Approximation to points in the plane by SL (2, Z)-orbits

Michel Laurent, Arnaldo Nogueira

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Abstract

The orbit $\mathrm{SL}(2,\mathbb{Z})\mathbf{x}$ is dense in $\mathbb{R}^2$ when the initial point $\mathbf{x}\in\mathbb{R}^2$ has irrational slope. We refine this result from a diophantine perspective. For any target point $\mathbf{y}\in\mathbb{R}^2$, we introduce two exponents $\mu(\mathbf{x},\mathbf{y})$ and $\hat{\mu}(\mathbf{x},\mathbf{y})$ that measure the approximation to $\mathbf{y}$ by elements $\gamma\mathbf{x}$ of the orbit in terms of the size of γ. We estimate both exponents under various conditions. Our results are optimal when the slope of the target point $\mathbf{y}$ is a rational number. In that case we express $\mu(\mathbf{x},\mathbf{y})$ and $\hat{\mu}(\mathbf{x},\mathbf{y})$ in terms of the irrationality measure of the slope of $\mathbf{x}$.

1 Introduction and results

We view the real plane $\mathbb{R}^2$ as a space of column vectors on which the group $\Gamma=\mathrm{SL}(2,\mathbb{Z})$ acts by left multiplication. Let $\mathbf{x}=\begin{pmatrix} x_1\\ x_2\end{pmatrix}$ be a point in $\mathbb{R}^2$ with irrational slope $\xi=x_1/x_2$. The orbit $\Gamma\mathbf{x}$ is then dense in $\mathbb{R}^2$. The assertion follows from density results of J. S. Dani [5] for lattice orbits in homogeneous spaces, see also [4] (Propriété 4.4 in Chapter V), as well as a more elementary proof in [6]. The study of lattice orbit distribution has been the subject of numerous works in a wide setting. In particular the articles [8, 11, 14, 15] are concerned in counting the number of elements $\gamma\mathbf{x}$ belonging to various sets under restriction on the size of γ, and [9] regards the approximation to a radius with rational slope. Here we are interested in the effective approximation of a given point $\mathbf{y}\in\mathbb{R}^2$ by points of the form $\gamma\mathbf{x}$, where $\gamma\in\Gamma$, in terms of the size of γ.

As a guide to our results, let us recall some classical results on inhomogeneous approximation in $\mathbb{R}$. The Minkowski Theorem, see for instance [3, Chapter III],

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asserts that for any irrational number ξ and any real number y not belonging to $\mathbb{Z}\xi + \mathbb{Z}$, there exist infinitely many pairs of integers (u, v), with $v \neq 0$, such that

$$|v\xi + u - y| \le \frac{1}{4|v|}. \tag{1.1}$$

Our first goal is to obtain an analogous result for the orbit $\Gamma \begin{pmatrix} \xi \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$. Let us equip $\mathbb{R}^2$ with the supremum norm $|\mathbf{x}| = \max(|x_1|, |x_2|)$, and for any matrix γ, denote as well by $|\gamma|$ the maximum of the absolute values of the entries of γ. Notice that any choice of norm on the algebra of matrices $\mathrm{M}_2(\mathbb{R})$ would lead to the same exponents with possibly different constants. We distinguish three cases, according as the target point $\mathbf{y}$ coincides with the origin $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or it lies on a line passing through the origin whose slope is either rational or irrational.

Theorem 1. *Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ with irrational slope.*
(i) *There exist infinitely many matrices $\gamma \in \Gamma$ such that*

$$|\gamma\mathbf{x}| \le \frac{|\mathbf{x}|}{|\gamma|}. \tag{1.2}$$

(ii) *Let $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be a point $\in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Assume that either the slope y_1/y_2 is a rational number a/b, where a and b are coprime integers, or that $y_2 = 0$ in which case we put $a = 1$ and $b = 0$. Then, there exist infinitely many matrices $\gamma \in \Gamma$ such that*

$$|\gamma\mathbf{x} - \mathbf{y}| \le \frac{c}{|\gamma|^{1/2}} \quad \text{with} \quad c = 2\sqrt{3}\max(|a|, |b|)|\mathbf{x}|^{1/2}|\mathbf{y}|^{1/2}. \tag{1.3}$$

(iii) *If the slope y_1/y_2 of the point $\mathbf{y}$ is irrational, there exist infinitely many matrices $\gamma \in \Gamma$ satisfying*

$$|\gamma\mathbf{x} - \mathbf{y}| \le \frac{c'}{|\gamma|^{1/3}} \quad \text{with} \quad c' = 7\sqrt{5}|\mathbf{x}|^{1/3}|\mathbf{y}|^{2/3}. \tag{1.4}$$

The exponents 1 and 1/2 of $|\gamma|$ occurring respectively in (1.2) and (1.3) are best possible. We are also interested in *uniform* versions of Theorem 1, in the sense of [2]. We first state the uniform version of Minkowski's Theorem. To this purpose, we need the standard notion of *irrationality measure* of an irrational number.

Definition 1. *For any irrational real number α, we denote by $\omega(\alpha)$ the supremum of the numbers ω such that the inequation*

$$|v\alpha + u| \le |v|^{-\omega}$$

has infinitely many integer solutions (v, u).

Then, for any real number $\mu < 1/\omega(\xi)$ and any positive real number T sufficiently large in terms of μ, there exist integers u, v such that

$$\max(|u|, |v|) \le T \quad \text{and} \quad |v\xi + u - y| \le T^{-\mu}. \tag{1.5}$$

See for instance the main theorem of [2], as well as the comments explaining the link with the claims (1.1) and (1.5). More information and results can be found in [1, 2, 3], including metrical theory and higher dimensional generalizations.

In view of the above results, let us define two exponents measuring respectively the usual and the uniform approximation to a point $\mathbf{y}$ by elements of the orbit $\Gamma\mathbf{x}$. We follow the notational conventions of [2].

Definition 2. *Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{R}^2$. We denote by $\mu(\mathbf{x}, \mathbf{y})$ the supremum of the real numbers μ for which there exist infinitely many matrices $\gamma \in \Gamma$ satisfying the inequality*

$$|\gamma\mathbf{x} - \mathbf{y}| \le |\gamma|^{-\mu}.$$

We denote by $\hat{\mu}(\mathbf{x}, \mathbf{y})$ the supremum of the exponents μ such that for any sufficiently large positive real number T, there exists a matrix $\gamma \in \Gamma$ satisfying

$$|\gamma| \le T \quad \textit{and} \quad |\gamma\mathbf{x} - \mathbf{y}| \le T^{-\mu}.$$

Clearly $\mu(\mathbf{x}, \mathbf{y}) \ge \hat{\mu}(\mathbf{x}, \mathbf{y}) \ge 0$, unless $\mathbf{y}$ belongs to the orbit $\Gamma\mathbf{x}$ in which case $\hat{\mu}(\mathbf{x}, \mathbf{y}) = +\infty$. We can now state the

Theorem 2. *Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ with irrational slope ξ.*
(i) *We have*

$$\mu(\mathbf{x}, \mathbf{0}) = 1 \quad \textit{and} \quad \hat{\mu}(\mathbf{x}, \mathbf{0}) = \frac{1}{\omega(\xi)}. \tag{1.6}$$

(ii) *Let $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be a point $\in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Assume that either the slope $y = y_1/y_2$ is rational or that $y_2 = 0$. Then, we have the equalities*

$$\mu(\mathbf{x}, \mathbf{y}) = \frac{\omega(\xi)}{\omega(\xi) + 1} \ge \frac{1}{2} \quad \textit{and} \quad \hat{\mu}(\mathbf{x}, \mathbf{y}) = \frac{1}{\omega(\xi) + 1}. \tag{1.7}$$

(iii) *If the slope y of the point $\mathbf{y}$ is an irrational number, then the following lower bounds hold*

$$\mu(\mathbf{x},\mathbf{y}) \geq \frac{1}{3} \quad \textit{and} \quad \hat{\mu}(\mathbf{x},\mathbf{y}) \geq \frac{\omega(y)+1}{2(2\omega(y)+1)\omega(\xi)} \geq \frac{1}{4\omega(\xi)}. \tag{1.8}$$

If ξ is a Liouville number, meaning that $\omega(\xi) = +\infty$, the equalities (1.7) obviously read $\mu(\mathbf{x},\mathbf{y}) = 1$ and $\hat{\mu}(\mathbf{x},\mathbf{y}) = 0$. When the slope y is rational, an explicit lower bound for the distance between $\gamma\mathbf{x}$ and $\mathbf{y}$ will be given in Theorem 4 of Section 8, which brings further information in terms of the convergents of ξ.

Subsequent to proving the results described in the present paper, we learned that Maucourant and Weiss [12] have obtained the weaker estimates

$$\mu(\mathbf{x},\mathbf{y}) \geq \frac{1}{144} \quad \text{and} \quad \hat{\mu}(\mathbf{x},\mathbf{y}) \geq \frac{1}{72(\omega(\xi)+1)},$$

as a consequence of effective equidistribution estimates for unipotent trajectories in $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ (use Corollary 1.9 in [12] and substitute $\delta_0 = 1/48$, which is an admissible value as mentioned in Remark 1.6). Our method is totally different. In another related work [9], Guilloux observes the existence of gaps around rational directions in the repartition of the *cloud* of points $\{\gamma\mathbf{x}\,;\gamma\in\Gamma, |\gamma| \leq T\}$ for large T. In our setting, he proves the upper bound $\mu(\mathbf{x},\mathbf{y}) \leq 1$ for any point $\mathbf{y}$ with rational slope.

We now discuss upper bounds for our exponents $\mu(\mathbf{x},\mathbf{y})$ and $\hat{\mu}(\mathbf{x},\mathbf{y})$. Applying Proposition 8 of [2] to the two inequalities of the form (1.5) determined by the two coordinates of $\gamma\mathbf{x}-\mathbf{y}$, we obtain the bound $\hat{\mu}(\mathbf{x},\mathbf{y}) \leq \omega(\xi)$ for any point $\mathbf{y}$ which does not belong to the orbit $\Gamma\mathbf{x}$. Moreover, the stronger upper bound

$$\hat{\mu}(\mathbf{x},\mathbf{y}) \leq \frac{1}{\omega(\xi)} \leq \omega(\xi)$$

holds for almost all (*) points $\mathbf{y}$, since the main theorem of [2] tells us that the exponent μ in (1.5) cannot be larger than $1/\omega(\xi)$ for almost all real number y. As for the exponent $\mu(\mathbf{x},\mathbf{y})$, it may be arbitrarily large when $\mathbf{y}$ is a point of *Liouville* type, meaning that $\mathbf{y}$ is the limit of a fast converging sequence $(\gamma_n\mathbf{x})_{n\geq 1}$ of points of the orbit. However, $\mu(\mathbf{x},\mathbf{y})$ is bounded almost everywhere. Projecting as above on both coordinates, the main theorem of [2] shows that the upper bound $\mu(\mathbf{x},\mathbf{y}) \leq 1$ holds for almost all points $\mathbf{y}$. Here is a stronger statement.

