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Hatem Hajri, Marc Arnaudon

APPLICATION OF STOCHASTIC FLOWS TO THE INTERFACE SDE ON A STAR GRAPH

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ABSTRACT. Using a perturbation method of stochastic flows of kernels, we prove some results on a particular coupling of solutions to the interface SDE on a star graph, recently introduced in [8]. This coupling consists in two solutions which are independent given the driving Brownian motion. As a consequence, we deduce that if the star graph contains 3 or more rays, the argument of the solution at a fixed time is independent of the driving Brownian motion.

1. INTRODUCTION AND MAIN RESULTS

Walsh Brownian motion [1] has acquired a particular interest since it was proved by Tsirelson that it can not be a strong solution to any SDE driven by a standard Brownian motion, although it satisfies the martingale representation property with respect to some Brownian motion [16]. Subsequently, Freidlin and Sheu [4] proved that a Walsh Brownian motion X satisfy the following SDE

$$df(X_t) = f'(X_t)dW_t + \frac{1}{2}f''(X_t)dt \tag{1}$$

where W is the Brownian motion given by the martingale part of $|X|$ and f runs over an appropriate domain of functions with an appropriate definition of its derivative. Freidlin and Sheu formula has led to the study of some variants of (1) in [5, 9, 8] based on stochastic flows as well as to the development of a general framework of stochastic calculus on graphs [11, 17].

The purpose of the present paper is to show by the study of a simple example some particularities of stochastic differential equations on graphs and some possibly unexpected results about their solutions. The example studied here is the interface SDE introduced in [8] and defined on a star graph G consisting of N half lines $(E_i)_{1\leq i\leq N}$ sharing the same origin. This SDE is driven by an N dimensional Brownian motion $W = (W^1, \cdots, W^N)$ and its solution X is a Walsh Brownian motion on G.

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While it moves inside E_i, X follows W^i so that the origin can be seen as an interface at the intersection of the half lines. For $N = 2$, the interface SDE is identified with

$$dX_t = 1_{\{X_t>0\}}dW_t^1 + 1_{\{X_t\leq 0\}}dW_t^2$$

which has a unique strong solution [15, 12] while for $N \geq 3$, solutions of the interface SDE are only weak by the general result of [16].

The main result proved in [8] was the existence of a stochastic flow of mappings, unique in law and a Wiener stochastic flow [13], unique up to a modification, which solve the interface SDE. The problem of finding the flows of kernels which "interpolate" between these two particular flows was left open in [8]. The answer to this question needs a complete understanding of weak solutions of this equation.

The main purpose of the present paper is to bring some light on the previous problem by establishing some results on weak solutions of the interface SDE in the case $N \geq 3$. These results are very different from the case $N = 2$. Our proofs strongly rely on Tsirelson perturbation method [16] and on the existence of the Wiener stochastic flow constructed in [8]. The present paper provides, in particular, a direct application of stochastic flows to the study of weak solutions (see [7] for another recent application by the same authors).

1.1. **Notations.**

This paragraph contains the main notations and definitions which will be used throughout the paper.

Let (G, d) be a metric star graph with a finite set of rays $(E_i)_{1\leq i\leq N}$ and origin denoted by 0. This means that (G, d) is a metric space, $E_i \cap E_j = \{0\}$ for all $i \neq j$ and for each i, there is an isometry $e_i : [0, \infty[\to E_i$. We assume that d is the geodesic distance on G in the sense that $d(x, y) = d(x, 0) + d(0, y)$ if x and y do not belong to the same E_i.

For any subset A of G, we will use the notation A^* for $A \setminus \{0\}$. Also, we define the function $\varepsilon : G^* \to \{1, \cdots, N\}$ by $\varepsilon(x) = i$ if $x \in E_i^*$.

Let $C_b^2(G^*)$ denote the set of all continuous functions $f : G \to \mathbb{R}$ such that for all $i \in [1, N]$, $f \circ e_i$ is C^2 on $]0, \infty[$ with bounded first and second derivatives both with finite limits at $0+$. For $x = e_i(r) \in G^*$, set $f'(x) = (f\circ e_i)'(r)$ and $f''(x) = (f\circ e_i)''(r)$.

Let $p_1, \cdots, p_N \in (0, 1)$ such that $\sum_{i=1}^N p_i = 1$ and define

$$\mathcal{D} = \left\{ f \in C_b^2(G^*) : \sum_{i=1}^N p_i(f \circ e_i)'(0+) = 0 \right\}.$$

For $f \in C_b^2(G^*)$, we will take the convention $f'(0) = \sum_{i=1}^N p_i(f \circ e_i)'(0+)$ and $f''(0) = \sum_{i=1}^N p_i(f\circ e_i)''(0+)$ so that $\mathcal{D}$ can be written as $\mathcal{D} = \{f \in C_b^2(G^*) : f'(0) = 0\}$. We are now in position to recall the following

Definition 1.1. *A solution of the interface SDE* (I) *on* G *with initial condition* $X_0 = x$ *is a pair* (X, W) *of processes defined on a filtered probability space* $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$ *such that*

(i) $W = (W^1, \ldots, W^N)$ *is a standard* $(\mathcal{F}_t)$*-Brownian motion in* $\mathbb{R}^N$*;*
(ii) X *is an* $(\mathcal{F}_t)$*-adapted continuous process on* G*;*
(iii) *For all* $f \in \mathcal{D}$*,*

$$f(X_t) = f(x) + \sum_{i=1}^{N} \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i + \frac{1}{2} \int_0^t f''(X_s) ds \tag{2}$$

It has been proved in [8] (Theorem 2.3) that for all $x \in G$, (I) admits a solution (X, W) with $X_0 = x$ and moreover the law of (X, W) is unique. We will denote this law by Q_x. Theorem 2.3 in [8] also states that X is a Walsh Brownian motion on G and X is $\sigma(W)$-measurable if and only if $N \leq 2$.

