv_g \leq \int_M (\Delta_g^{k/2} u)^2 dv_g + C \|u\|_{H_{k-1}^2}^2.$$

Assuming that (24) holds we have from (23)

$$(25) \quad \left(\int_M |u|^{2_k^\#} dv_g \right)^{2/2_k^\#} \leq (K_0(n, k) + \epsilon) \int_M (\Delta_g^{k/2} u)_g^2 dv_g + (K_0(n, k) + \epsilon) C \|u\|_{H_{k-1}^2}^2.$$

This proves (11), and therefore, with Lemma 2.1, this proves Theorem 8. We are now left with proving (24).

Step 3: Proof of (24): For any positive integer m , one can write that

$$(26) \quad \Delta_g^m (\sqrt{\eta_i} u) = \sqrt{\eta_i} \Delta_g^m u + \mathcal{P}_g^{(2m-1, 1)}(u, \sqrt{\eta_i}) + \mathcal{L}_{\sqrt{\eta_i}, g}^{2m-2}(u)$$

where

(27)

$$\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}) = \sum_{|l|=2m-1, |\beta|=1} (a_{l,\beta} \partial_\beta \sqrt{\eta_i}) \nabla^l u, \text{ and } \mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u) = \sum_{|l|=0}^{2m-2} a_l(\sqrt{\eta_i}) \nabla^l u$$

the coefficients $a_{l,\beta}$ and $a_l(\sqrt{\eta_i})$ are smooth functions on M . The $a_{l,\beta}$'s depends only on the metric g and on the manifold M and $a_l(\sqrt{\eta_i})$'s depends both on the metric g , the function $\sqrt{\eta_i}$ and its derivatives upto order $2m$. We shall use the same notations $\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})$, $\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)$ for any expression of the above form.

Step 3.1: k is even. We then write $k = 2m$, $m \geq 1$, and then

$$\begin{aligned} & \sum_{i=1}^N \int_M (\Delta_g^m(\sqrt{\eta_i}u))^2 dv_g = \sum_{i=1}^N \int_M \eta_i (\Delta_g^m u)^2 dv_g \\ & + \sum_{i=1}^N \int_M \left(\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}) \right)^2 dv_g + \sum_{i=1}^N \int_M \left(\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u) \right)^2 dv_g \\ & + 2 \sum_{i=1}^N \int_M \sqrt{\eta_i} \Delta_g^m u \mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}) dv_g + 2 \sum_{i=1}^N \int_M \sqrt{\eta_i} \Delta_g^m u \mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u) dv_g \\ (28) \quad & + 2 \sum_{i=1}^N \int_M \mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}) \mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u) dv_g \end{aligned}$$

We note that

(29)

$$\sum_{i=1}^N \int_M \left(\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}) \right)^2 dv_g \leq C \|u\|_{H_{2m-1}^2}^2 \text{ and } \sum_{i=1}^N \int_M \left(\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u) \right)^2 dv_g \leq C \|u\|_{H_{2m-2}^2}^2.$$

On the other hand

$$\begin{aligned} & \sum_{i=1}^N \int_M \sqrt{\eta_i} \Delta_g^m u \mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}) dv_g = \sum_{i=1}^N \sum_{|l|=2m-1} \sum_{|\beta|=1} \int_M (\sqrt{\eta_i} \Delta_g^m u) ((a_{l,\beta} \partial_\beta \sqrt{\eta_i}) \nabla^l u) dv_g \\ & = \frac{1}{2} \sum_{i=1}^N \sum_{|l|=2m-1} \sum_{|\beta|=1} \int_M (\Delta_g^m u) ((a_{l,\beta} \partial_\beta \eta_i) \nabla^l u) dv_g \\ & = \frac{1}{2} \sum_{|l|=2m-1} \sum_{|\beta|=1} \int_M (\Delta_g^m u) ((a_{l,\beta} \partial_\beta (\sum_{i=1}^N \eta_i)) \nabla^l u) dv_g = 0 \\ (30) \end{aligned}$$

while using the integration by parts formula we obtain

$$(31) \quad \sum_{i=1}^N \int_M \sqrt{\eta_i} \Delta_g^m u \mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u) dv_g \leq C \|u\|_{H_{2m-1}^2}^2$$

and by Hölder inequality

$$(32) \quad \sum_{i=1}^N \int_M \mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}) \mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u) dv_g \leq C \|u\|_{H_{2m-1}^2}^2$$

Hence if k is even, then

$$(33) \quad \sum_{i=1}^N \int_M (\Delta_g^m(\sqrt{\eta_i}u))^2 dv_g \leq \int_M (\Delta_g^m u)^2 dv_g + C \|u\|_{H_{2m-1}^2}^2.$$

So we have the claim for k even.

Step 3.2: k is odd. We then write $k = 2m + 1$ with $m \geq 0$. We have

$$(34) \quad \nabla (\Delta_g^m(\sqrt{\eta_i}u)) = \sqrt{\eta_i} \nabla (\Delta_g^m u) + (\Delta_g^m u) \nabla \sqrt{\eta_i} + \nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})) + \nabla (\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u))$$

and so

$$(35) \quad \begin{aligned} \sum_{i=1}^N \int_M |\nabla (\Delta_g^m(\sqrt{\eta_i}u))|^2 dv_g &= \sum_{i=1}^N \int_M \eta_i |\nabla (\Delta_g^m u)|^2 dv_g + \sum_{i=1}^N \int_M (\Delta_g^m u)^2 |\nabla \sqrt{\eta_i}|^2 dv_g \\ &+ \sum_{i=1}^N \int_M \left| \nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})) \right|^2 dv_g + \sum_{i=1}^N \int_M \left| \nabla (\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)) \right|^2 dv_g \\ &+ 2 \sum_{i=1}^N \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), (\Delta_g^m u) \nabla \sqrt{\eta_i}) dv_g + 2 \sum_{i=1}^N \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), \nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}))) dv_g \\ &+ 2 \sum_{i=1}^N \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), \nabla (\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u))) dv_g + 2 \sum_{i=1}^N \int_M ((\Delta_g^m u) \nabla \sqrt{\eta_i}, \nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}))) dv_g \\ (36) \quad &+ 2 \sum_{i=1}^N \int_M ((\Delta_g^m u) \nabla \sqrt{\eta_i}, \nabla (\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u))) dv_g + 2 \sum_{i=1}^N \int_M (\nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})), \nabla (\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u))) dv_g \end{aligned}$$

We have that

$$(37) \quad \sum_{i=1}^N \int_M \left| \nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})) \right|^2 dv_g \leq C \|u\|_{H_{2m}^2}^2 \quad \text{and} \quad \sum_{i=1}^N \int_M \left| \nabla (\mathcal{L}_{\sqrt{\eta_i},g}^{2m-2}(u)) \right|^2 dv_g \leq C \|u\|_{H_{2m-1}^2}^2$$

while

$$\sum_{i=1}^N \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), (\Delta_g^m u) \nabla \sqrt{\eta_i}) dv_g = \sum_{i=1}^N \int_M (\nabla (\Delta_g^m u), (\Delta_g^m u) (\sqrt{\eta_i} \nabla \sqrt{\eta_i})) dv_g$$

$$(38) \quad = \frac{1}{2} \sum_{i=1}^N \int_M (\nabla (\Delta_g^m u), (\Delta_g^m u) \nabla \eta_i) dv_g = \frac{1}{2} \int_M (\nabla (\Delta_g^m u), (\Delta_g^m u) \nabla (\sum_{i=1}^N \eta_i)) dv_g = 0$$

And we obtain

$$\begin{aligned}
& \left| \sum_{i=1}^N \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), \nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}))) dv_g \right| \\
&= \left| \sum_{i=1}^N \sum_{|l|=2m-1} \sum_{|\beta|=1} \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), \nabla ((a_{l,\beta} \partial_\beta \sqrt{\eta_i}) \nabla^l u)) dv_g \right| \\
&\leq \left| \sum_{i=1}^N \sum_{|l|=2m} \sum_{|\beta|=1} \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), (a_{l,\beta} \partial_\beta \sqrt{\eta_i}) \nabla^l u) dv_g \right| \\
&\quad + \left| \sum_{i=1}^N \sum_{|l|=2m-1} \sum_{|\beta|=1} \int_M (\nabla (\Delta_g^m u), (\sqrt{\eta_i} \nabla (a_{l,\beta} \partial_\beta \sqrt{\eta_i})) \nabla^l u) dv_g \right| \\
&\leq \left| \sum_{i=1}^N \sum_{|l|=2m} \sum_{|\beta|=1} \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), (a_{l,\beta} \partial_\beta \sqrt{\eta_i}) \nabla^l u) dv_g \right| \\
(39) \quad & + \sum_{i=1}^N \left| \sum_{|l|=2m-1} \sum_{|\beta|=1} \int_{B_{p_i}(\tau)} (\nabla (\Delta_g^m u), (\sqrt{\eta_i} \nabla (a_{l,\beta} \partial_\beta \sqrt{\eta_i})) \nabla^l u) dv_g \right|
\end{aligned}$$

Then we apply the integration by parts formula on each of the domains $\varphi^{-1}(B_{p_1}(\tau)) \subset \mathbb{R}^n$ to obtain

$$\begin{aligned}
& \left| \sum_{i=1}^N \sum_{|l|=2m} \sum_{|\beta|=1} \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), (a_{l,\beta} \partial_\beta \sqrt{\eta_i}) \nabla^l u) dv_g \right| \\
&+ \sum_{i=1}^N \left| \sum_{|l|=2m-1} \sum_{|\beta|=1} \int_{B_{p_i}(\tau)} (\nabla (\Delta_g^m u), (\sqrt{\eta_i} \nabla (a_{l,\beta} \partial_\beta \sqrt{\eta_i})) \nabla^l u) dv_g \right| \\
&\leq \left| \sum_{i=1}^N \sum_{|l|=2m} \sum_{|\beta|=1} \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), (a_{l,\beta} \partial_\beta \sqrt{\eta_i}) \nabla^l u) dv_g \right| + C \|u\|_{H_{2m}^2}^2 \\
&\leq \frac{1}{2} \left| \sum_{i=1}^N \sum_{|l|=2m} \sum_{|\beta|=1} \int_M (\nabla (\Delta_g^m u), (a_{l,\beta} \partial_\beta \eta_i) \nabla^l u) dv_g \right| + C \|u\|_{H_{2m}^2}^2 \\
&\leq \frac{1}{2} \left| \sum_{|l|=2m} \sum_{|\beta|=1} \int_M (\nabla (\Delta_g^m u), (a_{l,\beta} \partial_\beta (\sum_{i=1}^N \eta_i)) \nabla^l u) dv_g \right| + C \|u\|_{H_{2m}^2}^2 \\
(40) \quad & \leq C \|u\|_{H_{2m}^2}^2 \quad \text{since} \quad \sum_{i=1}^N \eta_i = 1
\end{aligned}$$

Similarly after integration by parts one obtains

$$(41) \quad \left| \sum_{i=1}^N \int_M (\sqrt{\eta_i} \nabla (\Delta_g^m u), \nabla (\mathcal{L}_{\sqrt{\eta_i}, g}^{2m-2}(u))) dv_g \right| \leq C \|u\|_{H_{2m}^2}^2$$

$$(42) \quad \left| \sum_{i=1}^N \int_M ((\Delta_g^m u) \nabla \sqrt{\eta_i}, \nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i}))) dv_g \right| \leq C \|u\|_{H_{2m}^2}^2$$

and

$$(43) \quad \begin{aligned} & \sum_{i=1}^N \int_M ((\Delta_g^m u) \nabla \sqrt{\eta_i}, \nabla (\mathcal{L}_{\sqrt{\eta_i}, g}^{2m-2}(u))) dv_g \\ & + \sum_{i=1}^N \int_M (\nabla (\mathcal{P}_g^{(2m-1,1)}(u, \sqrt{\eta_i})), \nabla (\mathcal{L}_{\sqrt{\eta_i}, g}^{2m-2}(u))) dv_g \leq C \|u\|_{H_{2m}^2}^2 \end{aligned}$$

Hence for k odd, we also obtain that

$$(44) \quad \sum_{i=1}^N \int_M (\nabla (\Delta_g^m(\sqrt{\eta_i} u)))^2 dv_g \leq \int_M (\nabla (\Delta_g^m u))_g^2 dv_g + C \|u\|_{H_{2m}^2}^2.$$

Hence we have the claim and this completes the proof.

