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Hatem Hajri, Marc Arnaudon

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Abstract. Inspired by Tsirelson proof of the non Brownian character of Walsh Brownian motion filtration on three or more rays, we prove some results on a particular coupling of solutions to the interface SDE on a star graph, recently introduced in [6]. This coupling consists in two solutions which are independent given the driving Brownian motion. As a consequence, we deduce that if the star graph contains 3 or more rays, the argument of the solution at a fixed time is independent of the driving Brownian motion.

1. Introduction and main results

A filtration $(\mathcal{F}_t)_t$ has the Brownian representation property (BRP) if there exists a Brownian motion B such that every $(\mathcal{F}_t)_t$-martingale is a stochastic integral of B. In 1979 Yor posed the reverse problem, i.e whether a filtration having the BRP is necessarily Brownian [13]. At the end of his paper [12], Walsh suggested the study of a Markov process with state space

$$G = \bigcup_{j=1}^{N} E_j; \ E_j = \{re^{i\theta_j} : r \geq 0\}$$

where θ_j are given angles. This process, called since then Walsh Brownian motion (WBM), behaves like a standard Brownian motion on each ray; and at 0 it makes excursions with probability p_j on $E_j \setminus \{0\}$. Later on, a detailed study of WBM was given in [1]. In particular, it was shown that WBM is a strong Markov process with Feller semigroup and that the natural filtration $(\mathcal{F}_t^Z)_t$ of a WBM Z has the BRP with respect to the Brownian motion B given by the martingale part of $|Z|$, the geodesic distance between Z and 0.

After nearly two decades a negative answer to Yor's question was finally given by Tsirelson [11]. The result proved by Tsirelson is the following

Theorem 1.1. *If $(\mathcal{G}_t)_t$ is a Brownian filtration, i.e a filtration generated by a finite or infinite family of independent standard Brownian motions, there does not exist any*

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$(\mathcal{G}_t)_t$-*WBM* ($(\mathcal{G}_t)_t$-*Markov process with semigroup* P, *the Feller semigroup of WBM*) *on a star graph with three or more rays.*

To prove Theorem 1.1, Tsirelson performs a beautiful reasoning by contradiction. Suppose there exists a Brownian motion B such $Z = F(B)$ is a WBM with $N \geq 3$ rays. Let $Z^r = F(B^r)$ where $B^r = rB + \sqrt{1-r^2}B'$ with B' an independent copy of B. Then, it is shown that $\mathbb{E}[d(Z_t^r, Z_t)]$ converges to 0. However, Tsirelson is able to prove that $\mathbb{E}[d(Z_t^r, Z_t)] > c > 0$ with c not depending on r.

In the present paper we are interested in a simple stochastic differential equation on G whose solutions are WBMs. This SDE is the interface SDE introduced in [6] and driven by an N dimensional Brownian motion $W = (W^1, \cdots, W^N)$. While moving inside E_i, a solution to this equation follows W^i so that the origin can be seen as an interface at the intersection of the half lines. For $N = 2$, the interface SDE is identified with

$$dX_t = 1_{\{X_t > 0\}} dW_t^1 + 1_{\{X_t \leq 0\}} dW_t^2 \tag{1}$$

Equation (1) has a unique strong solution [10, 7]. Not knowing Theorem 1.1, one could have the intuition, that similarly to $N = 2$, solutions are also strong ones for $N \geq 3$. The Theorem implies this cannot be the case.

The main result proved in [6] was the existence of a stochastic flow of mappings, unique in law and a Wiener stochastic flow [8] which solve the interface SDE. The problem of finding the flows of kernels which "interpolate" between these two particular flows was left open in [6]. The answer to this question needs a complete understanding of weak solutions of this equation.

The purpose of the present paper is to establish new results on weak solutions of the interface SDE in the case $N \geq 3$. These results are very different from the case $N = 2$. Our proofs are largely inspired by Tsirelson proof of Theorem 1.1.

1.1. **Notations.**

This paragraph contains the main notations and definitions which will be used throughout the paper.

Let (G, d) be a metric star graph with a finite set of rays $(E_i)_{1 \leq i \leq N}$ and origin denoted by 0. This means that (G, d) is a metric space, $E_i \cap E_j = \{0\}$ for all $i \neq j$ and for each i, there is an isometry $e_i : [0, \infty[\to E_i$. We assume d is the geodesic distance on G in the sense that $d(x, y) = d(x, 0) + d(0, y)$ if x and y do not belong to the same E_i.

For any subset A of G, we will use the notation A^* for $A \setminus \{0\}$. Also, we define the function $\varepsilon : G^* \to \{1, \cdots, N\}$ by $\varepsilon(x) = i$ if $x \in E_i^*$.

Let $C_b^2(G^*)$ denote the set of all continuous functions $f : G \to \mathbb{R}$ such that for all $i \in [1, N]$, $f \circ e_i$ is C^2 on $]0, \infty[$ with bounded first and second derivatives both with finite limits at $0+$. For $x = e_i(r) \in G^*$, set $f'(x) = (f \circ e_i)'(r)$ and $f''(x) = (f \circ e_i)''(r)$.

Let $p_1, \cdots, p_N \in (0,1)$ such that $\sum_{i=1}^N p_i = 1$ and define

$$\mathcal{D} = \left\{ f \in C_b^2(G^*) : \sum_{i=1}^N p_i (f \circ e_i)'(0+) = 0 \right\}.$$

For $f \in C_b^2(G^*)$, we will take the convention $f'(0) = \sum_{i=1}^N p_i (f \circ e_i)'(0+)$ and $f''(0) = \sum_{i=1}^N p_i (f \circ e_i)''(0+)$ so that $\mathcal{D}$ can be written as $\mathcal{D} = \{f \in C_b^2(G^*) : f'(0) = 0\}$. We are now in position to recall the following

Definition 1.2. *A solution of the interface SDE (I) on G with initial condition $X_0 = x$ is a pair of processes (X, W) defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$ such that*

(i) *$W = (W^1, \ldots, W^N)$ is a standard $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^N$;*
(ii) *X is an $(\mathcal{F}_t)$-adapted continuous process on G;*
(iii) *For all $f \in \mathcal{D}$,*

$$f(X_t) = f(x) + \sum_{i=1}^N \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i + \frac{1}{2} \int_0^t f''(X_s) ds \tag{2}$$

To emphasize on the filtration $(\mathcal{F}_t)_t$, we will sometimes say (X, W) is an $(\mathcal{F}_t)_t$-solution. It has been proved in [6] (Theorem 2.3) that for all $x \in G$, (I) admits a solution (X, W) with $X_0 = x$, the law of (X, W) is unique and X is an $(\mathcal{F}_t)$- WBM on G. We will denote by Q_x the law of a solution (X, W) with $X_0 = x$.