(*) Throughout the paper, the expression *almost all* refers to Lebesgue measure in the ambient space.

Theorem 3. *Let* $\mathbf{x}$ *be a point in* $\mathbb{R}^2$ *with irrational slope and let* y *be an irrational number having irrationality measure* $\omega(y) = 1$. *Then, the upper bound*

$$\mu(\mathbf{x}, \mathbf{y}) \leq \frac{1}{2}$$

holds for almost all points $\mathbf{y}$ *of the line* $\mathbb{R}\begin{pmatrix} y \\ 1 \end{pmatrix}$.

It follows from Theorems 2 and 3 that, $\mathbf{x}$ being fixed, we have the estimate

$$\frac{1}{3} \leq \mu(\mathbf{x}, \mathbf{y}) \leq \frac{1}{2}$$

for almost all points $\mathbf{y} \in \mathbb{R}^2$, since the assumption $\omega(y) = 1$ occurring in Theorem 3 is valid for almost all real numbers y. Moreover the maximal value $1/2$ is reached for any point $\mathbf{y} \neq \mathbf{0}$ lying on a radius with rational slope when the slope ξ of $\mathbf{x}$ has irrationality measure $\omega(\xi) = 1$. We address the problem of finding the generic value of the exponents $\mu(\mathbf{x}, \mathbf{y})$ and $\hat{\mu}(\mathbf{x}, \mathbf{y})$ on $\mathbb{R}^2 \times \mathbb{R}^2$. An heuristic (but optimistic) argument of equidistribution suggests that we should have

$$\mu(\mathbf{x}, \mathbf{y}) = \hat{\mu}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}$$

for almost all pairs of points $(\mathbf{x}, \mathbf{y})$. Note that generic values for both exponents do exist by the following remark due to B. Weiss. The natural linear action on $\mathbb{R}^2 \times \mathbb{R}^2$ of the group $\Gamma \times \Gamma$ is ergodic with respect to Lebesgue measure [13]. Therefore, the $(\Gamma \times \Gamma)$-invariant measurable functions $\mu(\mathbf{x}, \mathbf{y})$ and $\hat{\mu}(\mathbf{x}, \mathbf{y})$ are constant almost everywhere on $\mathbb{R}^2 \times \mathbb{R}^2$.

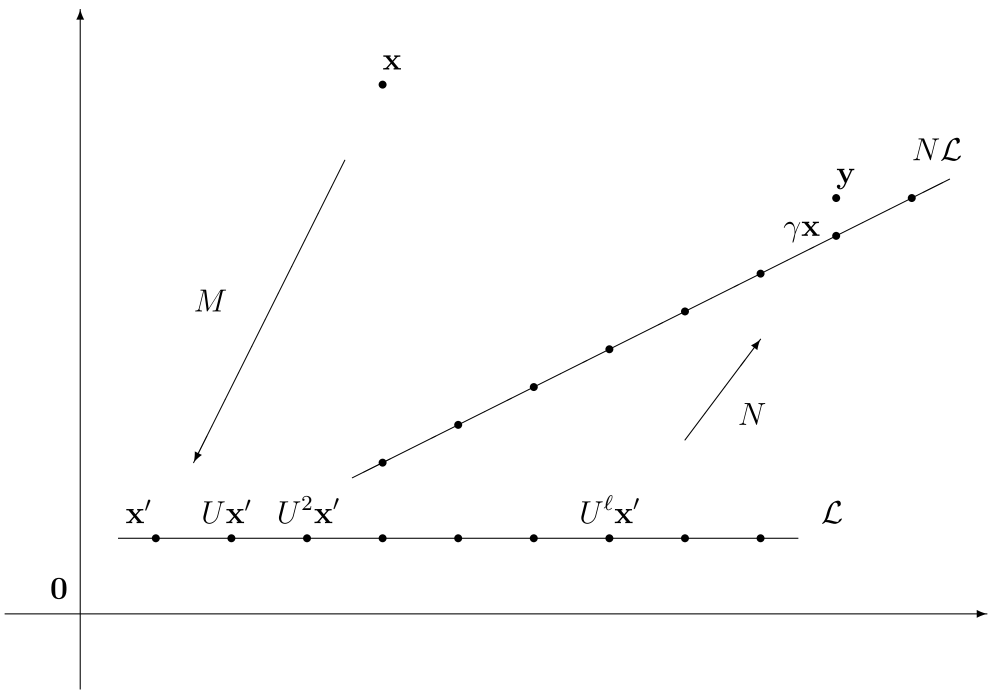
The proofs are based on an explicit construction of approximating matrices $\gamma \in \Gamma$. The process may be geometrically described as follows. The origin $\mathbf{0}$ plays a specific role in our approach. We first associate to every fixed irrational number ξ a sequence of matrices M_k in Γ, called *convergent matrices*, sending any point $\mathbf{x}$ with slope ξ towards the origin. Secondly, we introduce an other sequence of matrices N_j in Γ transforming the horizontal axis $\mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ into a line passing through $\mathbf{0}$ whose slope tends to the slope y of the target point $\mathbf{y}$ as j tends to infinity. These matrices N_j are essentially the inverse of the convergent matrices corresponding to the slope y. Set $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and fix two matrices $M = M_k$ and $N = N_j$ as above. We consider products of the form

$$\gamma = NU^{\ell}M,$$

where ℓ ranges over $\mathbb{Z}$. Therefore, we have the relation

$$\gamma \mathbf{x} = N U^{\ell} \mathbf{x}',$$

where $\mathbf{x}' = M\mathbf{x}$ is a point close to the origin. Observe now that the unipotent matrix U leaves invariant any horizontal line $\left\{ \begin{pmatrix} z \\ \epsilon \end{pmatrix} ; z \in \mathbb{R} \right\}$ and acts on this line as a translation with step $|\epsilon|$. Let $\mathcal{L}$ be the horizontal line passing through the point $\mathbf{x}'$. Then, the points $U^{\ell}\mathbf{x}'$, $\ell \in \mathbb{Z}$, form a lattice on $\mathcal{L}$ whose step is at most $|\mathbf{x}'|$. Transforming this lattice by the linear transformation N, we obtain a lattice on the line $N\mathcal{L}$ with step $\leq 2|N||\mathbf{x}'|$. On the other hand, the point $\mathbf{y}$ is close to the line $N\mathcal{L}$, since $\mathcal{L}$ is close to the axis $\mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ whose image by N is a line passing near to $\mathbf{y}$. Then, under suitable conditions, there exists an element $\gamma\mathbf{x} = NU^{\ell}\mathbf{x}'$ of the lattice contained in $N\mathcal{L}$, whose distance to $\mathbf{y}$ is not much larger than the step of this lattice. The argument will be quantified in Section 4 and translated in terms of inequalities. We thus obtain fairly good approximations in term of the norm $|\gamma|$, depending upon the diophantine nature of the slopes ξ and y. Here is a picture showing the motion from $\mathbf{x}$ to $\gamma\mathbf{x}$.



It would be interesting to extend our decomposition method to other lattices Γ in $\mathrm{SL}(2,\mathbb{R})$. Observe that the rational slopes, namely the set of cusps of the Fuchsian group $\mathrm{PSL}(2,\mathbb{Z})$, play a prominent role in our approach.

The paper is organized as follows. The convergent matrices are defined in Section 2 in terms of continued fractions of ξ. As a first application, the easy case $\mathbf{y}=\mathbf{0}$ is investigated in Section 3. In Section 4, we state two basic lemmas involving the above approximating matrices γ. We apply the method in Sections 5 and 6, thus obtaining various lower bounds for $\mu(\mathbf{x},\mathbf{y})$ and $\hat{\mu}(\mathbf{x},\mathbf{y})$ depending on whether the slope of the target point $\mathbf{y}$ is rational or not. Assuming now conversely that $\gamma\mathbf{x}$ is close to $\mathbf{y}$, it turns out that we necessarily have a decomposition of the form $\gamma = NGM$, where M and N are as above with an intermediate factor G of small norm. Such a statement will be provided by the important Lemma 7 in Section 7. Then, we deduce from this fact upper bounds for our exponents $\mu(\mathbf{x},\mathbf{y})$ and $\hat{\mu}(\mathbf{x},\mathbf{y})$, which are valid for almost all points $\mathbf{y}\in\mathbb{R}^2$, including all points $\mathbf{y}$ with rational slope. In the latter case, it turns out that the upper and lower bounds thus obtained coincide; hence we get the exact formulas (1.7) for $\mu(\mathbf{x},\mathbf{y})$ and $\hat{\mu}(\mathbf{x},\mathbf{y})$. The proofs are displayed in Sections 7 and 8. The final Section 9 deals with additional constraints of signs.

Throughout the paper, we write $A \ll B$ when there exists a positive constant c such that $A \le cB$ for all values of the parameters under consideration (usually the indices j and k). The coefficient c may possibly depend upon the points $\mathbf{x}$ and $\mathbf{y}$. As usual, the notation $A \asymp B$ means that $A \ll B$ and $A \gg B$.

2 Convergent matrices

Let ξ be an irrational number and let $(p_k/q_k)_{k\ge 0}$ be the sequence of convergents of ξ. We set $\epsilon_k = q_k\xi - p_k$. The theory of continued fractions, see for instance the monography [10], tells us that the sign of ϵ_k is alternatively positive or negative according to whether k is even or odd, and that the estimate

$$\frac{1}{2q_{k+1}} \le |\epsilon_k| \le \frac{1}{q_{k+1}} \tag{2.1}$$

holds for $k\ge 0$. For later use, note as a consequence of (2.1) that, when $\omega(\xi)$ is finite, we have the upper bound $q_{k+1}\le q_k^{\omega}$ for any real number $\omega > \omega(\xi)$ provided k is large enough, while if $\omega<\omega(\xi)$, the lower bound $q_{k+1}\ge q_k^{\omega}$ holds for infinitely many k.