Let us explain the meaning of the previous equation. Given a Walsh Brownian motion X started from 0, we will denote from now on the martingale part of $|X|$ by B^X. Comparing (2) with Freidlin-Sheu formula [4] (see also [10]), shows that

$$B_t^X = \sum_{i=1}^{N} \int_0^t 1_{\{X_s \in E_i\}} dW_s^i \tag{3}$$

Thus, while it moves inside E_i, X follows the Brownian motion W^i.

Let us now introduce the following

Definition 1.2. *We say that* (X, Y, W) *is a coupling of solutions to* (I) *if* (X, W) *and* (Y, W) *satisfy Definition 1.1 on the same filtered probability space* $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$.

A trivial coupling of solutions to (I) is given by (X, X, W) where (X, W) solves (I). This is also the law unique coupling of solutions to (I) if $N \leq 2$ as $\sigma(X) \subset \sigma(W)$ in this case. Let us now introduce another interesting coupling.

Definition 1.3. *A coupling* (X, Y, W) *of solutions to* (I) *is called the Wiener coupling if* X *and* Y *are independent given* W*.*

The existence of the Wiener coupling is easy to check. For this note there exists a law unique triplet (X, Y, W) such that (X, W) and (Y, W) are distributed respectively as Q_x and Q_y and moreover X and Y are independent given W. It remains to check that W is a standard $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^N$ where $\mathcal{F}_t = \sigma(X_u, Y_u, W_u, u \leq t)$. This holds from the conditional independence between X and Y given W and the fact that W is a Brownian motion with respect to the natural filtrations of (X, W) and (Y, W). The reason for choosing the name Wiener for this coupling will be justified in Section 2.2 in connection with stochastic flows of kernels.

1.2. **Main results.**

Given a Walsh Brownian motion X on G, we define the process $\overline{X}$ by

$$\overline{X}_t = 1_{\{X_t \neq 0\}} \sum_{i=1}^{N} 1_{\{\varepsilon(X_t)=i\}} \times e_i \left(\frac{e_i^{-1}(X_t)}{Np_i} \right)$$

Note that $\overline{X} = X$ if $p_i = \frac{1}{N}$ for all $1 \leq i \leq N$. Following the terminology used in [2], the process $\overline{X}$ is a spidermartingale ("martingale-araignée"). In fact, for all $1 \leq i \leq N$, define

$$\overline{X}^i_t = |\overline{X}_t| \text{ if } \overline{X}_t \in E_i \text{ and } \overline{X}^i_t = 0 \text{ if not} \tag{4}$$

Note that $\overline{X}^i_t = f^i(X_t)$, where $f^i(x) = \frac{|x|}{Np_i} 1_{\{x \in E_i\}}$. Applying Freidlin-Sheu formula for X and the function f^i shows that

$$\overline{X}^i_t = \frac{1}{Np_i} \int_0^t 1_{\{X_s \in E_i\}} dB^X_s + \frac{1}{N} L_t(|X|) \tag{5}$$

where $L_t(|X|)$ is the local at zero of the reflected Brownian motion $|X|$. In particular, $\overline{X}^i_t - \overline{X}^j_t$ is a martingale for all $i, j \in [1, N]$. Proposition 5 in [2] shows that $\overline{X}$ is a spidermartingale.

Our main result in this paper is the following

Theorem 1.4. *Suppose $N \geq 3$. Let (X, Y, W) be the Wiener coupling of solutions to (I) with $X_0 = Y_0 = 0$. Then*

(i) *$d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$ is a martingale. In particular,*

$$\mathbb{E}[d(\overline{X}_t, \overline{Y}_t)] = 2\frac{N-2}{N}\sqrt{\frac{2t}{\pi}}$$

(ii) *Call g^X_t and g^Y_t the last zeroes before t of X and Y, then for all $t > 0$, $\mathbb{P}(g^X_t = g^Y_t) = 0$ and $\mathbb{P}(X_t = Y_t) = 0$.*

(iii) *$\varepsilon(X_t)$ and $\varepsilon(Y_t)$ are independent for all $t > 0$.*

The claim (ii) says that common zeros of X and Y are rare. It has been proved in [8], that couplings (X, Y) to (I) have the same law before coalescence and that coalescence occurs in a finite time with probability one. The strong Markov property shows then that the set of common zeros of X and Y is infinite.

This theorem yields the following important

Corollary 1.5. *Suppose $N \geq 3$. Let (X, W) be a solution of (I) with $X_0 = 0$. Then for each $t > 0$, $\varepsilon(X_t)$ is independent of W.*

This corollary seems to us quite remarkable. In fact, admitting Tsirelson theorem [16] and using (3), it can be deduced that $\varepsilon(X_t)$ is not $\sigma(W)$-measurable (actually neither $\varepsilon(X_t)$ nor $|X_t|$ are $\sigma(W)$-measurable). However, Corollary (1.5) gives a much stronger result than this non-measurability. Comparing this with the case $N=2$ in which $\epsilon(X_t)$ is $\sigma(W)$-measurable, shows that stochastic differential equations on star graphs with $N \geq 3$ rays involve interesting "phase transitions".