Tsirelson theorem 1.1 combined with Theorem 2.3 in [6] show that X is $\sigma(W)$-measurable if and only if $N \leq 2$.

Let us give an intuitive description of solutions to the previous equation. Given a WBM X started from x, we will denote from now on by B^X the martingale part of $|X| - |x|$. Freidlin-Sheu formula [4] says that for all $f \in C_b^2(G^*)$

$$f(X_t) = f(x) + \int_0^t f'(X_s) dB_s^X + \frac{1}{2} \int_0^t f''(X_s) ds + \sum_{i=1}^N p_i (f \circ e_i)'(0+) L_t(|X)| \tag{3}$$

with $L_t(|X|)$ denoting the local time of $|X|$. Comparing (2) with (3), one gets

$$B_t^X = \sum_{i=1}^N \int_0^t 1_{\{X_s \in E_i\}} dW_s^i \tag{4}$$

Thus, while it moves inside E_i, X follows the Brownian motion W^i which shows that (2) extends (1) in a natural way.

Let us now introduce the following

Definition 1.3. *We say that (X, Y, W) is a coupling of solutions to (I) if (X, W) and (Y, W) satisfy Definition 1.2 on the same filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$.*

A trivial coupling of solutions to (I) is given by (X, X, W) where (X, W) solves (I). This is also the law unique coupling of solutions to (I) if $N \leq 2$ as $\sigma(X) \subset \sigma(W)$ in this case. Let us now introduce another interesting coupling.

Definition 1.4. *A coupling (X, Y, W) of solutions to (I) is called the Wiener coupling if X and Y are independent given W.*

The existence of the Wiener coupling is easy to check. For this note there exists a law unique triplet (X, Y, W) such that (X, W) and (Y, W) are distributed respectively as Q_x and Q_y and moreover X and Y are independent given W. It remains to check that W is a standard $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^N$ where $\mathcal{F}_t = \sigma(X_u, Y_u, W_u, u \leq t)$. This holds from the conditional independence between X and Y given W and the fact that W is a Brownian motion with respect to the natural filtrations of (X, W) and (Y, W). The reason for choosing the name Wiener for this coupling will be justified in Section 3 in connection with stochastic flows.

1.2. **Main results.**

Given a WBM X on G, we define the process $\overline{X}$ by

$$\overline{X}_t = 1_{\{X_t \neq 0\}} \sum_{i=1}^{N} 1_{\{\varepsilon(X_t)=i\}} \times e_i \left(\frac{e_i^{-1}(X_t)}{N p_i} \right)$$

Note that $\overline{X} = X$ if $p_i = \frac{1}{N}$ for all $1 \leq i \leq N$. Following the terminology used in [2], the process $\overline{X}$ is a spidermartingale ("martingale-araignée"). In fact, for all $1 \leq i \leq N$, define

$$\overline{X}^i_t = |\overline{X}_t| \ \text{ if } \ \overline{X}_t \in E_i \text{ and } \overline{X}^i_t = 0 \ \text{ if not} \tag{5}$$

Note that $\overline{X}^i_t = f^i(X_t)$, where $f^i(x) = \frac{|x|}{N p_i} 1_{\{x \in E_i\}}$. Applying Freidlin-Sheu formula (3) for X and the function f^i shows that

$$\overline{X}^i_t = \frac{1}{N p_i} \int_0^t 1_{\{X_s \in E_i\}} dB^X_s + \frac{1}{N} L_t(|X|) \tag{6}$$

In particular, $\overline{X}^i_t - \overline{X}^j_t$ is a martingale for all $i, j \in [1, N]$. Proposition 5 in [2] shows that $\overline{X}$ is a spidermartingale.

Our main result in this paper is the following.

Theorem 1.5. *Assume $N \geq 3$. Let (X, Y, W) be the Wiener coupling of solutions to (I) with $X_0 = Y_0 = 0$. Then*

(i) $d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$ *is a martingale. In particular,*

$$\mathbb{E}[d(\overline{X}_t, \overline{Y}_t)] = 2\frac{N-2}{N}\sqrt{\frac{2t}{\pi}}.$$

(ii) *Call* g_t^X *and* g_t^Y *the last zeroes before* t *of* X *and* Y*, then for all* $t > 0$*,* $\mathbb{P}(g_t^X = g_t^Y) = 0$ *and* $\mathbb{P}(X_t = Y_t) = 0$*.*

(iii) $\varepsilon(X_t)$ *and* $\varepsilon(Y_t)$ *are independent for all* $t > 0$*.*

Another important fact about (X, Y, W) proved in [6], also true for $N = 2$, says that (X, Y, W) is a strong Markov process associated with a Feller semigroup. This result will be sketched in Section 3 below.

The claim (ii) says that common zeros of X and Y are rare. It has been proved in [6], that couplings (X, Y) to (I) have the same law before $T = \inf\{t \geq 0 : X_t = Y_t\}$ and that $T < \infty$ with probability one. The strong Markov property shows then that the set of common zeros of X and Y is infinite.