For any positive integer k, we set

$$M_k = \begin{pmatrix} q_k & -p_k \\ -q_{k-1} & p_{k-1} \end{pmatrix} \quad \text{or} \quad M_k = \begin{pmatrix} q_k & -p_k \\ q_{k-1} & -p_{k-1} \end{pmatrix}$$

respectively, when k is even or odd. In both cases the matrix M_k belongs to Γ and has norm $|M_k| = \max(q_k, |p_k|)$. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a point with slope $\xi = x_1/x_2$. Then, we have

$$M_k\mathbf{x} = x_2 \begin{pmatrix} \epsilon_k \\ (-1)^{k-1}\epsilon_{k-1} \end{pmatrix} = x_2 \begin{pmatrix} \epsilon_k \\ |\epsilon_{k-1}| \end{pmatrix},$$

noting that the second coordinate $(-1)^{k-1}\epsilon_{k-1}$ is always positive and thus equals $|\epsilon_{k-1}|$.

The matrices M_k will be called *convergent matrices* of ξ. The name is justified by the fact that the numerator and the denominator of two consecutive convergents of ξ are given, up to a sign, by the columns of the inverse matrix

$$M_k^{-1} = \begin{pmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{pmatrix} \quad \text{or} \quad M_k^{-1} = \begin{pmatrix} -p_{k-1} & p_k \\ -q_{k-1} & q_k \end{pmatrix}.$$

3 Approximation to the origin

We first consider the easier case where the target point $\mathbf{y}$ equals the origin $\mathbf{0}$, and prove claims (1.2) and (1.6) in this section. We assume without loss of generality that $\mathbf{x} = \begin{pmatrix} \xi \\ 1 \end{pmatrix}$.

Lemma 1. *Let k be a positive integer and let $\gamma \in \Gamma$ with norm $|\gamma| \le q_{k+1}/2$. Then, we have the lower bound*

$$|\gamma\mathbf{x}| \ge \frac{1}{2q_k}.$$

Proof. We argue by contradiction. On the contrary, suppose that $|\gamma\mathbf{x}| < 1/(2q_k)$. Put $\gamma = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix}$ and $G = \gamma M_k^{-1}$. Assume first that k is even. We find the formula

$$G = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix} \begin{pmatrix} q_k & -p_k \\ -q_{k-1} & p_{k-1} \end{pmatrix}^{-1} = \begin{pmatrix} p_{k-1}v_1 + q_{k-1}u_1 & p_kv_1 + q_ku_1 \\ p_{k-1}v_2 + q_{k-1}u_2 & p_kv_2 + q_ku_2 \end{pmatrix}$$

$$= \begin{pmatrix} -v_1(q_{k-1}\xi - p_{k-1}) + q_{k-1}(v_1\xi + u_1) & -v_1(q_k\xi - p_k) + q_k(v_1\xi + u_1) \\ -v_2(q_{k-1}\xi - p_{k-1}) + q_{k-1}(v_2\xi + u_2) & -v_2(q_k\xi - p_k) + q_k(v_2\xi + u_2) \end{pmatrix}.$$

Bounding from above the norm of the second column of the above matrix gives

$$\max\Big(|-v_1(q_k\xi-p_k)+q_k(v_1\xi+u_1)|, |-v_2(q_k\xi-p_k)+q_k(v_2\xi+u_2)|\Big) \le \frac{|\gamma|}{q_{k+1}}+q_k|\gamma\mathbf{x}| < 1.$$

Since G has integer entries, it follows that the second column of G equals $\mathbf{0}$. The case k odd leads to the same conclusion. Contradiction with $\det G = 1$. $\square$

For any $\gamma \in \Gamma$ of norm $|\gamma| > q_1/2$, let k be the integer defined by the estimate

$$\frac{q_k}{2} < |\gamma| \leq \frac{q_{k+1}}{2}.$$

It follows from Lemma 1 that

$$|\gamma \mathbf{x}| \geq \frac{1}{2q_k} \geq \frac{1}{4|\gamma|}.$$

Therefore $\mu(\mathbf{x}, \mathbf{0}) \leq 1$. On the other hand, we have that

$$|M_k| = \max(|p_k|, q_k) \qquad \text{and} \qquad |M_k \mathbf{x}| = \max(|\epsilon_k|, |\epsilon_{k-1}|) = |\epsilon_{k-1}| \leq \frac{1}{q_k},$$

by (2.1). Observe that $p_k = q_k\xi - \epsilon_k$ has absolute value $\leq |\xi| q_k$ if ϵ_k and ξ have the same sign. Hence (1.2) holds for $\gamma = M_k$ when k is either odd or even.

It obviously follows from (1.2) that $\mu(\mathbf{x}, \mathbf{0}) = 1$, thus proving the first assertion of (1.6). The proof of the equality $\hat{\mu}(\mathbf{x}, \mathbf{0}) = 1/\omega(\xi)$ is similar. For any real number $\omega < \omega(\xi)$, there exist infinitely many k such that $q_{k+1} \geq q_k^{\omega}$. Put $T = q_{k+1}/2$. For all $\gamma \in \Gamma$ with norm $|\gamma| \leq T$, Lemma 1 gives the lower bound

$$|\gamma \mathbf{x}| \geq \frac{1}{2q_k} \geq \frac{1}{2(2T)^{1/\omega}}.$$

Therefore $\hat{\mu}(\mathbf{x}, \mathbf{0}) \leq 1/\omega$, and letting ω tend to $\omega(\xi)$, we obtain the upper bound $\hat{\mu}(\mathbf{x}, \mathbf{0}) \leq 1/\omega(\xi)$. On the other hand, the choice of the matrix $\gamma = M_k$ for $|M_k| \leq T < |M_{k+1}|$ shows that $\hat{\mu}(\mathbf{x}, \mathbf{0}) \geq 1/\omega(\xi)$. Hence the equality $\hat{\mu}(\mathbf{x}, \mathbf{0}) = 1/\omega(\xi)$ holds.

4 Construction of approximants

It is well known that the modular group Γ is generated by the two matrices

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Observe that the matrix J acts on $\mathbb{R}^2$ as a rotation by a right angle.

From now on, we assume that the target point $\mathbf{y}$ differs from $\mathbf{0}$. Note that $|J\mathbf{z}| = |\mathbf{z}|$ for all $\mathbf{z} \in \mathbb{R}^2$. Replacing possibly $\mathbf{x}$ by $J\mathbf{x}$ or $\mathbf{y}$ by $J\mathbf{y}$, we shall assume throughout the paper that

$$|\mathbf{x}| = |x_2| \quad \text{and} \quad |\mathbf{y}| = |y_2|,$$

so that the slopes $\xi = x_1/x_2$ and $y = y_1/y_2$ of the points $\mathbf{x}$ and $\mathbf{y}$ satisfy

$$0 < |\xi| < 1 \quad \text{and} \quad |y| \leq 1.$$

Recall the convergent matrices M_k associated to ξ. We consider approximating matrices of the form $\gamma = NU^\ell M_k$, where ℓ is an integer and N is a matrix in Γ, which will be specified later. We first estimate the norm of γ.

Lemma 2. *Let k be a positive integer, ℓ be an integer, and let $N = \begin{pmatrix} t & t' \\ s & s' \end{pmatrix}$ belong to Γ. Put $\gamma = NU^\ell M_k \in \Gamma$. Then*

$$\left|\ell q_{k-1} + (-1)^{k-1} q_k\right| |s| - |s'| q_{k-1} \leq |\gamma| \leq |\ell||N| q_{k-1} + 2|N| q_k.$$

Proof. Since $|\xi| < 1$, we have $|p_k| \leq q_k$ for all $k \geq 0$. When k is even, we have

$$\begin{aligned} \gamma = & \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_k & -p_k \\ -q_{k-1} & p_{k-1} \end{pmatrix} \\ = & \begin{pmatrix} -\ell t q_{k-1} + t q_k - t' q_{k-1} & \ell t p_{k-1} - t p_k + t' p_{k-1} \\ -\ell s q_{k-1} + s q_k - s' q_{k-1} & \ell s p_{k-1} - s p_k + s' p_{k-1} \end{pmatrix}. \end{aligned}$$

When k is odd, we find

$$\begin{aligned} \gamma = & \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_k & -p_k \\ q_{k-1} & -p_{k-1} \end{pmatrix} \\ = & \begin{pmatrix} \ell t q_{k-1} + t q_k + t' q_{k-1} & -\ell t p_{k-1} - t p_k - t' p_{k-1} \\ \ell s q_{k-1} + s q_k + s' q_{k-1} & -\ell s p_{k-1} - s p_k - s' p_{k-1} \end{pmatrix}. \end{aligned}$$

The required upper bound obviously holds in both cases. For the lower bound, look at the lower left entry of γ. □

The next lemma relates the two components of the point $\gamma \begin{pmatrix} \xi \\ 1 \end{pmatrix}$.

Lemma 3. *Let k be a positive integer, ℓ be an integer, let $N = \begin{pmatrix} t & t' \\ s & s' \end{pmatrix}$ belong to Γ and let y be any real number. Put*

$$\gamma = NU^\ell M_k = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix}, \quad \delta = |sy - t| \quad \textit{and} \quad \delta' = |s'y - t'|.$$

Then, we have the upper bound

$$|v_1\xi + u_1 - y(v_2\xi + u_2)| \le \frac{\delta|\ell|}{q_k} + \frac{\delta}{q_{k+1}} + \frac{\delta'}{q_k}.$$

Proof. It is a simple matter of bilinearity. We have the formula

$$\begin{aligned}
y(v_2\xi + u_2) - v_1\xi - u_1 = &\ \begin{pmatrix} -1 & y \end{pmatrix} \gamma \begin{pmatrix} \xi \\ 1 \end{pmatrix} \\
= &\ \begin{pmatrix} -1 & y \end{pmatrix} \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} M_k \begin{pmatrix} \xi \\ 1 \end{pmatrix} \\
= &\ \begin{pmatrix} sy - t & s'y - t' \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_k \\ |\epsilon_{k-1}| \end{pmatrix} \\
= &\ (sy - t)(\epsilon_k + \ell|\epsilon_{k-1}|) + (s'y - t')|\epsilon_{k-1}|.
\end{aligned}$$

Now the upper bound immediately follows from the estimate (2.1). □

We shall use Lemma 3 in the following way. Put

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \gamma\mathbf{x} - \mathbf{y} = \begin{pmatrix} x_2(v_1\xi + u_1) - y_1 \\ x_2(v_2\xi + u_2) - y_2 \end{pmatrix}$$

and let $y = y_1/y_2$ be the slope of the point $\mathbf{y}$, so that

$$\Lambda_1 - y\Lambda_2 = x_2\Big(v_1\xi + u_1 - y(v_2\xi + u_2)\Big).$$

Now, Lemma 3 provides us with a fine upper bound for $|\Lambda_1 - y\Lambda_2|$, as far as the quantities δ and δ' are small. Therefore to bound from above $|\gamma\mathbf{x} - \mathbf{y}|$, it suffices to bound one of its coordinates, say Λ_2. To that purpose, we use the expression

$$\Lambda_2 = x_2\Big(s\epsilon_k + (s\ell + s')|\epsilon_{k-1}|\Big) - y_2 = x_2 s|\epsilon_{k-1}|(\ell - \rho), \tag{4.1}$$

where

$$\rho = \frac{y_2}{x_2 s|\epsilon_{k-1}|} - \frac{\epsilon_k}{|\epsilon_{k-1}|} - \frac{s'}{s}. \tag{4.2}$$

4.1 Irrational slopes

We assume here that the slope $y = y_1/y_2$ is an irrational number and apply the key lemmas 2 and 3 for constructing matrices γ in Γ such that $\gamma\mathbf{x}$ is close to $\mathbf{y}$.

