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Inhomogeneous approximation with coprime integers and lattice orbits

Michel Laurent & Arnaldo Nogueira

ABSTRACT – Let \((\xi, y)\) be a point in \(\mathbb{R}^2\) and \(\psi : \mathbb{N} \to \mathbb{R}^+\) a function. We investigate the problem of the existence of infinitely many pairs \(p, q\) of coprime integers such that

\[ |q\xi + p - y| \leq \psi(|q|). \]

We give both unconditional results which are valid for every real pair \((\xi, y)\) with \(\xi\) irrational, and metrical results valid for almost all points \((\xi, y)\). We link the subject with density exponents of lattice orbits in \(\mathbb{R}^2\).

1 Introduction and results

Minkowski has proved that for every real irrational number \(\xi\) and every real number \(y\) not belonging to \(\mathbb{Z}\xi + \mathbb{Z}\), there exist infinitely many pairs of integers \(p, q\) such that

\[ |q\xi + p - y| \leq \frac{1}{4|q|}. \]

See for instance Theorem II in Chapter 3 of Cassels’ monograph [4]. The statement is optimal in the sense that the approximating function \(\ell \mapsto (4\ell)^{-1}\) cannot be decreased. Note that the restriction \(y \notin \mathbb{Z}\xi + \mathbb{Z}\) can be dropped at the cost of replacing the upper bound \((4|q|)^{-1}\) by \(c|q|^{-1}\) for any constant \(c\) greater than \(1/\sqrt{5}\). When \(y = 0\), the primitive point \((\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)})\) remains a solution to the above inequality, therefore we may moreover require that the pair of integers \(p, q\) be coprime. However, for a non-zero real number \(y\), this extra requirement is far from being obvious to satisfy. In this direction, Chalk and Erdos [6] have obtained the following result:

\[ 2010 \text{ Mathematics Subject Classification: } 11J20, 37A17. \]
Theorem (Chalk-Erdos). Let $\xi$ be an irrational real number and let $y$ be a real number. There exists an absolute constant $c$ such that the inequality

\begin{equation}
|q\xi + p - y| \leq \frac{c(\log q)^2}{q(\log \log q)^2}
\end{equation}

holds for infinitely many pairs of coprime integers $(p,q)$ with $q$ positive.

We study more generally the diophantine inequation

$$|q\xi + p - y| \leq \psi(|q|)$$

for coprime integers $p$ and $q$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a given function. Two types of questions naturally arise. First, finding unconditional results which are valid for every real pair $(\xi,y)$ with $\xi$ irrational as (1), and secondly getting metrical results valid for almost all points $(\xi,y)$. Here is an example of the first kind.

Theorem 1. Let $\xi$ be an irrational real number and let $y$ be a real number. There exist infinitely many integer quadruples $(p_1,q_1,p_2,q_2)$ satisfying

$$q_1p_2 - p_1q_2 = 1$$

and

\begin{equation}
|q_i\xi + p_i - y| \leq \frac{c}{\max(|q_1|,|q_2|)^{1/2}} \leq \frac{c}{\sqrt{|q_i|}}, \quad (i = 1, 2),
\end{equation}

with $c = 2\sqrt{3}\max(1,|\xi|^{1/2}|y|^{1/2})$.

Theorem 1 will be deduced in Section 2 from our results [10] of effective density for $SL(2,\mathbb{Z})$-orbits in $\mathbb{R}^2$. The estimate (2) is best possible, up to the value of the constant $c$. However, the optimality of (1) remains unclear. We address the following Problem. Can we replace the approximating function $\psi(\ell) = c(\log \ell)^2/\ell(\log \log \ell)^2$ occurring in (1) by a smaller one, possibly $\psi(\ell) = c\ell^{-1}$?

We shall further discuss this problem in Section 4 for the function $\psi(\ell) = 2\ell^{-1}$, offering some hints and indicating the difficulties which then arise. It turns out that the approximating function $\psi(\ell) = \ell^{-1}$ is permitted for almost all pairs $(\xi,y)$ of real numbers relatively to Lebesgue measure. The last assertion follows from the following metrical statement:
Theorem 2. Let $\psi : \mathbb{N} \mapsto \mathbb{R}^+$ be a function. Assume that $\psi$ is non-increasing, tends to 0 at infinity and that for every positive integer $c$ there exists a positive real number $c_1$ satisfying

$$\psi(c \ell) \geq c_1 \psi(\ell), \quad \forall \ell \geq 1.$$  

Furthermore assume that

$$\sum_{\ell \geq 1} \psi(\ell) = +\infty.$$  

Then, for almost all pairs $(\xi, y)$ of real numbers there exist infinitely many primitive points $(p, q)$ such that

$$q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q).$$

If $\sum_{\ell \geq 1} \psi(\ell)$ converges, the pairs $(\xi, y)$ satisfying (4) for infinitely many primitive points $(p, q)$ form a set of null Lebesgue measure.

