Fourth-order energy-preserving locally implicit time discretization for linear wave equations

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Abstract

A family of fourth order locally implicit schemes is presented as a special case of fourth order coupled implicit schemes for linear wave equations. The domain of interest is decomposed into several regions where different (explicit or implicit) fourth order time discretization are used. The coupling is based on a Lagrangian formulation on the boundaries between the several non conforming meshes of the regions. Fourth order accuracy follows from global energy identities. Numerical results in 1d and 2d illustrate the good behavior of the schemes and their potential for the simulation of realistic highly heterogeneous media or strongly refined geometries, for which using everywhere an explicit scheme can be extremely penalizing. Fourth order accuracy reduces the numerical dispersion inherent to implicit methods used with a large time step, and makes this family of schemes attractive compared to classical approaches.

Keywords: High-order numerical methods, Time discretization, Locally implicit schemes.

1 Continuous system

We want to solve for time $t > 0$, the system (closed with Neumann homogeneous boundary conditions):

$$\left\{\begin{array}{ll}\partial_t^2 u_0 - \nabla\cdot c^2(\mathbf{x})\,\nabla u_0 = s_0 \ \text{ in } \Omega_0, & \text{(1a)}\\ c^2(\mathbf{x})\,\nabla u_0\cdot \mathbf{n}_0 = \lambda \qquad \text{on } \Gamma, & \text{(1b)}\\ \partial_t^2 u_1 - \nabla\cdot c^2(\mathbf{x})\,\nabla u_1 = s_1 \ \text{ in } \Omega_1, & \text{(1c)}\\ c^2(\mathbf{x})\,\nabla u_1\cdot \mathbf{n}_1 = -\lambda \qquad \text{on } \Gamma, & \text{(1d)}\\ u_0 = u_1 \quad \text{on } \Gamma & \text{(1e)}\end{array}\right.$$

in a domain Ω composed by disjoint sets $\Omega = \Omega_0 \cup \Omega_1$ separated by $\Gamma = \overline{\Omega_0} \cap \overline{\Omega_1}$. s_0 and s_1 are given source terms, and $c(\mathbf{x}) > c_0 > 0$ is the inhomogeneous velocity of the waves. Any solution to (1) satisfies the energy identity:

$$\frac{d\mathcal{E}_{01}}{dt} = \int_{\Omega_0} s_0\,\partial_t u_0 + \int_{\Omega_1} s_1\,\partial_t u_1, \text{ where}$$

$$\mathcal{E}_{01} = \frac{1}{2}\,\|\partial_t u_0\|^2_{L^2(\Omega_0)} + \frac{1}{2}\,\|\partial_t u_1\|^2_{L^2(\Omega_1)} + \frac{1}{2}\,\|c\,\nabla u_0\|^2_{L^2(\Omega_0)} + \frac{1}{2}\,\|c\,\nabla u_1\|^2_{L^2(\Omega_1)} \quad (2)$$

2 Semi discrete system

We consider spatial meshes of Ω_0 and Ω_1 upon which are based finite dimensional finite element spaces: $\mathcal{V}_{h,0} \subset H^1(\Omega_0)$, $\mathcal{V}_{h,1} \subset H^1(\Omega_1)$ and $\Gamma_h \subset H^{-1/2}(\Gamma)$. One have leeway in the choice of $(\mathcal{V}_{h,0}, \mathcal{V}_{h,1})$ after which Γ_h must be chosen so that an inf-sup type condition is satisfied, see [1,3,4]. $(\widetilde{U}_{h,0}, \widetilde{U}_{h,1}, \widetilde{\Lambda}_h)$ is the solution of:

$$\left\{\begin{array}{ll}d_t^2 M_{h,0}\widetilde{U}_{h,0} + K_{h,0}\widetilde{U}_{h,0} - {}^tC_{h,0}\widetilde{\Lambda}_h = M_{h,0}\widetilde{S}_{h,0} & \text{(3a)}\\ d_t^2 M_{h,1}\widetilde{U}_{h,1} + K_{h,1}\widetilde{U}_{h,1} + {}^tC_{h,1}\widetilde{\Lambda}_h = M_{h,1}\widetilde{S}_{h,1} & \text{(3b)}\\ C_{h,0}\,\widetilde{U}_{h,0} = C_{h,1}\,\widetilde{U}_{h,1} & \text{(3c)}\end{array}\right.$$