The case $N = 2$**.** Point (i) in Theorem 1.5 is also true for $N = 2$ since $X = Y$ in this case [6]. This can also be deduced from the proofs below. In fact, Proposition 2.1 claims that Λ_t defined as the local time of the semimartingale $d(\overline{X}_t, \overline{Y}_t)$ is zero for all $N \geq 2$. By the usual Tanaka formula (see also Proposition 2.6),

$$d(\overline{X}_t, \overline{Y}_t) = M_t + \frac{1}{2}\Lambda_t$$

where M is a martingale. Taking the expectation shows that $\overline{X} = \overline{Y}$ and so $X = Y$. The same reasoning applies to any coupling (X, Y) and in particular pathwise uniqueness holds for (2) in the case $N = 2$. Since weak uniqueness is also satisfied, this yields the strong solvability of (2) when $N = 2$ (or equivalently (1)).

Theorem 1.5 yields the following important

Corollary 1.6. *Assume* $N \geq 3$*. Let* (X, W) *be a solution of* (I) *with* $X_0 = 0$*. Then for each* $t > 0$*,* $\varepsilon(X_t)$ *is independent of* W*.*

This corollary seems to us quite remarkable. In fact, admitting Tsirelson theorem 1.1 and using (4), it can be deduced that $\varepsilon(X_t)$ is not $\sigma(W)$-measurable (actually neither $\varepsilon(X_t)$ nor $|X_t|$ are $\sigma(W)$-measurable). However, Corollary (1.6) gives a much stronger result than this non-measurability. Comparing this with the case $N = 2$, in which $\epsilon(X_t)$ is $\sigma(W)$-measurable, shows that stochastic differential equations on star graphs with $N \geq 3$ rays involve interesting "phase transitions".

Corollary 1.6 is easy to deduce from Theorem 1.5. For this, define $C_t = \mathbb{P}(\varepsilon(X_t) = i|W)$. Since X and Y are independent given W and (X, W), (Y, W) have the same law,

$$\mathbb{P}(\varepsilon(X_t) = i)^2 = \mathbb{P}(\varepsilon(X_t) = i, \varepsilon(Y_t) = i) = \mathbb{E}[C_t^2]$$

Thus $\mathbb{E}[C_t] = \mathbb{E}[C_t^2]^{\frac{1}{2}}$ and so there exists a constant c_t such that $C_t = c_t$ a.s. Taking the expectation shows that $c_t = p_i$.

Let us now explain our arguments to prove Theorem 1.5.

In Section 2.1, we prove that for any coupling (X, Y, W) of solutions to (I) such that $X_0 = Y_0 = 0$, we have $L_t(D) = 0$ where $D_t = d(\overline{X}_t, \overline{Y}_t)$.

Next, inspired by Tsirelson arguments [11], we consider a perturbation $W^r = rW + \sqrt{1-r^2}\hat{W}, r < 1$ of W, $\hat{W}$ is an independent copy of W, and X^r, Y^r such that

- (X^r, W^r) and (Y^r, W) are solutions to (I).
- X^r and Y^r are independent given $(W, \hat{W})$.

The coupling (X^r, Y^r) satisfies

$$\frac{d}{dt}\langle |X^r|, |Y^r|\rangle_t \leq r < 1 \tag{7}$$

A crucial result proved in [11] (see also [2, 3]) which will be used below says that, since (7) holds, $L_t(|X^r|)$ and $L_t(|Y^r|)$ have rare common points of increase (see (ii) in Proposition 2.4 for more precision). The process (X^r, Y^r, W) is shown to converge in law as $r \to 1$ to the Wiener coupling (X, Y, W) described above. The passage to the limit $r \to 1$ allows to deduce the properties mentioned in Theorem 1.5.

Section 3 is a complement based on stochastic flows to the previous results. We consider the Wiener stochastic flow of kernels K constructed in [6] which is a strong solution to the flows of kernels version of (2) driven by a real white noise $(W_{s,t})_{s\leq t}$. The Wiener coupling (X, Y, W) is shown to be the strong Markov process associated to a Feller semigroup Q obtained from K.

2. Proofs

2.1. **The local time of the distance.**

The subject of this paragraph is to prove the following result which, in the case $N = 2$, is proved in [6].

Proposition 2.1. *Assume $N \geq 2$. Let (X, Y, W) be a coupling of two solutions to (I) with $X_0 = Y_0 = 0$ and let $D_t = d(\overline{X}_t, \overline{Y}_t)$. Then D is a semimartingale with $L_t(D) = 0$.*

Proof. The fact that D is a semimartingale is shown in [11] (see [2] and Proposition 2.6 below for more details). We follow the proof of Proposition 4.5 in [6] and first prove that a.s.

$$\int_{]0,+\infty]} L_t^a(D)\frac{da}{a} < \infty \tag{8}$$

where $L_t^a(D)$ is the local time of D at level a and time t. Recall that by the occupation formula

$$\int_{]0,+\infty]} L_t^a(D)\frac{da}{a} = \int_0^t 1_{\{D_s>0\}}\frac{d\langle D\rangle_s}{D_s}$$

By (6),

$$\begin{aligned} |\overline{X}_t| &= \sum_{i=1}^N \overline{X}_t^i = M_t^1 + L_t(|X|) \\ |\overline{Y}_t| &= \sum_{i=1}^N \overline{Y}_t^i = M_t^2 + L_t(|Y|) \end{aligned}$$

with

$$M_t^1 = \sum_{i=1}^N \frac{1}{Np_i}\int_0^t 1_{\{X_s\in E_i\}}dB_s^X,\ \ M_t^2 = \sum_{i=1}^N \frac{1}{Np_i}\int_0^t 1_{\{Y_s\in E_i\}}dB_s^Y$$

In particular,

$$\langle M^1\rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2}\int_0^t 1_{\{X_s\in E_i\}}ds,\ \ \langle M^2\rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2}\int_0^t 1_{\{Y_s\in E_i\}}ds$$

and

$$\langle M^1, M^2\rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2}\int_0^t 1_{\{X_s\in E_i,\ Y_s\in E_i\}}ds.$$