Note that we could have equivalently required in (4) that $q$ be negative. Such a refinement could as well be achieved in the frame of Theorem 1, with a weaker approximating function of the form $\psi(\ell) = \ell^{-\mu}$ for any given real number $\mu < 1/3$, by employing alternatively Theorem 5 in Section 9 of [10]. We leave the details of proof to the interested reader, arguing as in Section 2. For questions of density involving signs, see also [7].

The proof of Theorem 2 is given in Section 3. It combines standard tools from metrical number theory with the ergodic properties of the linear action of $SL(2, \mathbb{Z})$ on $\mathbb{R}^2$ [13]. We refer to Harman’s book [8] for closely related results. See also the recent overview [1] and the monographs [14], [15].

Theorem 2 is a metrical statement about pairs $(\xi, y)$ of real numbers. A natural question is to understand what happens on each fiber when we fix either $\xi$ or $y$. In this direction, here is a partial result which will be deduced from the explicit construction displayed in Section 4.

Theorem 3. Let $\xi$ be an irrational number and let $(p_k/q_k)_{k \geq 0}$ be the sequence of its convergents. Assume that the series

$$(5) \quad \sum_{k \geq 0} \frac{1}{\max(1, \log q_k)}$$

diverges. Then for almost every real number $y$ there exist infinitely many primitive points $(p, q)$ satisfying

$$|q\xi + p - y| \leq \frac{2}{|q|}.$$  

Moreover the series (5) diverges for almost every real number $\xi$.  

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We now turn to the second part of the paper devoted to density exponents for lattice orbits in $\mathbb{R}^2$. As already mentioned, the approximating function $\psi(\ell) = c \ell^{-1/2}$ occurring in Theorem 1 is directly connected to the density exponent $1/2$ for $SL(2, \mathbb{Z})$-orbits. We intend to show that this exponent $1/2$ is best possible in general.

We work in the more general setting of lattices $\Gamma$ in $SL(2, \mathbb{R})$. Recall that a lattice $\Gamma$ in $SL(2, \mathbb{R})$ is a discrete subgroup for which the quotient $\Gamma \backslash SL(2, \mathbb{R})$ has finite Haar measure. We view $\mathbb{R}^2$ as a space of column vectors on which the group of matrices $\Gamma$ acts by left multiplication. We equip $\mathbb{R}^2$ with the supremum norm $| |$, and for any matrix $\gamma \in \Gamma$, we denote as well by $| \gamma |$ the maximum of the absolute values of the entries of $\gamma$. Let us first give a

**Definition.** Let $x$ and $y$ be two points in $\mathbb{R}^2$. We denote by $\mu_{\Gamma}(x, y)$ the supremum, possibly infinite, of the exponents $\mu$ such that the inequality

\[(6) \quad |\gamma x - y| \leq |\gamma|^{-\mu}\]

has infinitely many solutions $\gamma \in \Gamma$.

Note that for a fixed $x \in \mathbb{R}^2$, the function $y \mapsto \mu_{\Gamma}(x, y)$ is $\Gamma$-invariant. By the ergodicity of the action of $\Gamma$ on $\mathbb{R}^2$, see [13], this function is therefore constant almost everywhere on $\mathbb{R}^2$. We denote by $\mu_{\Gamma}(x)$ its generic value and we call $\mu_{\Gamma}(x)$ the generic density exponent of the orbit $\Gamma x$.

**Theorem 4.** The upper bound $\mu_{\Gamma}(x) \leq 1/2$ holds true for any point $x \in \mathbb{R}^2$ such that the orbit $\Gamma x$ is dense in $\mathbb{R}^2$.

In an equivalent way, Theorem 4 asserts that the upper bound $\mu(x, y) \leq 1/2$ holds for almost all points $y \in \mathbb{R}^2$. This bound was already known in the case of the unimodular group $\Gamma = SL(2, \mathbb{Z})$ as a consequence of Theorem 3 in [10].

One may optimistically conjecture that $\mu_{\Gamma}(x) = 1/2$ for every point $x$ such that $\Gamma x$ is dense in $\mathbb{R}^2$, or at least for almost every point $x \in \mathbb{R}^2$. In this direction, it follows from [10] that the lower bound

\[\mu_{SL(2, \mathbb{Z})}(x) \geq \frac{1}{3}\]

holds for all points $x$ in $\mathbb{R}^2 \setminus \{0\}$ with irrational slope. Weaker lower bounds can as well be deduced from [12] which are valid for any lattice $\Gamma \subset SL(2, \mathbb{R})$. Note that the function $x \mapsto \mu_{\Gamma}(x)$ is $\Gamma$-invariant since the quantity $\mu_{\Gamma}(x)$ obviously depends only on the orbit $\Gamma x$. Thus, the generic density exponent $\mu_{\Gamma}(x)$ takes the same value for almost all points $x \in \mathbb{R}^2$.