3. BEST CONSTANT AND DIRECT MINIMIZATION

Let $\Omega_M \subset M$ be any smooth n -dimensional submanifold of M , possibly with boundary. In the sequel, we will either take $\Omega_M = M$, or $M \setminus \overline{B_{x_0}(\epsilon_0)}$ for some $\epsilon_0 > 0$ small enough. We define $H_{k,0}^2(\Omega_M) \subset H_k^2(M)$ as the completion of $C_c^\infty(\Omega_M)$ for the norm $\|\cdot\|_{H_k^2}$. In this section, we prove the following result in the spirit of Aubin [3]:

Theorem 3. *Let (M, g) be a compact Riemannian manifold of dimension $n > 2k$, with $k \geq 1$. $\Omega_M \subset M$ be any smooth n -dimensional submanifold of M as above. Let P be a differential operator as in (3) and let $f \in C^{0,\theta}(\Omega_M)$ be a Hölder continuous positive function. Assume that P is coercive on $H_{k,0}^2(\Omega_M)$. Suppose that*

$$(45) \quad \inf_{u \in \mathcal{N}_f} \int_{\Omega_M} u P(u) dv_g < \frac{1}{(\sup_{\Omega_M} f)^{\frac{2}{2_k}} K_0(n, k)},$$

where

$$(46) \quad \mathcal{N}_f := \{u \in H_{k,0}^2(\Omega_M) : \int_{\Omega_M} f |u|^{\frac{2}{2_k}} dv_g = 1\}.$$

Then there exists a minimizer $u \in \mathcal{N}_f$. Moreover, up to multiplication by a constant, $u \in C^{2k}(\overline{\Omega_M})$ is a solution to

$$\begin{cases} Pu = f |u|^{\frac{2}{2_k}-2} u & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1. \end{cases}$$

In addition, if the Green's function of P on Ω_M with Dirichlet boundary condition is positive, then any minimizer is either positive or negative. When $\Omega_M = M$, and the Green's function of P on M is positive, then up to changing sign, $u > 0$ is a solution to

$$Pu = fu^{2_k^\sharp - 1} \text{ in } M.$$

Proof of Theorem 3: This type of result is classical. We only sketch the proof. For simplicity, we take $\Omega_M = M$. The proof of the general case is similar. Here and in the sequel, we define (see (5))

$$I_P(u) := \int_M uP(u) dv_g \text{ for all } u \in H_k^2(M).$$

We start with the following lemma:

Lemma 3.1. *Let $(u_i) \in \mathcal{N}_f$ be a minimizing sequence for I_P on \mathcal{N}_f . Then*

- (i) *Either there exists $u_0 \in \mathcal{N}_f$ such that $u_i \rightarrow u_0$ strongly in $H_k^2(M)$, and u_0 is a minimizer of I_P on \mathcal{N}_f*
- (ii) *Or there exists $x_0 \in \overline{\Omega_M}$ such that $f(x_0) = \max_{\overline{\Omega_M}} f$ and $|u_i|^{2_k^\sharp} dv_g \rightharpoonup \delta_{x_0}$ as $i \rightarrow +\infty$ in the sense of measures. Moreover, $\inf_{u \in \mathcal{N}_f} I_P(u) = \frac{1}{K_0(n, k)(\max_M f)^{\frac{2}{2_k^\sharp}}}$.*

Proof of Lemma 3.1: We define $\alpha := \inf\{I_P(u)/u \in \mathcal{N}_f\}$. As the functional I_g is coercive so the sequence (u_i) is bounded in $H_k^2(M)$. We let $u_0 \in H_k^2(M)$ such that, up to a subsequence, $u_i \rightharpoonup u_0$ weakly in $H_k^2(M)$ as $i \rightarrow +\infty$, and $u_i(x) \rightarrow u_0(x)$ as $i \rightarrow +\infty$ for a.e. $x \in M$. Therefore,

$$(47) \quad \|u_0\|_{L^{2_k^\sharp}}^{2_k^\sharp} \leq \liminf_{i \rightarrow +\infty} \|u_i\|_{L^{2_k^\sharp}}^{2_k^\sharp} = 1.$$

We define $v_i := u_i - u_0$. Up to extracting a subsequence, we have that $(v_i)_i \rightarrow 0$ in $H_{k-1}^2(M)$. We define $\mu_i := (\Delta_g^{k/2} u_i)^2 dv_g$ and $\tilde{\nu}_i = |u_i|^{2_k^\sharp} dv_g$ and $\nu_i = f|u_i|^{2_k^\sharp} dv_g$ for all i . Up to a subsequence, we denote respectively by μ , $\tilde{\nu}$ and ν their limits in the sense of measures. It follows from the concentration-compactness Theorem 4 that,

$$(48) \quad \tilde{\nu} = |u_0|^{2_k^\sharp} dv_g + \sum_{j \in \mathcal{J}} \alpha_j \delta_{x_j} \text{ and } \mu \geq (\Delta_g^{k/2} u_0)^2 dv_g + \sum_{j \in \mathcal{I}} \beta_j \delta_{x_j}$$

where $\mathcal{J} \subset \mathbb{N}$ is at most countable, $(x_j)_{j \in \mathcal{J}} \in M$ is a family of points, and $(\alpha_j)_{j \in \mathcal{J}} \in \mathbb{R}_{\geq 0}$, $(\beta_j)_{j \in \mathcal{J}} \in \mathbb{R}_{\geq 0}$ are such that

$$(49) \quad \alpha_j^{2/2_k^\sharp} \leq K_0(n, k) \beta_j \text{ for all } j \in \mathcal{J}.$$

As a consequence, we get that

$$(50) \quad \nu = f|u_0|^{2_k^\sharp} dv_g + \sum_{j \in \mathcal{J}} f(x_j) \alpha_j \delta_{x_j}$$

Since $(u_i) \in \mathcal{N}_f$, and M is compact, we have that $\int_M d\nu = 1$ and then

$$(51) \quad 1 = \int_M f|u_0|^{2_k^\sharp} dv_g + \sum_{j \in \mathcal{J}} f(x_j) \alpha_j.$$

Since $(u_i)_i \rightarrow u_0$ strongly in $H_{k-1}^2(M)$, integrating (48) yields

$$(52) \quad \alpha \geq I_P(u_0) + \sum_{j \in \mathcal{J}} \beta_j \geq \alpha \|u_0\|_{2_k^\sharp}^2 + K_0(n, k)^{-1} \sum_{j \in \mathcal{J}} \alpha_j^{2/2_k^\sharp}.$$

Since $\alpha \leq K_0(n, k)^{-1}(\max_M f)^{-2/2_k^\sharp}$, we then get that

- (i) either $\|u_0\|_{2_k^\sharp} = 1$ and $\alpha_j = 0$ for all $j \in \mathcal{J}$,
- (ii) or $u_0 \equiv 0$, $f(x_{j_0})\alpha_{j_0} = 1$ for some $j_0 \in \mathcal{J}$, $f(x_{j_0}) = \max_M f$ and $\alpha_j = 0$ for all $j \neq j_0$.

In case (i), we get from the strong convergence to 0 of $(v_i)_i$ in $H_{k-1}^2(M)$ that $I_P(u_i) = \int_M (\Delta_g^{k/2} v_i)^2 dv_g + I_P(u_0) + o(1)$ as $i \rightarrow +\infty$. Since $u_0 \in \mathcal{N}_f$ and (u_i) is a minimizing sequence, we then get that $(v_i)_0$ goes to 0 strongly in $H_k^2(M)$, and therefore $u_i \rightarrow u_0$ strongly in $H_k^2(M)$.

In case (ii), (52) yields $\alpha = K_0(n, k)^{-1}(\max_M f)^{-2/2_k^\sharp}$ and $I_P(u_0) = 0$, which yields $u_0 \equiv 0$ since the operator is coercive.

This completes the proof of Lemma 3.1. \square

We go back to the proof of Theorem 3. Let $(u_i)_i$ be a minimizing sequence for I_P on \mathcal{N}_f . It follows from the assumption (45) that case (i) of Lemma 3.1 holds, and then, there exists a minimizer $u_0 \in \mathcal{N}_f$ that is a minimizer. Therefore, it is a weak solution to $P_g^k u_0 = \alpha f |u_0|^{2_k^\sharp - 2} u_0$ in M (see (145) for the definition). It then follows from the regularity Theorem 8.3 that $u \in C^{2k, \theta}(M)$.

We let $G : M \times M \setminus \{(x, x) / x \in M\}$ be the Green's function of P on M . We assume that $G(x, y) > 0$ for all $x \neq y \in M$. Green's representation formula yields

$$(53) \quad \varphi(x) = \int_M G(x, y) (P\varphi)(y) dv_g \text{ for all } x \in M \text{ and all } \varphi \in C^{2k}(M).$$

It follows from Proposition 8.2 that there exists $v \in H_k^2(M)$ such that

$$(54) \quad Pv = \alpha f |u_0|^{2_k^\sharp - 1} \quad \text{in } M.$$

Standard regularity (taking inspiration from Vand der Vorst [33]) yields $v \in C^{2k}(M)$. We have that $P(v \pm u_0) \geq 0$. Since $G > 0$, it follows from Green's formula (53) that $v \pm u_0 \geq 0$. So $v \geq |u_0|$ and therefore $v \neq 0$. Independently, since $Pv \geq 0$ and $v \neq 0$, Green's formula (53) yields $v > 0$. Using Hölder's inequality and $v \geq |u_0|$,

we get that

$$(55) \quad J_P(u) = \frac{\int_M v P(v) dv_g}{\left(\int_M f |v|^{2_k^\sharp} dv_g \right)^{2/2_k^\sharp}} = \frac{\alpha \int_M v f |u_0|^{2_k^\sharp - 1} dv_g}{\left(\int_M f |v|^{2_k^\sharp} dv_g \right)^{2/2_k^\sharp}}$$

$$(56) \quad \leq \frac{\alpha \left(\int_M f |v|^{2_k^\sharp} dv_g \right)^{\frac{1}{2_k^\sharp}} \left(\int_M f |u_0|^{2_k^\sharp} dv_g \right)^{\frac{2_k^\sharp - 1}{2_k^\sharp}}}{\left(\int_M f |v|^{2_k^\sharp} dv_g \right)^{2/2_k^\sharp}}$$

$$(57) \quad \leq \frac{\alpha \left(\int_M f |u_0|^{2_k^\sharp} dv_g \right)^{\frac{2_k^\sharp - 1}{2_k^\sharp}}}{\left(\int_M f |u_0|^{2_k^\sharp} dv_g \right)^{1/2_k^\sharp}} \leq \alpha \left(\int_M f |u_0|^{2_k^\sharp} dv_g \right)^{\frac{2_k^\sharp - 2}{2_k^\sharp}} \leq \alpha$$

since $\int_M f |u_0|^{2_k^\sharp} dv_g = 1$. Since α is the infimum of the functional, we get that $J_P(u) = \alpha$. Hence v attains the infimum and therefore it also solves the equation $Pv = \mu f v^{2_k^\sharp - 1}$ weakly in M , and $v \in H_{k,0}^2(M)$. Moreover, one has equality in all the inequalities above, and then $|u_0| = v > 0$, and therefore either $u_0 > 0$ or $u_0 < 0$ in M . This ends the proof of Theorem 3. \square

4. CONCENTRATION COMPACTNESS LEMMA

We now state and prove the concentration compactness lemma in the spirit of P.-L.Lions for the case of a closed manifold:

Theorem 4 (Concentration-compactness). *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Suppose (u_m) be a bounded sequence in $H_k^2(M)$. Up to extracting a subsequence, there exist two nonnegative Borel-regular measure μ, ν on M and $u \in H_k^2(M)$ such that*

- (a) $u_m \rightharpoonup u$ weakly in $H_k^2(M)$
- (b) $\mu_m := (\Delta_g^{k/2} u_m)^2 dv_g \rightharpoonup \mu$ weakly in the sense of measures
- (c) $\nu_m := |u_m|^{2_k^\sharp} dv_g \rightharpoonup \nu$ weakly in the sense of measures

Then there exists an at most countable index set \mathcal{I} , a family of distinct points $\{x_i \in M : i \in \mathcal{I}\}$, families of nonnegative weights $\{\alpha_i : i \in \mathcal{I}\}$ and $\{\beta_i : i \in \mathcal{I}\}$ such that

(i)

$$(58) \quad \nu = |u|^{2_k^\#} dv_g + \sum_{i \in \mathcal{I}} \alpha_i \delta_{x_i}$$

$$(59) \quad \mu \geq (\Delta_g^{k/2} u)^2 dv_g + \sum_{i \in \mathcal{I}} \beta_i \delta_{x_i}$$

where δ_x denotes the Dirac mass at $x \in M$ with mass equal to 1.

(ii) for all $i \in \mathcal{I}$, $\alpha_i^{2/2_k^\#} \leq K_0(n, k) \beta_i$. In particular $\sum_{i \in \mathcal{I}} \alpha_i^{2/2_k^\#} < \infty$.

Proof of Theorem 4: By the Riesz representation theorem (μ_m) , and (ν_m) are sequences of Radon measures on M .

