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Twisted waveguide with a Neumann window

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Dedicated to Pavel Exner on the occasion of his 70th birthday

Abstract

This paper is concerned with the study of the existence/non-existence of the discrete spectrum of the Laplace operator on a domain of $\mathbb{R}^3$ which consists in a twisted tube. This operator is defined by means of mixed boundary conditions. Here we impose Neumann Boundary conditions on a bounded open subset of the boundary of the domain (the Neumann window) and Dirichlet boundary conditions elsewhere.

classification Primary 81Q10; Secondary 47F05.

keywords Waveguide, mixed boundary conditions, twisting.

1 Introduction

In this work, we would like to study the influence of a geometric twisting on trapped modes which occur in certain waveguides. Here the waveguide consists in a straight tubular domain $\Omega_0 := \mathbb{R} \times \omega$ having a Neumann window on its boundary $\partial \Omega_0$.

The cross section $\omega$ is supposed to be an open bounded connected subset of $\mathbb{R}^2$ of diameter $d > 0$ which is not rotationally invariant. Moreover $\omega$ is supposed to have smooth boundary $\partial \omega$.

It can be shown that the Laplace operator associated to such a straight tube has bound states [8].

Let us introduce some notations. Denote by $\mathcal{N}$ the Neumann window. It is an open bounded subset of the boundary $\partial \Omega_0$. Let $\mathcal{D}$ be its complement set in $\partial \Omega_0$. When $\mathcal{N}$ is an annulus of size $l > 0$ we will denote it by,

$$\mathcal{A}_a(l) := I_0(l) \times \partial \omega, \; I_0(l) := (a, l + a), \; a \in \mathbb{R}.$$
Consider first the self-adjoint operator $H_{N0}^\mathcal{N}$ associated to the following quadratic form. Let $D(Q^\mathcal{N}) = \{ \psi \in H^1(\Omega_0) \mid \psi|_D = 0 \}$ and for $\psi \in D(Q^\mathcal{N})$,

$$Q^\mathcal{N}(\psi) = \int_{\Omega_0} |\nabla \psi|^2 dx$$

i.e. the Laplace operator defined on $\Omega_0$ with Neumann boundary conditions (NBC) on $\mathcal{N}$ and Dirichlet boundary conditions (DBC) on $D$ [5, 11].

It is actually shown in the Section 2 of this paper that if $\mathcal{N}$ contains an annulus of size $l$ large enough then $H_{N0}^\mathcal{N}$ has at least one discrete eigenvalue. In fact it is proved in [8] that this holds true if $\mathcal{N}$ contains an annulus of any size $l > 0$.

The question we are interested in is the following: is it possible that the discrete spectrum of $H_{N0}^\mathcal{N}$ disappears when we apply a geometric twisting on the guide? This question is motivated by the results of [6, 10] where it is shown that this phenomenon occurs in some bent tubes when they are subjected to a twisting defined from an angle function $\theta$ having a derivative $\dot{\theta}$ with a compact support. In this paper we consider the situation described above which is very different from the one of [6, 10].

Let us now define the twisting [4, 7]. Choose $\theta \in C^1_c(\mathbb{R})$ and introduce the diffeomorphism

$$\mathcal{L} : \Omega_0 \to \mathbb{R}^3, \quad (s, t_2, t_3) \mapsto \left( s, t_2 \cos \theta(s) - t_3 \sin \theta(s), t_2 \sin \theta(s) + t_3 \cos \theta(s) \right).$$

The twisted tube is given by $\Omega_\theta := \mathcal{L}(\Omega_0)$. Let $D(Q_{\theta}^\mathcal{N}) = \{ \psi \in H^1(\Omega_\theta) \mid \psi|_{\mathcal{L}(D)} = 0 \}$ and consider the following quadratic form

$$Q_{\theta}^\mathcal{N}(\psi) := \int_{\Omega_\theta} |\nabla \psi|^2 dx, \quad \psi \in D(Q_{\theta}^\mathcal{N}).$$

Through unitary equivalence, we then have to consider

$$q_{\theta}^\mathcal{N}(\psi) := Q_{\theta}^\mathcal{N}(\psi \mathcal{L}^{-1}) = \| \nabla' \psi \|^2 + \| \partial_s \psi + \dot{\theta} \partial_\tau \psi \|^2,$$

$\psi \in D(q_{\theta}^\mathcal{N}) := \{ \psi \in H^1(\Omega_\theta) \mid \psi|_D = 0 \}$ and where

$$\nabla' := l(\partial_{t_2}, \partial_{t_3}), \quad \partial_\tau := t_2 \partial_{t_3} - t_3 \partial_{t_2}.$$ (4)