Corollary 1.5 is easy to deduce from Theorem 1.4. For this, define $C_t = \mathbb{P}(\epsilon(X_t) = i|W)$. Since X and Y are independent given W and (X,W), (Y,W) have the same law,

$$\mathbb{P}\left(\varepsilon(X_t)=i\right)^2 = \mathbb{P}(\varepsilon(X_t)=i, \varepsilon(Y_t)=i) = \mathbb{E}[C_t^2]$$

Thus $\mathbb{E}[C_t] = \mathbb{E}[C_t^2]^{\frac{1}{2}}$ and so there exists a constant c_t such that $C_t = c_t$ a.s. Taking the expectation shows that $c_t = p_i$.

Let us now explain our arguments to prove Theorem 1.4. In Section 2.1, we prove that for any coupling (X,Y,W) of solutions to (I), $L_t(D)=0$ where $D_t = d(\overline{X}_t, \overline{Y}_t)$. Next, we consider the Wiener stochastic flow of kernels K constructed in [8] which is a strong solution to the flows of kernels version of (2). Inspired by Tsirelson arguments [16], we then consider a perturbation K^r of $K, r \to 1$. The semigroup

$$Q_t^r(f \otimes g)(x,y) = \mathbb{E}[K_{0,t}f(x)K_{0,t}^r g(y)]$$

on G^2 is Feller. Interesting results about the Markov process (X^r, Y^r) associated to Q^r are known [2, 3, 16]. This process also converges in law as $r \to 1$ to the Wiener coupling (X,Y) described above. The passage to the limit $r \to 1$ allows to deduce the properties of (X,Y) mentioned in Theorem 1.4.

2. Proofs

2.1. **The local time of the distance.**

The subject of this paragraph is to prove the following

Proposition 2.1. *Let (X,Y,W) be a coupling of two solutions to (I) with $X_0 = Y_0 = 0$ and let $D_t = d(\overline{X}_t, \overline{Y}_t)$. Then $L_t(D)=0$.*

Proof. We follow the proof of Proposition 4.5 in [8] and first prove that a.s.

$$\int_{]0,+\infty]} L_t^a(D)\frac{da}{a} = \int_0^t 1_{\{D_s>0\}}\frac{d\langle D\rangle_s}{D_s} < \infty \tag{6}$$

By (5),

$$\begin{aligned} |\overline{X}_t| &= \sum_{i=1}^{N} \overline{X}^i_t = M^1_t + L_t(|X|) \\ |\overline{Y}_t| &= \sum_{i=1}^{N} \overline{Y}^i_t = M^2_t + L_t(|Y|) \end{aligned}$$

with

$$M^1_t = \sum_{i=1}^{N} \frac{1}{Np_i} \int_0^t 1_{\{X_s \in E_i\}} dB^X_s,\ \ M^2_t = \sum_{i=1}^{N} \frac{1}{Np_i} \int_0^t 1_{\{Y_s \in E_i\}} dB^Y_s$$

In particular,

$$\langle M^1 \rangle_t = \sum_{i=1}^{N} \frac{1}{(Np_i)^2} \int_0^t 1_{\{X_s \in E_i\}} ds,\ \ \langle M^2 \rangle_t = \sum_{i=1}^{N} \frac{1}{(Np_i)^2} \int_0^t 1_{\{Y_s \in E_i\}} ds$$

and

$$\langle M^1, M^2 \rangle_t = \sum_{i=1}^{N} \frac{1}{(Np_i)^2} \int_0^t 1_{\{X_s \in E_i,\ Y_s \in E_i\}} ds.$$

Proposition 7 [2] tells us that

$$\begin{aligned} D_t &- \int_0^t 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} (dM^1_s + dM^2_s) \\ &- \int_0^t 1_{\{\varepsilon(X_s) = \varepsilon(Y_s)\}} \mathrm{sgn}(M^1_s - M^2_s)(dM^1_s - dM^2_s) \end{aligned}$$

is a continuous increasing process. Consequently,

$$d\langle D \rangle_s = \sum_{i=1}^{N} \frac{1}{(Np_i)^2} 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} (1_{\{X_s \in E_i\}} + 1_{\{Y_s \in E_i\}}) ds \leq C 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} ds$$

where C is a positive constant. Note there exists $C' > 0$ such that $D_s \geq C'(|X_s| + |Y_s|)$ for all s such that $\varepsilon(X_s) \neq \varepsilon(Y_s)$. Thus, to get (6), it is sufficient to prove

$$\int_0^t 1_{\{X_s \neq 0, Y_s \neq 0\}} 1_{\{\epsilon(X_s) \neq \epsilon(Y_s)\}} \frac{ds}{|X|_s + |Y_s|} < \infty$$

Let us prove for example that

$$(1) = \int_0^t \frac{1}{|X_s| + |Y_s|} 1_{\{X_s \in E_1^*, Y_s \notin E_1\}} ds < \infty$$

Define $f(z) = |z|$ if $z \in E_1$ and $f(z) = -|z|$ if not and set $x_t = f(X_t), y_t = f(Y_t)$. Clearly

$$\frac{1}{|X_s| + |Y_s|} 1_{\{X_s \in E_1^*, Y_s \notin E_1\}} = \frac{1}{2} \frac{|\mathrm{sgn}(x_s) - \mathrm{sgn}(y_s)|}{|x_s - y_s|} 1_{\{y_s < 0 < x_s\}}.$$