Proposition 7 [2] tells us that

$$\begin{aligned} D_t &- \int_0^t 1_{\{\varepsilon(X_s)\neq\varepsilon(Y_s)\}}(dM_s^1 + dM_s^2) \\ &- \int_0^t 1_{\{\varepsilon(X_s)=\varepsilon(Y_s)\}}\mathrm{sgn}(M_s^1 - M_s^2)(dM_s^1 - dM_s^2) \end{aligned}$$

is a continuous increasing process. Consequently,

$$d\langle D\rangle_s = \sum_{i=1}^N \frac{1}{(Np_i)^2}1_{\{\varepsilon(X_s)\neq\varepsilon(Y_s)\}}(1_{\{X_s\in E_i\}} + 1_{\{Y_s\in E_i\}})ds \le C1_{\{\varepsilon(X_s)\neq\varepsilon(Y_s)\}}ds$$

where C is a positive constant. Note there exists $C'>0$ such that $D_s \ge C'(|X_s|+|Y_s|)$ for all s such that $\varepsilon(X_s)\neq\varepsilon(Y_s)$. Thus, to get (8), it is sufficient to prove

$$\int_0^t 1_{\{X_s\neq 0, Y_s\neq 0\}}1_{\{\epsilon(X_s)\neq\epsilon(Y_s)\}}\frac{ds}{|X|_s+|Y_s|} < \infty$$

Let us prove for instance that

$$(1) = \int_0^t \frac{1}{|X_s|+|Y_s|} 1_{\{X_s\in E_1^*, Y_s\notin E_1\}} ds < \infty$$

Define $f(z) = |z|$ if $z \in E_1$ and $f(z) = -|z|$ if not and set $x_t = f(X_t), y_t = f(Y_t)$. Clearly

$$\frac{1}{|X_s|+|Y_s|} 1_{\{X_s\in E_1^*, Y_s\notin E_1\}} \;=\; \frac{1}{2}\frac{|\mathrm{sgn}(x_s)-\mathrm{sgn}(y_s)|}{|x_s-y_s|} 1_{\{y_s<0<x_s\}}.$$

As in [6], let $(f_n)_n \subset C^1(\mathbb{R})$ such that $f_n \to \mathrm{sgn}$ pointwise and $(f_n)_n$ is uniformly bounded in total variation. Defining $z_s^u = (1-u)x_s + uy_s$, we have by Fatou's Lemma

$$\begin{aligned}(1) \;&\leq\; \liminf_n \int_0^t 1_{\{y_s<0<x_s\}} \frac{|f_n(x_s)-f_n(y_s)|}{|x_s-y_s|}\frac{ds}{2} \\ &\leq\; \liminf_n \int_0^t 1_{\{y_s<0<x_s\}} \int_0^1 \big|f_n'(z_s^u)\big| du \frac{ds}{2}\end{aligned}$$

Writing Freidlin-Sheu formula for the function f applied to X and Y shows that on $\{y_s < 0 < x_s\}$,

$$\frac{d}{ds}\langle z^u\rangle_s = u^2 + (1-u)^2 \geq \frac{1}{2}$$

Thus

$$\begin{aligned}(1) \;&\leq\; \liminf_n \int_0^1 \int_0^t 1_{\{y_s<0<x_s\}} \big|f_n'(z_s^u)\big| d\langle z^u\rangle_s du \\ &\leq\; \liminf_n \int_0^1 \int_{\mathbb{R}} \big|f_n'(a)\big| L_t^a(z^u) da du\end{aligned}$$

So a sufficient condition for (1) to be finite is

$$\sup_{a\in\mathbb{R}, u\in[0,1]} \mathbb{E}\big[L_t^a(z^u)\big] < \infty$$

By Tanaka's formula

$$\begin{aligned}\mathbb{E}\big[L_t^a(z^u)\big] \;&=\; \mathbb{E}\big[\big|z_t^u - a\big|\big] - \mathbb{E}\big[\big|z_0^u - a\big|\big] - \mathbb{E}\Bigg[\int_0^t \mathrm{sgn}(z_s^u - a) dz_s^u\Bigg] \\ &\leq\; \mathbb{E}[\big|z_t^u - z_0^u\big|] - \mathbb{E}\Bigg[\int_0^t \mathrm{sgn}(z_s^u - a) dz_s^u\Bigg]\end{aligned}$$

Since x and y are two skew Brownian motions, it is easily seen that $\sup_{u\in[0,1]} \mathbb{E}[\big|z_t^u - z_0^u\big|] < \infty$. The same argument shows that

$$\mathbb{E}\Bigg[\int_0^t \mathrm{sgn}(z_s^u - a) dz_s^u\Bigg]$$

is uniformly bounded with respect to (u,a) and consequently (1) is finite. Finally $\int_{]0,+\infty]} L_t^a(D)\frac{da}{a}$ is finite a.s. Since $\lim_{a\downarrow 0} L^a(D) = L^0(D)$, we deduce $L_t^0(D)=0$. $\square$

2.2. **Proof of Theorem 1.5.**

This section gives the proof of our main result. First, we define the perturbation of the Wiener coupling as described in the introduction and then perform a passage to the limit.

Lemma 2.2. *For all $r\in[0,1]$, there exists a law unique process $(X,W,\hat{W})$ such that, denoting $\mathcal{F}_t=\sigma(X_u,W_u,\hat{W}_u,u\le t)$,*

- *W and $\hat{W}$ are two independent $(\mathcal{F}_t)_t$-Brownian motions in $\mathbb{R}^N$.*
- *(X,W^r) is an $(\mathcal{F}_t)_t$-solution to (I) with $X_0=0$ and where $W^r=rW+\sqrt{1-r^2}\hat{W}$.*

Proof. The proof of this lemma is similar to that of Theorem 2.3 in [6]. For the existence part, take independent processes $X,V^1,\cdots,V^N,\cdots,V^{2N}$ where X is a WBM started from 0 and each V^i is a standard Brownian motion. Denote by $(\mathcal{G}_t)_t$ the natural filtration of $(X,V^1,\cdots,V^N,\cdots,V^{2N})$ and for $1\le i\le N$, define

$$d\Gamma_t^i = 1_{\{X_t\in E_i\}}dB_t^X + 1_{\{X_t\notin E_i\}}dV_t^i,\ dW_t^i = rd\Gamma_t^i + \sqrt{1-r^2}dV_t^{i+N}$$

and

$$d\hat{W}_t^i = \sqrt{1-r^2}d\Gamma_t^i - rdV_t^{i+N}.$$

Then $W=(W^1,\cdots,W^N),\hat{W}=(\hat{W}^1,\cdots,\hat{W}^N)$ are two independent $(\mathcal{G}_t)_t$-Brownian motions in $\mathbb{R}^N$, $(X,\Gamma^1,\cdots,\Gamma^N)$ is a $(\mathcal{G}_t)_t$-solution to (I) and since $d\Gamma_t^i = rdW_t^i + \sqrt{1-r^2}d\hat{W}_t^i$, existence holds.