Denote by $H_{\theta}^\mathcal{N}$ the associated self-adjoint operator. It is defined as follows (see [5, 11]). Let $D(H_{\theta}^\mathcal{N}) = \{ \psi \in D(q_{\theta}^\mathcal{N}) \}, \quad H_{\theta}^\mathcal{N} \psi \in L^2(\Omega_\theta) \quad \frac{\partial \psi}{\partial n}[\mathcal{N} = 0]$ with

$$H_{\theta}^\mathcal{N} \psi = (-\Delta_\omega - (\dot{\theta} \partial_\tau + \partial_s)^2)\psi,$$ (5)

where the transverse Laplacian $\Delta_\omega := \partial_{t_2}^2 + \partial_{t_3}^2$. If $\mathcal{N} = A_\omega(l), l > 0$, we will denote these forms respectively as $Q_{\theta}^\mathcal{N}, q_{\theta}^\mathcal{N}$ and the corresponding operator as $H_{\theta}^\mathcal{N}$. If $\mathcal{N} = \emptyset$ we denote the associated operator by $H_{\theta}$.

Then the main result of this paper is
Theorem 1.1. i) Under conditions stated above on $\omega$ and $\theta$, there exists $l_{\min} := l_{\min}(\omega, d) > 0$ such as if for some $a \in \mathbb{R}$ and $l > l_{\min}$, $\mathcal{N} \cap \mathcal{A}_a(l)$ then

$$\sigma_d(H^N_{\theta}) \neq \emptyset.$$  

(6)

ii) Suppose $\theta$ is a non zero function satisfying the same conditions as in i) and has a bounded second derivative. Then there exists $d_{\max} := d_{\max}(\theta, \omega) > 0$ such that for all $0 < d < d_{\max}$ there exists $l_{\max} := l_{\max}(\omega, d, \theta)$ such as for all $0 < l < l_{\max}$, if $\mathcal{N} \subset \mathcal{A}_a(l)$ and supp$(\theta) \cap I_a(l) = \emptyset$ for some $a \in \mathbb{R}$ then

$$\sigma_d(H^N_{\theta}) = \emptyset.$$  

(7)

Roughly speaking this result implies that for $d$ small enough, the discrete spectrum disappears when the width of the Neumann window decreases.

Let us describe briefly the content of the paper. In the Section 2 we give the proof of the Theorem 1.1 i). The section 3 is devoted to the proof of the second part of the Theorem 1.1, this proof needs several steps. In particular we first establish a local Hardy inequality. This allows us to reduce the problem to the analysis of a one dimensional Schrödinger operator from which the Theorem 1.1 ii) follows. Finally in the Appendix of the paper we give partial results we use in previous sections.

2 Existence of bound states

First we prove the following. Denote by $E_1, E_2, \ldots$ the eigenvalues (transverse modes) of the Laplacian $-\Delta_\omega$ defined on $L^2(\omega)$ with DBC on $\partial \omega$. Let $\chi_1, \chi_2, \ldots$ be the associated eigenfunctions. Then we have

Proposition 2.1. $\sigma_{\text{ess}}(H^N_{\theta}) = [E_1, \infty)$.  

Proof. We know that $\sigma(H_0^\omega) = [E_1, \infty)$ see e.g. [2]. But by usual arguments [12], $H^N_{\theta} \leq H_\theta$, then

$$[E_1, \infty) \subset \sigma_{\text{ess}}(H^N_{\theta}).$$  

(8)

Let $a' \in \mathbb{R}$ and $l' > 0$ large enough such that $\mathcal{N} \subset \mathcal{A}_{a'}(l') = I_{a'}(l') \times \partial \omega$ and supp$(\theta) \subset I_{a'}(l')$. Let $H^e_{\theta}$ be the operator defined as in (5) but with additional Neumann boundary conditions on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. So $H^N_{\theta} \geq H^e_{\theta}$ and then $\sigma_{\text{ess}}(H^N_{\theta}) \subset \sigma_{\text{ess}}(H^e_{\theta})$ [12].

But $H^e_{\theta} = \bar{H}_i \oplus \bar{H}_e$. The interior operator $\bar{H}_i$ is the corresponding operator defined on $L^2(I_{a'}(l') \times \omega)$ with NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$ and DBC elsewhere on $\mathcal{A}_{a'}(l')$. By general arguments of [12] it has only discrete spectrum consequently $\sigma_{\text{ess}}(H^e_{\theta}) = \sigma_{\text{ess}}(\bar{H}_i)$.  

Now the exterior operator $\bar{H}_e$ is defined on $L^2((-\infty, a') \times \omega \cup (a' + l', \infty) \times \omega)$ with DBC on $(-\infty, a') \times \partial \omega \cup (a' + l', \infty) \times \partial \omega$ and NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. Since $\theta = 0$ for $x < a'$ and $x > a' + l'$, it is easy to see that

$$\bar{H}_e = \bigoplus_{n \geq 1} (-\partial^2 + E_n)(\chi_n \cdots)\chi_n.$$  

Hence $\sigma(\bar{H}_e) = \sigma_{\text{ess}}(\bar{H}_e) = [E_1, +\infty)$.  