As in [8], let $(f_n)_n \subset C^1(\mathbb{R})$ such that $f_n \to \mathrm{sgn}$ pointwise and $(f_n)_n$ is uniformly bounded in total variation. Defining $z_s^u = (1-u)x_s + uy_s$, we have

$$\begin{aligned}
(1) \quad &\leq \liminf_n \int_0^t 1_{\{y_s < 0 < y_s\}} \frac{|f_n(x_s) - f_n(y_s)|}{|x_s - y_s|} \frac{ds}{2} \\
&\leq \liminf_n \int_0^t 1_{\{y_s < 0 < y_s\}} \int_0^1 \left| f_n'(z_s^u) \right| du \frac{ds}{2}
\end{aligned}$$

Writing Freidlin-Sheu formula for the function f applied to X and Y shows that on $\{y_s < 0 < x_s\}$,

$$\frac{d}{ds} \langle z^u \rangle_s = u^2 + (1-u)^2 \geq \frac{1}{2}$$

Thus

$$\begin{aligned}
(1) \quad &\leq \liminf_n \int_0^1 \int_0^t 1_{\{y_s < 0 < x_s\}} \left| f_n'(z_s^u) \right| d\langle z^u \rangle_s du \\
&\leq \liminf_n \int_0^1 \int_{\mathbb{R}} \left| f_n'(a) \right| L_t^a(z^u) da du
\end{aligned}$$

So a sufficient condition for (1) to be finite is

$$\sup_{a \in \mathbb{R}, u \in [0,1]} \mathbb{E}\left[L_t^a(z^u) \right] < \infty$$

By Tanaka's formula

$$\begin{aligned}
\mathbb{E}\left[L_t^a(z^u) \right] &= \mathbb{E}\left[\left| z_t^u - a \right| \right] - \mathbb{E}\left[\left| z_0^u - a \right| \right] - \mathbb{E}\left[\int_0^t \mathrm{sgn}(z_s^u - a) dz_s^u \right] \\
&\leq \mathbb{E}\left[\left| z_t^u - z_0^u \right| \right] - \mathbb{E}\left[\int_0^t \mathrm{sgn}(z_s^u - a) dz_s^u \right]
\end{aligned}$$

Since x and y are two skew Brownian motions, it is easily seen that $\sup_{u \in [0,1]} \mathbb{E}\left[\left| z_t^u - z_0^u \right| \right] < \infty$. The same argument shows that

$$\mathbb{E}\left[\int_0^t \mathrm{sgn}(z_s^u - a) dz_s^u \right]$$

is uniformly bounded with respect to (u, a) and this shows that (1) is finite. Finally $\int_{]0,+\infty]} L_t^a(D) \frac{da}{a}$ is finite a.s. Since $\lim_{a \downarrow 0} L^a(D) = L^0(D)$, we deduce $L_t^0(D) = 0$. $\square$

2.2. **The Wiener stochastic flow of kernels solution of the interface SDE.**

Stochastic flows of kernels play a crucial role in our proofs. These objects were introduced in [13] and offer a robust method for the study of weak solutions as will be demonstrated in this paper (see also [7]). For K, a stochastic flow of kernels on G,

$$P_t^n f(x_1, \cdots, x_n) = \mathbb{E}\left[\int_G f(y_1, \cdots, y_n) K_{0,t}(x_1, dy_1) \cdots K_{0,t}(x_n, dy_n)\right] \tag{7}$$

defines a Feller semigroup on G^n. Moreover $(P^n)_{n\geq 1}$ is a compatible family (in a sense explained in [13]) of Feller semigroups acting respectively on $C_0(G^n)$ that uniquely characterize the law of K. Conversely, it has been proved in [13] that to each family of compatible Feller semigroups $(P^n)_{n\geq 1}$ is associated a (law unique) stochastic flow of kernels such that (7) holds for every $n \geq 1$.

Definition 2.2. *(Real white noise) A family $(W_{s,t})_{s\leq t}$ is called a real white noise if there exists a Brownian motion on the real line $(W_t)_{t\in\mathbb{R}}$, that is $(W_t)_{t\geq 0}$ and $(W_{-t})_{t\geq 0}$ are two independent standard Brownian motions such that for all $s \leq t$, $W_{s,t} = W_t - W_s$ (in particular, when $t \geq 0$, $W_t = W_{0,t}$ and $W_{-t} = -W_{-t,0}$).*

For a family of random variables $Z = (Z_{s,t})_{s\leq t}$, define $\mathcal{F}_{s,t}^Z = \sigma(Z_{u,v}, s \leq u \leq v \leq t)$ for all $s \leq t$. The extension to flows of kernels of the interface SDE is deduced from Definition (1.1) as follows.

Definition 2.3. *Let K be a stochastic flow of kernels on G and $\mathcal{W} = (\mathcal{W}^i, 1 \leq i \leq N)$ be a family of independent real white noises. We say that $(K, \mathcal{W})$ solves (I) if for all $s \leq t$, $f \in \mathcal{D}$ and $x \in G$, a.s.*

$$K_{s,t}f(x) = f(x) + \sum_{i=1}^N \int_s^t K_{s,u}(1_{E_i} f')(x) d\mathcal{W}_u^i + \frac{1}{2}\int_s^t K_{s,u} f''(x) du.$$

When $K = \delta_\varphi$, we only say (φ, W) solves (I). We say K is a Wiener solution if for all $s \leq t$, $\mathcal{F}_{s,t}^K \subset \mathcal{F}_{s,t}^{\mathcal{W}}$.

Note that when $K = \delta_\varphi$, we find the interface SDE as previously defined.