Now let $(X,W,\hat{W})$ and $(\mathcal{F}_t)_t$ be as in the lemma. Introduce a Brownian motion B independent of $(X,W,\hat{W})$ and define $\mathcal{G}_t=\mathcal{F}_t\vee\sigma(B_u,u\le t)$. Write $\hat{W}=(\hat{W}^1,\cdots,\hat{W}^N)$ and for $1\le i\le N$, define

$$d\Gamma_t^i = rdW_t^i + \sqrt{1-r^2}d\hat{W}_t^i,\quad dV_t^{i+N} = \sqrt{1-r^2}dW_t^i - rd\hat{W}_t^i$$

and

$$dV_t^i = 1_{\{X_t\notin E_i\}}d\Gamma_t^i + 1_{\{X_t\in E_i\}}dB_t.$$

Note that $V^1,\cdots,V^N,\cdots,V^{2N}$ are independent $(\mathcal{G}_t)_t$-Brownian motions. Using $1_{\{X_t\in E_i\}}dB_t^X = 1_{\{X_t\in E_i\}}d\Gamma_t^i$, simple calculations show that $(V^1,\cdots,V^N,\cdots,V^{2N})$ is independent of B^X. Since X is a $(\mathcal{G}_t)_t$-WBM, Lemma 4.3 in [6] claims that $X,V^1,\cdots,V^N,\cdots,V^{2N}$ are independent. Finally $(X^+,W^+,\hat{W}^+)$ constructed from $X,V^1,\cdots,V^N,\cdots,V^{2N}$ as in the existence part coincides with $(X,W,\hat{W})$. This finishes the proof.

$\square$

An immediate consequence of the previous lemma is the following

Lemma 2.3. *For all $r \in [0,1]$, there exists a law unique process $(X, Y, W, \hat{W})$ such that, denoting $\mathcal{F}_t = \sigma(X_u, Y_u, W_u, \hat{W}_u, u \le t)$,*

- *W and $\hat{W}$ are two independent $(\mathcal{F}_t)_t$-Brownian motions in $\mathbb{R}^N$.*
- *(X, W^r) and (Y, W) are two $(\mathcal{F}_t)_t$-solutions to (I) with $X_0 = Y_0 = 0$ and where $W^r = rW + \sqrt{1-r^2}\hat{W}$.*
- *X and Y are independent given $(W, \hat{W})$.*

The proof of this lemma is similar to the existence and law uniqueness of the Wiener coupling and is left as an exercise.

In the sequel, we will denote $(X, Y, W, \hat{W})$ by $(X^r, Y^r, W, \hat{W})$ and use the notation W^r to denote $rW + \sqrt{1-r^2}\hat{W}$.

Proposition 2.4. *The following assertions hold*

(i) $d\langle B^{X^r}, B^{Y^r}\rangle_t = r1_{\{\varepsilon(X_t^r)=\varepsilon(Y_t^r)\}}dt$.

(ii) $\int_0^t 1_{\{Y_s^r \neq 0\}} dL_s(|X^r|) = L_t(|X^r|)$ *and* $\int_0^t 1_{\{X_s^r \neq 0\}} dL_s(|Y^r|) = L_t(|Y^r|)$.

Proof. Write $W^r = (W^{r,1}, \cdots, W^{r,N})$. By the previous lemma

$$dB_t^{X^r} = \sum_{i=1}^{N} 1_{\{X_t^r \in E_i\}} dW_t^{r,i} \text{ and } dB_t^{Y^r} = \sum_{i=1}^{N} 1_{\{Y_t^r \in E_i\}} dW_t^{i}$$

which yields (i). (ii) is Lemma 4.12 in [11] (see also [2, 3]). □

The next lemma establishes the convergence in law of (X^r, Y^r, W) to the Wiener coupling (X, Y, W).

Lemma 2.5. *As $r \to 1$, (X^r, Y^r, W) converges in law to (X, Y, W), the Wiener coupling of solutions to (I) with $X_0 = Y_0 = 0$.*

Proof. Let $(r_n)_n$ be a sequence in $[0,1]$ such that $\lim_{n\to\infty} r_n = 1$. For any $p \ge 1$, $(f_i, g_i, h_i)_{1\le i\le p}$ bounded, $(t_i)_{1\le i\le p}$

$$\mathbb{E}\left[\prod_{i=1}^{p} f_i(X_{t_i}^{r_n}) g_i(Y_{t_i}^{r_n}) h_i(W_{t_i})\right] = \mathbb{E}\left[\prod_{i=1}^{p} \mathbb{E}[f_i(X_{t_i}^{r_n})|W, \hat{W}]\mathbb{E}[g_i(Y_{t_i}^{r_n})|W, \hat{W}] h_i(W_{t_i})\right]$$

Note that (Y^{r_n}, W) is independent of $\hat{W}$. This can be deduced from the uniqueness part in Lemma 2.2 by taking $r = 1$. Consequently $\mathbb{E}[g_i(Y_{t_i}^{r_n})|W, \hat{W}] = \mathbb{E}[g_i(Y_{t_i}^{r_n})|W]$ a.s. Now $\sigma(W, \hat{W}) = \sigma(W^{r_n}, \overline{W}^{r_n})$ where $\overline{W}^{r_n}$ is the independent complement to W^{r_n} given by

$$\overline{W}^{r_n} = \sqrt{1-r_n^2}W - r_n\hat{W}.$$

By the proof of Lemma 2.2, $(X^{r_n}, W^{r_n}, \overline{W}^{r_n})$ and $(Y^{r_n}, W, \hat{W})$ have the same law. Consequently (X^{r_n}, W^{r_n}) is also independent of $\overline{W}^{r_n}$ and so

$$\mathbb{E}[f_i(X^{r_n}_{t_i})|W, \hat{W}] = \mathbb{E}[f_i(X^{r_n}_{t_i})|W^{r_n}, \overline{W}^{r_n}] = \mathbb{E}[f_i(X^{r_n}_{t_i})|W^{r_n}].$$

