This document must be cited according to its final version which is published in a conference as:

C. Afri, V. Andrieu, L. Bako, P. Dufour,
“Identification of linear systems with nonlinear Luenberger Observers”,
2015 IEEE-IFAC American Control Conference (ACC), Chicago, IL, USA,
pp. 3373-3378, July 1-3, 2015. DOI:
10.1109/ACC.2015.7171833

You downloaded this document from the CNRS open archives server, on the webpages of Pascal Dufour:
http://hal.archives-ouvertes.fr/DUFOUR-PASCAL-C-3926-2008
Identification of linear systems with nonlinear Luenberger observers

Chouaib Afri$^1$  Vincent Andrieu$^2$  Laurent Bako $^3$
Pascal Dufour$^4$

LAGEP – AMPÈRE

LAGEP, UMR 5007, UCBL1-CNRS, ACC 2015 Chicago

$^1$Ph.D. student France MENRT funding since October 2013
$^2$Supervisor
$^3$Supervisor
$^4$Ph.D Director
Outline

1. Problem description
2. Solution by Luenberger observers approach
3. Numerical illustration
4. Perspectives
Outline

1. Problem description
2. Solution by Luenberger observers approach
3. Numerical illustration
4. Perspectives
We are looking for a mathematical model that describes the dynamic behaviour in order to better supervise, diagnose or control it.

It may belong to a class in the continuous time domain:

\[
\dot{x}(t) = f(x(t), u(t), t), \quad y(t) = g(x(t), t)
\]
## Different modelling approaches

<table>
<thead>
<tr>
<th>Output measurements knowledge</th>
<th>Input measurements knowledge</th>
<th>→</th>
<th><strong>white box model</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical model structure</td>
<td>Known parameters</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Identification of linear systems with nonlinear Luenberger observers
Different modelling approaches

Output measurements knowledge
Input measurements knowledge $\rightarrow$ white box model
Physical model structure
Known parameters

Output measurements knowledge
Input measurements knowledge $\rightarrow$ gray box model
Physical model structure
Unknown parameters
Different modelling approaches

Output measurements knowledge
Input measurements knowledge → white box model
Physical model structure
Known parameters

Output measurements knowledge
Input measurements knowledge → gray box model
Physical model structure
Unknown parameters

Output measurements knowledge
Input measurements knowledge → black box model
Assumed model structure
Unknown parameters
Assume that the process has a linear dynamic described by

\[
(\Sigma) \left\{ \begin{array}{l}
\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) \\
y(t) = C(\theta)x(t),
\end{array} \right.
\]

with:

- \( u \in \mathbb{R} \) in \( L^\infty_{loc}(\mathbb{R}_+) \).
- \( y \in \mathbb{R} \).
- \( x \in \mathbb{R}^n \).
- \( \theta \in \Theta \subset \mathbb{R}^q \).
- \( A(\cdot), B(\cdot) \) and \( C(\cdot) \) are sufficiently smooth and known.
Estimation problem

Goal

- Estimate on-line the state and parameters \((x(t), \theta)\) by knowing the inputs and outputs.
Estimation problem

Goal

- Estimate on-line the state and parameters \((x(t), \theta)\) by knowing the inputs and outputs.

Method

- Construct and asymptotic observer for the augmented system.

\[
\begin{align*}
\dot{x}(t) & = A(\theta)x(t) + B(\theta)u(t) \\
\dot{\theta} & = 0 \quad \leftarrow \text{added states} \\
y(t) & = C(\theta)x(t).
\end{align*}
\]
Outline

1. Problem description
2. Solution by Luenberger observers approach
3. Numerical illustration
4. Perspectives
Synthesis steps

- Introduce an extended dynamic system controlled by the known process inputs and outputs.

\[
\dot{z} = \Lambda z + Ly , \quad \dot{w} = g(w, u) , \quad z \in \mathbb{R}^r , \quad w \in \mathbb{R}^r .
\]
Synthesis steps

- Introduce an extended dynamic system controlled by the known process inputs and outputs.

\[ \dot{z} = \Lambda z + Ly , \dot{w} = g(w, u) , z \in \mathbb{R}^r , w \in \mathbb{R}^r . \]