Assume $(K, \mathcal{W})$ solves (I), then for all $s \leq t \leq \tau_s(x) = \inf\{u \geq s : |x| + \mathcal{W}_{s,u}^i = 0\}$ and $x \in E_i$, a.s $K_{s,t}(x) = \delta_{x+\mathcal{W}_{s,t}^i}$ [8]. Since this holds for all x, we deduce that $\mathcal{F}_{s,t}^{\mathcal{W}} \subset \mathcal{F}_{s,t}^K$ for all $s \leq t$. For this reason, we may only say K solves (I) as $\mathcal{W}$ is a function of K.

It has been proved in [8], that given $\mathcal{W}$ a real white noise, there exists a stochastic flow of mappings φ such that $(\varphi, \mathcal{W})$ solves (I). Filtering this flow with respect to $\mathcal{W}$ gives rise to a Wiener stochastic flow of kernels $K_{s,t}(x) = E[\delta_{\varphi_{s,t}(x)}|\mathcal{F}_{s,t}^{\mathcal{W}}]$ (to be more precise $K_{s,t}(x)$ is defined along a dense set $\{(s_i, t_i, x_i)\}$ as a regular conditional expectation of $\varphi_{s_i,t_i}(x_i)$ given $\mathcal{F}^{\mathcal{W}}_{s_i,t_i}$ and then extended to all values of (s,t,x) by a density argument, see Lemma 3.2 in [13]). The Wiener stochastic flow of kernels K is unique up to modification in the sense that if K' is another flow of kernels such that $(K',\mathcal{W})$ solves (I) and $\mathcal{F}^{K'}_{s,t}\subset\mathcal{F}^{\mathcal{W}}_{s,t}$ for all $s\le t$, then for all $x\in G$ and $s\le t$ a.s $K_{s,t}(x)=K'_{s,t}(x)$.

By considering the Feller semigroups $Q^n_t(f\otimes g)(x,w)=\mathbb{E}[K^{\otimes n}_{0,t}f(x)g(w+\mathcal{W}_t)]$, one can prove that stochastic flows of kernels satisfying Definition (2.3) are the projective limits of compatible weak solutions to the sticky equation satisfying Definition (1.1) (see Proposition 2.1 in [6] for more details in a similar context).

Note that in the case $N=2$, the Wiener flow and the flow of mappings coincide: $K=\delta_\varphi$ while $K\neq\delta_\varphi$ if $N\ge 3$ and other flows solving (I) may exist in this case.

From now on, $(K,\mathcal{W})$ will denote the Wiener stochastic flow which solves (I). Let Q be the Feller semigroup on $G^2\times\mathbb{R}^N$ defined by

$$Q_t(f\otimes g\otimes h)(x,y,w)=\mathbb{E}[K_{0,t}f(x)K_{0,t}g(y)h(w+\mathcal{W}_{0,t})].$$

Denote by (X,Y,W) the Markov process associated to $(Q_t)_t$ and started from $(x,y,0)$.

Proposition 2.4. *(X,Y,W) is the Wiener coupling solution of (I) with $X_0=x$ and $Y_0=y$.*

Proof. We follow the proof of Lemma 6.2 in [8]. Let $\mathcal{F}_t=\sigma(X_s,Y_s,W_s;s\le t)$. Clearly X and Y are two $(\mathcal{F}_t)$-Walsh Brownian motions. We first prove that (X,W) (and similarly (Y,W)) solves (I). By Freidlin-Sheu's formula, for all $f\in\mathcal{D}$,

$$f(X_t)=f(x)+\int_0^t f'(X_s)dB^X_s+\frac{1}{2}\int_0^t f''(X_s)ds. \tag{8}$$

Thus for (X,W) to solve (I), it will suffice to show $B^X_t=\sum_{i=1}^N\int_0^t 1_{\{X_s\in E_i\}}dW^i_s$. Define

$$\mathcal{D}_1=\{f\in\mathcal{D}:\ f,f',f''\in C_0(G)\} \tag{9}$$

By the condition $f',f''\in C_0(G)$, it is meant here that for all $i\in[1,N]$, the limits $\lim_{x\in E_i,x\neq 0}f'(x)$ are equal and the same holds for f'' (not to confuse this with the conventions taken for $f'(0),f''(0)$). As $f\in\mathcal{D}$, this yields $\lim_{x\in E_i,x\neq 0}f'(x)=0$ for all i. Denote by A the generator of $\widetilde{Q}_t(f\otimes g)=Q_t(f\otimes I\otimes g)$ and $\mathcal{D}(A)$ its domain, then $\mathcal{D}_1\otimes C^2_0(\mathbb{R}^N)\subset\mathcal{D}(A)$ and for all $f\in\mathcal{D}_1$ and $g\in C^2_0(\mathbb{R}^N)$,

$$A(f\otimes g)(x,w)=\frac{1}{2}f(x)\Delta g(w)+\frac{1}{2}f''(x)g(w)+\sum_{i=1}^N(f'1_{E_i})(x)\frac{\partial g}{\partial w^i}(w)$$

In particular, for all $f \in \mathcal{D}_1$ and $g \in C_0^2(\mathbb{R}^N)$,

$$f(X_t)g(W_t) - \int_0^t A(f \otimes g)(X_s, W_s)ds \text{ is a martingale.} \tag{10}$$

On another hand, using (8), it is possible to write Itô's formula for $f(X_t)g(W_t)$. Combining this Itô's formula with (10), it comes that

$$\sum_{i=1}^N \int_0^t (f' 1_{E_i})(X_s) \frac{\partial g}{\partial w^i}(W_s)ds = \sum_{i=1}^N \int_0^t (f' 1_{E_i})(X_s) \frac{\partial g}{\partial w^i}(W_s)d\langle B^X, W^i\rangle_s$$

Since this holds for all $f \in \mathcal{D}_1$ and $g \in C_0^2(\mathbb{R}^N)$, by an approximation argument, we deduce $\langle B^X, W^i\rangle_t = \int_0^t 1_{\{X_s \in E_i\}}ds$ for all i.