Slutsky lemma (see Theorem 1 in [2]) shows that for all $f : G \to \mathbb{R}$ measurable bounded and $t > 0$, as $n \to \infty$,

$$\mathbb{E}[f(X^{r_n}_t)|W^{r_n}] \longrightarrow Q_t f(W)$$

in probability where $Q_t f(W) = \int f(y) Q_t(W, dy)$ and $Q_t(W, dy)$ is a regular conditional expectation of X_t given W. Finally

$$\begin{aligned}\lim_n \mathbb{E}\left[\prod_{i=1}^p f_i(X^{r_n}_{t_i}) g_i(Y^{r_n}_{t_i}) h_i(W_{t_i})\right] &= \mathbb{E}\left[\prod_{i=1}^p Q_{t_i} f_i(W) Q_{t_i} g_i(W) h_i(W_{t_i})\right] \\ &= \mathbb{E}\left[\prod_{i=1}^p f_i(X_{t_i}) g_i(Y_{t_i}) h_i(W_{t_i})\right]\end{aligned}$$

and the lemma is proved.

□

Let us now recall Proposition 7 in [2].

Proposition 2.6. *Let Z^1 and Z^2 be two WBMs with respect to the same filtration such that $Z^1_0 = Z^2_0 = 0$. Denote by Λ the local time of $D_t = d(\overline{Z^1_t}, \overline{Z^2_t})$. Then*

$$D_t = M_t + \frac{1}{2}\Lambda_t + (N-2)\left(\int_0^t 1_{\{\overline{Z^1_s} \neq 0\}} dL^2_s + \int_0^t 1_{\{\overline{Z^2_s} \neq 0\}} dL^1_s\right)$$

with M a martingale, $M_0 = 0$ and L^1, L^2 are (see Proposition 5 in [2]) the bounded variation parts of $\overline{X}^i_t$ (defined by (5)) and $\overline{Y}^i_t$.

Note that $L^1_t = \frac{1}{N} L_t(|Z^1|)$ and $L^2_t = \frac{1}{N} L_t(|Z^2|)$ by (6).

Applying the previous proposition to $(Z^1, Z^2) = (X^r, Y^r)$ and using Proposition 2.4 (ii), we get

$$d(\overline{X^r_t}, \overline{Y^r_t}) = M^r_t + \frac{1}{2}\Lambda^r_t + \frac{(N-2)}{N}\left(L_t(|X^r|) + L_t(|Y^r|)\right)$$

with M^r a martingale and Λ^r the local time of $d(\overline{X^r_t}, \overline{Y^r_t})$. In particular,

$$\mathbb{E}[d(\overline{X^r_t}, \overline{Y^r_t})] \geq 2\frac{(N-2)}{N}\mathbb{E}[R_t] \tag{9}$$

with R a reflected Brownian motion started from 0.

Proposition 2.6 applied to the Wiener coupling $(Z^1, Z^2) = (X, Y)$ and the result of Section 2.1 show that

$$d(\overline{X_t}, \overline{Y_t}) = M_t + \frac{(N-2)}{N}\left(\int_0^t 1_{\{X_s \neq 0\}} dL_s(|Y|) + \int_0^t 1_{\{Y_s \neq 0\}} dL_s(|X|)\right) \tag{10}$$

with M a martingale. By the Balayage formula (see [9] on page 111 or the proof of Proposition 8 in [2]) and the fact that $L_t(D) = 0$,

$$d(\overline{X_t}, \overline{Y_t}) = \text{Martingale} + \frac{N-2}{N}\left(1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}| + 1_{\{\overline{Y}_{g^1} \neq 0\}}|\overline{X_t}|\right) \tag{11}$$

where $g^1 := g_t^{\overline{X}}$ and $g^2 := g_t^{\overline{Y}}$. Admit for a moment that $\mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})]$ converges to $\mathbb{E}[d(\overline{X_t}, \overline{Y_t})]$. It comes from (9), (11), (X, Y) has the same law as (Y, X), that

$$2\frac{N-2}{N}\mathbb{E}\left[1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}|\right] \geq 2\frac{(N-2)}{N}\mathbb{E}[R_t]$$

Consequently

$$\mathbb{E}\left[|\overline{Y_t}|\right] \geq \mathbb{E}\left[1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}|\right] \geq \mathbb{E}[R_t]$$

Note that this consequence is true only if $N \geq 3$. But $\mathbb{E}\left[|\overline{Y_t}|\right] = \mathbb{E}[R_t]$ and so $\overline{X}_{g^2} \neq 0$ a.s. By symmetry $\overline{Y}_{g^2} \neq 0$. Returning back to (11), we deduce that $d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$ is a martingale which proves Theorem 1.5 (i).

Note that $g^1 = g_t^X$, $g^2 = g_t^Y$ and for Z a WBM, the sets of zeros of Z and $\overline{Z}$ are equal. Consequently $X_{g_t^Y} \neq 0$ and $Y_{g_t^X} \neq 0$ a.s. In particular $g_t^X \neq g_t^Y$ a.s and since $\{X_t = Y_t\} \subset \{g_t^X = g_t^Y\}$ (as X, Y follow the same Brownian motion on the same ray), Theorem 1.5 (ii) is also proved.

Remark 2.7. *Using the convergence of $\mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})]$ to $\mathbb{E}[d(\overline{X_t}, \overline{Y_t})]$, (9) and (10), we easily deduce that*

$$\int_0^t 1_{\{X_s \neq 0\}} dL_s(|Y|) = L_t(|Y|); \int_0^t 1_{\{Y_s \neq 0\}} dL_s(|X|) = L_t(|X|)$$

which is similar to Proposition 2.4 (ii).