Denote by $(t_j/s_j)_{j\geq 0}$ the sequence of convergents of y, and put

$$N_j = \begin{pmatrix} t_j & t'_j \\ s_j & s'_j \end{pmatrix}, \quad \text{where} \quad s'_j = (-1)^{j-1} s_{j-1} \quad \text{and} \quad t'_j = (-1)^{j-1} t_{j-1},$$

for any $j \geq 1$. Observe that JN_j^{-1} coincides with the convergent matrix M_j associated to the irrational number y as in Section 2. Hence N_j belongs to Γ.

Lemma 4. *Let j and k be positive integers. There exists a matrix $\gamma \in \Gamma$, of the form $N_j U^\ell M_k$ for some integer ℓ, such that*

$$\left| \frac{|y_2|}{|x_2|} q_{k-1} q_k - s_j q_k \right| - 4 s_j q_{k-1} \leq |\gamma| \leq \frac{2|y_2|}{|x_2|} q_{k-1} q_k + 4 s_j q_k \tag{4.3}$$

and

$$|\gamma \mathbf{x} - \mathbf{y}| \leq \frac{2|y_2|}{s_j s_{j+1}} + \frac{5|x_2| s_j}{q_k}. \tag{4.4}$$

Proof. Since $|y| < 1$, we have $|t_j| \leq s_j$ and $|t'_j| \leq |s'_j| < s_j$. The matrix N_j has thus norm $|N_j| = s_j$. The theory of continued fractions gives the upper bounds

$$\delta = |s_j y - t_j| \leq s_{j+1}^{-1} \quad \text{and} \quad \delta' = |s'_j y - t'_j| = |s_{j-1} y - t_{j-1}| \leq s_j^{-1}. \tag{4.5}$$

Recall the definition of ρ given in (4.2), and substitute s_j to s and s'_j to s'. Bounding $|\epsilon_k/\epsilon_{k-1}| \leq 1$, $s_{j-1}/s_j \leq 1$, and $q_k \leq |\epsilon_{k-1}|^{-1} \leq 2q_k$ by (2.1), we find

$$\frac{|y_2| q_k}{|x_2| s_j} - 2 \leq |\rho| \leq \frac{2|y_2| q_k}{|x_2| s_j} + 2.$$

Define ℓ to be the unique integer such that

$$|\ell - \rho| < 1 \quad \text{and} \quad |\ell| \leq |\rho|.$$

We set

$$\gamma = N_j U^\ell M_k \quad \text{and} \quad \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \gamma \mathbf{x} - \mathbf{y}.$$

Therefore

$$\frac{|y_2| q_k}{|x_2| s_j} - 3 \leq |\ell| \leq \frac{2|y_2| q_k}{|x_2| s_j} + 2, \tag{4.6}$$

and it follows from (4.1) that

$$|\Lambda_2| = |x_2| s_j |\epsilon_{k-1}| |\ell - \rho| \leq \frac{|x_2| s_j}{q_k}.$$

Now, we apply Lemma 3 to bound $\Lambda_1 - y\Lambda_2$. Using (4.5) and (4.6), we find

$$|\Lambda_1 - y\Lambda_2| \le |x_2| \left(\frac{|\ell|}{s_{j+1}q_k} + \frac{1}{s_{j+1}q_{k+1}} + \frac{1}{s_j q_k} \right) \le |x_2| \left(\frac{2|y_2|}{|x_2| s_j s_{j+1}} + \frac{4}{s_j q_k} \right).$$

Since $|y| < 1$, adding the two above upper bounds gives

$$|\Lambda_1| \le |\Lambda_2| + |\Lambda_1 - y\Lambda_2| \le |x_2| \left(\frac{2|y_2|}{|x_2| s_j s_{j+1}} + \frac{5 s_j}{q_k} \right).$$

We have obtained the upper bound

$$|\gamma \mathbf{x} - \mathbf{y}| = \max(|\Lambda_1|, |\Lambda_2|) \le \frac{2|y_2|}{s_j s_{j+1}} + \frac{5|x_2| s_j}{q_k}$$

claimed in (4.4). On the other hand, Lemma 2 combined with (4.6) gives the estimate of norm (4.3). □

4.2 Rational slopes

We consider here a target point $\mathbf{y}$ with rational slope y. Writing the rational $y = a/b$ in reduced form, the integers a and b are coprime and we have $|a| \le b$, since we have assumed that $|y| \le 1$.

Lemma 5. *For any sufficiently large integer k, there exists a matrix $\gamma \in \Gamma$ such that*

$$\frac{|y_2|}{2|x_2|} q_{k-1} q_k \le |\gamma| \le \frac{3|y_2|}{|x_2|} q_{k-1} q_k \quad \text{and} \quad |\gamma \mathbf{x} - \mathbf{y}| \le \frac{2b|x_2|}{q_k}.$$

Proof. We now use the number $y = a/b$ itself as a best rational approximation to y. Let us complete the primitive point $\begin{pmatrix} a \\ b \end{pmatrix}$ into an unimodular matrix $N = \begin{pmatrix} a & a' \\ b & b' \end{pmatrix}$, with norm $|N| = b$. The matrix N is thus fixed, independently of k, and we have

$$\delta = |by - a| = 0 \quad \text{and} \quad \delta' = |b'y - a'| = \frac{1}{b}. \tag{4.7}$$

We use lemmas 2 and 3 with this choice of matrix N. Recall the definition of ρ given in (4.2), with s and s' respectively replaced by b and b'. As previously, define ℓ as the unique integer verifying $|\ell| \le |\rho|$ and $|\ell - \rho| < 1$. We have the estimate

$$\left(\frac{|y_2|}{b|x_2|} \right) q_k - 3 \le |\ell| \le \left(\frac{2|y_2|}{b|x_2|} \right) q_k + 2, \tag{4.8}$$

and

$$|\Lambda_2| = |x_2|b|\epsilon_{k-1}||\ell - \rho| \le \frac{|x_2|b}{q_k}. \tag{4.9}$$

Substituting the values of δ and δ' given by (4.7), Lemma 3 now gives

$$|\Lambda_1 - y\Lambda_2| \le \frac{|x_2|}{bq_k}. \tag{4.10}$$

We deduce from (4.9), (4.10) and the triangle inequality that

$$|\gamma\mathbf{x} - \mathbf{y}| \le \frac{2b|x_2|}{q_k},$$

as claimed. Finally, taking (4.8) into account, Lemma 2 gives

$$|\gamma| \le |\ell|bq_{k-1} + 2bq_k \le 2\frac{|y_2|}{|x_2|}q_{k-1}q_k + 2bq_{k-1} + 2bq_k \le 3\frac{|y_2|}{|x_2|}q_{k-1}q_k$$

and

$$|\gamma| \ge |\ell|bq_{k-1} - 2bq_k \ge \frac{|y_2|}{|x_2|}q_{k-1}q_k - 5bq_k \ge \frac{|y_2|}{2|x_2|}q_{k-1}q_k,$$

for large k. □

5 Proof of Theorem 1

We apply lemmas 4 and 5 in order to prove respectively the claims (1.3) and (1.4). We first deal with an irrational slope y and prove (1.4) in the sections 5.1 and 5.2 below. The argument splits into two parts depending on whether the value of the irrationality measure $\omega(\xi)$ is smaller than 3 or greater than 2.

5.1 The case $\omega(\xi) < 3$

Let us define infinitely many pairs of integers j and k in the following way. Let j_0 be an arbitrarily large integer. We determine k by the estimate

$$\left(\frac{|y_2|q_{k-1}}{|x_2|}\right)^{1/3} < s_{j_0} \le \left(\frac{|y_2|q_k}{|x_2|}\right)^{1/3}.$$

Let j be the largest integer such that s_j belongs to the above interval. We thus have the inequalities

$$\left(\frac{|y_2|q_{k-1}}{|x_2|}\right)^{1/3} < s_j \le \left(\frac{|y_2|q_k}{|x_2|}\right)^{1/3} < s_{j+1}. \tag{5.1}$$

We use Lemma 4 for any pair j and k verifying (5.1). It provides us with a matrix γ satisfying (4.3) and (4.4). Combining (4.4) and (5.1), we find the upper bound

$$|\gamma\mathbf{x} - \mathbf{y}| \le |y_2|^{1/3}|x_2|^{2/3}\left(\frac{2}{q_{k-1}^{1/3}q_k^{1/3}} + \frac{5}{q_k^{2/3}}\right) \le \frac{7|y_2|^{1/3}|x_2|^{2/3}}{(q_{k-1}q_k)^{1/3}}. \tag{5.2}$$

Observe now that for any real number ω satisfying $\omega(\xi) < \omega < 3$, we have $q_{k-1} \ge q_k^{1/\omega}$ for all k sufficiently large. Since $s_j \ll q_k^{1/3}$, the second term $4s_jq_k$ occurring on the right hand side of (4.3) is much smaller than the first one, as k tends to infinity. Thus, for any sufficiently large k, we have the norm bound

$$|\gamma| \le 3\frac{|y_2|}{|x_2|}q_{k-1}q_k. \tag{5.3}$$

Combining then (5.2) and (5.3), we obtain

$$|\gamma\mathbf{x} - \mathbf{y}| \le 7\sqrt[3]{3}|x_2|^{1/3}|y_2|^{2/3}|\gamma|^{-1/3} \le c'|\gamma|^{-1/3}.$$

