High Order Absorbing Boundary Conditions for the 2D Helmholtz Equation

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Abstract

We present high order absorbing boundary conditions (ABC) for the 2D Helmholtz equation that can adapt to any regular shaped surface. The new ABCs are derived by using the technique of micro-diagonalisation to approximate the Dirichlet-to-Neumann map. Numerical results on different shapes illustrate the behavior of the new ABCs along with high-order finite elements.

Keywords: absorbing boundary conditions, Helmholtz equation, high-order approximation.

1 Introduction

Local boundary conditions called "absorbing boundary conditions" (ABC) are often used to simulate outgoing waves in a artificially truncated numerical domain. The aim of the present work is to develop high order ABCs for the Helmholtz equation that can adapt to regular shaped surfaces.

To obtain efficient conditions, Taylor's micro-diagonalisation method (see [1]) can be used for hyperbolic systems (see [2]). The use of this technique is followed by an asymptotic truncation to make the ABC local. During the process, while increasing the degree of the pseudo differential operator decomposition along with the order of asymptotic truncation, we retrieve classical ABCs that have been found with other techniques by other authors (see [3]).

2 General Approach

The Helmholtz equation in local coordinates system near the artificial boundary Σ reads:

$$\partial_r U = LU \tag{1}$$

where $U = (u, v)^t$ is the vectorial unknown, and the symbol of the pseudo-differential operator L is given by

$$\mathcal{L} = \sigma(L) = \begin{pmatrix} 0 & -i\omega \\ -\dfrac{i\omega}{c^2} - \dfrac{\partial_s h}{h^3}\dfrac{\xi}{\omega} - \dfrac{\xi^2}{i\omega h^2} & -\kappa_r \end{pmatrix} \tag{2}$$

where ξ is the dual variables associated to the tangent coordinate s, $\kappa(s)$ is the curvature of Σ, $h = 1 + r\,\kappa(s)$ and $\kappa_r = \kappa(s)/h$, r being the radial coordinate.

For a given $m \in \mathbb{N}$, our aim is to find a diagonal pseudo-differential operator Λ such that

$$\Lambda = \Lambda_1 + \Lambda_0 + \Lambda_{-1} + \ldots + \Lambda_{-m}$$

where $\sigma(\Lambda_j) = \mathcal{D}_j$ is homogeneous of degree exactly j if $j > -m$, of degree j if $j = -m$, and a pseudo-differential operator P such that

$$P = P_0 + P_{-1} + P_{-2} + \ldots + P_{-m-1}$$

where $\sigma(P_j) = \mathcal{P}_j$ is homogeneous of degree exactly j if $j \geq -m$, of degree j if $j = -m-1$, so that

$$\begin{cases} V = PU \\ (1) \Longleftrightarrow \partial_r V = \Lambda V \end{cases}$$

When all the operators have an explicit expression, we end up with a diagonal system. As the first component of V corresponds to the ingoing wave, and the second component stands for the outgoing wave, we state that the first component of V must vanish on the boundary. To obtain a local ABC, we take a Taylor expansion up to an order n with two possible asymptotics: small "angle of incidence" ($\delta = \xi/\omega \to 0$) or "high frequency" ($\omega \to \infty$).

3 Obtained ABCs

The non-local ABC for $m = 1$ writes

$$\left[1 + \frac{\gamma}{\lambda_1}\right]\left(\frac{\lambda_1}{i\omega}\hat{u} + \frac{\partial_r \hat{u}}{i\omega}\right) + \frac{\kappa\omega}{4i\lambda_1^2 c^2}\left(\frac{\partial_r \hat{u}}{\lambda_1} - \hat{u}\right) = 0 \tag{3}$$

where $\lambda_1 = i\sqrt{k^2 - \xi^2/h^2}$, $k = \omega/c$ and γ is a parameter that can be arbitrarily fixed.