Now it remains to prove the following

Lemma 2.8. *We have*

$$\lim_{r \to 1} \mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})] = \mathbb{E}[d(\overline{X_t}, \overline{Y_t})].$$

Proof. From the convergence in law given in Lemma 2.5, it is easily seen that $(\overline{X^r}, \overline{Y^r})$ converges in law to $(\overline{X}, \overline{Y})$. This is because $\overline{Z}$ is a continuous function of Z. Let r_n be a sequence converging to 1. Skorokhod representation theorem says that it is possible to construct on some probability space $(\Omega', \mathcal{A}', \mathbb{P}')$, random variables $(X^n, Y^n)_{n \geq 1}$

and (X^∞, Y^∞) such that for each n, (X^n, Y^n) has the same law as $(\overline{X^{r_n}}, \overline{Y^{r_n}})$ and (X^∞, Y^∞) has the same law as $(\overline{X}, \overline{Y})$ and moreover (X^n, Y^n) converges a.s. to (X^∞, Y^∞). The lemma holds as soon as we prove

$$\lim_{n\to\infty} \mathbb{E}[d(X_t^n, Y_t^n)] = \mathbb{E}[d(X_t^\infty, Y_t^\infty)].$$

For each $\epsilon > 0$,

$$\begin{aligned}
\mathbb{E}[d(X_t^n, X_t^\infty)] &\leq \epsilon + \mathbb{E}[d(X_t^n, X_t^\infty)1_{\{d(X_t^n, X_t^\infty)>\epsilon\}}] \\
&\leq \epsilon + \mathbb{E}[d(X_t^n, X_t^\infty)^2]^{1/2}\mathbb{P}[d(X_t^n, X_t^\infty) > \epsilon]^{1/2} \\
&\leq \epsilon + C \times \mathbb{P}[d(X_t^n, X_t^\infty) > \epsilon]^{1/2}
\end{aligned}$$

for some finite constant C. Thus, $\limsup_n \mathbb{E}[d(X_t^n, X_t^\infty)] = 0$ and similarly $\limsup_n \mathbb{E}[d(Y_t^n, Y_t^\infty)] = 0$. The lemma follows now using the triangle inequality. □

Let us now prove Theorem 1.5 (iii).

Denote by $\mathcal{G}$ the natural filtration of the Wiener coupling (X, Y). For a random time R, let us recall the following σ-fields (see [2] on page 286)

$$\begin{aligned}
\mathcal{G}_R &= \sigma(U_R : U \text{ is a } \mathcal{G}- \text{ optional process}), \\
\mathcal{G}_{R+} &= \sigma(U_R : U \text{ is a } \mathcal{G}- \text{ progressive process}).
\end{aligned}$$

In the sequel, we will always consider the completions of these sigma-fields by null sets. Let $g^1 = g_t^X, g^2 = g_t^Y$. It is known (see for example Proposition 19 in [2]), that $\varepsilon(X_t)$ is independent of $\mathcal{G}_{g^1}$ and $\varepsilon(X_t)$ is $\mathcal{G}_{g^1+}$ measurable (the same holds for Y). The event $\{g^1 < g^2\} \in \mathcal{G}_{g^2}$ (see Proposition 13 in [2]) and on this event, $\varepsilon(X_t) = \limsup_{\epsilon\to 0+} \varepsilon(X_{(g^1+\epsilon)\wedge g^2})$. Since $(g^1 + \epsilon) \wedge g^2 \leq g^2$, by Proposition 13 in [2] again, $\mathcal{G}_{(g^1+\epsilon)\wedge g^2} \subset \mathcal{G}_{g^2}$ and so $\limsup_{\epsilon\to 0+} \varepsilon(X_{(g^1+\epsilon)\wedge g^2})$ is $\mathcal{G}_{g^2}$-measurable. Take f an indicator function on a subset of $\{1, \cdots, N\}$. By conditioning with respect to $\mathcal{G}_{g^2}$, we deduce

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^1<g^2\}}] = \mathbb{E}[f(\varepsilon(Y_t)]\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1<g^2\}}]$$

and

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^2<g^1\}}] = \mathbb{E}[f(\varepsilon(X_t)]\mathbb{E}[f(\varepsilon(Y_t))1_{\{g^2<g^1\}}]$$

Summing and using $\mathbb{P}(g^1 = g^2) = 0$, we get

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = \mathbb{E}[f(\varepsilon(X_t)]\big(\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1<g^2\}}] + \mathbb{E}[f(\varepsilon(Y_t))1_{\{g^2<g^1\}}]\big)$$

But $\{g^1 < g^2\} = \{g^2 < g^1\}^c$ a.s. Since $\mathcal{G}_{g^1}$ is complete, $\{g^1 < g^2\} \in \mathcal{G}_{g^1}$ which is independent of $\varepsilon(X_t)$ so that

$$\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1<g^2\}}] = \frac{1}{2}\mathbb{E}[f(\varepsilon(X_t))].$$

Using the symmetry, we arrive at $\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = \mathbb{E}[f(\varepsilon(X_t))]\mathbb{E}[f(\varepsilon(Y_t))]$.

3. Interpretation using stochastic flows

This section gives an interpretation of the Wiener coupling using the Wiener stochastic flow of kernels solving the generalized interface equation considered in [6]. For basic definitions of stochastic flows of mappings, kernels and real white noises, the reader is referred to [8].

For a family of doubly indexed random variables $Z = (Z_{s,t})_{s\le t}$, define $\mathcal{F}^Z_{s,t} = \sigma(Z_{u,v}, s \le u \le v \le t)$ for all $s \le t$. The extension to flows of kernels of the interface SDE is the following.