The Theorem 1.1 i) follows from
**Proposition 2.2.** Under conditions of the Theorem 1.1 i), there exists \( l_{\text{min}} := l_{\text{min}}(\omega, d) > 0 \) such as for all \( l > l_{\text{min}} \) we have

\[
\sigma_d(H_l^{\theta}) \neq \emptyset.
\] (9)

**Proof.** Let \( \varphi_{l,a} \) be the following function

\[
\varphi_{l,a}(s) := \begin{cases} 
\frac{10}{l}(s-a), & \text{on } [a, a + \frac{l}{10}); \\
1, & \text{on } [a + \frac{l}{10}, a + \frac{9l}{10}); \\
\frac{10}{l}(s-l-a), & \text{on } [a + \frac{9l}{10}, a + l); \\
0, & \text{elsewhere.}
\end{cases}
\]

It is easy to see that \( \varphi_{l,a} \in D(q_l^{\theta}) \) and \( \| \varphi_{l,a} \|_2^2 = \frac{100}{l^2} | \omega | \). Let us calculate

\[
q_l^{\theta}(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|_2^2 = \| \nabla' \varphi_{l,a} \|_2^2 + \| \partial_s \varphi_{l,a} \|_2^2 - E_1 \| \varphi_{l,a} \|_2^2. \] (10)

Evidently the first term on the r.h.s of (10) is zero. For the second term on the r.h.s of (10) we get,

\[
\| \partial_s \varphi_{l,a} \|_2^2 = \frac{20}{l} | \omega |.
\]

Then

\[
q_l^{\theta}(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|_2^2 = \| \partial_s \varphi_{l,a} \|_2^2 = \frac{20}{l} | \omega |.
\] (11)

and thus if \( l \geq l_{\text{min}} := \sqrt{\frac{300}{1132}} \) we have \( q_l^{\theta}(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|_2^2 \leq 0 \) \( \square \)

**2.1 Proof of the Theorem 1.1 i)\)**

Using the same notation as in the Theorem 1.1 i), then \( H_N^{\theta} \leq H_l^{\theta} \). Moreover these operators have the same essential spectrum, then by the min-max principle the assertion follows.

**3 Absence of bound state**

In this section we want to prove the second part of the Theorem 1.1. Denote by \( \theta_m = \inf(\text{supp}(\hat{\theta})) \), \( \theta_M = \sup(\text{supp}(\hat{\theta})) \) and \( L = \theta_M - \theta_m \). Here \( L > 0 \). We first consider the case where the Neumann window is an annulus, \( \mathcal{A}_a(l) = I_a(l) \times \omega \).

**Proposition 3.1.** Suppose \( \mathcal{A}_a(l) \) is such that \( a \geq \theta_M \). Assume that conditions of the Theorem 1.1 ii) hold. Then there exists \( d_{\max} := d_{\max}(\omega, \theta) > 0 \), such that for all \( 0 < d \leq d_{\max} \) there exists \( l_{\max}(d, \theta, \omega) > 0 \) such as for all \( 0 < l \leq l_{\max} \) we have

\[
\sigma_d(H_d^{\theta}) = \emptyset.
\] (12)

**Remark 3.2.** the case where \( l + a \leq \theta_m \) follows from same arguments developed below.

This proof is based on the fact that under conditions of the Proposition 3.1, for every \( \psi \in D(q_d^{\theta}) \) it holds,

\[
Q(\psi) := q_d^{\theta}(\psi) - E_1 \| \psi \|_2^2 \geq 0.
\] (13)

The proof of (13) involves several steps.
3.1 A local Hardy inequality

The aim of this paragraph is to show a Hardy type inequality needed for the proof of the Proposition 3.1. It is the first step of the proof of (13). Let \( g \) be the following function

\[
g(s) := \begin{cases} 
0, & \text{on } I_0(l); \\
E_1, & \text{elsewhere.}
\end{cases}
\]

Choose \( p \in (\theta_m, \theta_M) \) s.t. \( \hat{\theta}(p) \neq 0 \) and let

\[
\rho(s) := \begin{cases} 
\frac{1}{1+(s-p)^2}, & \text{on } (-\infty, p]; \\
0, & \text{elsewhere.}
\end{cases}
\]