Step 1: First we assume that $u \equiv 0$. Let $\varphi \in C^\infty(M)$, then from (2) we have that, given any $\varepsilon > 0$ there exists $B_\varepsilon \in \mathbb{R}$ such that

$$(60) \quad \left(\int_M |\varphi u_m|^{2_k^\#} dv_g \right)^{2/2_k^\#} \leq (K_0(n, k) + \varepsilon) \int_M (\Delta_g^{k/2}(\varphi u_m))^2 dv_g + B_\varepsilon \|\varphi u_m\|_{H_{k-1}^2}^2.$$

Since $u_m \rightharpoonup 0$ in $H_k^2(M)$, letting $m \rightarrow +\infty$ and then taking the limit $\varepsilon \rightarrow 0$, it follows that

$$(61) \quad \left(\int_M |\varphi|^{2_k^\#} d\nu \right)^{2/2_k^\#} \leq K_0(n, k) \int_M \varphi^2 d\mu.$$

By regularity of the Borel measure ν , (61) holds for any Borel measurable function φ and in particular for any Borel set $E \subset M$ we have

$$(62) \quad \nu(E)^{2/2_k^\#} \leq K_0(n, k) \mu(E).$$

Therefore the measure ν is absolutely continuous with respect to the measure μ and hence by the Radon-Nikodyn theorem, we get

$$(63) \quad d\nu = f d\mu \text{ and } d\mu = g d\nu + d\sigma$$

where $f \in L^1(M, \mu)$ and $g \in L^1(M, \nu)$ are nonnegative functions, σ is a positive Borel measure on M and $d\nu \perp d\sigma$.

Let $S = M \setminus (\text{supp } \sigma)$. Then for any $\varphi \in C(M)$ with support $\text{supp}(\varphi) \subset S$ one has

$$(64) \quad \int_M \varphi d\nu = \int_M \varphi f d\mu = \int_M \varphi f g d\nu.$$

By regularity of the Borel measures μ and ν (64) holds for any Borel measurable function φ . This implies that $fg = 1$ a.e with respect to ν . So, in particular $g > 0$

ν a.e in S . Let $\psi \in C(M)$, taking $\varphi = \psi\chi_S$ in (61) we have

$$(65) \quad \left(\int_M |\psi|^{2_k^\#} \mathcal{X}_S d\nu \right)^{2/2_k^\#} \leq K_0(n, k) \int_M \psi^2 \mathcal{X}_S d\mu$$

$$= K_0(n, k) \int_M \psi^2 \mathcal{X}_S [g d\nu + d\sigma] = K_0(n, k) \int_M \psi^2 g \mathcal{X}_S d\nu$$

Since $d\nu \perp d\sigma$ and $\text{supp } \nu \subset S$, we get that

$$(66) \quad \left(\int_M |\psi|^{2_k^\#} d\nu \right)^{2/2_k^\#} \leq K_0(n, k) \int_M \psi^2 g d\nu$$

By regularity of the Borel measure ν the above relation holds for any Borel measurable function ψ .

Let $\phi \in C(M)$ and let $\psi = \phi g^{\frac{1}{2_k^\#-2}} \mathcal{X}_{\{g \leq N\}}$, $d\nu_N = g^{\frac{2_k^\#}{2_k^\#-2}} \mathcal{X}_{\{g \leq N\}} d\nu$. Then we have

$$(67) \quad \left(\int_M |\phi|^{2_k^\#} d\nu_N \right)^{2/2_k^\#} \leq K_0(n, k) \int_M \phi^2 d\nu_N.$$

By regularity of the Borel measure ν the above relation holds for any Borel measurable function ϕ .

It follows from Proposition 4.1 below that for each N there exist a finite set \mathcal{I}_N , a finite set of distinct points $\{x_i : i \in \mathcal{I}_N\}$ and a finite set of weights $\{\tilde{\alpha}_i : i \in \mathcal{I}_N\}$ such that

$$(68) \quad d\nu_N = \sum_{i \in \mathcal{I}_N} \tilde{\alpha}_i \delta_{x_i}$$

Let $\mathcal{I} = \bigcup_{N=1}^{\infty} \mathcal{I}_N$. Then \mathcal{I} is a countable set. For a Borel set E , then one has by monotone convergence theorem

$$(69) \quad \int_M \chi_E g^{\frac{2_k^\#}{2_k^\#-2}} d\nu = \lim_{N \rightarrow \infty} \int_M \chi_E d\nu_N.$$

So $g^{\frac{2_k^\#}{2_k^\#-2}} d\nu = \sum_{i \in \mathcal{I}} \tilde{\alpha}_i \delta_{x_i}$. Since $g > 0$ ν a.e, there exists $\alpha_i > 0$ such that we have $d\nu = \sum_{i \in \mathcal{I}} \alpha_i \delta_{x_i}$. Since $\mu = g d\nu + d\sigma \geq g d\nu$, we get that

$$(70) \quad \mu \geq \sum_{i \in \mathcal{I}} \beta_i \delta_{x_i} \quad \text{where } \beta_i = g(x_i) \alpha_i$$

Taking $\psi = \mathcal{X}_{\{x_i\}}$ in (66) we have for all $i \in \mathcal{I}$

$$(71) \quad \alpha_i^{2/2_k^\#} \leq K_0(n, k) g \alpha_i = K_0(n, k) \beta_i$$

and

$$(72) \quad \frac{1}{K_0(n, k)} \sum_{i \in \mathcal{I}} \alpha_i^{2/2_k^\#} \leq \sum_{i \in \mathcal{I}} \beta_i \leq \mu(M) < +\infty.$$

This proves the theorem for $u \equiv 0$. This ends Step 1.

Step 2: Assume $u \not\equiv 0$ and let $v_m := u_m - u$. Then $v_m \rightharpoonup 0$ weakly in $H_k^2(M)$. Therefore, as one checks, $\tilde{\mu}_m := (\Delta_g^{k/2} v_m)^2 dv_g \rightharpoonup \mu - (\Delta_g^{k/2} u)^2 dv_g$ and $\tilde{\nu}_m := |v_m|^{2_k^\#} dv_g \rightharpoonup \nu - |u|^{2_k^\#} dv_g$ weakly in the sense of measures. Applying Step 1 to the measures $\tilde{\mu}_m$ and $\tilde{\nu}_m$ yields Theorem 4. \square

We now prove the reversed Hölder inequality that was used in the proof.

Proposition 4.1. *Let μ be a finite Borel measure on M and suppose that for any Borel measurable function φ one has*

$$(73) \quad \left(\int_M |\varphi|^q d\mu \right)^{1/q} \leq C \left(\int_M |\varphi|^p d\mu \right)^{1/p}$$

for some $C > 0$ and $1 \leq p < q < +\infty$. Then there exists j points $x_1, \dots, x_j \in M$, and j positive real numbers c_1, \dots, c_j such that

$$(74) \quad \mu = \sum_{i=1}^j c_i \delta_{x_i}$$

where δ_x denotes the Dirac measure concentrated at $x \in M$ with mass equal to 1. Moreover $c_i \geq (\frac{1}{C})^{\frac{pq}{q-p}}$.

Proof. Let E be a Borel set in M . Taking $\varphi = \chi_E$ we obtain that, either $\mu(E) = 0$ or $\mu(E) \geq (\frac{1}{C})^{\frac{pq}{q-p}}$.

We define $\mathcal{O} := \{x \in M : \text{for some } r > 0 \ \mu(B_x(r)) = 0\}$. Then \mathcal{O} is open. Now if $K \subset \mathcal{O}$ is compact, then K can be covered by a finite number of balls each of which has measure 0, therefore $\mu(K) = 0$. By the regularity of the measure hence it follows that $\mu(\mathcal{O}) = 0$. If $x \in M \setminus \mathcal{O}$, then for all $r > 0$ one has $\mu(B_x(r)) \geq (\frac{1}{C})^{\frac{pq}{q-p}}$. Then

$$(75) \quad \mu(\{x\}) = \lim_{m \rightarrow +\infty} \mu(B_x(1/m)) \geq \left(\frac{1}{C} \right)^{\frac{pq}{q-p}}.$$

Since the measure μ is finite, this implies that the set $M \setminus \mathcal{O}$ is finite. So let $M \setminus \mathcal{O} = \{x_1, \dots, x_j\}$, therefore for any borel set E in M

$$(76) \quad \mu(E) = \mu(E \cap \{x_1, \dots, x_j\}) = \sum_{x_i \in E} \mu(\{x_i\}) = \sum_{i=1}^j \mu(\{x_i\}) \delta_{x_i}(E).$$

Hence the lemma follows with $c_i = \mu(\{x_i\})$. \square

5. TOPOLOGICAL METHOD OF CORON

In this section we obtain higher energy solutions by Coron's topological method if the functional J_P does not have a minimizer, for the case $f \equiv 1$. This will complete the proof of the first part of Theorem 1, that is the existence of solutions to (4) with no sign-restriction. For $\mu > 0$ and $y_0 \in \mathbb{R}^n$, we define

$$(77) \quad \mathcal{B}_{y_0, \mu}(y) = \alpha_{n,k} \left(\frac{\mu}{\mu^2 + |y - y_0|^2} \right)^{\frac{n-2k}{2}}$$

where the choice of $\alpha_{n,k}$'s are such that for all μ , $\|\mathcal{B}_{y_0, \mu}\|_{L^{2_k^*}} = 1$ and $\|\mathcal{B}_{y_0, \mu}\|_{\mathcal{D}^{k,2}}^2 = \frac{1}{K_0(n,k)}$. These functions are the extremal functions of the Euclidean Sobolev Inequality (10) and they satisfy the equation

$$(78) \quad \Delta^k \mathcal{B}_{y_0, \mu} = \frac{1}{K_0(n,k)} \mathcal{B}_{y_0, \mu}^{2_k^* - 1} \quad \text{in } \mathbb{R}^n.$$

Let $\tilde{\eta}_r \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \tilde{\eta}_r \leq 1$ be a smooth cut-off function, such that $\tilde{\eta}_r = 1$ for $x \in B_0(r)$ and $\tilde{\eta}_r = 0$ for $x \in \mathbb{R}^n \setminus B_0(2r)$. Let $\iota_g > 0$ be the injectivity radius of (M, g) . For any $p \in M$, we let η_p be a smooth cut-off function on M such that

$$(79) \quad \eta_p(x) = \begin{cases} \tilde{\eta}_{\frac{\iota_g}{10}}(\exp_p^{-1}(x)) & \text{for } x \in B_p(\iota_g) \subset M \\ 0 & \text{for } x \in M \setminus B_p(\iota_g). \end{cases}$$

For any $x \in M$, we define

$$(80) \quad \mathcal{B}_{p, \mu}^M(x) = \eta_p(x) \mathcal{B}_{0, \mu}(\exp_p^{-1}(x)).$$

$\mathcal{B}_{p, \mu}^M$ is the standard bubble centered at the point $p \in M$ and with radius μ

$$(81) \quad \mathcal{B}_{p, \mu}^M(x) = \alpha_{n,k} \eta_p(x) \left(\frac{\mu}{\mu^2 + d_g(p, x)^2} \right)^{\frac{n-2k}{2}}.$$

We have

Proposition 5.1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Consider the functional J_P on the space $H_k^2(M) \setminus \{0\}$. Then the sequence of functions $(\mathcal{B}_{p, \mu}^M) \in C^\infty(M)$ defined above is such that:*

- (a) $\lim_{\mu \rightarrow 0} J_P(\mathcal{B}_{p, \mu}^M) = \frac{1}{K_0(n,k)}$ uniformly for $p \in M$
- (b) $\lim_{\mu \rightarrow 0} \|\mathcal{B}_{p, \mu}^M\|_{L^{2_k^*}} = 1$ uniformly for $p \in M$
- (c) $\mathcal{B}_{p, \mu}^M \rightharpoonup 0$ weakly in $H_k^2(M)$, as $\mu \rightarrow 0$

Proof of Proposition 5.1: We claim that (c) holds. We first prove that $\mathcal{B}_{p,\mu}^M$ is uniformly bounded in $H_k^2(M)$. Indeed,

$$\begin{aligned}
\sum_{\alpha \leq k} \int_M \left(\Delta_g^{\alpha/2} \mathcal{B}_{p,\mu}^M \right)^2 dv_g &\leq \sum_{\alpha \leq k} \int_{B_p(\iota_g/5)} \left(\Delta_g^{\alpha/2} \mathcal{B}_{p,\mu}^M \right)^2 dv_g \\
&\leq C \sum_{l \leq k} \int_{B_0(\iota_g/5)} \left| \nabla^l \mathcal{B}_{p,\mu}^M \circ \exp_p \right|^2 dx \\
&\leq \sum_{l \leq k} \int_{B_0(\iota_g/5)} \left| \nabla^l \left(\frac{\mu}{\mu^2 + |x|^2} \right)^{\frac{n-2k}{2}} \right|^2 dx \\
&\leq \sum_{l \leq k} \int_{B_0(\iota_g/(5\mu))} \mu^{2(k-l)} \left| \nabla^l (1 + |x|^2)^{-\frac{n-2k}{2}} \right|^2 dx.
\end{aligned}$$

As one checks, the right-hand-side is uniformly bounded wrt $\mu \rightarrow 0$, so $(\mathcal{B}_{p,\mu}^M)$ is uniformly bounded wrt p and $\mu \rightarrow 0$. Moreover, the above computations yield $\int_M (\mathcal{B}_{p,\mu}^M)^2 dv_g \rightarrow 0$ as $\mu \rightarrow 0$. Therefore, $\mathcal{B}_{p,\mu}^M \rightarrow 0$ as $\mu \rightarrow 0$ uniformly wrt $p \in M$. This proves the claim.