The upper bound (1.4) is therefore established. It remains to show that our construction produces infinitely many solutions of (1.4). To that purpose, it suffices to bound from below the norm of γ. The estimate (4.3) in Lemma 4 gives indeed

$$|\gamma| \asymp \frac{|y_2|}{|x_2|}q_{k-1}q_k.$$

5.2 The case $\omega(\xi) > 2$

Let us fix a real number ω satisfying $2 < \omega < \omega(\xi)$. There exist infinitely many k such that $q_{k-1}^{\omega} \le q_k$. For any such integer k, let j be the integer defined by the inequality

$$s_j \le \left(\frac{|y_2|q_k}{|x_2|}\right)^{1/2} < s_{j+1}. \tag{5.4}$$

Applying Lemma 4 and using (5.4), we obtain the upper bounds

$$|\gamma| \le \frac{2|y_2|}{|x_2|}q_{k-1}q_k + 4s_jq_k \le \frac{2|y_2|}{|x_2|}q_{k-1}q_k + 4\frac{|y_2|^{1/2}}{|x_2|^{1/2}}q_k^{3/2} \tag{5.5}$$

and

$$\begin{aligned} |\gamma\mathbf{x} - \mathbf{y}| &\le \frac{2|y_2|}{s_js_{j+1}} + \frac{5|x_2|s_j}{q_k} \le \left(\frac{2}{s_j} + 5\right)|x_2|^{1/2}|y_2|^{1/2}q_k^{-1/2} \\ &\le 7|x_2|^{1/2}|y_2|^{1/2}q_k^{-1/2}. \end{aligned} \tag{5.6}$$

$$(5.6)\quad |\gamma\mathbf{x}-\mathbf{y}| \leq \frac{2|y_2|}{s_j s_{j+1}} + \frac{5|x_2|s_j}{q_k} \leq \left(\frac{2}{s_j}+5\right)\left(\frac{|x_2||y_2|}{q_k}\right)^{1/2} \leq 7\left(\frac{|x_2||y_2|}{q_k}\right)^{1/2}.$$

Recall that k has been chosen satisfying $q_{k-1} \leq q_k^{1/\omega}$, where $\omega > 2$. Consequently, the first term $(2|y_2|/|x_2|)q_{k-1}q_k$ occurring on the right hand side of (5.5) is much smaller than the second one, as k tends to infinity. The upper bound

$$(5.7)\qquad |\gamma| \leq 5\frac{|y_2|^{1/2}}{|x_2|^{1/2}}q_k^{3/2},$$

is thus valid for k large enough. Combining (5.6) and (5.7), we obtain

$$|\gamma\mathbf{x}-\mathbf{y}| \leq 7\sqrt[3]{5}|x_2|^{1/3}|y_2|^{2/3}|\gamma|^{-1/3} = c'|\gamma|^{-1/3}.$$

Note that (5.7) turns out to be an estimate

$$|\gamma| \asymp \frac{|y_2|^{1/2}}{|x_2|^{1/2}}q_k^{3/2},$$

using (4.3). Hence the norm of γ tends to infinity with k, and here again, our construction furnishes infinitely many solutions of the inequation (1.4).

The assertion (1.4) of Theorem 1 is finally established for any point $\mathbf{y}$ with irrational slope.

5.3 Rational slopes

We deduce from Lemma 5 the claim (1.3) of Theorem 1. For any large integer k, it furnishes a matrix $\gamma \in \Gamma$ satisfying the inequalities

$$|\gamma| \leq 3\frac{|y_2|}{|x_2|}q_{k-1}q_k \leq 3\frac{|y_2|}{|x_2|}q_k^2 \quad \text{and} \quad |\gamma\mathbf{x}-\mathbf{y}| \leq \frac{2b|x_2|}{q_k},$$

which imply

$$|\gamma\mathbf{x}-\mathbf{y}| \leq 2\sqrt{3}b|x_2|^{1/2}|y_2|^{1/2}|\gamma|^{-1/2} = c|\gamma|^{-1/2}.$$

Using the lower bound for $|\gamma|$ given in Lemma 5, we find the estimate

$$|\gamma| \asymp \frac{|y_2|}{|x_2|}q_{k-1}q_k.$$

Therefore, our construction produces infinitely many solutions γ of the inequation (1.3). The proof of Theorem 1 is complete.

6 Lower bounds of exponents

Applying further Lemmas 4 and 5, we now estimate from below the exponents $\mu(\mathbf{x}, \mathbf{y})$ and $\hat{\mu}(\mathbf{x}, \mathbf{y})$.

6.1 Lower bounds for irrational slopes

We assume here that the slope y of the point $\mathbf{y}$ is an irrational number. As an immediate consequence of (1.4), we get the lower bound $\mu(\mathbf{x}, \mathbf{y}) \geq 1/3$.

We prove in this section the lower bound

$$\hat{\mu}(\mathbf{x}, \mathbf{y}) \geq \frac{\omega(y)+1}{2(2\omega(y)+1)\omega(\xi)},$$

claimed in (1.8). The irrationality measure $\omega(y)$ of the slope of the point $\mathbf{y}$ is taken into account thanks to the following.

Lemma 6. *Set*

$$\tau = \frac{\omega(y)}{2\omega(y)+1}.$$

For any $\varepsilon > 0$ and any integer k sufficiently large in terms of ε, there exists $\gamma \in \Gamma$ such that

$$|\gamma| \leq C q_k^2 \quad \text{and} \quad |\gamma \mathbf{x} - \mathbf{y}| \leq q_k^{\tau - 1 + \varepsilon},$$

where $C = C(\mathbf{x}, \mathbf{y}, \varepsilon)$ does not depend upon k.

Proof. Once again, we apply Lemma 4. Let j be the integer defined by the inequality

$$s_j \leq q_k^{\tau} < s_{j+1}. \tag{6.1}$$

Observe that $1/3 \leq \tau \leq 1/2$, since $\omega(y) \geq 1$. Therefore j tends to infinity, as k tends to infinity. When $\omega(y)$ is finite, the lower bound $s_j \geq s_{j+1}^{1/\omega}$ holds for any $\omega > \omega(y)$ provided that j is large enough. Selecting properly ω close to $\omega(y)$, it follows from (6.1) that

$$s_j \geq q_k^{\tau/\omega(y) - \varepsilon}, \tag{6.2}$$

for all sufficiently large integers k. When $\omega(y) = +\infty$, we read (6.2) as the obvious lower bound $s_j \geq q_k^{-\varepsilon}$. Now, Lemma 4 provides a matrix $\gamma \in \Gamma$ satisfying

$$|\gamma| \ll q_{k-1} q_k + s_j q_k \leq C q_k^2,$$

and

$$|\gamma\mathbf{x}-\mathbf{y}| \ll \frac{1}{s_j s_{j+1}} + \frac{s_j}{q_k} \ll q_k^{-\tau-\tau/\omega(y)+\varepsilon} + q_k^{\tau-1},$$

by (6.1) and (6.2). Note that the exponents $-\tau-\tau/\omega(y)$ and $\tau-1$ arising above, are equal by the definition of τ. Therefore, we obtain the bound

$$|\gamma\mathbf{x}-\mathbf{y}| \ll q_k^{\tau-1+\varepsilon},$$

and, possibly increasing ε, Lemma 6 is proved. □

For any sufficiently large real number T, let k be the integer defined by the inequalities

$$Cq_k^2 \le T < Cq_{k+1}^2. \tag{6.3}$$

Clearly, k tends to infinity when T tends to infinity. For any $\varepsilon > 0$, we can bound further

$$T \le Cq_{k+1}^2 \le q_k^{2\omega(\xi)+\varepsilon}, \tag{6.4}$$

when T is large enough. Then, Lemma 6 gives a matrix $\gamma \in \Gamma$ satisfying

$$|\gamma| \le Cq_k^2 \le T \quad \text{and} \quad |\gamma\mathbf{x}-\mathbf{y}| \le q_k^{\tau-1+\varepsilon} \le T^{-(1-\tau-\varepsilon)/(2\omega(\xi)+\varepsilon)},$$

by (6.3) and (6.4). Therefore

$$\hat{\mu}(\mathbf{x},\mathbf{y}) \ge \frac{1-\tau-\varepsilon}{2\omega(\xi)+\varepsilon},$$

and letting ε tend to 0, we obtain the claimed lower bound

$$\hat{\mu}(\mathbf{x},\mathbf{y}) \ge \frac{1-\tau}{2\omega(\xi)} = \frac{\omega(y)+1}{2(2\omega(y)+1)\omega(\xi)}.$$

6.2 Lower bounds for rational slopes

In this section, we prove that the lower bounds

$$\mu(\mathbf{x},\mathbf{y}) \ge \frac{\omega(\xi)}{\omega(\xi)+1} \quad \text{and} \quad \hat{\mu}(\mathbf{x},\mathbf{y}) \ge \frac{1}{\omega(\xi)+1}$$

hold for any point $\mathbf{y}$ with rational slope y, or when $y_2 = 0$. As in Section 4.2, we assume that $y_2 \neq 0$ and that $y = a/b$, where a and b are coprime integers with $|a| \le b$.

We start with the inequality $\mu(\mathbf{x},\mathbf{y}) \geq \omega(\xi)/(\omega(\xi)+1)$. For any $\omega < \omega(\xi)$ there exist infinitely many integers k satisfying $q_{k-1} \leq q_k^{1/\omega}$. Using Lemma 5 for such an index k, we get $\gamma \in \Gamma$ such that

$$|\gamma| \ll q_{k-1}q_k \ll q_k^{1+1/\omega} \quad \text{and} \quad |\gamma\mathbf{x}-\mathbf{y}| \ll q_k^{-1}.$$

Then $|\gamma\mathbf{x}-\mathbf{y}| \ll |\gamma|^{-\omega/(\omega+1)}$ for infinitely many γ. Hence

$$\mu(\mathbf{x},\mathbf{y}) \geq \frac{\omega(\xi)}{\omega(\xi)+1},$$

by letting ω tend to $\omega(\xi)$.

As for the lower bound $\hat{\mu}(\mathbf{x},\mathbf{y}) \geq 1/(\omega(\xi)+1)$, we briefly take again the argumentation given in Section 6.1. We may obviously assume that $\omega(\xi)$ is finite. For any real number T sufficiently large, let k be the integer uniquely determined by

$$3\frac{|y_2|}{|x_2|}q_{k-1}q_k \leq T < 3\frac{|y_2|}{|x_2|}q_kq_{k+1}.$$

For any $\varepsilon > 0$, we bound from above

$$T < 3\frac{|y_2|}{|x_2|}q_kq_{k+1} \leq q_k^{\omega(\xi)+1+\varepsilon},$$

when k is large enough. Lemma 5 gives us a matrix $\gamma \in \Gamma$ satisfying

$$|\gamma| \leq 3\frac{|y_2|}{|x_2|}q_{k-1}q_k \leq T \quad \text{and} \quad |\gamma\mathbf{x}-\mathbf{y}| \leq \frac{2b|x_2|}{q_k} \leq 2b|x_2|T^{-1/(\omega(\xi)+1+\varepsilon)}.$$

Therefore $\hat{\mu}(\mathbf{x},\mathbf{y}) \geq 1/(\omega(\xi)+1+\varepsilon)$ for any $\varepsilon > 0$.