**Proposition 3.3.** Under same conditions of the Proposition 3.1, then there exists a constant \( C > 0 \) depending on \( p \) and \( \omega \) and \( \hat{\theta} \) such that for any \( \psi \in D(q_0^A) \),

\[
\| \nabla' \psi \|^2 + \| \hat{\theta} \partial_r \psi + \partial_s \psi \|^2 - \int_{\Omega_0} g(s) |\psi|^2 \, ds \, dt \geq C \int_{\Omega_0} \rho(s) |\psi|^2 \, ds \, dt.
\]

We first show the following lemma. Denote by \( \Omega_p := (-\infty, p) \times \omega. \)

**Lemma 3.4.** Under same conditions of the Proposition 3.3. Then for any \( \psi \in D(q_0^A) \) we have

\[
\int_{\Omega_p} \left| \nabla' \psi \right|^2 + \left| \hat{\theta} \partial_r \psi + \partial_s \psi \right|^2 - E_1 |\psi|^2 \, ds \, dt \geq C \int_{\Omega_p} \rho(s) |\psi|^2 \, ds \, dt.
\]

In the following we will use notations suggested in [6]. For \( A \subset \mathbb{R} \) denote by \( \chi_A \) the characteristic function of \( A \times \omega. \) Let \( \psi \in D(q_0^A) \) and define,

\[
q_A^1(\psi) := \| \chi_A \nabla' \psi \|^2 - E_1 \| \chi_A \psi \|^2, \quad q_A^2(\psi) := \| \chi_A \partial_s \psi \|^2, \quad q_{2,A}(\psi) := 2 \Re(\chi_A \hat{\theta} \partial_r \psi).
\]

Denote also by \( Q_A(\psi) = q_A^1(\psi) + q_A^2(\psi) + q_{2,A}(\psi). \) Here and hereafter we often use the fact that for any \( \psi \in D(q_0^A) \)

\[
q_A^1(\psi) \geq 0,
\]

for every \( A \subset \mathbb{R} \) such that \( A \cap I_0(l) = \emptyset. \)

**Proof.** Choose \( r > 0 \) such that \( \hat{\theta}(s) \neq 0 \) for any \( s \in [p-r,p]. \) Let \( f \) be the following function:

\[
f(s) := \begin{cases} 
0, & \text{on } (p, \infty); \\
\frac{p-s}{r}, & \text{on } (p-r, p]; \\
1, & \text{elsewhere.}
\end{cases}
\]

For any \( \psi \in D(q_0^A) \), simple estimates lead to:

\[
\int_{\Omega_p} \frac{|\psi(s,t)|^2}{1 + (s-p)^2} \, ds \, dt = \int_{\Omega_p} \frac{|\psi(s,t)f(s) + (1 - f(s))\psi(s,t)|^2}{1 + (s-p)^2} \, ds \, dt \]

\[
\leq 2 \left( \int_{\Omega_p} \frac{|f(s)\psi(s,t)|^2}{(s-p)^2} \, ds \, dt + \|\chi_{[p-r,p]}\psi\|^2 \right).
\]
Since \( f(p)\psi(p,.) = 0 \), we can use the usual Hardy inequality (see e.g. [9]), then we get,

\[
\int_{\Omega_p} \frac{\mid \psi(s, t) \mid^2}{1 + (s - p)^2} dsdt \leq 8q_2^{(-\infty, p)}(f\psi) + 2\|\chi_{(p-r, p)}\psi\|^2.
\]

(22)

Note that with our choice \([p - r, p] \cap [a, a + t] = \emptyset\). Hence to estimate the second term on the r.h.s of (22) we use the Theorem 6.5 of [10], then there exists \( \lambda_0 = \lambda_0(\theta, p, r) > 0 \) s.t. for any \( \psi \in \mathcal{D}(q_1^p) \) we have

\[
\|\chi_{(p-r, p)}\psi\|^2 \leq \frac{1}{\lambda_0} Q^{(p-r, p)}(\psi) \leq \frac{1}{\lambda_0} Q^{(-\infty, p)}(\psi).
\]

(23)

We now want to estimate the first term on the right hand side of (22). We have

\[
q_2^{(-\infty, p)}(f\psi) = \int_{\Omega_p} |\partial_s(f\psi)|^2 dsdt = q_2^{(-\infty, \theta_m)}(f\psi) + q_2^{(\theta_m, p)}(f\psi).
\]

(24)

Evidently since \( \dot{\theta} = 0 \) and \( f = 1 \) in \((-\infty, \theta_m)\), from (19), we have

\[
q_2^{(-\infty, \theta_m)}(f\psi) \leq Q^{(-\infty, \theta_m)}(\psi).
\]

(25)

In the other hand since \( f(p)\psi(p,.) = 0 \), we can apply the Lemma 4.1 of the Appendix. So for any \( 0 < \alpha < 1 \) there exists \( \gamma_{\alpha, 1} > 0 \) such that

\[
|q_2^{(\theta_m, p)}(f\psi)| \leq \gamma_{\alpha, 1} q_1^{(\theta_m, p)}(f\psi) + \alpha q_2^{(\theta_m, p)}(f\psi) + q_3^{(\theta_m, p)}(f\psi).
\]