The space $H_k^2(M)$ is compactly embedded in $H_{k-1}^2(M)$. Therefore $\mathcal{B}_{p,\mu}^M \rightarrow 0$ in $H_{k-1}^2(M)$ as $\mu \rightarrow 0$. Hence

$$(82) \quad \lim_{\mu \rightarrow 0} \left(\sum_{l=0}^{k-1} \int_M A_l(g) (\nabla^l \mathcal{B}_{p,\mu}^M, \nabla^l \mathcal{B}_{p,\mu}^M) dv_g \right) = 0.$$

Now we estimate the term $\int_M |\mathcal{B}_{p,\mu}^M|^{2_k^\sharp} dv_g$. We claim that

$$(83) \quad \lim_{R \rightarrow +\infty} \lim_{\mu \rightarrow 0} \int_{M \setminus B_p(\mu R)} |\mathcal{B}_{p,\mu}^M|^{2_k^\sharp} dv_g = 0.$$

We fix R . Now for μ sufficiently small

$$\begin{aligned}
\int_{M \setminus B_p(\mu R)} |\mathcal{B}_{p,\mu}^M|^{2_k^\sharp} dv_g &= \int_{B_p(\iota_g) \setminus B_p(\mu R)} |\mathcal{B}_{p,\mu}^M|^{2_k^\sharp} dv_g \\
&= \int_{B_0(\iota_g) \setminus B_0(\mu R)} |\mathcal{B}_{p,\mu}^M(\exp_p(y))|^{2_k^\sharp} \sqrt{|g(\exp_p(y))|} dy \\
(84) \quad &\leq \int_{B_0(\frac{\iota_g}{\mu}) \setminus B_0(R)} |\mathcal{B}_{0,1}(y)|^{2_k^\sharp} \sqrt{|g(\exp_p(\mu y))|} dy.
\end{aligned}$$

Since $\mathcal{B}_{0,1} \in L^{2_k^\sharp}(\mathbb{R}^n)$, this yields the claim.

Similarly, for μ sufficiently small

$$(85) \quad \int_{B_p(\mu R)} |\mathcal{B}_{p,\mu}^M|^{2_k^\sharp} dv_g = \int_{B_0(\mu R)} |\mathcal{B}_{p,\mu}^M(\exp_p(y))|^{2_k^\sharp} \sqrt{|g(\exp_p(y))|} dy$$

$$(86) \quad = \int_{B_0(R)} |\mathcal{B}_{0,1}|^{2_k^\sharp} \sqrt{|g(\exp_p(\mu y))|} dy$$

$$(87) \quad = \int_{B_0(R)} |\mathcal{B}_{0,1}|^{2_k^\sharp} dy + o\left(\|\mathcal{B}_{0,1}\|_{L^{2_k^\sharp}}\right) \quad \text{as } \mu \rightarrow 0$$

Therefore

$$(88) \quad \lim_{\mu \rightarrow 0} \int_M |\mathcal{B}_{p,\mu}^M|^{2_k^\sharp} dv_g = \int_{\mathbb{R}^n} |\mathcal{B}_{0,1}|^{2_k^\sharp} dv_g.$$

So we have (b).

Finally we estimate the term $\int_M (\Delta_g^{k/2} \mathcal{B}_{p,\mu}^M)^2 dv_g$. We fix $R > 0$. By calculating in terms of the local coordinates given by \exp_p , we get for μ sufficiently small

$$(89) \quad \int_{B_p(\mu R)} (\Delta_g^{k/2} \mathcal{B}_{p,\mu}^M)^2 dv_g = \int_{B_0(R)} (\Delta^{k/2} \mathcal{B}_{0,1})^2 dy + o(1) \quad \text{as } \mu \rightarrow 0.$$

We claim that

$$(90) \quad \lim_{R \rightarrow +\infty} \lim_{\mu \rightarrow 0} \int_{M \setminus B_p(\mu R)} (\Delta_g^{k/2} \mathcal{B}_{p,\mu}^M)^2 dv_g = 0.$$

We prove the claim. Indeed, via the exponential map at p , we have that

$$(91) \quad \int_{M \setminus B_p(\mu R)} (\Delta_g^{k/2} \mathcal{B}_{p,\mu}^M)^2 dv_g = \int_{B_p(\iota_g) \setminus B_p(\mu R)} (\Delta_g^{k/2} \mathcal{B}_{p,\mu}^M)^2 dv_g$$

$$(92) \quad = \int_{B_0(\iota_g) \setminus B_0(\mu R)} (\Delta_{\exp_p^* g}^{k/2} \mathcal{B}_{0,\mu})^2 dv_{\exp_p^* g}$$

$$(93) \quad \leq C \sum_{|\alpha|=0}^k \int_{B_0(\iota_g) \setminus B_0(\mu R)} |D^\alpha(\tilde{\eta}_{10}^{\iota_g} \mathcal{B}_{0,\mu})|^2 dx$$

Since $\mathcal{B}_{0,\mu} \rightarrow 0$ strongly in $H_{k-1,loc}^2(\mathbb{R}^n)$, then, as $\mu \rightarrow 0$, we have that

$$(94) \quad \int_{M \setminus B_p(\mu R)} (\Delta_g^{k/2} \mathcal{B}_{p,\mu}^M)^2 dv_g \leq C \int_{B_0(\iota_g) \setminus B_0(\mu R)} \tilde{\eta}_{10}^2 |D^k \mathcal{B}_{0,\mu}|^2 dx + o(1)$$

$$(95) \quad \leq C \int_{B_0(\iota_g/\mu) \setminus B_0(R)} |D^k \mathcal{B}_{0,1}|^2 dx + o(1) \leq C \int_{\mathbb{R}^n \setminus B_0(R)} |D^k \mathcal{B}_{0,1}|^2 dx + o(1).$$

Since $D^k \mathcal{B}_{0,1} \in L^2(\mathbb{R}^n)$, this yields (90). This proves the claim.

Equations (89) and (90) yield (a) and (b) of Proposition 5.1 for any fixed $p \in M$. Since the manifold M is compact, we note that in the above calculations there is no

dependence on the point p of the closed manifold M . So the convergence is uniform for all points $p \in M$. This ends the proof of Proposition 5.1. \square

Fix some θ such that $\frac{1}{K_0(n,k)} + 4\theta < 2^{2k/n} \frac{1}{K_0(n,k)}$. Then from (5.1) it follows that, there exists μ_0 small, such that for all $\mu \in (0, \iota_g \mu_0)$ and for all $p \in M$ we have

$$(96) \quad J_P(\mathcal{B}_{p,\mu}^M) \leq \frac{1}{K_0(n,k)} + \theta.$$

We fix $x_0 \in M$, and we assimilate isometrically $T_{x_0}M$ to \mathbb{R}^n , and we define the sphere $S^{n-1} := \{x \in \mathbb{R}^n / \|x\| = 1\}$. For $(\sigma, t) \in S^{n-1} \times [0, \iota_g/2]$, we define $\sigma_t^M := \exp_{x_0}(t\sigma)$ and

$$(97) \quad u_t^\sigma(x) = \alpha_{n,k} \eta_{\sigma_t^M}(x) \left[\frac{\mu_0(\iota_g/2 - t)}{(\mu_0(\iota_g/2 - t))^2 + d_g(\sigma_t^M, x)^2} \right]^{\frac{n-2k}{2}} = \mathcal{B}_{\sigma_t^M, \mu_0(\iota_g/2-t)}^M$$

It then follows from our previous step and the choice of μ_0 in (96)

$$(98) \quad J_P(u_t^\sigma) \leq \frac{1}{K_0(n,k)} + \theta \quad \forall (\sigma, t) \in S^{n-1} \times [0, \iota_g/2].$$

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a smooth, nonnegative, cut-off function such that $\eta(x) = 1$ for $|x| \geq 1/2$ and $\eta(x) = 0$ for $|x| < 1/4$. For $R \geq 1$, let η_R be a smooth, nonnegative, cut-off function, such that

$$(99) \quad \eta_R(x) = \begin{cases} 1 & \text{if } d_g(x_0, x) \geq \frac{\iota_g}{10R} \\ \eta\left(\frac{10R}{\iota_g} \exp_{x_0}^{-1}(x)\right) & \text{if } d_g(x_0, x) < \frac{\iota_g}{10R} \end{cases}$$

Then the functions η_R are such that $\eta_R(x) = 1$ if $d_g(x_0, x) \geq \frac{\iota_g}{20R}$ and $\eta_R(x) = 0$ if $d_g(x_0, x) < \frac{\iota_g}{40R}$. We define

$$(100) \quad v_{t,R}^\sigma(x) := \eta_R(x) u_t^\sigma(x) \text{ for all } x \in M.$$

Then we have

Proposition 5.2.

$$(101) \quad \lim_{R \rightarrow +\infty} v_{t,R}^\sigma = u_t^\sigma \text{ in } H_k^2(M) \text{ uniformly } \forall (\sigma, t) \in S^{n-1} \times [0, \iota_g/2].$$

Proof of Proposition 5.2: We first note that for all $(\sigma, t) \in S^{n-1} \times [0, \iota_g/2]$ the functions u_t^σ are uniformly bounded in C^{2k} -norm in the ball $B_{x_0}(\frac{\iota_g}{20}) \subset M$. And for any nonnegative integer α , one has $|\nabla_g^\alpha \eta_R|_g \leq CR^\alpha$. Therefore

$$(102) \quad \|v_{t,R}^\sigma - u_t^\sigma\|_{H_k^2}^2 = \sum_{\alpha=0}^k \int_M (\Delta_g^{\alpha/2} (v_{t,R}^\sigma - u_t^\sigma))^2 dv_g$$

$$(103) \quad = \sum_{\alpha=0}^k \int_{B_{x_0}(\frac{\iota_g}{20R})} (\Delta_g^{\alpha/2} (v_{t,R}^\sigma - u_t^\sigma))^2 dv_g$$

$$(104) \quad = \sum_{\alpha=0}^k \int_{B_{x_0}(\frac{\iota_g}{20R})} (\Delta_g^{\alpha/2} ((\eta_R - 1)u_t^\sigma))^2 dv_g = O\left(\frac{1}{R}\right) \quad (\text{ as } n \geq 2k+1)$$

The above convergence is uniform *w.r.t* $(\sigma, t) \in S^{n-1} \times [0, \iota_g/2]$. This proves Proposition 5.2. \square

So it follows that, there exists $R_0 > 0$, large, such that for all $R \geq R_0$ one has

$$(105) \quad J_P(v_{t,R}^\sigma) \leq J_P(u_t^\sigma) + 2\theta < 2^{2k/n} \frac{1}{K_0(n,k)} \quad \forall (\sigma, t) \in S^{n-1} \times [0, \iota_g/2].$$

As one checks for any $(\sigma, t) \in S^{n-1} \times [0, \iota_g/2]$, the functions $v_{t,R_0}^\sigma \neq 0$, and has support in $M \setminus B_{x_0}(\iota_g/40R_0)$. Let $\epsilon_0 > 0$ be such that $M \setminus B_{x_0}(\iota_g/40R_0) \subset M \setminus B_{x_0}(\epsilon_0)$ and we define

$$(106) \quad \Omega_{\epsilon_0} := M \setminus \overline{B_{x_0}(\epsilon_0)}.$$

Then for any $(\sigma, t) \in S^{n-1} \times [0, \iota_g/2]$ the functions $v_{t,R_0}^\sigma \in H_{k,0}^2(\Omega_{\epsilon_0}) \setminus \{0\}$. Propositions 5.1 and 5.2 yield

$$(107) \quad \lim_{t \rightarrow \iota_g/2} J_P(v_{t,R_0}^\sigma) = \frac{1}{K_0(n,k)} \quad \text{uniformly for all } \sigma \in S^{n-1}.$$

Also v_{0,R_0}^σ is a fixed function independent of σ and

$$(108) \quad v_{t,R_0}^\sigma \rightharpoonup \delta_{\exp_{x_0}(\frac{\iota_g}{2}\sigma)} \quad \text{weakly in the sense of measures as } t \rightarrow \iota_g/2.$$

We define $S_k := K_0(n,k)^{-1}$. For any $c \in \mathbb{R}$, we define the sublevel sets of the functional I_P on \mathcal{N}_{ϵ_0}

$$(109) \quad \mathcal{I}_c := \{u \in \mathcal{N}_{\epsilon_0} : I_P(u) < c\}.$$

where $\mathcal{N}_{\epsilon_0} := \{u \in H_{k,0}^2(\Omega_{\epsilon_0}) / \|u\|_{2_k^\sharp} = 1\}$.

