Fourth-order energy-preserving locally implicit time discretization for linear wave equations
Juliette Chabassier, Sebastien Imperiale

To cite this version:
Juliette Chabassier, Sebastien Imperiale. Fourth-order energy-preserving locally implicit time discretization for linear wave equations. GDRE, Dec 2015, Aussois, France. 10.1002/nme.5130. hal-01264048

HAL Id: hal-01264048
https://hal.archives-ouvertes.fr/hal-01264048
Submitted on 28 Jan 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Fourth order energy-preserving locally implicit discretization for linear wave equations

Juliette Chabassier and Sébastien Imperiale

Abstract Time domain simulation of realistic highly heterogeneous media or strongly refined geometries can be a computational challenge when using explicit schemes because they impose a time step restriction that can be extremely penalizing. In this work, we present fourth order locally implicit schemes. The domain of interest is decomposed into several regions where different (explicit or implicit) fourth order time discretization are used. Whilst implicit schemes tolerate the use of larger time steps, they can induce greater numerical dispersion. Fourth order accuracy reduces this lack of precision, and makes this family of schemes attractive compared to other approaches as local time stepping.

1 Continuous system

We want to solve for time \( t > 0 \), the system (closed with Neumann homogeneous boundary conditions):

\[
\begin{align*}
\partial_t^2 u_0 - \nabla \cdot c^2(x) \nabla u_0 &= s_0 \quad \text{in } \Omega_0, \\
\partial_t^2 u_1 - \nabla \cdot c^2(x) \nabla u_1 &= s_1 \quad \text{in } \Omega_1, \\
u_0 &= u_1 \quad \text{on } \Gamma
\end{align*}
\] (1)

in a domain \( \Omega \) composed by disjoint sets \( \Omega = \Omega_0 \cup \Omega_1 \) separated by \( \Gamma = \partial \Omega_0 \cap \partial \Omega_1 \). \( s_0 \) and \( s_1 \) are given source terms, and \( c(x) > c_0 > 0 \) is the inhomogeneous velocity of the waves. Any solution to (1) satisfies the energy identity \( \frac{dE_{01}}{dt} = \int_{\Omega_0} s_0 \partial_t u_0 + \int_{\Omega_1} s_1 \partial_t u_1 \), where :

Juliette Chabassier
Magique 3D team, Inria Bordeaux Sud Ouest, e-mail: juliette.chabassier@inria.fr

Sébastien Imperiale
Inria, University Paris Saclay, e-mail: sebastien.imperiale@inria.fr
A semi discrete energy identity can be obtained, which satisfies
\[
\mathcal{E}_{01} = \frac{1}{2} \| \partial_t u_0 \|^2_{L^2(\Omega_0)} + \frac{1}{2} \| \partial_t u_1 \|^2_{L^2(\Omega_1)} + \frac{1}{2} \| c \nabla u_0 \|^2_{L^2(\Omega_0)} + \frac{1}{2} \| c \nabla u_1 \|^2_{L^2(\Omega_1)} \tag{2}
\]

\section{Semi discrete system}

We consider spatial meshes of \( \Omega_0 \) and \( \Omega_1 \) upon which are based finite dimensional finite element spaces: \( \mathcal{V}_{h,0} \subset H^1(\Omega_0) \), \( \mathcal{V}_{h,1} \subset H^1(\Omega_1) \) and \( I_h \subset H^{-1/2}(\Gamma) \). One has leeway in the choice of \( (\mathcal{V}_{h,0}, \mathcal{V}_{h,1}) \) after which \( I_h \) must be chosen so that an inf-sup type condition is satisfied, see \[4, 3, 1\]. \((\bar{u}_{h,0}, \bar{u}_{h,1}, \bar{A}_h)\) is the solution of:

\[
\begin{aligned}
&\left\{
\begin{align*}
&\partial_t^2 M_{h,0} \bar{u}_{h,0} + K_{h,0} \bar{u}_{h,0} - c_{h,0} \bar{A}_h = M_{h,0} \bar{S}_{h,0} \\
&\partial_t^2 M_{h,1} \bar{u}_{h,1} + K_{h,1} \bar{u}_{h,1} + c_{h,1} \bar{A}_h = M_{h,1} \bar{S}_{h,1}
\end{align*}
\right.
\end{aligned} \tag{3a}
\]

\[
C_{h,0} \bar{u}_{h,0} = C_{h,1} \bar{U}_{h,1} \tag{3b}
\]