(26)

Let \( \gamma := \max(1, \gamma_{\alpha, 1}) \). Then

\[
\gamma^{-1} |q_2^{(\theta_m, p)}(f\psi)| \leq q_1^{(\theta_m, p)}(f\psi) + \alpha \gamma^{-1} q_2^{(\theta_m, p)}(f\psi) + \gamma^{-1} q_3^{(\theta_m, p)}(f\psi).
\]

(27)

Hence with the decomposition, \( q_2^{(\theta_m, p)} = \gamma^{-1} q_2^{(\theta_m, p)} + (1 - \gamma^{-1}) q_2^{(\theta_m, p)} \) and (27) we have,

\[
Q^{(\theta_m, p)}(f\psi) \geq (1 - \gamma^{-1}) \left( q_1^{(\theta_m, p)}(f\psi) + q_2^{(\theta_m, p)}(f\psi) + q_3^{(\theta_m, p)}(f\psi) \right)
\]

\[
+ \gamma^{-1} (1 - \alpha) q_2^{(\theta_m, p)}(f\psi)
\]

(28)

and since \( q_3^{(\theta_m, p)} + q_2^{(\theta_m, p)} + q_2^{(\theta_m, p)} \geq 0 \), we arrive at,

\[
q_2^{(\theta_m, p)}(f\psi) \leq \frac{\gamma}{1 - \alpha} Q^{(\theta_m, p)}(f\psi).
\]

(29)

Now by using that \( q_1^{(\theta_m, p)}(f\psi) \leq q_1^{(\theta_m, p)}(\psi) \),

\[
\|\chi_{(\theta_m, p)}(\partial_s + \dot{\theta}\partial_r)(f\psi)\|^2 \leq 2\|\chi_{(\theta_m, p)}(\partial_s + \dot{\theta}\partial_r)\psi\|^2 + \frac{1}{r^2} \|\chi_{(p-r, p)}\psi\|^2
\]

and (23), we get,

\[
q_2^{(\theta_m, p)}(f\psi) \leq \frac{2\gamma}{1 - \alpha} Q^{(\theta_m, p)}(\psi) + \frac{1}{\lambda_0 r^2} Q^{(p-r, p)}(\psi) \leq c' Q^{(\theta_m, p)}(\psi)
\]

(30)
with \( c' = \frac{2\gamma}{(1-\alpha)(1+\lambda_0 r^2)} \). Then (25) and (30) imply
\[
q_2^{(-\infty,p)}(f\psi) \leq (1 + c') Q^{(-\infty,p)}(\psi).
\] (31)
Hence (31) and (23) prove the lemma with
\[
C^{-1} = 8(1 + c') + \frac{2}{\lambda_0}.
\] (32)

Proof of the proposition 3.3. To prove the proposition we note that for any \( \psi \in D(q'_0) \) and for \( p' \in \mathbb{R} \) we have
\[
\int_\omega \int_\rho' |\nabla\psi|^2 + |\hat{\partial}_t \psi + \partial_s \psi|^2 \, ds \, dt \geq \int_\omega \int_\rho' g(s) |\psi|^2 \, ds \, dt.
\] (33)
Then (33) with \( p' = p \) and Lemma 3.4 imply (16).

3.2 Reduction to a one dimensional problem

We now want to prove the following result,

Proposition 3.5. Under conditions of the Proposition 3.1, then a sufficient condition in order to get (13) is given by
\[
\int_\mathbb{R} |\psi'(s)|^2 + 2C\rho(s) |\psi(s)|^2 \, ds - 4E_1 \int_a^{a+l} |\psi(s)|^2 \, ds \geq 0,
\] (34)
for any \( \psi \in H^1(\mathbb{R}) \) where the constant \( C \) is defined in (32).

Remark 3.6. This proposition means that the positivity needed here is given by the positivity of the effective one dimensional Schrödinger operator on \( L^2(\mathbb{R}) \),
\[
-\frac{d^2}{ds^2} + 2C\rho(s) - 4E_1 1_{I_a(l)}.
\] (35)
where \( 1_{I_a(l)} \) is the characteristic function of \( I_a(l) \).