Proposition 5.3. *Suppose $I_P(u) > \frac{1}{K_0(n,k)}$ for all $u \in \mathcal{N}_{\epsilon_0}$, then there exists $\sigma_0 > 0$ for which there exists a continuous map*

$$(110) \quad \Gamma : \mathcal{I}_{S_k + \sigma_0} \longrightarrow \overline{\Omega}_{\epsilon_0}$$

such that if $(u_i) \in \mathcal{I}_{S_k + \sigma_0}$ is a sequence such that $|u_i|_{2_k^\sharp}^2 dv_g \rightharpoonup \delta_{p_0}$ weakly in the sense of measures, for some point $p_0 \in \overline{\Omega}_{\epsilon_0}$, then

$$(111) \quad \lim_{i \rightarrow +\infty} \Gamma(u_i) = p_0.$$

Proof of Proposition 5.3: By the Whitney embedding theorem, the manifold M admits a smooth embedding into \mathbb{R}^{2n+1} . If we denote this embedding by $\mathcal{F} : M \rightarrow \mathbb{R}^{2n+1}$, then M is diffeomorphic to $\mathcal{F}(M)$ where $\mathcal{F}(M)$ is an embedded submanifold of \mathbb{R}^{2n+1} . For $u \in \mathcal{N}_{\epsilon_0}$, we define

$$(112) \quad \tilde{\Gamma}(u) := \int_{\Omega_M} \mathcal{F}(x) |u(x)|_{2_k^\sharp}^2 dv_g(x).$$

Then $\tilde{\Gamma} : \mathcal{N}_{\epsilon_0} \rightarrow \mathbb{R}^{2n+1}$ is continuous. Next we claim that for every $\epsilon > 0$ there exists a $\sigma > 0$ such that

$$(113) \quad u \in \mathcal{I}_{S_k + \sigma} \Rightarrow \text{dist}(\tilde{\Gamma}(u), \mathcal{F}(\overline{\Omega}_{\epsilon_0})) < \epsilon.$$

Suppose that the claim is not true, then there exists an $\epsilon' > 0$ and a sequence $(u_i) \in \mathcal{N}_{\epsilon_0}$, such that $\lim_{i \rightarrow +\infty} I_P(u_i) = S_k$ and $\text{dist}(\tilde{\Gamma}(u), \mathcal{F}(\bar{\Omega}_{\epsilon_0})) \geq \epsilon'$. Since there is no minimizer for I_P on \mathcal{N}_{ϵ_0} , it follows from Lemma 3.1 that for such a sequence (u_i) there exists a point $p_0 \in \bar{\Omega}_{\epsilon_0}$ such that $|u_i|^{2_k^\#} dv_g \rightharpoonup \delta_{p_0}$ weakly in the sense of measures. So $\tilde{\Gamma}(u_i) \rightarrow \mathcal{F}(p_0)$, a contradiction since $\text{dist}(\tilde{\Gamma}(u), \mathcal{F}(\bar{\Omega}_{\epsilon_0})) \geq \epsilon'$. This proves our claim.

By the Tubular Neighbourhood Theorem, the embedded submanifold $\mathcal{F}(M)$ has a tubular neighbourhood \mathcal{U} in \mathbb{R}^{2n+1} and there exists a smooth retraction

$$(114) \quad \pi : \mathcal{U} \longrightarrow \mathcal{F}(M).$$

Choose an $\epsilon_0 > 0$ small so that $\{y \in \mathbb{R}^{2n+1} : \text{dist}(y, \mathcal{F}(M)) < \epsilon_0\} \subset \mathcal{U}$. Then from our previous claim it follows that, there exists $\sigma_0 > 0$ such that

$$(115) \quad u \in \mathcal{I}_{S_k + \sigma_0} \Rightarrow \tilde{\Gamma}(u) \in \mathcal{U}.$$

We define

$$(116) \quad \Gamma_M(u) = \mathcal{F}^{-1} \circ \pi \left(\int_M \mathcal{F}(x) |u(x)|^{2_k^\#} dv_g(x) \right).$$

Then the map $\Gamma_M : \mathcal{I}_{S_k + \sigma_0} \rightarrow M$ is continuous. Similarly as in our previous claim we have: for every $\epsilon > 0$ small there exists $\delta > 0$ such that

$$(117) \quad u \in \mathcal{I}_{S_k + \delta} \Rightarrow d_g(\Gamma_M(u), \bar{\Omega}_{\epsilon_0}) < \epsilon.$$

Let $\pi^{\bar{\Omega}_{\epsilon_0}} : M \setminus B_{x_0}(\epsilon_0/2) \longrightarrow \bar{\Omega}_{\epsilon_0}$ be a retraction. Choose an $\epsilon' > 0$ small so that $\{p \in M : d_g(p, \bar{\Omega}_{\epsilon_0}) < \epsilon'\} \subset M \setminus B_{x_0}(\epsilon_0/2)$. Then from our claim it follows that there exists a $\delta_0 > 0$ such that $\Gamma_M(u) \in M \setminus B_{x_0}(\epsilon_0/2)$ for all $u \in \mathcal{I}_{S_k + \delta_0}$. So for $u \in \mathcal{I}_{S_k + \delta_0}$ we define $\Gamma(u) := \pi^{\bar{\Omega}_{\epsilon_0}} \circ \Gamma_M(u)$. Then the map Γ satisfies the hypothesis of the proposition. This proves Proposition 5.3. This proposition is in the spirit of Proposition 4.4 of [25] \square

Now we proceed to prove the first part of Theorem 1. By the regularity result obtained in Theorem 8.3, it is sufficient to show the existence of a non-trivial $H_{k,0}^2(\Omega_{\epsilon_0})$ weak solution to the equation (see (145) for the definition)

$$(118) \quad \begin{cases} Pu = |u|^{2_k^\# - 2} u & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1 \end{cases}$$

Suppose on the contrary the above equation only admits trivial solutions, we will show that this leads to a contradiction.

Now suppose that the functional I_P has no critical point in \mathcal{N}_{ϵ_0} , that is there is not weak solution to (118). This is equivalent to the assertion that the functional

$$(119) \quad F_P(u) = \frac{1}{2} \int_{\Omega_{\epsilon_0}} u P(u) dv_g - \frac{1}{2_k^\#} \int_{\Omega_{\epsilon_0}} |u|^{2_k^\#} dv_g.$$

does not admit a nontrivial critical point in $H_{k,0}^2(\Omega_{\epsilon_0})$.

Proposition 5.4. *If equation (118) admits only the trivial solution $u \equiv 0$, then the functional I_P satisfies the $(P.S)_c$ condition for $c \in (S_k, 2^{\frac{2k}{n}} S_k)$.*

Proof of Proposition 5.4: Let $(v_i) \in \mathcal{N}_{\epsilon_0}$ be a Palais-Smale sequence for the functional I_P such that $\lim_{i \rightarrow +\infty} I_P(v_i) = c \in (S_k, 2^{\frac{2k}{n}} S_k)$, if this exists. Define

$u_i := (I_P(v_i))^{\frac{1}{2^{\frac{2k}{n}}-2}} v_i$. Then (u_i) is a Palais-Smale sequence for the functional F_P on the space $H_{k,0}^2(\Omega_{\epsilon_0})$ such that $\lim_{i \rightarrow +\infty} F_P(u_i) \in (\frac{k}{n} S_k^{n/2k}, \frac{2k}{n} S_k^{n/2k})$. Since there is no nontrivial solution to (118), it follows from the Struwe-decomposition for polyharmonic operators by the author [26] that there exists $d \in \mathbb{N}$ non-trivial functions $u^j \in \mathcal{D}^{k,2}(\mathbb{R}^n)$, $j = 1, \dots, d$, such that upto a subsequence the following holds

$$(120) \quad F_P(u_i) = \sum_{j=1}^d E(u^j) + o(1) \quad \text{as } i \rightarrow +\infty$$

where $E(u) := \frac{1}{2} \int_{\mathbb{R}^n} (\Delta^{k/2} u)^2 dx - \frac{1}{2^{\frac{2k}{n}}} \int_{\mathbb{R}^n} |u|^{2^{\frac{2k}{n}}} dx$. The u^j 's are nontrivial solutions

in $\mathcal{D}^{k,2}(\mathbb{R}^n)$ to $\Delta^k u = |u|^{2^{\frac{2k}{n}}-2} u$ on \mathbb{R}^n or on $\{x \in \mathbb{R}^n / x_1 < 0\}$ with Dirichlet boundary condition (we refer to [26] for details). It then follows from Lemma 3 and 5 of Ge-Wei-Zhou [17] that for any j , either u^j has fixed sign and $E(u) = \frac{k}{n} S_k^{n/2k}$, or u^j changes sign and $E(u) \geq \frac{2k}{n} S_k^{n/2k}$, contradicting $\lim_{i \rightarrow +\infty} F_P(u_i) \in (\frac{k}{n} S_k^{n/2k}, \frac{2k}{n} S_k^{n/2k})$. Therefore the Palais-Smale condition holds at level $c \in (S_k, 2^{\frac{2k}{n}} S_k)$. More precisely, there is even no Palais-Smale sequence at this level. This ends the proof of Proposition 5.4. \square

Proof of Theorem 1: By the Deformation Lemma (see Theorem II.3.11 and Remark II.3.12 in the monograph by Struwe [31]), there exists a retraction $\beta : \mathcal{I}_{S_k+4\theta} \rightarrow \mathcal{I}_{S_k+\sigma_0}^k$, where σ_0 is as given in Proposition 5.3. Let $r_{\mathcal{N}_{\epsilon_0}} : H_{k,0}^2(\Omega_{\epsilon_0}) \setminus \{0\} \rightarrow \mathcal{N}_{\epsilon_0}$ be the projection given by $u \mapsto \frac{u}{\|u\|_{L^{2^{\frac{2k}{n}}}}}$. Consider the map $h : S^{n-1} \times [0, \iota_g/2] \rightarrow \overline{\Omega}_{\epsilon_0}$ given by

$$(121) \quad h(\sigma, t) := \begin{cases} \Gamma \circ \beta(r_{\mathcal{N}_{\epsilon_0}}(v_{t,R_0}^\sigma)) & \text{for } t < \iota_g/2 \\ \sigma_{\iota_g/2}^M & \text{for } t = \iota_g/2 \end{cases}$$

where $\sigma_t^M := \exp_{x_0}(t\sigma)$. This map is well defined and continuous by Proposition 5.3 and there exists $p_0 \in \overline{\Omega}_{\epsilon_0}$ such that

$$(122) \quad h(\sigma, t) = \begin{cases} p_0 & \text{for } t = 0 \\ \exp_{x_0}(\frac{\iota_g}{2}\sigma) & \text{for } t = \iota_g/2 \end{cases}$$

So we obtain a homotopy of the embedded $(n-1)$ -dimensional sphere $\{\exp_{x_0}(\frac{\iota_g}{2}\sigma) : \sigma \in S^{n-1}\}$ to a point in Ω_{ϵ_0} , which is a contradiction to our topological assumption. This proves Theorem 1 for potentially sign-changing solutions.