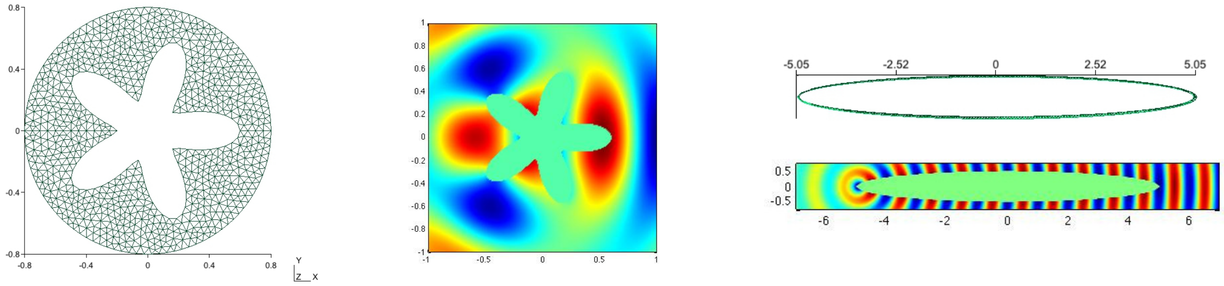

Figure 1: Mesh and reference solution for the star-shaped case (left) and the ellipse case (right)

– Taking the Taylor expansion of order $n = 1$ in δ, we obtain the following local ABC

$$(\partial_r \hat{u} + ik\, \hat{u}) + \left(\gamma + \frac{\kappa}{4}\right)\, \hat{u} + \left(\gamma - \frac{\kappa}{4}\right)\, \frac{\partial_r \hat{u}}{ik} = 0 \quad (4)$$

We notice that for the specific value $\gamma = \kappa/4$, we retrieve the classical C-ABC.

– Taking the Taylor expansion of order $n = 1$ in $1/\omega$, we obtain the following local ABC

$$\left[1 + \frac{\gamma - \frac{\kappa}{4}}{ik}\right] \partial_r \hat{u} + \left[ik + \gamma + \frac{\kappa}{4}\right] \hat{u} + \frac{\xi^2}{2\, ik} \hat{u} = 0 \quad (5)$$

We notice that only the specific value $\gamma(s) = \kappa(s)/4$ leads to a symmetric ABC that can be used in a variational context.

More generally, all the obtained ABCs for $m = 1, 2$ and $n = 0, 1, 2$ can be written under the following form:

$$\begin{aligned} &\left[a_0(ik) + a_1(ik)\, i\xi + a_2(ik)\, \xi^2\right] \partial_r u \\ &+ \left[b_0(ik) + b_1(ik)\, i\xi + b_2(ik)\, \xi^2\right] u = 0 \end{aligned} \quad (6)$$

where the coefficients a_j and b_j contains parameters that have to be fixed. When need is and when it is possible, specific parameters are used so that the ABC can be used in a variational context. A particular choice of the other parameters may optimize the ABC in terms of L^2 error, but a thorough study of their impact has to be done.

4 Numerical Illustrations

We consider an obstacle in a convex domain with different ABCs placed on the boundary. In Fig. 1, two geometries tested are displayed: a star-shaped obstacle in a circular domain and an elliptic obstacle placed in an elliptic domain. Frequency is taken equal to 1Hz. Solutions are computed with with Galerkin finite elements $\mathbb{Q}_8$. The reference solution displayed on Fig. 1 is computed with Galerkin finite elements $\mathbb{Q}_8$ and with an artificial boundary carrying a transparent condition far from the obstacle. Only variational ABCs are tested and the remaining parameters are all set equal to $\kappa(s)/4$. Tab. 1 summarizes the relative L^2 error of the Dirichlet trace between the reference solution and each solution obtained with the different ABCs tested.

Table 1: Relative L^2-error of the Dirichlet trace for the ellipse and the star-shaped obstacle

(m,n) asymptotic	Ellipse	Star
$(1,1)\ \delta$	12,70%	8,55%
$(1,1)\ \omega$	5,56%	3,03%
$(2,1)\ \delta$	12,69%	8,49 %
$(2,1)\ \omega$	7,15%	3,31 %
$(2,2)\ \omega$	7,31%	1,62 %

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