7 Proof of Theorem 3

Recall the matrices M_k and N_j introduced in Sections 2 and 4.1. We intend to show that if an element $\gamma\mathbf{x}$ of the orbit is close to a given point $\mathbf{y}$, then γ can be factorized in the form $\gamma = N_jGM_k$, with a good estimate of the norm $|G|$ for suitable indices j and k. Without loss of generality, we may assume that $\mathbf{x} = \begin{pmatrix} \xi \\ 1 \end{pmatrix}$.

Lemma 7. *Let k be a positive integer, μ and T be real numbers such that*

$$0 \leq \mu \leq 1 \quad \textit{and} \quad q_{k-1}q_k \leq T \leq q_kq_{k+1},$$

and let $\gamma \in \Gamma$ satisfy

$$|\gamma| \leq 2T \qquad \text{and} \qquad |\gamma \mathbf{x} - \mathbf{y}| \leq T^{-\mu}.$$

Let j be a positive integer such that $s_j \geq T^{\mu/2}$. Then γ can be decomposed as a product $\gamma = N_j G M_k$, where the two columns of the matrix $G = \begin{pmatrix} m & \ell \\ m' & \ell' \end{pmatrix} \in \Gamma$ satisfy the norm bound

$$\max(|m|, |m'|) \leq \frac{c s_j T^{1-\mu}}{q_k} \qquad \text{and} \qquad \max(|\ell|, |\ell'|) \leq c s_j q_k T^{-\mu},$$

with $c = 10 \max(|\mathbf{y}|, |\mathbf{y}|^{-1})$.

Proof. Write $\gamma = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix}$ and put

$$\Lambda_1 = v_1\xi + u_1 - y_1, \quad \Lambda_2 = v_2\xi + u_2 - y_2.$$

The upper bound $|\gamma \mathbf{x} - \mathbf{y}| \leq T^{-\mu}$ means that

$$\max(|\Lambda_1|, |\Lambda_2|) \leq T^{-\mu}. \tag{7.1}$$

We have the identities

$$\begin{aligned} v_1y_2 - v_2y_1 &= \begin{vmatrix} v_1 & y_1 \\ v_2 & y_2 \end{vmatrix} = \begin{vmatrix} v_1 & v_1\xi + u_1 - \Lambda_1 \\ v_2 & v_2\xi + u_2 - \Lambda_2 \end{vmatrix} = 1 + \Lambda_1 v_2 - \Lambda_2 v_1, \\ u_1y_2 - u_2y_1 &= \begin{vmatrix} u_1 & y_1 \\ u_2 & y_2 \end{vmatrix} = \begin{vmatrix} u_1 & v_1\xi + u_1 - \Lambda_1 \\ u_2 & v_2\xi + u_2 - \Lambda_2 \end{vmatrix} = -\xi + \Lambda_1 u_2 - \Lambda_2 u_1. \end{aligned} \tag{7.2}$$

By (7.1), they imply the upper bound

$$\max\Big(|u_1y_2 - u_2y_1|, |v_1y_2 - v_2y_1|\Big) \leq 1 + 4T^{1-\mu}. \tag{7.3}$$

We first factorize N_j. Define

$$\begin{aligned} \gamma' = \; & N_j^{-1}\gamma = \begin{pmatrix} t_j & t'_j \\ s_j & s'_j \end{pmatrix}^{-1} \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix} = \begin{pmatrix} s'_jv_1 - t'_jv_2 & s'_ju_1 - t'_ju_2 \\ -s_jv_1 + t_jv_2 & -s_ju_1 + t_ju_2 \end{pmatrix} \\ = \; & \frac{1}{y_2} \begin{pmatrix} s'_j(v_1y_2 - v_2y_1) + v_2(s'_jy_1 - t'_jy_2) & s'_j(u_1y_2 - u_2y_1) + u_2(s'_jy_1 - t'_jy_2) \\ -s_j(v_1y_2 - v_2y_1) - v_2(s_jy_1 - t_jy_2) & -s_j(u_1y_2 - u_2y_1) - u_2(s_jy_1 - t_jy_2) \end{pmatrix}. \end{aligned}$$

Using (7.3) and the estimate $|s_jy - t_j| \leq |s'_jy - t'_j| \leq 1/s_j$, we deduce from the above expression the upper bound for the norm

$$|\gamma'| \leq \frac{s_j(1 + 4T^{1-\mu})}{|y_2|} + \frac{2T}{s_j} \leq (5|y_2|^{-1} + 2)s_jT^{1-\mu}, \tag{7.4}$$

since $s_j \geq T^{\mu/2}$. Now, put $\gamma' = \begin{pmatrix} v_1' & u_1' \\ v_2' & u_2' \end{pmatrix}$ and write

$$\begin{pmatrix} v_1'\xi + u_1' \\ v_2'\xi + u_2' \end{pmatrix} = \gamma'\mathbf{x} = N_j^{-1}\gamma\mathbf{x} = N_j^{-1}\begin{pmatrix} y_1 + \Lambda_1 \\ y_2 + \Lambda_2 \end{pmatrix} = \begin{pmatrix} y_1 s_j' - y_2 t_j' + s_j'\Lambda_1 - t_j'\Lambda_2 \\ -y_1 s_j + y_2 t_j - s_j\Lambda_1 + t_j\Lambda_2 \end{pmatrix}.$$

It follows that

$$\max\Big(|v_1'\xi + u_1'|, |v_2'\xi + u_2'|\Big) = |\gamma'\mathbf{x}| \leq \frac{|y_2|}{s_j} + 2s_jT^{-\mu} \leq (|y_2| + 2)s_jT^{-\mu}. \tag{7.5}$$

Now, we multiply γ' on the right by M_k^{-1} and set

$$G = N_j^{-1}\gamma M_k^{-1} = \gamma' M_k^{-1}.$$

Suppose first that k is even. We find the formula

$$G = \begin{pmatrix} v_1' & u_1' \\ v_2' & u_2' \end{pmatrix}\begin{pmatrix} q_k & -p_k \\ -q_{k-1} & p_{k-1} \end{pmatrix}^{-1}\begin{pmatrix} p_{k-1}v_1' + q_{k-1}u_1' & p_kv_1' + q_ku_1' \\ p_{k-1}v_2' + q_{k-1}u_2' & p_kv_2' + q_ku_2' \end{pmatrix}.$$

Write next

$$\begin{aligned}
\ell &= p_kv_1' + q_ku_1' = -v_1'(q_k\xi - p_k) + q_k(v_1'\xi + u_1'), \\
\ell' &= p_kv_2' + q_ku_2' = -v_2'(q_k\xi - p_k) + q_k(v_2'\xi + u_2'), \\
m &= p_{k-1}v_1' + q_{k-1}u_1' = -v_1'(q_{k-1}\xi - p_{k-1}) + q_{k-1}(v_1'\xi + u_1'), \\
m' &= p_{k-1}v_2' + q_{k-1}u_2 = -v_2'(q_{k-1}\xi - p_{k-1}) + q_{k-1}(v_2'\xi + u_2').
\end{aligned}$$

We deduce from (2.1), (7.4) and (7.5) that

$$\begin{aligned}
\max(|\ell|, |\ell'|) &\leq (5|y_2|^{-1} + 2)\frac{s_jT^{1-\mu}}{q_{k+1}} + (|y_2| + 2)q_ks_jT^{-\mu} \leq cs_jq_kT^{-\mu}, \\
\max(|m|, |m'|) &\leq (5|y_2|^{-1} + 2)\frac{s_jT^{1-\mu}}{q_k} + (|y_2| + 2)q_{k-1}s_jT^{-\mu} \leq \frac{cs_jT^{1-\mu}}{q_k},
\end{aligned}$$

since $q_{k-1}q_k \leq T \leq q_kq_{k+1}$. The case k odd leads to the same upper bound. $\square$

We are now able to prove Theorem 3. Let $\mathcal{C}$ be a compact subset of the punctured line $(\mathbb{R} \setminus \{0\})\begin{pmatrix} y \\ 1 \end{pmatrix}$, and let μ be a real number greater than 1/2. Denote by $\mathcal{C}_\mu$ the subset consisting of the points $\mathbf{y} \in \mathcal{C}$ for which the inequation

$$|\gamma\mathbf{x} - \mathbf{y}| \leq |\gamma|^{-\mu} \tag{7.6}$$

has infinitely many solutions $\gamma \in \Gamma$. We have to show that $\mathcal{C}_\mu$ has null Lebesgue measure.

Let $\gamma \in \Gamma$ and $\mathbf{y} \in \mathcal{C}_\mu$ satisfy (7.6). Assuming that $|\gamma|$ is large enough, let $k \geq 1$ and $n \geq 0$ be the integers defined by the inequalities

$$q_{k-1}q_k < |\gamma| \leq q_k q_{k+1} \quad \text{and} \quad 2^n q_{k-1}q_k < |\gamma| \leq 2^{n+1}q_{k-1}q_k. \tag{7.7}$$

Put $T = 2^n q_{k-1}q_k$. It follows from (7.6) and (7.7) that

$$|\gamma| \leq 2T \quad \text{and} \quad |\gamma\mathbf{x} - \mathbf{y}| \leq |\gamma|^{-\mu} \leq T^{-\mu}. \tag{7.8}$$

Let j be the smallest integer such that $s_j \geq T^{\mu/2}$. Since we have assumed that $\omega(y) = 1$, for any positive real number ε, we can bound from above $s_j \leq T^{\mu/2+\varepsilon}$ when j is large enough. Note that j is arbitrarily large if we take γ of sufficiently large norm. Then, Lemma 7 provides us with the decomposition $\gamma = N_j G M_k$ for some matrix $G = \begin{pmatrix} m & \ell \\ m' & \ell' \end{pmatrix}$ in Γ whose columns satisfy the bound of norm

$$\begin{aligned} \max(|m|, |m'|) &\leq \frac{c s_j T^{1-\mu}}{q_k} \leq \frac{c T^{1-\mu/2+\varepsilon}}{q_k} = B_1, \\ \max(|\ell|, |\ell'|) &\leq c s_j q_k T^{-\mu} \leq c q_k T^{-\mu/2+\varepsilon} = B_2, \end{aligned} \tag{7.9}$$

where the coefficient $c = 10 \max_{\mathbf{y}\in\mathcal{C}}(|\mathbf{y}|, |\mathbf{y}|^{-1})$ depends only upon $\mathcal{C}$.