A semi discrete energy identity can be obtained, which satisfies

\[
\frac{d\mathcal{E}_{01,h}}{dt} = M_{h,0} \bar{S}_{h,0} \cdot \partial_t \bar{U}_{h,0} + M_{h,1} \bar{S}_{h,1} \cdot \partial_t \bar{U}_{h,1}, \quad \text{where}
\]

\[
\mathcal{E}_{01,h} = \frac{1}{2} \| \partial_t \bar{U}_{h,0} \|^2_{M_{h,0}} + \frac{1}{2} \| \partial_t \bar{U}_{h,1} \|^2_{M_{h,1}} + \frac{1}{2} \| \bar{U}_{h,0} \|^2_{K_{h,0}} + \frac{1}{2} \| \bar{U}_{h,1} \|^2_{K_{h,1}} \tag{4}
\]

where \( \| X \|_M^2 = MX \cdot X \) for any nonnegative matrix \( M \). In the following, \( I_h \) will denote the identity matrix.

\section{Discrete system}

The proposed numerical discretization is based on the following definitions:

\[
D^2_{\Delta t, h} U^n_h := (U^{n+1}_h - 2U^n_h + U^{n-1}_h) / \Delta t^2, \quad \{U_h\}_0^n := \theta U^{n+1}_h + (1 - 2\theta) U^n_h + \theta U^{n-1}_h
\]

The consistency analysis of the fourth order family of schemes \[2\] applied to each equation of system (3) instigates the following scheme:

\[
\begin{aligned}
&\left\{
\begin{align*}
&M_{h,0} D^2_{\Delta t, h} U^n_{h,0} + K_{h,0} \{U_{h,0}\}_0^n - c_{h,0} \bar{A}_h = M_{h,0} \bar{S}_{h,0} + \Delta t^2 c_{h,0} K_{h,0} M_{h,0}^{-1} [-K_{h,0} \{U_{h,0}\}_0^n + \bar{A}_h] \\
&M_{h,1} D^2_{\Delta t, h} U^n_{h,1} + K_{h,1} \{U_{h,1}\}_0^n + c_{h,1} \bar{A}_h = M_{h,1} \bar{S}_{h,1} + \Delta t^2 c_{h,1} K_{h,1} M_{h,1}^{-1} [-K_{h,1} \{U_{h,1}\}_0^n - \bar{A}_h]
\end{align*}
\right.
\end{aligned} \tag{5}
\]

where \( c_{i,j} = \theta_j - 1/12 \). Any solution to (5) satisfies the energy identity:
Fourth order energy-preserving locally implicit discretization for linear wave equations

(a) Relative energy deviation for a 1D case: the energy is preserved up to machine precision.

(b) Relative L2 error with the analytical solution as the time step tends to zero.

Fig. 1 Numerical illustrations in 1D.

\[ \frac{\epsilon^{n+1/2}_{01.4,h} - \epsilon^{n-1/2}_{01.4,h}}{\Delta t} = \bar{I}_{h,0}^{-1} M_{h,0} s^n_{h,0} \cdot \frac{U^{n+1}_{h,0} - U^{n-1}_{h,0}}{2\Delta t} + \bar{I}_{h,1}^{-1} M_{h,1} s^n_{h,1} \cdot \frac{U^{n+1}_{h,1} - U^{n-1}_{h,1}}{2\Delta t}, \]

where the modified mass matrices \( \bar{M}_{h,i} \) are defined by \( \bar{M}_{h,i} = \bar{I}_{h,i}^{-1} M_{h,i} \), where \( \bar{I}_{h,i} = \mathcal{I}_{h,i} + \Delta t^2 \left( \theta_i - \frac{1}{4} \right) K_{h,i}^{-1} M_{h,i}^{-1} \) and \( \bar{M}_{h,i} = M_{h,i} + \Delta t^2 \left( \theta_i - \frac{1}{4} \right) K_{h,i}^{-1} M_{h,i}^{-1} K_{h,i}. \)

The positivity of the energy can be proven under standard CFL condition that depend on the parameters \((\theta_i, \phi_i)\). Despite the non standard form of this energy, stability in L2-norm can be proved via non standard estimates. Fig 1(b) shows that the coupling of second order implicit and explicit schemes only provides second order accuracy (as expected), while our scheme provides fourth order accuracy. Numerical illustrations in 2D as well as details about stability and consistency of scheme (5) will be presented.

References