- Find a mapping \((x, \theta, w) \mapsto T(x, \theta, w)\) in \(C^1\) which satisfies the following ODE

\[ \dot{T}(x, \theta, w) = \Lambda T(x, \theta, w) + Ly . \]
Synthesis steps

- Introduce an extended dynamic system controlled by the known process inputs and outputs.

\[ \dot{z} = \Lambda z + Ly, \quad \dot{w} = g(w, u), \quad z \in \mathbb{R}^r, \quad w \in \mathbb{R}^r. \]

- Find a mapping \((x, \theta, w) \mapsto T(x, \theta, w)\) in \(C^1\) which satisfies the following ODE

\[ \dot{T}(x, \theta, w) = \Lambda T(x, \theta, w) + Ly. \]

- Implies the following equation

\[ \dot{e}(t) = \Lambda e(t), \]

where

\[ e(t) = z(t) - T(x(t), \theta, w(t)) \]
Synthesis steps

- If $\Lambda$ is a Hurwitz matrix then $z$ is an estimate of $T$
  \[
  \lim_{t \to +\infty} |z(t) - T(x(t), \theta, w(t))| = 0.
  \]
Synthesis steps

- If $\Lambda$ is a Hurwitz matrix then $z$ is an estimate of $T$

\[ \lim_{t \to +\infty} |z(t) - T(x(t), \theta, w(t))| = 0. \]

The nonlinear Luenberger observer is given as

\[ \begin{align*}
\dot{z} & = \Lambda z + Ly \\
\dot{w} & = g(w, u) \\
(\hat{x}, \hat{\theta}) & = T^*(z(t), w(t))
\end{align*} \]

$T^*$: is the left inverse of $T$. 
Difficulties

1- Is there an explicit expression for $T$?
### Difficulties

1. Is there an explicit expression for $T$?
2. Is $T$ injective and full rank?
Difficulties

1- Is there an explicit expression for $T$?

2- Is $T$ injective and full rank?

3- Is there an explicit expression for $T^*$?
Question 1: Existence of an explicit expression for $T$

**Theorem 1**

For any $r$-uplet of real negative elements $(\lambda_1, \ldots, \lambda_r)$ such that

$$\lambda_i \notin \left( \bigcup_{\theta \in \Theta} \sigma\{A(\theta)\} \right) \quad i = 1, \ldots, r$$

The mapping $T(x, \theta, w) = [T_1(x, \theta, w_1) \ldots T_r(x, \theta, w_r)]^\top$ is a solution.

with

$$T_i : \mathbb{R}^n \times \Theta \times \mathbb{R} \rightarrow \mathbb{R}
\begin{align*}
(x, \theta, w_i) & \rightarrow T_i(x, \theta, w_i) = C(\theta)(A(\theta) - \lambda_i I_n)^{-1}[x - B(\theta)w_i],
\end{align*}$$

and

$$g : \mathbb{R}^r \times \mathbb{R} \mapsto \mathbb{R}^r
\begin{align*}
(w, u) & \mapsto g(w, u) = \Lambda w + Lu,
\end{align*}$$

$\Lambda = \text{Diag}\{\lambda_1, \ldots, \lambda_r\}; \quad L = 1_r.$
Question 2: Is $T$ injective and full rank?

$$T(x, \theta, w) = \begin{bmatrix} T_1(x, \theta, w_1) & \cdots & T_r(x, \theta, w_r) \end{bmatrix}^\top$$
Question 2: Is $T$ injective and full rank?

$$T(x, \theta, w) = \left[ T_1(x, \theta, w_1) \ldots T_r(x, \theta, w_r) \right]^\top$$

$T_i : \mathbb{R}^n \times \Theta \times \mathbb{R} \rightarrow \mathbb{R}$

$(x, \theta, w_i) \rightarrow T_i(x, \theta, w_i) = C(\theta)(A(\theta) - \lambda_i I_n)^{-1}[x - B(\theta)w_i]$,
Question 2: Is $T$ injective and full rank?