Proof. Evidently we have
\[
Q(\psi) = \frac{1}{2} (Q(\psi) - \int_{\Omega_0} (E_1 - g(s)) |\psi|^2 \, ds \, dt + q_0^l(\psi) - \int_{\Omega_0} g(s) |\psi|^2 \, ds \, dt),
\] (36)
where \( g \) is defined in (14). By using (16), then
\[
Q(\psi) \geq \frac{1}{2} (q_0^l(\psi) - E_1 |\psi|^2 + C \int_{\Omega_0} \rho(s) |\psi|^2 \, ds \, dt - E_1 \| \chi_{(a,a+l)} \psi \|^2)
\] (37)
Rewrite the expression of \( q_0^l \) given by (3) as follows:
\[
q_0^l(\psi) = \| \nabla' \psi \|^2 + \| \partial_s \psi \|^2 + \| \hat{\partial}_t \psi \|^2 + 2\Re (\partial_s \psi, \hat{\partial}_t \psi).
\] (38)
We estimate the last term of the r.h.s. of (38). By using the formula (49) of the Appendix, 
\[ |q_{23}(\psi)| \leq q_{23}^{(\theta_m, \theta_M)}(\psi) \leq \gamma_{\frac{1}{2}, 4}q_1^{(\theta_m, \theta_M)}(\psi) + \frac{1}{2}\tilde{q}_2^{(\theta_m, \theta_M)}(\psi) + \frac{1}{2}\tilde{q}_3^{(\theta_m, \theta_M)}(\psi) \] 
where 
\[ \gamma_{\frac{1}{2}, 4} := \gamma_{\frac{1}{2}, 4} + 4d^2 \| \hat{\theta} \|_\infty^2 \] 
with \[ \gamma_{\frac{1}{2}, 4} := \max \left\{ \frac{d\|\hat{\theta}\|_\infty}{\theta_0\sqrt{\lambda}}, \frac{d^2\|\hat{\theta}\|_\infty^2 f(L)}{\lambda_0\lambda'}, 2d^2 \| \hat{\theta} \|_\infty^2 f(L) \right\} \] 
for some constant \( \lambda > 0 \) depending only on the section \( \omega \) and \( f(L) := \max \{2 + \frac{4d^2}{L^2}, 4L^2\} \).

Hence (38) together with (39) give:
\[ q_0^l(\psi) \geq \| \nabla' \psi \|^2 + \frac{1}{2} \| \partial_s \psi \|^2 + \frac{1}{2} \| \hat{\theta}\partial_s \psi \|^2 - \gamma_{\frac{1}{2}, 4}q_1^{(\theta_m, \theta_M)}(\psi). \] 
In view of (19) we have
\[ \| \nabla' \psi \|^2 - E_1 \| \psi \|^2 \geq q_1^{(\theta_m, \theta_M)}(\psi) + q_1^{I_a(l)}(\psi) \geq q_1^{(\theta_m, \theta_M)}(\psi) - E_1 \| \chi_{(a, a+l)} \psi \|^2. \]
Thus this last inequality together with (41) in (37) give
\[ Q(\psi) \geq \frac{1}{2} \left( \frac{1}{2} \| \partial_s \psi \|^2 + \frac{1}{2} \| \hat{\theta}\partial_s \psi \|^2 \right) + C \int_{\Omega_0} \rho(s) | \psi |^2 \, ds \, dt - 2E_1 \| \chi_{(a, a+l)} \psi \|^2 \\
+ \left( 1 - \gamma_{\frac{1}{2}, 4}^l \right)q_1^{(\theta_m, \theta_M)}(\psi). \]
Now if \( 0 < d \leq d_{\text{max}} \) then \( \gamma_{\frac{1}{2}, 4} \leq 1 \) so the Proposition 3.5 follows.

### 3.3 The one dimensional Schrödinger operator

In this part, under our conditions, we want to show that the one dimensional Schrödinger operator (35) is a positive operator. In view of the Proposition 3.5 this will imply the Proposition 3.1. Here we follow a similar strategy as in [1].

**Proposition 3.7.** for all \( \varphi \in \mathcal{H}^1(\mathbb{R}) \), then there exists \( t_{\text{max}} > 0 \) such that for any \( 0 < t \leq t_{\text{max}} \) we have
\[ \int_{\mathbb{R}} | \varphi'(s) |^2 + 2C \rho(s) | \varphi(s) |^2 \, ds \geq 4E_1 \int_{I_a(l)} | \varphi(s) |^2 \, ds. \] 

**Proof.** Introduce the following function:
\[ \Phi(s) := \begin{cases} \left( \frac{\pi}{2} + \arctan (s - p) \right), & \text{if } s < p; \\
\left( \frac{\pi}{2} \right), & \text{if } s \geq p. \end{cases} \] 
where \( p \) is the same real number as in (15). So clearly \( \Phi' = \rho \). For any \( t \in I_a(l) \) and \( \varphi \in \mathcal{H}^1(\mathbb{R}) \), we have:
\[ \frac{\pi}{2} \varphi(t) = \Phi(t)\varphi(t) = \int_{-\infty}^{t} (\Phi(s)\varphi(s))' \, ds \\
= \int_{-\infty}^{t} \rho(s)\varphi(s) \, ds + \int_{-\infty}^{t} \Phi(s)\varphi'(s) \, ds \]
and since $\rho(s) = 0$ for any $s \in (p, \infty)$, we get,

$$\frac{\pi}{2} \varphi(t) = \int_{-\infty}^{p} \rho(s)\varphi(s)ds + \int_{-\infty}^{t} \Phi(s)\varphi'(s)ds.$$ \hspace{1cm} (45)