Now it remains to prove that X and Y are independent given W. We will check that

$$\mathbb{E}\left[\prod_{i=1}^n f_i(X_{t_i})g_i(Y_{t_i})h_i(W_{t_i})\right] = \mathbb{E}\left[\prod_{i=1}^n E[f_i(X_{t_i})|W]\mathbb{E}[g_i(Y_{t_i})|W]h_i(W_{t_i})\right]$$

for all measurable and bounded test functions $(f_i, g_i, h_i)_i$. Since K is a measurable function of $\mathcal{W}$, we may assume K (and so $\mathcal{W}$) is defined on the same space as X and Y and that $W_t = \mathcal{W}_{0,t}$. By an easy induction (see the proof of Proposition 4.1 in [6]),

$$\mathbb{E}\left[\prod_{i=1}^n f_i(X_{t_i})g_i(Y_{t_i})h_i(W_{t_i})\right] = \mathbb{E}\left[\prod_{i=1}^n K_{0,t_i}f_i(x)K_{0,t_i}g_i(y)h_i(W_{t_i})\right] \tag{11}$$

From (11), we also deduce $K_{0,t_i}f_i(x) = E[f_i(X_{t_i})|\mathcal{F}^W_{0,t_i}]$ and $K_{0,t_i}g_i(y) = \mathbb{E}[g_i(Y_{t_i})|\mathcal{F}^W_{0,t_i}]$. This completes the proof. □

Since $(Q_t)_t$ is Feller, the proposition shows in particular that (X, Y) is a Feller process on G^2.

2.3. **Perturbation of the Wiener flow.**

We now fix a measurable function F such that if $\mathcal{V}$ is a real white noise on $\mathbb{R}^N$, then $K = F(\mathcal{V})$ solves (I). Let $\mathcal{W}$ and $\mathcal{W}'$ be two independent real white noises on $\mathbb{R}^N$ and for $r > 0$, define the real white noise on $\mathbb{R}^N$

$$\mathcal{W}^r = r\mathcal{W} + \sqrt{1-r^2}\mathcal{W}'$$

Let $K = F(\mathcal{W})$ and $K^r = F(\mathcal{W}^r)$ and note that (K, K^r) and (K^r, K) have the same law since this is also true for $(\mathcal{W}, \mathcal{W}^r)$ and $(\mathcal{W}^r, \mathcal{W})$. Define now

$$Q_t^r(f \otimes g \otimes h)(x, y, w) = \mathbb{E}[K_{0,t}f(x)K^r_{0,t}g(y)h(w + \mathcal{W}_{0,t})]$$

Then Q^r is a Feller semigroup on $G^2 \times \mathbb{R}^N$. Call (X^r, Y^r, W^r) the associated Markov process started from $(0,0,0)$, then again (X^r, Y^r) and (Y^r, X^r) have the same law.

Proposition 2.5. *The following assertions hold*

(i) $d\langle B^{X^r}, B^{Y^r}\rangle_t = r1_{\{\varepsilon(X_t^r)=\varepsilon(Y_t^r)\}}dt$.

(ii) $\int_0^t 1_{\{Y_s^r\neq 0\}} dL_s(|X^r|) = L_t(|X^r|)$ *and* $\int_0^t 1_{\{X_s^r\neq 0\}} dL_s(|Y^r|) = L_t(|Y^r|)$.

Proof. The proof of (i) follows similar arguments as in the proof of Proposition 2.4. Denote by A^r, the generator of $Q_t^r(f\otimes g\otimes \mathrm{Id})$, then $\mathcal{D}_1 \otimes \mathcal{D}_1 \subset \mathcal{D}(A^r)$ where $\mathcal{D}_1$ is given by (9). Moreover for all $f, g \in \mathcal{D}_1$,

$$A^r(f\otimes g)(x,y) = \frac{1}{2}f(x)g''(y) + \frac{1}{2}f''(x)g(y) + rf'(x)g'(y)1_{\{\varepsilon(x)=\varepsilon(y)\}}$$

Writing Itô's formula for $f(X^r)$ and $g(Y^r)$ and using martingales as in the proof of Proposition 2.4, we easily deduce (i). (ii) is Lemma 4.12 in [16] (see also [2, 3]).

□

2.4. **Proof of Theorem 1.4.**

Let (X, Y, W) be a Wiener coupling solutions of (I) with $X_0 = Y_0 = 0$.