It is easily seen that the set of matrices $G \in \Gamma$ whose first and second columns have norm respectively bounded by B_1 and B_2, has at most $4(2B_1+1)(2B_2+1)$ elements. Of course, if either B_1 or B_2 is smaller than 1, no such matrix exists. Hence, there are at most

$$36B_1B_2 = 36c^2T^{1-\mu+2\varepsilon}$$

matrices G in Γ satisfying (7.9). The second upper bound of (7.8) means that $\mathbf{y}$ belongs to the intersection of the line $\mathbb{R}\begin{pmatrix} y \\ 1 \end{pmatrix}$ with the square centered at the point $N_jGM_k\mathbf{x}$ with side $2T^{-\mu}$. This intersection is a segment of Euclidean length $\leq 2\sqrt{2}T^{-\mu}$. For fixed k and n, at most $36B_1B_2$ such segments may thus appear. It follows that $\mathbf{y}$ belongs to some subset of the line $\mathbb{R}\begin{pmatrix} y \\ 1 \end{pmatrix}$ whose Lebesgue measure does not exceed

$$(36B_1B_2)(2\sqrt{2}T^{-\mu}) = 72\sqrt{2}c^2(2^n q_{k-1}q_k)^{1-2\mu+2\varepsilon}.$$

Note that the sequence q_k of denominators of convergents of the irrational number ξ is bounded from below by the Fibonacci sequence $1, 1, 2, \ldots$, which grows geometrically. Therefore, the series

$$\sum_{k\geq 1}\sum_{n\geq 0}(2^n q_{k-1}q_k)^{1-2\mu+2\varepsilon}$$

converges when ε is small enough, since the exponent $1-2\mu$ is negative. By the Borel-Cantelli Lemma, the lim sup set $\mathcal{C}_\mu$ has null Lebesgue measure.

8 Upper bounds for rational slopes

Here we prove that the upper bounds

$$\mu(\mathbf{x},\mathbf{y}) \leq \frac{\omega(\xi)}{\omega(\xi)+1} \qquad \text{and} \qquad \hat{\mu}(\mathbf{x},\mathbf{y}) \leq \frac{1}{\omega(\xi)+1}$$

hold for any point $\mathbf{y} \neq \mathbf{0}$ with rational slope y. Since the reverse inequalities have been established in Section 6.2, the proof of (1.7) will then be complete. To that purpose, we adapt to rational slopes the factorisation method displayed in the preceding section. We obtain the following explicit lower bound of distance which may have its own interest.

Theorem 4. *Let* $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ *be a point having rational slope* $y_1/y_2 = a/b$, *where* a *and* b *are coprime integers with* $|a| \leq b$, *and let* k *be a positive integer such that* $q_k \geq 12b/|y_2|$. *Then, for any* $\gamma \in \Gamma$ *with norm*

$$|\gamma| \leq \frac{|y_2|}{4} q_k q_{k+1},$$

we have the lower bound

$$\left|\gamma \begin{pmatrix} \xi \\ 1 \end{pmatrix} - \mathbf{y}\right| \geq \frac{1}{4bq_k}.$$

Proof. Recall the matrix $N = \begin{pmatrix} a & a' \\ b & b' \end{pmatrix}$ in Γ introduced in Section 4.2. Notice that N^{-1} maps the line $\mathbb{R}\begin{pmatrix} a \\ b \end{pmatrix}$ on to the horizontal axis $\mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore any point close to the line $\mathbb{R}\begin{pmatrix} a \\ b \end{pmatrix}$ is sent by the map N^{-1} to a point close to the horizontal axis. We insert this additional information into the proof of Lemma 7 with $\mu = 1/2$.

Set $\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \gamma \begin{pmatrix} \xi \\ 1 \end{pmatrix} - \mathbf{y}$ and suppose on the contrary that $\max(|\Lambda_1|, |\Lambda_2|) < (4bq_k)^{-1}$. Put

$$\gamma' = N^{-1}\gamma = \begin{pmatrix} v'_1 & u'_1 \\ v'_2 & u'_2 \end{pmatrix}.$$

Noting that

$$by_1 - ay_2 = 0 \quad \text{and} \quad b'y_1 - a'y_2 = \frac{y_2}{b},$$

we obtain as in Section 7 the expressions

$$\gamma' = \begin{pmatrix} \dfrac{b'(v_1y_2 - v_2y_1)}{y_2} + \dfrac{v_2}{b} & \dfrac{b'(u_1y_2 - u_2y_1)}{y_2} + \dfrac{u_2}{b} \\ -\dfrac{b(v_1y_2 - v_2y_1)}{y_2} & -\dfrac{b(u_1y_2 - u_2y_1)}{y_2} \end{pmatrix} \tag{8.1}$$

and

$$\gamma'\mathbf{x} = \begin{pmatrix} \frac{y_2}{b} + b'\Lambda_1 - a'\Lambda_2 \\ -b\Lambda_1 + a\Lambda_2 \end{pmatrix}. \tag{8.2}$$

Using the formulas (7.2), we have that

$$|v_1y_2 - v_2y_1| \le 1 + 2\max(|\Lambda_1|, |\Lambda_2|)|\gamma| \le 1 + \frac{|y_2|}{8b}q_{k+1} \le \frac{|y_2|}{4b}q_{k+1}, \tag{8.3}$$

since we have assumed that $q_k \ge 12b/|y_2|$. Then, we deduce from the expressions (8.1), (8.2) and from the upper bound (8.3) that

$$|v'_2| < \frac{q_{k+1}}{4} \quad \text{and} \quad |v'_2\xi + u'_2| < \frac{1}{2q_k}. \tag{8.4}$$

Set now

$$G = N^{-1}\gamma M_k^{-1} = \gamma' M_k^{-1}.$$

Assuming that k is even (the case k odd is similar), we use again the expressions

$$G = \begin{pmatrix} -v'_1(q_{k-1}\xi - p_{k-1}) + q_{k-1}(v'_1\xi + u'_1) & -v'_1(q_k\xi - p_k) + q_k(v'_1\xi + u'_1) \\ -v'_2(q_{k-1}\xi - p_{k-1}) + q_{k-1}(v'_2\xi + u'_2) & -v'_2(q_k\xi - p_k) + q_k(v'_2\xi + u'_2) \end{pmatrix}$$

obtained in Section 7. We deduce from (2.1) and (8.4) the upper bound

$$\Big| - v'_2(q_k\xi - p_k) + q_k(v'_2\xi + u'_2)\Big| \le \frac{|v'_2|}{q_{k+1}} + q_k|v'_2\xi + u'_2| \le \frac{1}{4} + \frac{1}{2} < 1,$$

for the absolute value of the lower right entry of the matrix G, which therefore vanishes. It follows that G has the form

$$G = \pm \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix},$$

where m is an integer. Hence

$$\begin{pmatrix} \frac{y_2}{b} + b'\Lambda_1 - a'\Lambda_2 \\ -b\Lambda_1 + a\Lambda_2 \end{pmatrix} = \gamma'\mathbf{x} = GM_k\mathbf{x} = \pm \begin{pmatrix} m\epsilon_k - |\epsilon_{k-1}| \\ \epsilon_k \end{pmatrix}.$$

Looking at the first component of the above vectorial equality, we find the estimates

$$\frac{|y_2|}{b} - \frac{1}{2q_k} \leq \left| \frac{y_2}{b} + b'\Lambda_1 - a'\Lambda_2 \right| = \Big| m\epsilon_k - |\epsilon_{k-1}| \Big| \leq \frac{|m|}{q_{k+1}} + \frac{1}{q_k}.$$

We thus obtain the lower bound

$$|m| \geq \frac{|y_2| q_{k+1}}{2b} \geq 6, \tag{8.5}$$

since $q_k \geq 12b/|y_2|$. Now, write

$$\begin{aligned} \gamma = {} & \pm \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_k & -p_k \\ -q_{k-1} & p_{k-1} \end{pmatrix} \\ = {} & \pm \begin{pmatrix} amq_k + aq_{k-1} + a'q_k & -amp_{k-1} - ap_{k-1} - a'p_k \\ \\ bmq_k + bq_{k-1} + b'q_k & -bmp_{k-1} - bp_{k-1} - b'p_k \end{pmatrix}. \end{aligned}$$

Hence, taking (8.5) into account, we find the lower bound

$$|\gamma| \geq b(|m| - 2)q_k \geq \frac{|y_2|}{3} q_k q_{k+1},$$

which contradicts the assumption $|\gamma| \leq (|y_2|/4) q_k q_{k+1}$. □

We first deduce from Theorem 4 that $\mu(\mathbf{x}, \mathbf{y}) \leq \omega(\xi)/(\omega(\xi) + 1)$. For any matrix γ in Γ with norm $|\gamma|$ large enough, let k be the integer defined by the inequality

$$\frac{|y_2|}{4} q_{k-1} q_k < |\gamma| \leq \frac{|y_2|}{4} q_k q_{k+1}.$$

In the case where $\omega(\xi)$ is finite, let ω be a real number greater than $\omega(\xi)$. We then bound from below $q_{k-1} \geq q_k^{1/\omega}$, if k is large enough in terms of ω. In the case $\omega(\xi) = +\infty$, we simply bound from below $q_{k-1} \geq 1$. Now, Theorem 4 gives us the lower bound

$$|\gamma\mathbf{x} - \mathbf{y}| \geq \frac{1}{4bq_k} \geq \frac{1}{4b}\frac{1}{(4|\gamma|/|y_2|)^{1/(1+1/\omega)}},$$

where the exponent $1/(1+1/\omega)$ is understood to be 1 when $\omega(\xi) = +\infty$. The latter lower bound of distance is valid for any $\gamma \in \Gamma$ with large norm. It thus implies the upper bound

$$\mu(\mathbf{x}, \mathbf{y}) \leq \frac{1}{1+\frac{1}{\omega}} = \frac{\omega}{\omega+1}.$$

Letting ω tend to $\omega(\xi)$, we have proved the claim.