6. POSITIVE SOLUTIONS

This section is devoted to the second part of Theorem 1, that is the existence of positive solutions. The proof is very similar to the proof of Theorem 1 with no restriction on the sign. We just stress on the specificities and refer to the proof of Theorem 1 everytime it is possible. We let $\Omega_M \subset M$ be any smooth n -dimensional submanifold of M , possibly with boundary. In the sequel, we will

either take $\Omega_M = M$, or $M \setminus \overline{B_{x_0}(\epsilon_0)}$. For $u \in H_{k,0}^2(\Omega_M)$, we define $u^+ := \max\{u, 0\}$, $u^- := \max\{-u, 0\}$ and

$$(123) \quad \mathcal{N}_+ := \{u \in H_{k,0}^2(\Omega_M) : \int_{\Omega_M} (u^+)^{2_k^\#} dv_g = 1\}.$$

which is a codimension 1 submanifold of $H_{k,0}^2(\Omega_M)$. Any critical point $u \in H_{k,0}^2(\Omega_M)$ of I_g on \mathcal{N}_+ is a weak solution to

$$(124) \quad Pu = u_+^{2_k^\# - 1} \text{ in } \Omega_M ; D^\alpha u = 0 \text{ on } \partial\Omega_M \text{ for } |\alpha| \leq k-1.$$

Consider the Green's function G_P associated to the operator P with Dirichlet boundary condition on the smooth domain $\Omega_M \subsetneq M$, which is a function $G_P : \Omega_M \times \Omega_M \setminus \{(x, x) : x \in \Omega_M\} \rightarrow \mathbb{R}$ such that

- (i) For any $x \in \Omega_M$, the function $G_P(x, \cdot) \in L^1(\Omega_M)$
- (ii) For any $\varphi \in C^\infty(\overline{\Omega_M})$ such that $D^\alpha \varphi = 0$ on $\partial\Omega_M$ for all $|\alpha| \leq k-1$, we have that

$$(125) \quad \varphi(x) = \int_{\Omega_M} G_P(x, y) P\varphi(y) dv_g(y).$$

Lemma 6.1. *Let $(u_i) \in \mathcal{N}_+$ be a minimizing sequence for I_g on \mathcal{N}_+ . Then*

- (i) *Either there exists $u_0 \in \mathcal{N}_+$ such that $u_i \rightarrow u_0$ strongly in $H_{k,0}^2(\Omega_M)$, and u_0 is a minimizer of I_P on \mathcal{N}_+*
- (ii) *Or there exists $x_0 \in \overline{\Omega_M}$ such that $|u_i|^{2_k^\#} dv_g \rightharpoonup \delta_{x_0}$ as $i \rightarrow +\infty$ in the sense of measures. Moreover, $\inf_{u \in \mathcal{N}_+} I_P(u) = \frac{1}{K_0(n,k)}$.*

Proof of Lemma 6.1: As the functional I_g is coercive so the sequence (u_i) is bounded in $H_{k,0}^2(\Omega_M)$. We let $u_0 \in H_{k,0}^2(\Omega_M)$ such that, up to a subsequence, $u_i \rightharpoonup u_0$ weakly in $H_{k,0}^2(\Omega_M)$ as $i \rightarrow +\infty$, and $u_i(x) \rightarrow u_0(x)$ as $i \rightarrow +\infty$ for a.e. $x \in \Omega_M$. As the sequences $(u_i^+), (u_i^-)$ is bounded in $L^{2_k^\#}(\Omega_M)$ and $u_i^+(x) \rightarrow u_0^+(x)$, $u_i^-(x) \rightarrow u_0^-(x)$ for a.e. $x \in \Omega_M$, integration theory yields

$$(126) \quad u_i^+ \rightharpoonup u_0^+ \text{ and } u_i^- \rightharpoonup u_0^- \quad \text{weakly in } L^{2_k^\#}(\Omega_M) \text{ as } i \rightarrow +\infty.$$

Therefore,

$$(127) \quad \|u_0^+\|_{L^{2_k^\#}}^{2_k^\#} \leq \liminf_{i \rightarrow +\infty} \|u_i^+\|_{L^{2_k^\#}}^{2_k^\#} = 1 \quad \text{and} \quad \|u_0^-\|_{L^{2_k^\#}}^{2_k^\#} \leq \liminf_{i \rightarrow +\infty} \|u_i^-\|_{L^{2_k^\#}}^{2_k^\#}.$$

We claim that

$$(128) \quad u_i^- \rightarrow u_0^- \quad \text{strongly in } L^{2_k^\#}(\Omega_M).$$

We prove the claim. We define $v_i := u_i - u_0$. Up to extracting a subsequence, we have that $(v_i)_i \rightarrow 0$ in $H_{k-1}^2(M)$. Therefore, as $i \rightarrow +\infty$,

$$(129) \quad I_P(u_i) = \int_{\Omega_M} (\Delta_g^{k/2} v_i)^2 dv_g + I_P(u_0) + o(1).$$

And then, letting $\alpha := \inf_{u \in \mathcal{N}_+} I_P(u)$, we have that

$$\alpha = I_P(u_i) + o(1) \geq \int_{\Omega_M} (\Delta_g^{k/2} v_i)^2 dv_g + \alpha \|u_0^+\|_{L^{2_k^\#}}^2 + o(1)$$

and then

$$(130) \quad \alpha \left(1 - \|u_0^+\|_{L^{2_k^\#}}^2\right) \geq \int_{\Omega_M} (\Delta_g^{k/2} v_i)^2 dv_g + o(1)$$

as $i \rightarrow +\infty$. We fix $\epsilon > 0$. It then follows from (11) and $(v_i)_i \rightarrow 0$ in $H_{k-1}^2(M)$ that

$$(131) \quad \alpha (K_0(n, k) + \epsilon) \left(1 - \|u_0^+\|_{L^{2_k^\#}}^2\right) \geq \|v_i\|_{L^{2_k^\#}}^2 + o(1).$$

Since $1 - a^{2_k^\#/2} \geq (1 - a^2)^{2_k^\#/2}$ for $1 \geq a \geq 0$, we get that

$$(132) \quad (\alpha (K_0(n, k) + \epsilon))^{2_k^\#/2} \left(1 - \|u_0^+\|_{L^{2_k^\#}}^2\right) \geq \|v_i\|_{L^{2_k^\#}}^{2_k^\#} + o(1).$$

Integration theory yields $\|u_i\|_{L^{2_k^\#}}^{2_k^\#} = \|v_i\|_{L^{2_k^\#}}^{2_k^\#} + \|u_0\|_{L^{2_k^\#}}^{2_k^\#} + o(1)$ as $i \rightarrow +\infty$. Therefore

$$\begin{aligned} & (\alpha (K_0(n, k) + \epsilon))^{2_k^\#/2} \left(1 - \|u_0^+\|_{L^{2_k^\#}}^2\right) + o(1) \geq \|u_i\|_{L^{2_k^\#}}^{2_k^\#} - \|u_0\|_{L^{2_k^\#}}^{2_k^\#} \\ & = \|u_i^+\|_{L^{2_k^\#}}^{2_k^\#} - \|u_0^+\|_{L^{2_k^\#}}^{2_k^\#} + \|u_i^-\|_{L^{2_k^\#}}^{2_k^\#} - \|u_0^-\|_{L^{2_k^\#}}^{2_k^\#} = 1 - \|u_0^+\|_{L^{2_k^\#}}^{2_k^\#} + \|u_i^-\|_{L^{2_k^\#}}^{2_k^\#} - \|u_0^-\|_{L^{2_k^\#}}^{2_k^\#} \end{aligned}$$

Then $\|u_i^-\|_{L^{2_k^\#}}^{2_k^\#} = \|u_i^- - u_0^-\|_{L^{2_k^\#}}^{2_k^\#} + \|u_0^-\|_{L^{2_k^\#}}^{2_k^\#} + o(1)$ as $i \rightarrow +\infty$ yields

$$\begin{aligned} (133) \quad & \left((\mu (K_0(n, k) + \epsilon))^{2_k^\#/2} - 1\right) \left(1 - \|u_0^+\|_{L^{2_k^\#}}^2\right) + o(1) \geq \|u_i^-\|_{L^{2_k^\#}}^{2_k^\#} - \|u_0^-\|_{L^{2_k^\#}}^{2_k^\#} \\ (134) \quad & = \|u_i^- - u_0^-\|_{L^{2_k^\#}}^{2_k^\#} + o(1). \end{aligned}$$

Since $\alpha K_0(n, k) \leq 1$ and $\epsilon > 0$ is arbitrary small, we get (128). This proves the claim.

We define $\mu_i := (\Delta_g^{k/2} u_i)^2 dv_g$ and $\nu_i = |u_i|^{2_k^\#} dv_g$ for all i . Up to a subsequence, we denote respectively by μ and ν their limits in the sense of measures. It follows from the concentration-compactness Theorem 4 that,

$$(135) \quad \nu = |u_0|^{2_k^\#} dv_g + \sum_{j \in \mathcal{J}} \alpha_j \delta_{x_j} \text{ and } \mu \geq (\Delta_g^{k/2} u_0)^2 dv_g + \sum_{j \in \mathcal{I}} \beta_j \delta_{x_j}$$

where $J \subset \mathbb{N}$ is at most countable, $(x_j)_{j \in J} \in M$ is a family of points, and $(\alpha_j)_{j \in J} \in \mathbb{R}_{\geq 0}$, $(\beta_j)_{j \in J} \in \mathbb{R}_{\geq 0}$ are such that $\alpha_j^{2/2_k^\#} \leq K_0(n, k) \beta_j$ for all $j \in J$. Since $u_i^- \rightarrow u_0^-$ strongly in $L^{2_k^\#}(M)$, we then get that

$$(136) \quad |u_i^+|^{2_k^\#} dv_g \rightharpoonup |u_0^+|^{2_k^\#} dv_g + \sum_{j \in \mathcal{J}} \alpha_j \delta_{x_j}$$

as $i \rightarrow +\infty$ in the sense of measures. The sequel is similar to the proof of Lemma 3.1. We omit the details. This completes the proof of Lemma 6.1. \square

Lemma 6.2. *We assume that there is no nontrivial solution to (124). Then the functional I_P satisfies the $(P.S)_c$ condition on \mathcal{N}_+ for $c \in (S_k, 2^{\frac{2k}{n}} S_k)$ if the equation.*

Proof of Lemma 6.2: This is equivalent to prove that the functional

$$(137) \quad F_P^+(u) = \frac{1}{2} \int_{\Omega_{\epsilon_0}} u P(u) dv_g - \frac{1}{2_k^\#} \int_{\Omega_{\epsilon_0}} (u^+)^{2_k^\#} dv_g.$$

satisfies the $(P.S)_c$ condition on $H_{k,0}^2(\Omega_{\epsilon_0})$ for $c \in (\frac{k}{n} S_k^{n/2k}, \frac{2k}{n} S_k^{n/2k})$. Let (u_i) be a Palais-Smale sequence for the functional F_P^+ on the space $H_{k,0}^2(\Omega_{\epsilon_0})$. Then, as $v \in H_{k,0}^2(\Omega_{\epsilon_0})$ goes to 0,

$$(138) \quad \int_{\Omega_{\epsilon_0}} u_i P_g^k(v) dv_g - \int_{\Omega_{\epsilon_0}} (u_i^+)^{2_k^\#-1} v dv_g = o(\|v\|_{H_k^2}).$$

Without loss of generality we can assume that $u_i \in C_c^\infty(\Omega_{\epsilon_0})$ for all i . Let $\varphi_i \in C^\infty(\overline{\Omega_{\epsilon_0}})$ be the unique solution of the equation

$$(139) \quad \begin{cases} P_g^k \varphi_i = (u_i^+)^{2_k^\#-1} & \text{in } \Omega_{\epsilon_0} \\ D^\alpha \varphi_i = 0 & \text{on } \partial\Omega_{\epsilon_0} \text{ for } |\alpha| \leq k-1. \end{cases}$$

The existence of such φ_i is guaranteed by Theorem 8.2. It then follows from Green's representation formula that

$$(140) \quad \varphi_i(x) = \int_{\Omega_M} G_P(x, y) (u_i^+(y))^{2_k^\#-1} dv_g(y) \geq 0.$$

for all $x \in \Omega_M$. Note that the sequence (φ_i) is bounded in $H_{k,0}^2(\Omega_{\epsilon_0})$. It follows from (138) that $\varphi_i = u_i + o(1)$, where $o(1) \rightarrow 0$ in $H_{k,0}^2(\Omega_{\epsilon_0})$ as $i \rightarrow +\infty$. And so (φ_i) is Palais-Smale sequences for the functional F_P^+ on the space $H_{k,0}^2(\Omega_{\epsilon_0})$. Therefore, since $\varphi_i \geq 0$, it is also a Palais-Smale sequence for F_P defined in (119). Since there is no nontrivial critical point for F_P^+ , using the Struwe decomposition [26] as in the proof of Proposition 5.4, we then get that $(\varphi)_i$ is relatively compact in $H_{k,0}^2(\Omega_{\epsilon_0})$, and so is (u_i) . This ends the proof of Lemma 6.1. \square

Proof of Theorem 1, positive solutions: this goes essentially as in the proof of Theorem 1, the key remark being that the functions $v_{t,R}^\sigma$ defined in (100) are nonnegative. We define $\mathcal{N}_+^{\epsilon_0} = \{u \in H_{k,0}^2(\Omega_{\epsilon_0}) : \|u^+\|_{L^{2_k^\#}} = 1\}$, where $\Omega_{\epsilon_0} = M \setminus \overline{B_{\epsilon_0}}(x_0)$ and $\epsilon_0 > 0$ was defined in (106). For $c \in \mathbb{R}$ we define the sublevel sets of the functional I_P on $\mathcal{N}_+^{\epsilon_0}$ as $\mathcal{I}_c^+ := \{u \in \mathcal{N}_+^{\epsilon_0} : I_g^k(u) < c\}$. Arguing as in the proof of Proposition 5.3, it follows from Lemma 6.1 that there exists a $\delta_0 > 0$ such that there exists $\Gamma : \mathcal{I}_{S_k+\delta_0}^+ \rightarrow \overline{\Omega_{\epsilon_0}}$ a continous map such that: If $(u_i) \in \mathcal{I}_{S_k+\delta_0}^+$ is a sequence such that $|u_i^+|^{2_k^\#} dv_g \rightharpoonup \delta_{p_0}$ weakly in the sense of measures, for some point $p_0 \in \overline{\Omega_{\epsilon_0}}$, then $\lim_{i \rightarrow +\infty} \Gamma(u_i) = p_0$.