Lemma 2.6. *As $r \to 1$, (X^r, Y^r, W^r) converges in law to (X, Y, W).*

Proof. Let $(r_n)_n$ be a sequence in $[0,1]$ such that $\lim_{n\to\infty} r_n = 1$. Slutsky lemma (see Theorem 1 in [2]) shows that for all $g: G \to \mathbb{R}$ measurable bounded, $y \in G$ and $t > 0$, $K_{0,t}^{r_n}g(y)$ converges in probability to $K_{0,t}g(y)$ and for any measurable function L, $L(\mathcal{W}^{r_n})$ converges in probability to $L(\mathcal{W})$. Now for any $p \geq 1$, $(f_i, g_i, h_i)_{1\leq i\leq p}$ bounded, $(t_i)_{1\leq i\leq p}$

$$\begin{aligned}\lim_n \mathbb{E}\left[\prod_{i=1}^p f_i(X_{t_i}^{r_n})g_i(Y_{t_i}^{r_n})h_i(W_{t_i}^{r_n})\right] &= \lim_n \mathbb{E}\left[\prod_{i=1}^p K_{0,t}^{r_n}f_i(0)K_{0,t}^{r_n}g_i(0)h_i(\mathcal{W}_{0,t_i}^{r_n})\right]\\ &= \mathbb{E}\left[\prod_{i=1}^p K_{0,t}f_i(0)K_{0,t}g_i(0)h_i(\mathcal{W}_{0,t_i})\right]\end{aligned}$$

This gives the desired result. □

Let us now recall Proposition 7 in [2].

Proposition 2.7. *Let Z^1 and Z^2 be two Walsh Brownian motion with respect to the same filtration. Denote by Λ the local time of $D_t = d(\overline{Z_t^1}, \overline{Z_t^2})$. Then*

$$D_t = M_t + \frac{1}{2}\Lambda_t + (N-2)\left(\int_0^t 1_{\{\overline{Z_s^1}\neq 0\}} dL_s^2 + \int_0^t 1_{\{\overline{Z_s^2}\neq 0\}} dL_s^1\right)$$

with M a martingale and L^1, L^2 are (see Proposition 5 in [2]) the bounded variation parts of $\overline{X}_t^i$ (defined by (4)) and $\overline{Y}_t^i$.

Note that $L_t^1 = \frac{1}{N} L_t(|Z^1|)$ and $L_t^2 = \frac{1}{N} L_t(|Z^2|)$ by (5).

Applying the previous proposition to (X^r, Y^r) and using Proposition 2.5 (ii), we get

$$d(\overline{X_t^r}, \overline{Y_t^r}) = M_t^r + \frac{1}{2}\Lambda_t^r + \frac{(N-2)}{N}\left(L_t(|X^r|) + L_t(|Y^r|)\right)$$

with M^r a martingale and Λ^r the local time of $d(\overline{X_t^r}, \overline{Y_t^r})$. In particular,

$$\mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})] \geq 2\frac{(N-2)}{N}\mathbb{E}[R_t] \tag{12}$$

with R a reflected Brownian motion started from 0.

The previous proposition applied to the Wiener coupling (X, Y) and the result of Section 2.1 show that

$$d(\overline{X_t}, \overline{Y_t}) = M_t + \frac{(N-2)}{N}\left(\int_0^t 1_{\{X_s \neq 0\}} dL_s(|Y|) + \int_0^t 1_{\{Y_s \neq 0\}} dL_s(|X|)\right) \tag{13}$$

with M a martingale. By the Balayage formula (see [14] on page 111 or the proof of Proposition 8 in [2]) and the fact that $L_t(D) = 0$,

$$d(\overline{X_t}, \overline{Y_t}) = \text{Martingale} + \frac{N-2}{N}\left(1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}| + 1_{\{Y_{g^1} \neq 0\}}|\overline{X_t}|\right) \tag{14}$$

where $g^1 := g_t^{\overline{X}}$ and $g^2 := g_t^{\overline{Y}}$. Admit for a moment that $\mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})]$ converges to $\mathbb{E}[d(\overline{X_t}, \overline{Y_t})]$. It comes from (12), (14), (X, Y) has the same law as (Y, X), that

$$2\frac{N-2}{N}\mathbb{E}\left[1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}|\right] \geq 2\frac{(N-2)}{N}\mathbb{E}[R_t]$$

Consequently

$$\mathbb{E}\left[|\overline{Y_t}|\right] \geq \mathbb{E}\left[1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y_t}|\right] \geq \mathbb{E}[R_t]$$

But $\mathbb{E}\left[|\overline{Y_t}|\right] = \mathbb{E}[R_t]$ and so $\overline{X}_{g^2} \neq 0$ a.s. By symmetry $\overline{Y}_{g^2} \neq 0$. Returning back to (14), we deduce that $d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$ is a martingale which proves Theorem 1.4 (i).

Note that $g^1 = g_t^X$, $g^2 = g_t^Y$ and Z has the same set of zeros as $\overline{Z}$ for Z a Walsh Brownian motion. This shows that $X_{g_t^Y} \neq 0$ and $Y_{g_t^X} \neq 0$ a.s. In particular $g_t^X \neq g_t^Y$ a.s and since $\{X_t = Y_t\} \subset \{g_t^X = g_t^Y\}$ (as X, Y follow the same Brownian motion on the same ray), Theorem 1.4 (ii) is also proved.

Remark 2.8. *Using the convergence of* $\mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})]$ *to* $\mathbb{E}[d(\overline{X_t}, \overline{Y_t})]$, *(12) and (13), we easily deduce that*

$$\int_0^t 1_{\{X_s \neq 0\}} dL_s(|Y|) = L_t(|Y|); \int_0^t 1_{\{Y_s \neq 0\}} dL_s(|X|) = L_t(|X|)$$

which is similar to Proposition 2.5 (ii).

