Fourth-order energy-preserving locally implicit time discretization for linear wave equations

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Abstract Time domain simulation of realistic highly heterogeneous media or strongly refined geometries can be a computational challenge when using explicit schemes because they impose a time step restriction that can be extremely penalizing. In this work, we present fourth order locally implicit schemes. The domain of interest is decomposed into several regions where different (explicit or implicit) fourth order time discretization are used. Whilst implicit schemes tolerate the use of larger time steps, they can induce greater numerical dispersion. Fourth order accuracy reduces this lack of precision, and makes this family of schemes attractive compared to other approaches as local time stepping.

1 Continuous system

We want to solve for time $t > 0$, the system (closed with Neumann homogeneous boundary conditions):

$$\left\{\begin{array}{lll} \partial_t^2 u_0 - \nabla\cdot c^2(\mathbf{x})\,\nabla u_0 = s_0 \text{ in } \Omega_0, & c^2(\mathbf{x})\,\nabla u_0\cdot\mathbf{n}_0 = \lambda & \text{on } \Gamma, \quad (1a)\\ \partial_t^2 u_1 - \nabla\cdot c^2(\mathbf{x})\,\nabla u_1 = s_1 \text{ in } \Omega_1, & c^2(\mathbf{x})\,\nabla u_1\cdot\mathbf{n}_1 = -\lambda & \text{on } \Gamma, \quad (1b)\\ u_0 = u_1 \quad \text{on } \Gamma & & (1c)\end{array}\right.$$

in a domain Ω composed by disjoint sets $\Omega = \Omega_0 \cup \Omega_1$ separated by $\Gamma = \overline{\Omega_0} \cap \overline{\Omega_1}$. s_0 and s_1 are given source terms, and $c(\mathbf{x}) > c_0 > 0$ is the inhomogeneous velocity of the waves. Any solution to (1) satisfies the energy identity $\frac{d\mathscr{E}_{01}}{dt} = \int_{\Omega_0} s_0\,\partial_t u_0 + \int_{\Omega_1} s_1\,\partial_t u_1$, where :

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$$\mathscr{E}_{01} = \frac{1}{2}\,\|\partial_t u_0\|^2_{L^2(\Omega_0)} + \frac{1}{2}\,\|\partial_t u_1\|^2_{L^2(\Omega_1)} + \frac{1}{2}\,\|c\,\nabla u_0\|^2_{L^2(\Omega_0)} + \frac{1}{2}\,\|c\,\nabla u_1\|^2_{L^2(\Omega_1)} \quad (2)$$

2 Semi discrete system

We consider spatial meshes of Ω_0 and Ω_1 upon which are based finite dimensional finite element spaces: $\mathscr{V}_{h,0} \subset H^1(\Omega_0)$, $\mathscr{V}_{h,1} \subset H^1(\Omega_1)$ and $\Gamma_h \subset H^{-1/2}(\Gamma)$. One have leeway in the choice of $(\mathscr{V}_{h,0}, \mathscr{V}_{h,1})$ after which Γ_h must be chosen so that an inf-sup type condition is satisfied, see [4, 3, 1]. $(\widetilde{U}_{h,0}, \widetilde{U}_{h,1}, \widetilde{\Lambda}_h)$ is the solution of:

$$\begin{cases} d_t^2 M_{h,0}\widetilde{U}_{h,0} + K_{h,0}\widetilde{U}_{h,0} - {}^tC_{h,0}\widetilde{\Lambda}_h = M_{h,0}\widetilde{S}_{h,0} & \text{(3a)} \\ d_t^2 M_{h,1}\widetilde{U}_{h,1} + K_{h,1}\widetilde{U}_{h,1} + {}^tC_{h,1}\widetilde{\Lambda}_h = M_{h,1}\widetilde{S}_{h,1} & \text{(3b)} \\ C_{h,0}\,\widetilde{U}_{h,0} = C_{h,1}\,\widetilde{U}_{h,1} & \text{(3c)} \end{cases}$$

A semi discrete energy identity can be obtained, which satisfies

$$\frac{d\mathscr{E}_{01,h}}{dt} = M_{h,0}\widetilde{S}_{h,0}\cdot d_t\widetilde{U}_{h,0} + M_{h,1}\widetilde{S}_{h,1}\cdot d_t\widetilde{U}_{h,1}, \qquad \text{where}$$

$$\mathscr{E}_{01,h} \;=\; \frac{1}{2}\|d_t\widetilde{U}_{h,0}\|^2_{M_{h,0}} \;+\; \frac{1}{2}\|d_t\widetilde{U}_{h,1}\|^2_{M_{h,1}} \;+\; \frac{1}{2}\|\widetilde{U}_{h,0}\|^2_{K_{h,0}} \;+\; \frac{1}{2}\|\widetilde{U}_{h,1}\|^2_{K_{h,1}} \quad (4)$$

where $\|X\|^2_M = MX\cdot X$ for any nonnegative matrix M. In the following, I_h will denote the identity matrix.