Then some straightforward estimates lead to,

$$\frac{\pi^2}{4} \varphi^2(t) \leq 2 \left( \left( \int_{-\infty}^{\rho(s)\varphi(s)ds} \right)^2 + \left( \int_{-\infty}^{t} \Phi(s)\varphi'(s)ds \right)^2 \right) \hspace{1cm} (46)$$

$$\leq 2 \left( \int_{-\infty}^{p} \rho(s)ds \int_{-\infty}^{p} \rho(s)\varphi^2(s)ds + \int_{-\infty}^{t} \Phi^2(s)ds \int_{-\infty}^{t} \varphi^2(s)ds \right).$$

By direct calculation $\int_{-\infty}^{p} \rho(s)ds = \frac{\pi}{2}$ and $\int_{-\infty}^{p} \Phi^2(s)ds + \int_{p}^{t} \Phi^2(s)ds = \pi \ln 2 + \frac{\pi^2}{4}(t - p)$. Hence we get,

$$| \varphi(t) |^2 \leq \frac{4}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \left( \frac{8 \ln 2}{\pi} + 2(t - p) \right) \int_{\mathbb{R}} | \varphi'(s) |^2 ds.$$ \hspace{1cm} (47)

We integrate both sides of (47) over $I_{a}(l)$, then

$$\int_{I_{a}(l)} | \varphi(t) |^2 dt \leq \frac{4l}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \left( \frac{8 \ln 2}{\pi} + 2(a - p) \right) l^2 \int_{\mathbb{R}} | \varphi'(s) |^2 ds$$

$$\leq c'' \int_{\mathbb{R}} 2C \rho(s)\varphi^2(s) + | \varphi'(s) |^2 ds$$

where $c'' = 2l(\frac{\pi}{4} + \frac{4 \ln 2}{\pi} + a - p) + l^2$. Finally we get,

$$4E_1 \int_{a}^{l+a} | \varphi(t) |^2 dt \leq 4E_1 c'' \int_{\mathbb{R}} 2C \rho(s) | \varphi(s) |^2 + | \varphi'(s) |^2 ds.$$ \hspace{1cm} (48)

So choose $0 < l \leq l_{\max}$ with

$$l_{\max} := \left( \frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a - p \right) + \sqrt{\left( \frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a - p \right)^2 + (4E_1)^{-1}}$$

then $4E_1 c'' \leq 1$ and the proposition 3.7 follows.

\begin{proof}

\end{proof}

\subsection{3.4 proof of the Theorem 1.1 ii) }

Under assumptions of the Theorem 1.1 ii) $H_0^{\psi} \geq H_p$. These two operators have the same essential spectrum so the Theorem 1.1 ii) is proved by applying the Proposition 3.1 and the min-max principle.

\section{4 Appendix}

In this appendix we give a slight extension of the lemma 3 of [6] which states that under our conditions, for all $\psi \in D(q_0^\alpha)$ we have for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha, \beta} > 0$ such that:

$$| q_{2,3}(\psi) | \leq \gamma_{\alpha, \beta}q_1(\psi) + \alpha q_2(\psi) + \beta q_3(\psi).$$ \hspace{1cm} (49)

Then we have
Lemma 4.1. Let \( p \in (\theta_m, \theta_M) \). For all \( \psi \in D(q_0^p) \) such that \( \psi(p,.) = 0 \), then for any \( \alpha, \beta > 0 \) there exists \( \gamma_{\alpha, \beta} > 0 \) such that:

\[
| q_{2,3}^{(\theta_m,p)}(\psi) | \leq \gamma_{\alpha, \beta} q_1^{(\theta_m,p)}(\psi) + \alpha q_2^{(\theta_m,p)}(\psi) + \beta q_3^{(\theta_m,p)}(\psi). \tag{50}
\]