Now it remains to prove the following

Lemma 2.9. *We have*

$$\lim_{r \to 1} \mathbb{E}[d(\overline{X_t^r}, \overline{Y_t^r})] = \mathbb{E}[d(\overline{X_t}, \overline{Y_t})].$$

Proof. From the convergence in law given in Lemma 2.6, it is easily seen that $(\overline{X^r}, \overline{Y^r})$ converges in law to $(\overline{X}, \overline{Y})$. This is because $\overline{Z}$ is a continuous of Z. Let r_n be a sequence converging to 1. Skorokhod representation theorem says that it is possible to construct on some probability space $(\Omega', \mathcal{A}', \mathbb{P}')$, random variables $(X^n, Y^n)_{n \geq 1}$ and (X^∞, Y^∞) such that for each n, (X^n, Y^n) has the same law as $(\overline{X^{r_n}}, \overline{Y^{r_n}})$ and (X^∞, Y^∞) has the same law as $(\overline{X}, \overline{Y})$ and moreover (X^n, Y^n) converges a.s. to (X^∞, Y^∞). The lemma holds as soon as we prove

$$\lim_{n \to \infty} \mathbb{E}[d(X_t^n, Y_t^n)] = \mathbb{E}[d(X_t^\infty, Y_t^\infty)].$$

For each $\epsilon > 0$,

$$\begin{aligned}
\mathbb{E}[d(X_t^n, X_t^\infty)] &\leq \epsilon + \mathbb{E}[d(X_t^n, X_t^\infty) 1_{\{d(X_t^n, X_t^\infty) > \epsilon\}}] \\
&\leq \epsilon + \mathbb{E}[d(X_t^n, X_t^\infty)^2]^{1/2} \mathbb{P}[d(X_t^n, X_t^\infty) > \epsilon]^{1/2} \\
&\leq \epsilon + C \times \mathbb{P}[d(X_t^n, X_t^\infty) > \epsilon]^{1/2}
\end{aligned}$$

for some finite constant C. Thus, $\limsup_n \mathbb{E}[d(X_t^n, X_t^\infty)] = 0$ and similarly $\limsup_n \mathbb{E}[d(Y_t^n, Y_t^\infty)] = 0$. The lemma follows now using the triangle inequality. $\square$

Let us now prove Theorem 1.4 (iii).

Denote by $\mathcal{G}$ the natural filtration of the Wiener coupling (X, Y). For a random time R, let us recall the following σ-fields (see [2] on page 286)

$$\begin{aligned}
\mathcal{G}_R &= \sigma(X_R : X \text{ is a } \mathcal{G} - \text{ optional process}) \\
\mathcal{G}_{R+} &= \sigma(X_R : X \text{ is a } \mathcal{G} - \text{ progressive process})
\end{aligned}$$

Let $g^1 = g_t^X, g^2 = g_t^Y$. It is known (see for example Proposition 19 in [2]), that $\varepsilon(X_t)$ is independent of $\mathcal{G}_{g^1}$ and $\varepsilon(X_t)$ is $\mathcal{G}_{g^1+}$ measurable (the same holds for Y). The event $\{g^1 < g^2\} \in \mathcal{G}_{g^2}$ (see Proposition 13 in [2]) and on this event, $\varepsilon(X_t) = \limsup_{\epsilon \to 0+} \varepsilon(X_{(g^1+\epsilon) \wedge g^2})$. Since $(g^1 + \epsilon) \wedge g^2 \leq g^2$, by Proposition 13 in [2] again,

$\mathcal{G}_{(g^1+\epsilon)\wedge g^2} \subset \mathcal{G}_{g^2}$ and so $\limsup_{\epsilon\to 0+} \varepsilon(X_{(g^1+\epsilon)\wedge g^2})$ is $\mathcal{G}_{g^2}$-measurable. Take f an indicator function on a subset of $\{1, \cdots, N\}$. By conditioning with respect to $\mathcal{G}_{g^2}$, we deduce

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^1<g^2\}}] = \mathbb{E}[f(\varepsilon(Y_t)]\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1<g^2\}}]$$

and

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^2<g^1\}}] = \mathbb{E}[f(\varepsilon(X_t)]\mathbb{E}[f(\varepsilon(Y_t))1_{\{g^2<g^1\}}]$$

Summing, we get

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = \mathbb{E}[f(\varepsilon(X_t)]\big(\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1<g^2\}}] + \mathbb{E}[f(\varepsilon(Y_t))1_{\{g^2<g^1\}}]\big)$$

But $\{g^1 < g^2\} = \{g^2 < g^1\}^c$ belongs to $\mathcal{G}_{g^1}$ which is independent of $\varepsilon(X_t)$ so that

$$\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1<g^2\}}] = \frac{1}{2}\mathbb{E}[f(\varepsilon(X_t))]$$

By symmetry, we get $\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = \mathbb{E}[f(\varepsilon(X_t))]\mathbb{E}[f(\varepsilon(Y_t))]$.

Final remarks and open problems

There are several interesting open problems related to the interface SDE. Let us mention some of them.

- What is the conditional law of $|X_t|$ (and more generally of X_t) given W?
- What are the couplings which "interpolate" between the coalescing coupling and the Wiener one?
- What are the stochastic flows which "interpolate" between the coalescing flow and the Wiener one?

Let us finish with the following remark regarding the first question. Let W be a standard Brownian motion and let $X^1, X^2, \cdots$ be Walsh Brownian motions started from 0 such that (X^i, W) is solution of (I) with $X_0^i = 0$ for all i and $X^1, X^2, \cdots$ are independent given W. Then by the law of the large numbers for all $f \in C_0(G)$, a.s $\mathbb{E}[f(X_t^1)|W] = \lim_n \frac{1}{n}\sum_{i=1}^n f(X_t^i)$. (see Section 2.6 in [13]).

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