A semi discrete energy identity can be obtained.

$$\frac{d\mathcal{E}_{01,h}}{dt} = M_{h,0}\widetilde{S}_{h,0}\cdot d_t\widetilde{U}_{h,0} + M_{h,1}\widetilde{S}_{h,1}\cdot d_t\widetilde{U}_{h,1}, \quad \text{where}$$

$$\mathcal{E}_{01,h} = \frac{1}{2}\|d_t\widetilde{U}_{h,0}\|^2_{M_{h,0}} + \frac{1}{2}\|d_t\widetilde{U}_{h,1}\|^2_{M_{h,1}} + \frac{1}{2}\|\widetilde{U}_{h,0}\|^2_{K_{h,0}} + \frac{1}{2}\|\widetilde{U}_{h,1}\|^2_{K_{h,1}} \quad (4)$$

where $\|X\|^2_M = MX\cdot X$ for any nonnegative matrix M.

3 Discrete system

The proposed numerical discretization is based on the following definitions:

$$D^2_{\Delta t}U^n_h := \left(U^{n+1}_h - 2U^n_h + U^{n-1}_h\right)/\Delta t^2$$

$$\{U_h\}^n_\theta := \theta\,U^{n+1}_h + (1-2\theta)U^n_h + \theta\,U^{n-1}_h$$

The consistency analysis of the fourth order family of schemes [2] applied to each equation of system (3) instigates the following scheme:

$$\left\{\begin{array}{ll}
M_{h,0}D^2_{\Delta t}U^n_{h,0} + K_{h,0}\{U_{h,0}\}^n_{\theta_0} - {}^tC_{h,0}\Pi^n_h = M_{h,0}\, S^n_{h,0} & \\
+\Delta t^2 \alpha_0 K_{h,0}M^{-1}_{h,0}\left[-K_{h,0}\{U_{h,0}\}^n_{\varphi_0} + {}^tC_{h,0}\Pi^n_h\right] & \text{(5a)} \\[2mm]
M_{h,1}D^2_{\Delta t}U^n_{h,1} + K_{h,1}\{U_{h,1}\}^n_{\theta_1} + {}^tC_{h,1}\Pi^n_h = M_{h,1}\, S^n_{h,1} & \\
+\Delta t^2 \alpha_1 K_{h,1}M^{-1}_{h,1}\left[-K_{h,1}\{U_{h,1}\}^n_{\varphi_1} - {}^tC_{h,1}\Pi^n_h\right] & \text{(5b)} \\[2mm]
C_{h,0}\,\dfrac{U^{n+1}_{h,0} - U^{n-1}_{h,0}}{2\Delta t} - C_{h,1}\,\dfrac{U^{n+1}_{h,1} - U^{n-1}_{h,1}}{2\Delta t} = 0 & \text{(5c)}
\end{array}\right.$$

where $\alpha_i = \theta_i - 1/12$. Any solution to (5) satisfies the energy identity:

$$\frac{\mathcal{E}^{n+1/2}_{01,4,h} - \mathcal{E}^{n+1/2}_{01,4,h}}{\Delta t} = \widetilde{I}^{-1}_{h,0}\, M_{h,0}\, S^n_{h,0} \cdot \frac{U^{n+1}_{h,0} - U^{n-1}_{h,0}}{2\Delta t} + \widetilde{I}^{-1}_{h,1}\, M_{h,1}\, S^n_{h,1} \cdot \frac{U^{n+1}_{h,1} - U^{n-1}_{h,1}}{2\Delta t},$$