Definition 3.1. *Let K be a stochastic flow of kernels on G and $\mathcal{W} = (\mathcal{W}^i, 1 \le i \le N)$ be a family of independent real white noises. We say that $(K, \mathcal{W})$ solves (I) if for all $s \le t$, $f \in \mathcal{D}$ and $x \in G$, a.s.*

$$K_{s,t}f(x) = f(x) + \sum_{i=1}^{N} \int_s^t K_{s,u}(1_{E_i} f')(x) d\mathcal{W}^i_{s,u} + \frac{1}{2}\int_s^t K_{s,u} f''(x) du.$$

We say K is a Wiener solution if for all $s \le t$, $\mathcal{F}^K_{s,t} \subset \mathcal{F}^{\mathcal{W}}_{s,t}$. When K is induced by a stochastic flow of mappings φ ($K = \delta_\varphi$), we say $(\varphi, \mathcal{W})$ is a solution of (I).

Note that when $K = \delta_\varphi$, the flow φ defines a system of solutions to the interface SDE (1.2) for all possible time and position initial conditions.

If $(K, \mathcal{W})$ solves (I), then $\mathcal{F}^{\mathcal{W}}_{s,t} \subset \mathcal{F}^K_{s,t}$ for all $s \le t$ [6]. Therefore Wiener solutions are characterized by $\mathcal{F}^{\mathcal{W}}_{s,t} = \mathcal{F}^K_{s,t}$ for all $s \le t$.

It has been proved in [6] that there exists a law unique stochastic flow of mappings φ and a real white noise $\mathcal{W}$ such that $(\varphi, \mathcal{W})$ solves (I). Filtering this flow with respect to $\mathcal{W}$ gives rise to a Wiener stochastic flow of kernels $K_{s,t}(x) = \mathbb{E}[\delta_{\varphi_{s,t}(x)} | \mathcal{F}^{\mathcal{W}}_{s,t}]$ solution of (I) which is unique up to modification.

In the case $N = 2$ the Wiener flow and the flow of mappings coincide ($K = \delta_\varphi$) while $K \neq \delta_\varphi$ if $N \ge 3$ and other flows solving (I) may exist [6].

Let $(K, \mathcal{W})$ be the Wiener stochastic flow which solves (I). Then

$$Q_t(f \otimes g \otimes h)(x, y, w) = \mathbb{E}[K_{0,t}f(x) K_{0,t}g(y) h(w + \mathcal{W}_{0,t})]$$

defines a Feller semigroup on $G^2 \times \mathbb{R}^N$. Denote by (X, Y, W) the Markov process associated to $(Q_t)_t$ and started from $(x, y, 0)$.

Proposition 3.2. *(X, Y, W) is the Wiener coupling solution of (I) with $X_0 = x$ and $Y_0 = y$.*

Proof. Note that

$$\widetilde{Q}_t(f \otimes h)(x, w) := Q_t(f \otimes I \otimes h)(x, w) = \mathbb{E}[f(\varphi_{0,t}(x)) h(w + \mathcal{W}_{0,t})]$$

In particular (X, W) has the same law as $(\varphi_{0,t}(x), \mathcal{W}_{0,t})_{t\geq 0}$ and so it is a solution to (I). The same holds for (Y, W). Now it remains to prove that X and Y are independent given W. We will check that

$$\mathbb{E}\left[\prod_{i=1}^{n} f_i(X_{t_i}) g_i(Y_{t_i}) h_i(W_{t_i})\right] = \mathbb{E}\left[\prod_{i=1}^{n} \mathbb{E}[f_i(X_{t_i})|W] \mathbb{E}[g_i(Y_{t_i})|W] h_i(W_{t_i})\right]$$

for all measurable and bounded test functions $(f_i, g_i, h_i)_i$. Since K is a measurable function of $\mathcal{W}$, we may assume K (and so $\mathcal{W}$) is defined on the same space as X and Y and that $W_t = \mathcal{W}_{0,t}$. By an easy induction (see the proof of Proposition 4.1 in [5]),

$$\mathbb{E}\left[\prod_{i=1}^{n} f_i(X_{t_i}) g_i(Y_{t_i}) h_i(W_{t_i})\right] = \mathbb{E}\left[\prod_{i=1}^{n} K_{0,t_i} f_i(x) K_{0,t_i} g_i(y) h_i(W_{t_i})\right] \tag{12}$$

From (12), we also deduce $K_{0,t_i} f_i(x) = \mathbb{E}[f_i(X_{t_i})|\mathcal{F}^W_{0,t_i}]$ and $K_{0,t_i} g_i(y) = \mathbb{E}[g_i(Y_{t_i})|\mathcal{F}^W_{0,t_i}]$. This completes the proof. $\square$

Let $(\mathcal{W}_{s,t})_{s\leq t}$ and $(\hat{\mathcal{W}}_{s,t})_{s\leq t}$ be two independent real white noises and set $\mathcal{W}^r_{s,t} = r\mathcal{W}_{s,t} + \sqrt{1-r^2}\hat{\mathcal{W}}_{s,t}$. Denote by K and K^r the Wiener flows solutions of (I) respectively driven by $\mathcal{W}$ and $\mathcal{W}^r$ and define

$$Q^r_t(f \otimes g \otimes h)(x, y, w) = \mathbb{E}[K^r_{0,t} f(x) K_{0,t} g(y) h(w + \mathcal{W}_{0,t})] \tag{13}$$

Then Q^r is a Feller semigroup. Following the proof of Proposition 3.2, one can prove that (X^r, Y^r, W) given in Lemma 2.5 is the Markov process associated to Q^r and starting from $(0, 0, 0)$. In particular this is also a Feller process.

Final remarks and open problems.

There are interesting open problems related to the interface SDE. Let us mention some of them.

- What is the conditional law of $|X_t|$ (and more generally of X_t) given W?
- What are the couplings which "interpolate" between the coalescing coupling and the Wiener one?
- What are the stochastic flows which "interpolate" between the coalescing flow and the Wiener one? (see [6] for more details).

Let us finish with the following remark regarding the first question. Let W be a standard Brownian motion and let $X^1, X^2, \cdots$ be WBMs started from 0 such that (X^i, W) is solution to (I) with $X^i_0 = 0$ for all i and $X^1, X^2, \cdots$ are independent given W. Then by the law of the large numbers for all $f \in C_0(G)$, a.s $\mathbb{E}[f(X^1_t)|W] = \lim_n \frac{1}{n}\sum_{i=1}^{n} f(X^i_t)$ (see Section 2.6 in [8]).

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