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Abstract

Solar acoustic waves are continuously excited by turbulent convection (a random process). The forward problem of local helioseismology was specified at the Waves 2013 conference: computing the cross-covariance of the wave field between any two locations on the solar surface. Here we solve the problem in the frequency domain using the finite element solver *Montjoie*. One of the specificities of propagation/scattering problems in the Sun is the very sharp decrease of sound speed and density with radius near the surface. We show that the problem simplifies considerably under the assumption that the covariance function of the source of excitation is proportional to the attenuation.

Keywords: acoustics, forward problem, local helioseismology

1 Scalar Acoustics

Rather than solving the forward problem in all its complexity, we neglect gravity, assume that the medium is steady, and consider linear adiabatic waves only. Under these approximations the linearized equations of motion reduce to a single equation for the scalar quantity

$$\psi = c\,\mathrm{div}\boldsymbol{\xi}, \tag{1}$$

where $\boldsymbol{\xi}$ is the wave displacement vector and c is the sound speed. If we further assume that waves are excited by a stationary random process (source function s), we only need to solve the problem one frequency ω at a time:

$$L\psi = s \tag{2}$$

with the wave operator

$$\begin{aligned} L &= -\omega^2 - 2\mathrm{i}\omega\gamma - 2\mathrm{i}\omega\boldsymbol{u}\cdot\nabla + H & (3)\\ H\psi &= -c\,\mathrm{div}\left(\frac{1}{\rho}\nabla(\rho c\psi)\right), & (4)\end{aligned}$$

where ρ is density, $\gamma > 0$ is the attenuation, and $\boldsymbol{u}$ is the background flow. The factor c in the definition of ψ (eq. 1) is chosen such that the spatial operator H is Hermitian symmetric under free surface boundary conditions ($\psi = 0$ on ∂V). Since mass is conserved the advection operator is also Hermitian, while the attenuation operator is anti-Hermitian. Note that the computational domain V ends approximately 500 km above the solar photosphere, i.e. above the observation height.

2 Cross-Covariance Function

Consider ψ measured at positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ near the solar surface (inside the computational domain). At frequency ω, the cross-covariance function is

$$C(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) = \mathbb{E}[\psi^*(\boldsymbol{r}_1, \omega)\psi(\boldsymbol{r}_2, \omega)], \tag{5}$$

as defined by Duvall et al. 1996. We study the forward problem, i.e. how a change in solar structure,

$$\begin{aligned} c(\boldsymbol{r}) &\to c(\boldsymbol{r}) + \delta c(\boldsymbol{r}),\\ \rho(\boldsymbol{r}) &\to \rho(\boldsymbol{r}) + \delta\rho(\boldsymbol{r}), \qquad (6)\\ \boldsymbol{u}(\boldsymbol{r}) &\to \boldsymbol{u}(\boldsymbol{r}) + \delta\boldsymbol{u}(\boldsymbol{r}),\end{aligned}$$

will affect the cross-covariance function,

$$C \to C + \delta C. \tag{7}$$

Using the first-order Born approximation and the assumption that sources are spatially uncorrelated, Gizon (2013) wrote

$$\begin{aligned}\delta C(\boldsymbol{r}_1, \boldsymbol{r}_2) &= -\int_V G(\boldsymbol{r}_2, \boldsymbol{r})\, \delta L[C(\boldsymbol{r}_1, \boldsymbol{r})]\, \rho d\boldsymbol{r} \\ &\quad - \int_V G^*(\boldsymbol{r}_1, \boldsymbol{r})\, \delta L^*[C^*(\boldsymbol{r}_2, \boldsymbol{r})]\, \rho d\boldsymbol{r},\end{aligned} \tag{8}$$

where δL is the perturbation to the wave operator L caused by the perturbations to the medium, and G is the Green's function

$$LG(\boldsymbol{r}, \boldsymbol{r}', \omega) = \frac{1}{\rho}\delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{9}$$

Thus the two quantities that really matter in local helioseismology are the functions C and G computed in the reference model.

3 Convenient Source of Excitation

It is well known that under appropriate conditions the expectation value of the cross-covariance is related to the imaginary part of the Green's function. If we could write such a simple relationship, our problem would simplify considerably. In particular, for all practical purposes, the problem would become deterministic and the Green's function would be the only remaining quantity in our problem.

Starting from the definition of the Green's functions $G(\boldsymbol{r}, \boldsymbol{r}_1)$ and $G(\boldsymbol{r}, \boldsymbol{r}_2; -\boldsymbol{u})$, where the latter is for a medium with opposite background flow, one can show

$$\begin{aligned}&G(\boldsymbol{r}_2, \boldsymbol{r}_1) - G^*(\boldsymbol{r}_2, \boldsymbol{r}_1; -\boldsymbol{u}) = \\ &\qquad 4\mathrm{i}\omega \int_V \gamma(\boldsymbol{r}) G^*(\boldsymbol{r}_1, \boldsymbol{r}) G(\boldsymbol{r}_2, \boldsymbol{r})\, \rho d\boldsymbol{r}.\end{aligned} \tag{10}$$

In order to obtain this result, generalized seismic reciprocity was used:

$$G(\boldsymbol{r}, \boldsymbol{r}') = G(\boldsymbol{r}', \boldsymbol{r}; -\boldsymbol{u}). \tag{11}$$

Note that an extra surface integral should be included above if the boundary condition is not Dirichlet. By identification with equation (10), we see that the choice of source covariance

$$\mathbb{E}[s^*(\boldsymbol{r}, \omega) s(\boldsymbol{r}', \omega)] = P_s(\omega) \frac{\gamma(\boldsymbol{r})}{\rho(\boldsymbol{r})} \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{12}$$

implies

$$\mathbb{E}[C(\boldsymbol{r}_1, \boldsymbol{r}_2)] = \frac{P_s}{4\mathrm{i}\omega} \left[G(\boldsymbol{r}_2, \boldsymbol{r}_1) - G^*(\boldsymbol{r}_2, \boldsymbol{r}_1; -\boldsymbol{u})\right]. \tag{13}$$

Thus the cross-covariance can be written as a sum of causal and anti-causal Green's functions. The volume sources must be proportional to the local attenuation to enforce energy equipartition between the modes (see [4] for a discussion).

4 2.5D FEM Forward Solver

To check if equation (13) is a good approximation, we compute the Green's function for a standard solar model (Model S) using the FEM direct solver *Montjoie* from INRIA Pau, and compare the cross-covariance and oscillation power spectra with observations from the Solar Dynamics Observatory (NASA). The results are very encouraging.

To speed up the computations, we consider a solar background model that is symmetric about an axis. The computational domain is a 2D generating section of the geometry, which is meshed in quadrilateral elements.

5 Inverse problem

The inverse problem consists of inferring the properties of the medium (ρ, c and $\boldsymbol{u}$) from measurements of the cross-covariance function (or travel times). An iterative inversion is possible since all the tools are in place to compute the perturbation to the cross-covariance starting from a completely general background medium. Damien Fournier et al. will discuss the inverse problem of local helioseismology at the conference.

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