where the discrete energy reads

$$\mathcal{E}^{n+1/2}_{01,4,h} = \frac{1}{2}\left\|\frac{U^{n+1}_{h,0} - U^n_{h,0}}{\Delta t}\right\|^2_{\widehat{M}_{h,0}} + \frac{1}{2}\left\|\frac{U^{n+1}_{h,1} - U^n_{h,1}}{\Delta t}\right\|^2_{\widehat{M}_{h,1}} + \frac{1}{2}\left\|\frac{U^{n+1}_{h,0} + U^n_{h,0}}{2}\right\|^2_{K_{h,0}} + \frac{1}{2}\left\|\frac{U^{n+1}_{h,1} + U^n_{h,1}}{2}\right\|^2_{K_{h,1}}$$

where the modified mass matrices $\widehat{M}_{h,i}$ are defined by $\widehat{M}_{h,i} = \widetilde{I}^{-1}_{h,i}\, \widetilde{M}_{h,i}$ where

$$\left\{\begin{array}{l}
\widetilde{I}_{h,i} = I_{h,i} + \Delta t^2\left(\theta_i - \dfrac{1}{12}\right) K_{h,i}\, M^{-1}_{h,i} \\[2mm]
\widetilde{M}_{h,i} = M_{h,i} + \Delta t^2\left(\theta_i - \dfrac{1}{4}\right) K_{h,i} \\[2mm]
\qquad + \Delta t^4\left(\theta_i - \dfrac{1}{12}\right)\left(\varphi_i - \dfrac{1}{4}\right) K_{h,i}M^{-1}_{h,i}K_{h,i}
\end{array}\right.$$

The positivity of the energy can be proven under standard CFL condition that depend on the parameters (θ_i, φ_i). Despite the non standard form of this energy, stability in $L2$-norm can be proved in the case $\theta_i \geq 1/4$ and $\varphi_i \geq 1/4$. $L2$-Stability in the other cases is not proven yet but show good numerical behavior.

A 1d numerical experiment is performed where the segment $[0, 1]$ is cut in two intervals $\Omega_0 = [0, 0.5]$ and $\Omega_1 = [0.5, 1]$. Ω_0 and Ω_1 are respectively divided into 7 and 13 elements. Sixth order spectral elements are implemented. A fourth order explicit scheme is used on Ω_0 ($\theta_0 = \varphi_0 = 0$) while an unconditionally stable implicit scheme is used on Ω_1 ($\theta_1 = \varphi_1 = 1/4$). A gaussian initial condition is set on the left interval and crosses the middle point around time 0.3. Fig. 1 shows that the energy is preserved up to machine precision. Fig. 2 shows that the coupling of second order implicit and explicit schemes only provides second order accuracy (as expected), while our scheme provides fourth order accuracy.

Figure 1: Relative energy deviation

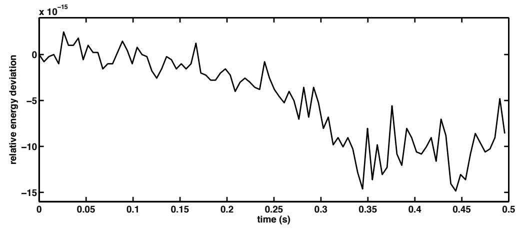
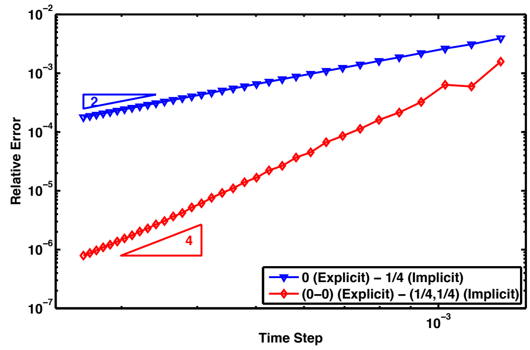


Figure 2: Convergence curve



Numerical illustrations in 2D as well as details about stability and consistency of scheme (5) will be presented at the oral session.

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