Let μ be a positive real number such that the inequations

$$|\gamma| \leq T \quad \text{and} \quad |\gamma\mathbf{x} - \mathbf{y}| \leq T^{-\mu} \tag{8.6}$$

have a solution $\gamma \in \Gamma$ for any large real number T. Let ω be a real number smaller than $\omega(\xi)$. There exist infinitely many integer k such that $q_{k+1} \geq q_k^{\omega}$. Choose $T = (|y_2|/4)q_kq_{k+1}$ for such an integer k. Thus $T \geq (|y_2|/4)q_k^{1+\omega}$, and Theorem 4 now gives the lower bound

$$|\gamma\mathbf{x} - \mathbf{y}| \geq \frac{1}{4bq_k} \geq \frac{1}{4b}\frac{1}{(4T/|y_2|)^{1/(1+\omega)}},$$

for any $\gamma \in \Gamma$ with norm $|\gamma| \leq T$. Comparing with (8.6), we find that $\mu \leq 1/(1+\omega)$. Letting ω tend to $\omega(\xi)$, we obtain the expected bound $\hat{\mu}(\mathbf{x}, \mathbf{y}) \leq 1/(\omega(\xi)+1)$.

9 Approximation with signs

Let us first state a theorem due to Davenport and Heilbronn which gives a version of Minkowski's Theorem with prescribed signs [7].

Theorem. (Davenport–Heilbronn) *Let ξ be an irrational number and let y be a real number not belonging to the subgroup $\mathbb{Z}\xi + \mathbb{Z}$. There exist infinitely many pairs of integers (v, u) such that*

$$v > 0 \quad \textit{and} \quad 0 < v\xi + u - y \leq \frac{1}{v}.$$

Here is an analogous statement for Γ-orbits. For simplicity, we assume that the target point $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ belongs to the positive quadrant $\mathbb{R}^2_+$.

Theorem 5. *Let ξ be an irrational number and let y_1, y_2 be two positive real numbers such that the ratio $y = y_1/y_2$ is an irrational number with irrationality measure $\omega(y) = 1$. Then, for any positive real number $\mu < 1/3$, there exist infinitely many matrices $\gamma = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix} \in \Gamma$ satisfying*

$$v_1 > 0, \quad v_2 > 0 \quad \text{and} \quad 0 < v_1\xi + u_1 - y_1 \le |\gamma|^{-\mu}, \quad 0 < v_2\xi + u_2 - y_2 \le |\gamma|^{-\mu}.$$

Remark. Other constraints of signs are admissible. Notice however that v_1 and v_2 have necessarily the same sign whenever y_1 and y_2 have the same sign, if we assume that $\left|\gamma \begin{pmatrix} \xi \\ 1 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right| = \mathcal{O}(|\gamma|^{-\mu})$ with $\mu > 0$. That follows from the estimate

$$\begin{aligned} v_1y_2 - v_2y_1 = \begin{vmatrix} v_1 & y_1 \\ v_2 & y_2 \end{vmatrix} &= \begin{vmatrix} v_1 & v_1\xi + u_1 \\ v_2 & v_2\xi + u_2 \end{vmatrix} - \begin{vmatrix} v_1 & v_1\xi + u_1 - y_1 \\ v_2 & v_2\xi + u_2 - y_2 \end{vmatrix} \\ &= 1 + \mathcal{O}\left(|\gamma|^{1-\mu}\right) \end{aligned}$$

already mentioned in (7.2). Theorem 5 is a sample of statements that could be obtained by reworking the previous sections and controling all signs.

Denote by Γ_+ the semi-group of Γ consisting of the matrices γ with non-negative entries. Theorem 5 enables us to recover in a constructive way the following result from [6]:

Corollary. (Dani-Nogueira) *Let ξ be a negative irrational number. Then, the intersection with $\mathbb{R}^2_+$ of the semi-orbit $\Gamma_+ \begin{pmatrix} \xi \\ 1 \end{pmatrix}$ is dense in $\mathbb{R}^2_+$.*

Proof. The points $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2_+$ for which the slope $y = y_1/y_2$ has irrationality measure $\omega(y) = 1$ form a full set in $\mathbb{R}^2_+$ (*i.e.* the complementary set has null Lebesgue measure), hence dense. For any such point $\mathbf{y}$, Theorem 5 provides us with a sequence of points in $\Gamma_+ \begin{pmatrix} \xi \\ 1 \end{pmatrix}$ tending to $\mathbf{y}$, since the second column $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of γ has necessarily positive entries when $v_1 > 0$, $v_2 > 0$, $\xi < 0$ and $\left|\gamma \begin{pmatrix} \xi \\ 1 \end{pmatrix} - \mathbf{y}\right|$ is sufficiently small. □

Proof of Theorem 5. We take again the construction of Section 4.1. In order to prescribe positive signs, we need to introduce a variant $\tilde{N}_j$ of the matrices N_j which induces slight modifications in the estimates.

Recall that $(t_j/s_j)_{j\geq 0}$ stands for the sequence of convergents of y. For any $j \geq 1$, we set

$$\tilde{N}_j = \begin{pmatrix} t_{j-1} & t_j \\ s_{j-1} & s_j \end{pmatrix} \qquad \text{or} \qquad \tilde{N}_j = N_j = \begin{pmatrix} t_j & t_{j-1} \\ s_j & s_{j-1} \end{pmatrix},$$

respectively when j is even or odd. The matrix $\tilde{N}_j$ belongs to Γ_+ and has norm

$$|\tilde{N}_j| = \max(s_j, t_j) \asymp s_j.$$

Notice that if we put

$$\tilde{N}_j = \begin{pmatrix} t & t' \\ s & s' \end{pmatrix} \qquad \text{and} \qquad \delta = sy - t, \quad \delta' = s'y - t'$$

then δ is negative, and we now have the (weaker) estimates

$$\frac{1}{2s_{j+1}} < -\delta \leq \frac{1}{s_j} \qquad \text{and} \qquad |\delta'| \leq \frac{1}{s_j} \tag{9.1}$$

for any $j \geq 1$. We consider matrices of the form $\gamma = \tilde{N}_j U^\ell M_k$, where k and ℓ are positive integers and k is odd. Observe that the matrix M_k has positive entries on its first column precisely when k is odd. We find the formula

$$\gamma = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix} = \begin{pmatrix} \ell t q_{k-1} + t q_k + t' q_{k-1} & -\ell t p_{k-1} - t p_k - t' p_{k-1} \\ \ell s q_{k-1} + s q_k + s' q_{k-1} & -\ell s p_{k-1} - s p_k - s' p_{k-1} \end{pmatrix}.$$

It follows that the first column $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ of the matrix γ has positive entries, and that we have the bound of norm

$$|\gamma| \leq (\ell + 2)|\tilde{N}_j||M_k| \ll \ell s_j q_k. \tag{9.2}$$

Denote as usual

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \begin{pmatrix} v_1\xi + u_1 - y_1 \\ v_2\xi + u_2 - y_2 \end{pmatrix}.$$

Taking again the computations of Lemma 3, we find the formulas

$$\Lambda_1 - y\Lambda_2 = -\delta(\epsilon_k + \ell|\epsilon_{k-1}|) - \delta'|\epsilon_{k-1}| \tag{9.3}$$

and

$$\Lambda_2 = s|\epsilon_{k-1}|(\ell - \rho) \quad \text{with} \quad \rho = \frac{y_2}{s|\epsilon_{k-1}|} - \frac{\epsilon_k}{|\epsilon_{k-1}|} - \frac{s'}{s}. \tag{9.4}$$

For any odd large index k, let j be the integer defined by the estimate

$$s_{j-1} < q_k^{1/3} \le s_j.$$

Since we have assumed that $\omega(y) = 1$, the inequalities

$$q_k^{1/3-\varepsilon} \le s_{j-1} < q_k^{1/3} \le s_j \le q_k^{1/3+\varepsilon} \quad \text{and} \quad s_{j+1} \le q_k^{1/3+2\varepsilon} \tag{9.5}$$

hold for any $\varepsilon > 0$, provided that j is large enough. We deduce from the expression for ρ, given in (9.4), the estimate

$$y_2 q_k^{2/3-\varepsilon} - 1 - q_k^{2\varepsilon} \le \rho \le 2y_2 q_k^{2/3+\varepsilon} + 1, \tag{9.6}$$

using (2.1), (9.5), and noting that $0 \le s'/s \le s_j/s_{j-1} \le q_k^{2\varepsilon}$ by (9.5). It follows that the real number ρ is positive, when k is large enough. Let ℓ be the smallest integer larger or equal to ρ. We deduce from (2.1) and (9.5) that

$$0 < \Lambda_2 \le s|\epsilon_{k-1}| \le \frac{s_j}{q_k} \le q_k^{-2/3+\varepsilon}. \tag{9.7}$$

Moreover, ℓ is a positive integer satisfying

$$q_k^{2/3-\varepsilon} \ll \ell \ll q_k^{2/3+\varepsilon}, \tag{9.8}$$

according to the estimate (9.6). Using (9.5) and (9.8), observe now that the leading term on the right hand side of formula (9.3) giving $\Lambda_1 - y\Lambda_2$ is $-\delta\ell|\epsilon_{k-1}|$, which is positive. We thus find the estimate

$$0 < \Lambda_1 - y\Lambda_2 \ll \frac{\ell|\epsilon_{k-1}|}{s_j} \ll q_k^{-2/3+\varepsilon}, \tag{9.9}$$

making use of the inequalities (2.1), (9.1), (9.5) and (9.8). Since y is positive, it follows that Λ_1 is positive as well. Moreover, we deduce from (9.7) and (9.9) that

$$\max(\Lambda_1, \Lambda_2) \ll q_k^{-2/3+\varepsilon}. \tag{9.10}$$

Next, the bound of norm

$$|\gamma| \ll \ell s_j q_k \ll q_k^{2+2\varepsilon}.$$

follows from (9.5) and (9.8). Now, we deduce from (9.10) that

$$\max(\Lambda_1, \Lambda_2) \ll |\gamma|^{-(2/3-\varepsilon)/(2+2\varepsilon)} \leq |\gamma|^{-\mu},$$

provided $\mu < (2-3\varepsilon)/(6+6\varepsilon)$. Since $\mu < 1/3$, this last inequality is satisfied by choosing ε small enough.

Finally, observe that we have the estimate of norm

$$|\gamma| \asymp \ell s q_{k-1} \gg q_k^{1-2\varepsilon} q_{k-1},$$

by (9.5) and (9.8). Therefore, $|\gamma|$ may be arbitrarily large when k is large enough, and our construction produces infinitely many matrices γ verifying Theorem 5. □

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