3 Discrete system

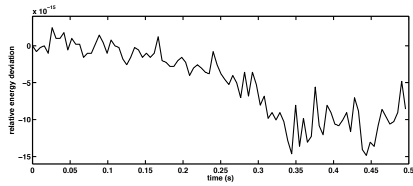
The proposed numerical discretization is based on the following definitions:

$$D^2_{\Delta t}U^n_h := \left(U^{n+1}_h - 2U^n_h + U^{n-1}_h\right)/\Delta t^2, \qquad \{U_h\}^n_\theta := \theta\,U^{n+1}_h + (1-2\theta)U^n_h + \theta\,U^{n-1}_h$$

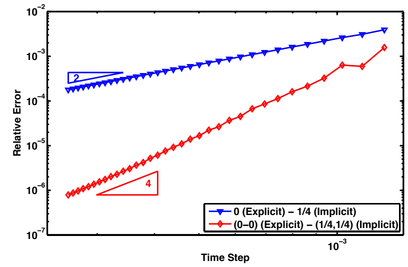
The consistency analysis of the fourth order family of schemes [2] applied to each equation of system (3) instigates the following scheme:

$$\begin{cases} M_{h,0}D^2_{\Delta t}U^n_{h,0} + K_{h,0}\{U_{h,0}\}^n_{\theta_0} - {}^tC_{h,0}\Pi^n_h = M_{h,0}\,S^n_{h,0} + \Delta t^2\alpha_0 K_{h,0}M^{-1}_{h,0}\left[-K_{h,0}\{U_{h,0}\}^n_{\varphi_0} + {}^tC_{h,0}\Pi^n_h\right] \\ M_{h,1}D^2_{\Delta t}U^n_{h,1} + K_{h,1}\{U_{h,1}\}^n_{\theta_1} + {}^tC_{h,1}\Pi^n_h = M_{h,1}\,S^n_{h,1} + \Delta t^2\alpha_1 K_{h,1}M^{-1}_{h,1}\left[-K_{h,1}\{U_{h,1}\}^n_{\varphi_1} - {}^tC_{h,1}\Pi^n_h\right] \\ C_{h,0}\,\frac{U^{n+1}_{h,0}-U^{n-1}_{h,0}}{2\Delta t} - C_{h,1}\,\frac{U^{n+1}_{h,1}-U^{n-1}_{h,1}}{2\Delta t} = 0 \end{cases} \quad (5)$$

where $\alpha_i = \theta_i - 1/12$. Any solution to (5) satisfies the energy identity:



(a) Relative energy deviation for a 1D case: the energy is preserved up to machine precision.



(b) Relative L2 error with the analytical solution as the time step tends to zero.

Fig. 1 Numerical illustrations in 1D.

$$\frac{\mathscr{E}_{01,4,h}^{n+1/2}-\mathscr{E}_{01,4,h}^{n-1/2}}{\Delta t}=\widetilde{I}_{h,0}^{-1}\,M_{h,0}\,S_{h,0}^{n}\cdot\frac{U_{h,0}^{n+1}-U_{h,0}^{n-1}}{2\Delta t}+\widetilde{I}_{h,1}^{-1}\,M_{h,1}\,S_{h,1}^{n}\cdot\frac{U_{h,1}^{n+1}-U_{h,1}^{n-1}}{2\Delta t},$$

where the discrete energy reads

$$\mathscr{E}_{01,4,h}^{n+1/2}=\frac{1}{2}\left\|\frac{U_{h,0}^{n+1}-U_{h,0}^{n}}{\Delta t}\right\|_{\widehat{M}_{h,0}}^{2}+\frac{1}{2}\left\|\frac{U_{h,1}^{n+1}-U_{h,1}^{n}}{\Delta t}\right\|_{\widehat{M}_{h,1}}^{2}+\frac{1}{2}\left\|\frac{U_{h,0}^{n+1}+U_{h,0}^{n}}{2}\right\|_{K_{h,0}}^{2}+\frac{1}{2}\left\|\frac{U_{h,1}^{n+1}+U_{h,1}^{n}}{2}\right\|_{K_{h,1}}^{2}$$

where the modified mass matrices $\widehat{M}_{h,i}$ are defined by $\widehat{M}_{h,i}=\widetilde{I}_{h,i}^{-1}\,\widetilde{M}_{h,i}$ where $\widetilde{I}_{h,i}=I_{h,i}+\Delta t^{2}\left(\theta_i-\frac{1}{12}\right)K_{h,i}M_{h,i}^{-1}$ and $\widetilde{M}_{h,i}=M_{h,i}+\Delta t^{2}\left(\theta_i-\frac{1}{4}\right)K_{h,i}+\Delta t^{4}\left(\theta_i-\frac{1}{12}\right)\left(\varphi_i-\frac{1}{4}\right)K_{h,i}M_{h,i}^{-1}K_{h,i}$. The positivity of the energy can be proven under standard CFL condition that depend on the parameters (θ_i,φ_i). Despite the non standard form of this energy, stability in $L2$-norm can be proved vie non standard estimates. Fig 1(b) shows that the coupling of second order implicit and explicit schemes only provides second order accuracy (as expected), while our scheme provides fourth order accuracy. Numerical illustrations in 2D as well as details about stability and consistency of scheme (5) will be presented.

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