$$T(x, \theta, w) = \begin{bmatrix} T_1(x, \theta, w_1) & \ldots & T_r(x, \theta, w_r) \end{bmatrix}^T$$

$$T_i : \mathbb{R}^n \times \Theta \times \mathbb{R} \rightarrow \mathbb{R}$$

$$(x, \theta, w_i) \rightarrow T_i(x, \theta, w_i) = C(\theta)(A(\theta) - \lambda_i I_n)^{-1}[x - B(\theta)w_i],$$

Is it an injective and full rank function?
Question 2: Is $T$ injective and full rank?

\[
T(x, \theta, w) = \left[ T_1(x, \theta, w_1) \cdots T_r(x, \theta, w_r) \right]^\top
\]

$T_i : \mathbb{R}^n \times \Theta \times \mathbb{R} \rightarrow \mathbb{R}$

$(x, \theta, w_i) \rightarrow T_i(x, \theta, w_i) = C(\theta)(A(\theta) - \lambda_i I_n)^{-1}[x - B(\theta)w_i],$

Is it an injective and full rank function?

**Answer**

If the input makes the extended system observable, then by choosing the eigenvalues $\lambda_i$ sufficiently large the function $T$ is injective and full rank after a transient.
Question 2: Is $T$ injective and full rank?

Let the mapping $H_r : \Theta \rightarrow \mathbb{R}^{r \times n}$ be defined as

$$
\theta \mapsto H_r(\theta) = \begin{bmatrix}
C(\theta) \\
C(\theta)A(\theta) \\
\vdots \\
C(\theta)A(\theta)^{r-1}
\end{bmatrix},
$$

Question 2: Is $T$ injective and full rank?

Let the mapping $H_r : \Theta \rightarrow \mathbb{R}^{r \times n}$ be defined as

$$\theta \mapsto H_r(\theta) = \begin{bmatrix} C(\theta) \\ C(\theta)A(\theta) \\ \vdots \\ C(\theta)A(\theta)^{r-1} \end{bmatrix},$$

and $G_r : \Theta \rightarrow \mathbb{R}^{r \times r}$ be defined as

$$\theta \mapsto G_r(\theta) = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ C(\theta)B(\theta) & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ C(\theta)A(\theta)B(\theta) & C(\theta)B(\theta) & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & 0 \\ C(\theta)A(\theta)^{r-2}B(\theta) & C(\theta)A(\theta)^{r-3}B(\theta) & \cdots & \cdots & C(\theta)B(\theta) & 0 \end{bmatrix},$$

then

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(r-1)}(t) \end{bmatrix} = H_r(\theta)x + G_r(\theta)v$$
Question 2: Is $T$ injective and full rank?

Assumption (Uniform differential observability)

There exist two compact subsets $C_{\theta} \in \Theta$, $C_{x} \in \mathbb{R}^n$, an integer $r$ and $U_r$ a bounded subset of $\mathbb{R}^{r-1}$ such that the mapping $\mathcal{H}_r$ is injective and full rank

$$\mathcal{H}_r(x, \theta, v) = H_r(\theta)x + G_r(\theta)v,$$

for all $(x, \theta)$ and $(x^*, \theta^*)$ both in $\text{Cl}(C_{\theta}) \times \text{Cl}(C_{x})$ and all $v$ in $U_r$.

Or, there exist $L > 0$ such that

$$|\mathcal{H}_r(x^*, \theta^*, v) - \mathcal{H}_r(x, \theta, v)| \geq L \left| \begin{bmatrix} x - x^* \\ \theta - \theta^* \end{bmatrix} \right|.$$
Question 2: Is $T$ injective and full rank?

**Theorem 2**

- Assume the assumption holds. Let $u(\cdot)$ be a bounded $C^{r-2}([0, +\infty])$ function with bounded $r-2$ first derivatives $\bar{u}^{(r-2)}(\cdot)$.
- For all $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_r)$, for all positive time $\tau$ and all $t_1 \geq \tau$, if $\bar{u}(t_1)$ is in $U_r$ then for all $(x, \theta)$ and $(x^*, \theta^*)$ in $C_x \times C_\theta$, $T$ is injective and full rank.