Let $r_{\mathcal{N}_+^{\epsilon_0}} : H_{k,0}^2(\Omega_{\epsilon_0}) \setminus \{\|u^+\|_{L^{2_k^\#}} = 0\} \rightarrow \mathcal{N}_+^{\epsilon_0}$ be the map given by $u \mapsto \frac{u}{\|u^+\|_{L^{2_k^\#}}}$.

Consider the map $h : S^{n-1} \times [0, \iota_g/2] \rightarrow \overline{\Omega_{\epsilon_0}}$ given by

$$(141) \quad h(\sigma, t) = \begin{cases} \Gamma \circ \beta(r_{\mathcal{N}_+^{\epsilon_0}}(v_{t,R_0}^\sigma)) & \text{for } t < \iota_g/2 \\ \sigma_{\iota_g/2}^M & \text{for } t = \iota_g/2 \end{cases}$$

where $\beta : \mathcal{I}_{S_k+4\theta}^+ \rightarrow \mathcal{I}_{S_k+\delta_0}^+$ is a retract (we have used Lemma 6.2) and $\sigma_t^M = \exp_{x_0}(t\sigma)$. Note here that we use that $v_{t,R_0}^\sigma \geq 0$. As in the proof of Theorem 1, h is an homotopy of the embedded $(n-1)$ -dimensional sphere $\{\exp_{x_0}(\frac{t}{2}\sigma) : \sigma \in S^{n-1}\}$ to a point in Ω_{ϵ_0} , which is a contradiction to our topological assumption. So there exists a nontrivial critical point u for the functional I_P on $\mathcal{N}_+^{\epsilon_0}$, which yields a weak solution to (124). It then follows from the regularity theorem 8.3 that $u \in C^\infty(\overline{\Omega_{\epsilon_0}})$, $u > 0$, is a solution to (2). This ends the proof of Theorem 1 for positive solutions. \square

7. AN IMPORTANT REMARK

We remark that the topological condition of Theorem 1 is in general a necessary condition. Consider the n -dimensional unit sphere \mathbb{S}^n endowed with its standard round metric h_r and let P_{h_r} be the conformally invariant GJMS operator on \mathbb{S}^n . By the stereographic projection it follows that $\mathbb{S}^n \setminus \{x_0\}$ is conformal to \mathbb{R}^n . Also one has that $\mathbb{S}^n \setminus \{x_0\}$ is contractible to a point. Let Ω_{ϵ_0} be the domain in $\mathbb{S}^n \setminus \{x_0\}$ constructed as earlier in (1), and let $u \in H_{k,0}^2(\Omega_{\epsilon_0})$, $u \neq 0$ solve the equation

$$(142) \quad \begin{cases} P_{h_r} u = (u^+)^{2_k^\sharp - 1} & \text{in } \Omega_{\epsilon_0} \\ D^\alpha u = 0 & \text{on } \partial\Omega_{\epsilon_0} \quad \text{for } |\alpha| \leq k-1. \end{cases}$$

Then by the stereographic projection it follows that there exists a ball of radius R , $B_0(R)$ such that there is a nontrivial solution $v \in H_{k,0}^2(B_0(R))$ to the equation

$$(143) \quad \begin{cases} \Delta^k v = (v^+)^{2_k^\sharp - 1} & \text{in } B_0(R) \\ D^\alpha v = 0 & \text{on } \partial B_0(R) \quad \text{for } |\alpha| \leq k-1. \end{cases}$$

By a result of Boggio[6], the Green's function for the Dirichlet problem above is positive. Therefore, we get that $v > 0$ is a smooth classical solution to

$$(144) \quad \begin{cases} \Delta^k v = v^{2_k^\sharp - 1} & \text{in } B_0(R) \\ D^\alpha v = 0 & \text{on } \partial B_0(R) \quad \text{for } |\alpha| \leq k-1. \end{cases}$$

This is impossible by Pohozaev identity, see Lemma 3 of Ge-Wei-Zhou [17].

8. APPENDIX: REGULARITY

Let $f \in L^1(\Omega_M)$. We say that $u \in H_{k,0}^2(\Omega_M)$ is a weak solution of the equation $Pu = f$ in Ω_M and $D^\alpha u = 0$ on $\partial\Omega_M$ for $|\alpha| \leq k-1$, if for all $\varphi \in C_c^\infty(\Omega_M)$

$$(145) \quad \int_{\Omega_M} \Delta_g^{k/2} u \Delta_g^{k/2} \varphi \, dv_g + \sum_{\alpha=0}^{k-1} \int_{\Omega_M} A_l(g)(\nabla^l u, \nabla^l \varphi) \, dv_g = \int_{\Omega_M} f \varphi \, dv_g.$$

Now let the operator P be coercive on the space $H_{k,0}^2(\Omega_M)$, i.e there exists a constant $C > 0$ such that for all $u \in H_{k,0}^2(\Omega_M)$

$$(146) \quad \int_{\Omega_M} u P(u) \, dv_g \geq C \|u\|_{H_{k,0}^2(\Omega_M)}^2.$$

We then have

Proposition 8.1 ($(H_k^p\text{-coercivity})$).

$$(147) \quad \inf_{u \in H_k^p(\Omega_M) \setminus \{0\}} \frac{\|Pu\|_p}{\|u\|_{H_k^p}} > 0.$$

Proof of Proposition 8.1: We proceed by contradiction. If not, then there exists a sequence $(u_i) \in C_c^\infty(\Omega_M)$ such that $\|u_i\|_{H_k^p} = 1$ and $\lim_{i \rightarrow +\infty} \|Pu_i\|_p = 0$. It follows from classical estimates (see Agmon-Douglis-Nirenberg [1]) that

$$(148) \quad \|u_i\|_{H_{2k}^p(\Omega_M)} \leq C_p \left(\|Pu_i\|_{L^p} + \|u_i\|_{H_k^p} \right) = O(1).$$

So there exists $u_0 \in H_{2k,0}^p(\Omega_M)$ such that upto a subsequence $u_i \rightharpoonup u_0$ weakly in $H_{2k,0}^p(\Omega_M)$. Then $u_i \rightarrow u_0$ strongly in $H_{k,0}^p(\Omega_M)$ and so $\|u_0\|_{H_k^p} = 1$. Also u_0 weakly solves the equation $Pu_0 = 0$ in Ω_M and $D^\alpha u_0 = 0$ on $\partial\Omega_M$ for $|\alpha| \leq k-1$. It follows from standard elliptic estimates (see Agmon-Douglis-Nirenberg [1]) that $u_0 \in C^\infty(\overline{\Omega}_M)$. Then, multiplying the equation by u_0 and integrating over M , coercivity yields

$$(149) \quad C \|u_0\|_{H_k^2(\Omega_M)}^2 \leq \int_M u_0 Pu_0 \, dv_g = 0.$$

and hence $u_0 \equiv 0$, a contradiction since we have also obtained that $\|u_0\|_{H_k^p} = 1$. This proves Proposition 8.1. \square

Proposition 8.2 (Existence and Uniqueness). *Let the operator P be coercive. Then given any $f \in L^p(\Omega_M)$, $1 < p < +\infty$, there exists a unique weak solution $u \in H_{k,0}^p(\Omega_M) \cap H_{2k}^p(\Omega_M)$ to*

$$(150) \quad \begin{cases} Pu = f & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1. \end{cases}$$

The proof is classical and we only sketch it here. For $p = 2$, existence and uniqueness follows from the Riesz representation theorem in Hilbert spaces. For arbitrary $p > 1$, we approximate f in L^p by smooth compactly supported function on Ω_M . For each of these smooth functions, there exists a solution to the pde with the approximation as a right-hand-side. The coercivity and the Agmon-Douglis-Nirenberg estimates yield convergence of these solutions to a solution of the original equation. Coercivity yields uniqueness.

We now proceed to prove our regularity results. The proof is based on ideas developed by Van der Vorst [33], and also employed by Djadli-Hebey-Ledoux [12] for the case $k = 2$.

Theorem 5. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Let Ω_M be a smooth domain in M and suppose $u \in H_{k,0}^2(\Omega_M)$ be a weak solution of the equation*

$$(151) \quad \begin{cases} Pu = f(x, u) & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1 \end{cases}$$

where $|f(x, u)| \leq C|u|(1 + |u|^{2_k^\sharp - 2})$ for some positive constant C , then

$$(152) \quad u \in L^p(\Omega_M) \quad \text{for all } 1 < p < +\infty.$$

Proof of 5: We write $f(x, u) = bu$ where $|b| \leq C(1 + |u|^{2_k^\sharp - 2})$. Then $b \in L^{n/2k}(\Omega_M)$ and u solves weakly the equation

$$(153) \quad \begin{cases} Pu = bu & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1. \end{cases}$$

Step 1: We claim that for any $\epsilon > 0$ there exists $q_\epsilon \in L^{n/2k}(\Omega_M)$ and $f_\epsilon \in L^\infty(\Omega_M)$ such that

$$(154) \quad bu = q_\epsilon u + f_\epsilon, \quad \|q_\epsilon\|_{L^{n/2k}(\Omega_M)} < \epsilon.$$

Now $\lim_{i \rightarrow +\infty} \int_{\{|u| \geq i\}} |b|^{n/2k} dv_g = 0$, so given any $\epsilon > 0$ we can choose i_0 such that

$$\int_{\{|u| \geq i_0\}} |b|^{n/2k} dv_g < \epsilon^{n/2k}.$$

We define $q_\epsilon := \chi_{\{|u| \geq i_0\}} b$ and $f_\epsilon := (b - q_\epsilon)u = \chi_{\{|u| < i_0\}} b$. Then, since $|b| \leq C(1 + |u|^{2_k^\sharp - 2})$, we have that $\|q_\epsilon\|_{L^{n/2k}(\Omega_M)} < \epsilon$ and $f_\epsilon \in L^\infty(M)$. This proves our claim and ends Step 1.

We rewrite (153) as

$$(155) \quad \begin{cases} Pu = q_\epsilon u + f_\epsilon & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1. \end{cases}$$

Let \mathcal{H}_ϵ be the operator defined formally as

$$(156) \quad \mathcal{H}_\epsilon u = (P_g^k)^{-1}(q_\epsilon u).$$

Then $Pu = q_\epsilon u + f_\epsilon$ becomes $u - \mathcal{H}_\epsilon u = (P_g^k)^{-1}(f_\epsilon)$.

Step 2: we claim that for any $s > 1$, \mathcal{H}_ϵ maps $L^s(\Omega_M)$ to $L^s(\Omega_M)$.

We prove the claim. Let $v \in L^s(\Omega_M)$, $s \geq 2_k^\sharp$, then $q_\epsilon v \in L^{\hat{s}}(\Omega_M)$ where $\hat{s} = \frac{ns}{n + 2ks}$, and we have by Hölder inequality

$$(157) \quad \|q_\epsilon v\|_{L^{\hat{s}}(\Omega_M)} \leq \|q_\epsilon\|_{L^{n/2k}(\Omega_M)} \|v\|_{L^s(\Omega_M)}$$

Since $\|q_\epsilon\|_{L^{n/2k}(\Omega_M)} < \epsilon$, so we have

$$(158) \quad \|q_\epsilon v\|_{L^{\hat{s}}(\Omega_M)} \leq \epsilon \|v\|_{L^s(\Omega_M)}$$

From (8.2) it follows that there exists a unique $v_\epsilon \in H_{2k}^{\hat{s}}(\Omega_M)$ such that

$$(159) \quad \begin{cases} Pv_\epsilon = q_\epsilon v & \text{in } \Omega_M \\ D^\alpha v_\epsilon = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1 \end{cases}$$

weakly. Further we have for a positive constant $C(s)$

$$(160) \quad \|v_\epsilon\|_{H_{2k}^{\hat{s}}(\Omega_M)} \leq C(s) \|q_\epsilon v\|_{L^{\hat{s}}(\Omega_M)}$$

So we obtained that

$$(161) \quad \|v_\epsilon\|_{H_{2k}^{\hat{s}}(\Omega_M)} \leq C(s) \epsilon \|v\|_{L^s(\Omega_M)}.$$

By Sobolev embedding theorem $H_{2k}^{\hat{s}}(\Omega_M)$ is continuously imbedded in $L^s(\Omega_M)$ so $v_\epsilon \in L^s(\Omega_M)$ and we have

$$(162) \quad \|v_\epsilon\|_{L^s(\Omega_M)} \leq C(s)\epsilon \|v\|_{L^s(\Omega_M)}.$$

In other words, for any $s \geq 2_k^\sharp$ the operator \mathcal{H}_ϵ acts from $L^s(\Omega_M)$ into $L^s(\Omega_M)$, and its norm $\|\mathcal{H}_\epsilon\|_{L^s \rightarrow L^s} \leq C(s)\epsilon$. This proves the claim and ends Step 2.