Proof. Let \( \psi \in D(q_0^p) \) such that \( \psi(p,.) = 0 \). Then \( \psi \in H_0^1(\Omega_p) \). We know that we may first consider vectors \( \psi(s, t) = \chi_1(t)\phi(s, t) \), where \( \phi \in C_0^\infty(\Omega_p) \). For such a vector \( \psi \) we have,

\[
q_1^{(\theta_m,p)}(\psi) = \| \chi(\theta_m,p)\chi_1\nabla^\prime \phi \|^2, \quad q_2^{(\theta_m,p)}(\psi) = \| \chi(\theta_m,p)\chi_1\partial_s \phi \|^2 \tag{51}
\]

and

\[
q_3^{(\theta_m,p)}(\psi) = 2(\theta \chi(\theta_m,p)\chi_1\partial_r \phi, \chi_1\partial_s \phi) + 2(\theta \chi(\theta_m,p)\phi \partial_r \chi_1, \chi_1\partial_s \phi) \tag{52}
\]

By using simple estimates the first term on the r.h.s of \( (52) \) is estimated as:

\[
| 2(\theta \chi(\theta_m,p)\chi_1\partial_r \phi, \chi_1\partial_s \phi) | \leq 2 \| \theta \|_\infty \| \chi(\theta_m,p)\chi_1\nabla^\prime \phi \| \| \chi(\theta_m,p)\chi_1\partial_s \phi \|
\]

then

\[
| 2(\theta \chi(\theta_m,p)\chi_1\partial_r \phi, \chi_1\partial_s \phi) | \leq c_1 q_1^{(\theta_m,p)}(\psi) + \frac{\alpha}{2} q_2^{(\theta_m,p)}(\psi), \tag{53}
\]

where \( c_1 := \frac{\alpha}{d} d^2 \| \theta \|_\infty^2 \) and \( \alpha > 0 \).

Integrating by parts twice and using the fact that \( \theta(\theta_m) = \phi(p,.) = 0 \), the second term of the r.h.s of \( (52) \) is written as

\[
2(\theta \chi(\theta_m,p)\phi \partial_r \chi_1, \chi_1\partial_s \phi) = (\chi(\theta_m,p)\theta \phi \chi_1, \chi_1\partial_r \phi). \tag{54}
\]

Then the Cauchy Schwartz inequality implies,

\[
| (\chi(\theta_m,p)\theta \phi \chi_1, \chi_1\partial_r \phi) |^2 \leq d^2 \| \theta \|_\infty^2 q_1^{(\theta_m,p)} \| \chi(\theta_m,p)\chi_1\nabla^\prime \phi \|^2. \tag{55}
\]

Let \( p' \in \mathbb{R} \) and \( r' > 0 \) such that \( (p' - r, p') \subset (\theta_m,p) \) and for \( s \in (p' - r, p') \), \( |\theta(s)| \geq \theta_0 \) for some \( \theta_0 > 0 \). As in the proof of the Lemma 3 of [6] we have,

\[
\| \chi(\theta_m,p)\chi_1 \phi \|^2 \leq c_2 (q_2^{(\theta_m,p)}(\psi) + \theta_0^{-2} \| \chi(\theta_m,p)\chi_1 \phi \|^2) \tag{56}
\]

where \( c_2 := \max \left\{ 2 + 16(\frac{\theta_m - \theta_0}{\theta_0})^2, 4(p - \theta_m)^2 \right\} \).

Moreover, for any \( s \in \mathbb{R} \), \( \theta(s) \chi_1(\phi(.,s) \in H_0^1(\Omega_p) \), then by using the Lemma 1 of [6] there exists \( \lambda > 0 \) depending on \( \omega \) such that :

\[
\| \chi(\theta_m,p)\theta \chi_1 \phi \|^2 \leq \lambda^{-1} \left( q_1^{(\theta_m,p)}(\psi) + \| \theta \|_\infty^2 q_1^{(\theta_m,p)}(\psi) \right). \tag{57}
\]

Hence \((56)\), \((57)\) and \((54)\) give

\[
| (\chi(\theta_m,p)\theta \phi \chi_1, \chi_1\partial_r \phi) |^2 \leq \left( c_3 q_1^{(\theta_m,p)}(\psi) + \frac{\alpha}{2} q_2^{(\theta_m,p)}(\psi) + \beta q_3^{(\theta_m,p)}(\psi) \right)^2 \tag{58}
\]

where \( c_3 := \max \left\{ d^2 \| \theta \|_\infty^2 \| \text{grad} \alpha \| \alpha^{-2} c_2, d^2 \| \theta \|_\infty^2 \| \text{grad} \alpha \| \alpha^{-2} c_2, \frac{d^2 \| \theta \|_\infty^2 \alpha^{-2} c_2}{2 \theta_0^2 \lambda} \right\} \). Then \((53)\) and \((58)\) imply \((50)\) with \( \gamma_{\alpha, \beta} := c_1 + c_3 \).

Note that we can choose \( \lambda > 0 \) on \( \omega \). So that \((50)\) holds for every \( \psi \in C_0^\infty(\Omega_p) \) and by a density argument this is even true for \( \psi \in H_0^1(\Omega_p) \).
References