Or, there exist two positive real numbers $k^*$ and $L_T$ such that for all $k > k^*$

$$|T(x, \theta, w(t_1)) - T(x^*, \theta^*, w(t_1))| \geq \frac{L_T}{k^r} \left| \begin{bmatrix} x - x^* \\ \theta - \theta^* \end{bmatrix} \right|.$$  

With the mapping $T$ is defined by taking $\lambda_i = k\tilde{\lambda}_i$.  

Question 2: Is $T$ injective and full rank?
Question 3: Existence of an explicit expression for $T^*$

The general form of $T^*$ is the optimization of the following criteria

$$(\hat{x}(t), \hat{\theta}(t)) = \arg\min_{x, \theta} \| T(x(t), \theta, w(t)) - z(t) \|_2^2$$

$T(x, \theta, w)$ is nonlinear $\Rightarrow$ iterative optimization methods.
Question 3: Existence of an explicit expression for $T^*$

The general form of $T^*$ is the optimization of the following criteria

$$(\hat{x}(t), \hat{\theta}(t)) = \arg\min_{x, \theta} \| T(x(t), \theta, w(t)) - z(t) \|^2_2$$

$T(x, \theta, w)$ is nonlinear $\Rightarrow$ iterative optimization methods.

We are looking for an explicit expression of $T^*$. 
Question 3: Existence of an explicit expression for $T^*$

Let the canonical observable structure

$$A(\theta) = \begin{bmatrix}
-\theta_{a1} & 1 & 0 & \cdots & 0 \\
-\theta_{a2} & 0 & 1 & \cdots & 0 \\
& \vdots & \vdots & \ddots & \vdots \\
& \vdots & \vdots & \ddots & 1 \\
-\theta_{an} & 0 & 0 & \cdots & 0 \\
\end{bmatrix} \in \mathbb{R}^{n \times n},$$

$$C(\theta) = C = \begin{bmatrix}
1 & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{R}^{1 \times n},$$

$$B(\theta) = B(\theta_b) = \begin{bmatrix}
\theta_{b1} & \theta_{b2} & \cdots & \theta_{bn}
\end{bmatrix}^\top \in \mathbb{R}^{n \times 1},$$

All strictly proper linear SISO systems can be written in this form.
Question 3: Existence of an explicit expression for $T^*$

For all $i = 1, \ldots, r$, we have

$$z_i(t) = T_i(\hat{x}(t), \hat{\theta}, w_i(t)) = C(A(\hat{\theta}) - \lambda_i I_n)^{-1}[\hat{x}(t) - B(\hat{\theta})w_i(t)]$$

By using the **Kronecker algebra**, we can get the following expression

$$z_i = \begin{bmatrix} z_i V_i^T & V_i^T \\ \end{bmatrix} - (w_i^T \otimes V_i^T) P_i(z_i, w_i) \begin{bmatrix} \hat{\theta}_a \\ \hat{x} \\ \hat{\theta}_b \end{bmatrix}$$

The solution is given by

$$(\hat{\theta}_a(t), \hat{x}(t), \hat{\theta}_b(t))^T = (P(z(t), w(t))^\top P(z(t), w(t)))^{-1} P(z(t), w(t))^\top z(t)$$

where: $P(z, w) = [P_1(z_1, w_1)^\top, \ldots, P_r(z_r, w_r)^\top]^\top \in \mathbb{R}^{r \times (2n+1)}$ and

$$V_i = \begin{bmatrix} 1 \\ \vdots \\ \lambda_i \\ \ldots \\ 1 \\ \lambda_i^n \end{bmatrix}^\top \in \mathbb{R}^n, \quad r \geq 4n - 1$$
Question 3: Existence of an explicit expression for $T^*$

**Question?**

Under which conditions on $T(x, \theta, w)$ and $\lambda_i$ is the matrix $P$ full rank column?
Question 3: Existence of an explicit expression for $T^*$

**Question?**

Under which conditions on $T(x, \theta, w)$ and $\lambda_i$ is the matrix $P$ full rank column?