Step 3: Now let $s \geq 2_k^\sharp$ be given, then for $\epsilon > 0$ sufficiently small one has

$$(163) \quad \|\mathcal{H}_\epsilon\|_{L^s \rightarrow L^s} \leq \frac{1}{2}$$

and so the operator $I - \mathcal{H}_\epsilon : L^s(\Omega_M) \rightarrow L^s(\Omega_M)$ is invertible. We have

$$(164) \quad u - \mathcal{H}_\epsilon u = (P_g^k)^{-1}(f_\epsilon)$$

Since $u \in L^{2_k^\sharp}(\Omega_M)$ and $f_\epsilon \in L^\infty(\Omega_M)$, so $u \in L^p(\Omega_M)$ for all $1 < p < +\infty$.

This ends the proof of Theorem 5. \square

Proposition 8.3. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be a positive integer such that $2k < n$. Let $f \in C^{0,\theta}(\Omega_M)$ a Hölder continuous function. Let Ω_M be a smooth domain in M and suppose $u \in H_{k,0}^2(\Omega_M)$ be a weak solution of the equation*

$$(165) \quad \begin{cases} Pu = f|u|^{2_k^\sharp-2}u \text{ or } f(u^+)^{2_k^\sharp-1} & \text{in } \Omega_M \\ D^\alpha u = 0 & \text{on } \partial\Omega_M \text{ for } |\alpha| \leq k-1. \end{cases}$$

Then $u \in C^{2k}(\Omega_M)$, and is a classical solution of the above equation. Further if $u > 0$ and $f \in C^\infty(\Omega_M)$, then $u \in C^\infty(\Omega_M)$.

Proof of Proposition 8.3: It follows from (5) that $u \in H_{2k}^p(\Omega_M)$ for all $1 < p < +\infty$. By Sobolev imbedding theorem this implies $u \in C^{2k-1,\gamma}(\overline{\Omega}_M)$ for all $0 < \gamma < 1$. $|u|^{2_k^\sharp-2}u, (u^+)^{2_k^\sharp-1} \in C^1(\overline{\Omega}_M)$. The Schauder estimates (here again, we refer to Agmon-Douglis-Nirenberg [1]) then yield $u \in C^{2k,\gamma}(\overline{\Omega}_M)$ for all $\gamma \in (0, 1)$, and u is a classical solution.

If $u > 0$, then the right-hand-side is $u^{2_k^\sharp-1}$ and has the same regularity as u . Therefore, iterating the Schauder estimates yields $u \in C^\infty(\overline{\Omega}_M)$. This ends the proof of Proposition 8.3. \square

9. APPENDIX: LOCAL COMPARISON OF THE RIEMANNIAN NORM WITH THE EUCLIDEAN NORM

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 1$. For any point $p \in M$ there exists a local coordinate around p , $\varphi_p^{-1} : \Omega \subset \mathbb{R}^n \rightarrow M$, $\varphi(p) = 0$, such that in these local coordinates one has for all indices $i, j, k = 1, \dots, n$

$$\begin{cases} (1-\epsilon)\delta_{ij} \leq g_{ij}(x) \leq (1+\epsilon)\delta_{ij} & \text{as bilinear forms.} \\ |g_{ij}(x) - \delta_{ij}| \leq \epsilon \end{cases}$$

Here we have identified $T_p M \cong \mathbb{R}^n$ for any point $p \in M$. For example, one can take the exponential map at p : \exp_p , which is normal at p . We will let ι_g be the injectivity radius of M . Using the above local comparison of the Riemannian metric with the Euclidean metric one obtains

Lemma 9.1. *Let (M, g) be a smooth, compact Riemannian manifold of dimension n and let k be positive integer such that $2k < n$. We fix $s \geq 1$. Let $\varphi_p^{-1} : \Omega \subset \mathbb{R}^n \rightarrow M$, $\varphi(p) = 0$ be a local coordinate around p with the above mentioned properties. Then given any $\epsilon_0 > 0$ there exists $\tau \in (0, \iota_g)$, such that for any point $p \in M$, and $u \in C_c^\infty(B_0(\tau))$ one has*

$$(166) \quad (1 - \epsilon_0) \int_{\mathbb{R}^n} (\Delta^{k/2} u)^2 dx \leq \int_M (\Delta_g^{k/2} (u \circ \varphi_p))^2 dv_g \leq (1 + \epsilon_0) \int_{\mathbb{R}^n} (\Delta^{k/2} u)^2 dx$$

and

$$(167) \quad (1 - \epsilon_0) \int_{\mathbb{R}^n} |u|^s dx \leq \int_M |u \circ \varphi_p|^s dv_g \leq (1 + \epsilon_0) \int_{\mathbb{R}^n} |u|^s dx$$

Proof of Lemma 9.1: In terms of the coordinate map $\varphi_p^{-1} : \Omega \subset \mathbb{R}^n \rightarrow M$, for any $f \in C^2(M)$ we have

$$(168) \quad \Delta_g f(x) = -g^{ij}(x) \left(\frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_i \partial x_j}(x) - \Gamma_{ij}^k(\varphi(x)) \frac{\partial (f \circ \varphi^{-1})}{\partial x_k}(x) \right).$$

Since the manifold M is compact, then given any $\epsilon > 0$ there exists a $\tau \in (0, \iota_g)$ depending only on (M, g) , such that for any point $p \in M$ and for any $x \in B_0(\tau) \subset \mathbb{R}^n$ one has for all indices $i, j, k = 1, \dots, n$

$$\begin{cases} (1 - \epsilon)\delta_{ij} \leq g_{ij}(x) \leq (1 + \epsilon)\delta_{ij} & \text{as bilinear forms.} \\ |g_{ij}(x) - \delta_{ij}| \leq \epsilon \end{cases}$$

Without loss of generality we can assume that $\tau < 1$. We let $u \in C_c^\infty(\mathbb{R}^n)$ be such that $\text{supp}(u) \subset B_0(\tau)$. In the sequel, the constant C will denote any positive constant depending only on (M, g) and τ : the same notation C may apply to different constants from line to line, and even in the same line. All integrals below are taken over $B_0(\tau)$, and we will therefore omit to write the domain for the sake of clearness.

Case 1: k is even. We then write $k = 2m$, $m \geq 1$. Then calculating in terms of local coordinates we obtain

$$(169) \quad |\Delta_g^m (u \circ \varphi_p)(\varphi_p^{-1}(x)) - \Delta^m u(x)| \leq \epsilon |\nabla^{2m} u(x)| + C \sum_{\beta=1}^{2m-1} |\nabla^{(2m-\beta)} u(x)|$$

where C_g is a constant depending only on the metric g on M . Then we have

$$(170) \quad \left| \int (\Delta_g^m (u \circ \varphi_p)(\varphi_p^{-1}(x)))^2 dx - \int (\Delta^m u)^2 dx \right| \leq 2^2 \epsilon^2 \int |\nabla^{2m} u|^2 dx +$$

$$(171) \quad C \sum_{\beta=1}^{2m-1} \int |\nabla^{(2m-\beta)} u|^2 dx + 2\epsilon \int |\nabla^{2m} u| |\Delta^m u| dx + C \sum_{\beta=1}^{2m-1} \int |\Delta^m u| |\nabla^{(2m-\beta)} u| dx.$$

Now for any β such that $\beta \leq 2m - 1$ we have $\nabla^{(2m-\beta)} u \in \mathcal{D}^{\beta,2}(\mathbb{R}^n)$ and by Sobolev embedding theorem this implies that $|\nabla^{(2m-\beta)} u|^2 \in L^{2_\beta^*}(\mathbb{R}^n)$. Applying

the Hölder inequality we obtain

$$(172) \quad \sum_{\beta=1}^{2m-1} \int |\nabla^{(2m-\beta)} u|^2 dx \leq C \left(\sum_{\beta=1}^{2m-1} \tau^{2\beta} \right) \left(\int |\nabla^{(2m-\beta)} u|^{2_{\beta}^{\sharp}} dx \right)^{2/2_{\beta}^{\sharp}}$$

And then the Sobolev inequality gives us

$$(173) \quad \left(\int |\nabla^{(2m-\beta)} u|^{2_{\beta}^{\sharp}} dx \right)^{2/2_{\beta}^{\sharp}} \leq C \int |\nabla^{2m} u|^2 dx.$$

Applying the integration by parts formula, we obtain

$$(174) \quad \int |\nabla^{2m} u|^2 dx = \int (\Delta^m u)^2 dx.$$

So we have, since $\tau < 1$

$$(175) \quad \sum_{|\beta|=1}^{2m-1} \int |\nabla^{2m-\beta} u|^2 dx \leq C\tau \int (\Delta^m u)^2 dx.$$

Therefore, we get that

$$(176) \quad \left| \int (\Delta_g^m(u \circ \varphi_p)(\varphi_p^{-1}(x)))^2 dx - \int (\Delta^m u)^2 dx \right| \leq C(\epsilon + \tau) \int (\Delta^m u)^2 dx.$$

Now in these local coordinates one has

$$(177) \quad (1 - \epsilon)^{n/2} \int (\Delta_g^m(u \circ \varphi_p)(\varphi_p^{-1}(x)))^2 dx \leq \int_M (\Delta_g^m(u \circ \varphi_p))^2 dv_g$$

$$(178) \quad \leq (1 + \epsilon)^{n/2} \int (\Delta_g^m(u \circ \varphi_p)(\varphi_p^{-1}(x)))^2 dx.$$

So given an $\epsilon_0 > 0$ small, we first choose ϵ small and then choose a sufficiently small τ , so that for any $u \in C_c^\infty(B_0(\tau))$ we have

$$(179) \quad \left| \int (\Delta_g^m(u \circ \varphi_p))^2 dv_g - \int (\Delta^m u)^2 dx \right| \leq \epsilon_0 \int (\Delta^m u)^2 dx.$$

So we have the lemma for k even.

Case 2: k is odd. We then write $k = 2m + 1$ with $m \geq 0$. Calculating in terms of local coordinates, like in the even case, we obtain

$$(180) \quad \left| |\nabla(\Delta_g^m(u \circ \varphi_p))|^2(\varphi_p^{-1}(x)) - |\nabla(\Delta^m u)|^2(x) \right| \leq \epsilon |\nabla(\Delta^m u)|^2(x)$$

$$(181) \quad + C\epsilon |\nabla^{2m+1} u|^2(x) + C \sum_{\beta=1}^{2m} \left| \nabla^{(2m+1-\beta)} u \right|^2(x)$$

$$(182) \quad + C\epsilon |\nabla^{2m+1} u|(x) |\nabla(\Delta^m u)|(x) + C \sum_{\beta=1}^{2m} \left| \nabla^{(2m+1-\beta)} u \right|(x) |\nabla(\Delta^m u)|(x)$$

for all $x \in B_0(\tau)$. Therefore

$$\begin{aligned}
 & \left| \int |\nabla(\Delta_g^m(u \circ \varphi_p))|^2(\varphi_p^{-1}(x)) \, dx - \int |\nabla(\Delta^m u)|^2(x) \, dx \right| \leq \epsilon \int |\nabla(\Delta^m u)|^2 \, dx \\
 & + C\epsilon \int |\nabla^{2m+1} u|^2 \, dx + C \sum_{\beta=1}^{2m} \int |\nabla^{(2m+1-\beta)} u|^2 \, dx \\
 (183) \quad & + C\epsilon \int |\nabla^{2m+1} u| |\nabla(\Delta^m u)| \, dx + C \sum_{\beta=1}^{2m} \int |\nabla^{(2m+1-\beta)} u| |\nabla(\Delta^m u)| \, dx
 \end{aligned}$$

And then by calculations similar to the even case, along with the integration by parts formula, we obtain

$$(184) \quad \left| \int |\nabla(\Delta_g^m(u \circ \varphi_p))|^2(\varphi_p^{-1}(x)) \, dx - \int |\nabla(\Delta^m u)|^2(x) \, dx \right| \leq \tilde{C}_g (\epsilon + \sqrt{\tau}) \int |\nabla(\Delta^m u)|^2 \, dx.$$

Now given an $\epsilon_0 > 0$ small, we first choose ϵ small and then choose a sufficiently small τ , so that for any $u \in C_c^\infty(B_0(\tau))$ we have

$$(185) \quad \left| \int_M |\nabla(\Delta_g^m(u \circ \varphi_p))|^2 \, dv_g - \int |\nabla(\Delta^m u)|^2 \, dx \right| \leq \epsilon_0 \int |\nabla(\Delta^m u)|^2 \, dx.$$

Then one has the lemma for k odd. This ends the proof of Lemma 9.1. \square

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