**Proposition**

If the matrices $A(\theta)$, $B(\theta)$ and $C(\theta)$ have the observable form, if the $\lambda_i$’s are different from $A(\theta)$ eigenvalues and if the dimension of $T$ $r \geq 4n - 1$, then for any $(z, x, \theta, w)$ such that $z = T(x, \theta, w)$ and

$$\text{rank} \left( \frac{\partial T}{\partial (x, \theta)}(x, \theta, w) \right) = r$$

the matrix $P$ is a full rank column.
Question 3: Existence of an explicit expression for $T^*$

Result

- When $z(t)$ is in $\text{Im}(T)$, the Luenberger observer

\[
\begin{cases}
\dot{z}(t) = \Lambda z(t) + Ly(t) \\
\dot{w}(t) = \Lambda w(t) + Lu(t) \\
(\hat{x}(t), \hat{\theta}(t)) = (P^T(w(t), z(t))P(w(t), z(t)))^{-1} P^T z(t),
\end{cases}
\]

is well defined.
Question 3: Existence of an explicit expression for $T^*$

Result

- When $z(t)$ is in $\text{Im}(T)$, the Luenberger observer

$$
\begin{align*}
\dot{z}(t) &= \Lambda z(t) + Ly(t) \\
\dot{w}(t) &= \Lambda w(t) + Lu(t) \\
(\hat{x}(t), \hat{\theta}(t)) &= (P^T(w(t), z(t))P(w(t), z(t)))^{-1} P^T z(t),
\end{align*}
$$

is well defined.

- In the transient phase we can not guarantee that $P$ is of full rank column. If it is numerically not the case, we keep the old values of the unknown parameters.
Outline

1. Problem description
2. Solution by Luenberger observers approach
3. Numerical illustration
4. Perspectives
Black box model

\[ \dot{x}(t) = \begin{bmatrix} -\theta_{a1} & 1 \\ -\theta_{a2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} \theta_{b1} \\ \theta_{b2} \end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \]

We want to estimate \( \theta_{a1}, \theta_{a2}, \theta_{b1}, \theta_{b2} \) and state \( x_2 \) from knowledge of signals \( u(t) \) and \( y(t) \).

| (z(0), w(0)) = (0, 0) |
| (x_1(0), x_2(0), \theta_{a1}(0), \theta_{a2}(0), \theta_{b1}(0), \theta_{b2}(0)) = (0, 0.5, 2, 3, 1, -1) |
| (\hat{x}_1(0), \hat{x}_2(0), \hat{\theta}_{a1}(0), \hat{\theta}_{a2}(0), \hat{\theta}_{b1}(0), \hat{\theta}_{b2}(0)) = (0, 0, 0, 0, 0, 0) |

**Table:** Initial configuration of the system and the observer states.
Simulation without noise effect

Figure: Estimation of parameters $\theta_{a1}$ and $\theta_{a2}$.
Simulation without noise effect

Figure: Estimation of parameters $\theta_{b1}$ and $\theta_{b2}$. 
Simulation without noise effect

**Figure:** Comparison between estimated output $\hat{y}(t)$ and real output $y(t)$. 

\[ \text{Output } y(t) \text{ vs } \hat{y}(t) \]

\[ \text{State } \hat{x}_2(t) \]

\[ \text{Time (s)} \]

\[ \text{Estim} \]

\[ \text{Real} \]
Simulation with output added noise of 5\%

Figure: Estimation of parameters $\theta_{a1}$ and $\theta_{a2}$. 

<table>
<thead>
<tr>
<th>Matrix A(\theta) parameters</th>
<th>Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\theta}_{a1}$</td>
<td></td>
</tr>
<tr>
<td>$\hat{\theta}_{a2}$</td>
<td></td>
</tr>
<tr>
<td>$\theta_{a1}$</td>
<td></td>
</tr>
<tr>
<td>$\theta_{a2}$</td>
<td></td>
</tr>
</tbody>
</table>
Simulation with output added noise of 5%

Figure: Estimation of parameters $\theta_{b1}$ and $\theta_{b2}$. 
Simulation with output added noise of 5%

Figure: Comparison between estimated output $\hat{y}(t)$ and real output $y(t)$. 
Outline

1. Problem description
2. Solution by Luenberger observers approach
3. Numerical illustration
4. Perspectives
Perspectives

- Numerical comparison with other approaches in literature.
- Study of the observer robustness with respect to noise.
- Study of the persistency excitation of input and observer order.
- Application of this approach on a real system.
THANK YOU