**Acknowledgement.** We are grateful to Martin Windmer for calling our attention on the Chalk-Erdos Theorem in [6].
2 Proof of Theorem 1

We first state a result obtained in [10]. In this section, we denote by $\Gamma$ the lattice $\text{SL}(2, \mathbb{Z})$. For any point $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}^2$ with irrational slope $x_1/x_2$, the orbit $\Gamma x$ is dense in $\mathbb{R}^2$. We have obtained in [10] effective results concerning the density of such an orbit. In particular, our estimates are essentially optimal when the target point $y$ has rational slope.

Lemma 1. Let $x$ be a point in $\mathbb{R}^2$ with irrational slope and $y = \begin{pmatrix} y \\ y \end{pmatrix}$ a point on the diagonal with $y \neq 0$. Then, there exist infinitely many matrices $\gamma \in \Gamma$ such that
\begin{equation}
|\gamma x - y| \leq \frac{c}{|\gamma|^{1/2}} \quad \text{with} \quad c = 2\sqrt{3}|x|^{1/2}|y|^{1/2}.
\end{equation}

Proof. The point $y$ has rational slope 1. Apply Theorem 1 (ii) of [10] with $a = b = 1$.

Put $x = \begin{pmatrix} \xi \\ 1 \end{pmatrix}$. The point $x$ has irrational slope $\xi$ so that Lemma 1 may be applied. Write $\gamma = \begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix}$ a matrix provided by Lemma 1. Then, the inequality (7) gives
\[\max(|q_1 \xi + p_1 - y|, |q_2 \xi + p_2 - y|) \leq \frac{c}{\max(|p_1|, |p_2|, |q_1|, |q_2|)^{1/2}} \leq \frac{c}{\max(|q_1|, |q_2|)^{1/2}}.\]

Therefore, both points $(p_1, q_1)$ and $(p_2, q_2)$ satisfy (2), and since the determinant $q_1 p_2 - q_2 p_1 = 1$, the two integer points $(p_1, q_1)$ and $(p_2, q_2)$ are primitive. As there exist infinitely many matrices $\gamma$ verifying (7), we thus find infinitely many solutions to (2).

Assume now that the irrational number $\xi$ has bounded partial quotients. Then, Theorem 4 in [10] gives us in the opposite direction a lower bound of the form
\[|\gamma x - y| \geq \frac{c'}{|\gamma|^{1/2}},\]
for some positive constant $c'$ depending only upon $(\xi, y)$. Since $|\gamma| \leq c'' \max(|q_1|, |q_2|)$ when (2) holds, the estimate (2) is optimal up to the value of $c$.

Remark. The single inequality $|q_1 \xi + p_1 - y| \leq \psi(|q_1|)$ geometrically means that the point $\gamma x$ falls inside a neighborhood of the vertical line $x_1 = y$. A better understanding of the shrinking target problem for the dense orbit $\Gamma x$, not to a point $y$ as in [10] but to a line in $\mathbb{R}^2$, may possibly lead to a refinement of (1).
3 Proof of Theorem 2

It is convenient to view the pairs \((\xi, y)\) occurring in Theorem 2 as column vectors \(\begin{pmatrix} \xi \\ y \end{pmatrix}\) in \(\mathbb{R}^2\). We are concerned with the set \(E(\psi)\) of vectors \(\begin{pmatrix} \xi \\ y \end{pmatrix}\) \(\in\mathbb{R}^2\) for which there exist infinitely many primitive integer points \((p, q)\) such that

\[
q \geq 1 \text{ and } |q\xi + p - y| \leq \psi(q).
\]

For fixed \(p, q\), denote by \(E_{p,q}(\psi)\) the strip

\[
E_{p,q}(\psi) := \left\{ \begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathbb{R}^2; \quad |q\xi + p - y| \leq \psi(q) \right\},
\]

and for every positive integer \(q\), let

\[
E_q(\psi) := \bigcup_{\substack{p \in \mathbb{Z} \\ \gcd(p, q) = 1}} E_{p,q}(\psi)
\]

be the union of all relevant strips involved in (8) for fixed \(q\). Without loss of generality, we shall assume that \(\psi(q) \leq 1/2\), so that the above union is disjoint. Then \(E(\psi)\) is equal to the lim sup set

\[
E(\psi) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} E_q(\psi).
\]

As usual when dealing with lim sup set in metrical theory, we first estimate Lebesgue measure of pairwise intersections of the subsets \(E_q(\psi), q \geq 1\). We establish next a new kind of zero-one law.

3.1 Measuring intersections

In this section, we restrict our attention to points located in the unit square \([0, 1]^2\). We denote by \(\varphi\) the Euler totient function and by \(\lambda\) the Lebesgue measure on \(\mathbb{R}^2\).

**Lemma 2.** Let \(\psi : \mathbb{N} \to [0, 1/2]\) be a function.

(i) For every positive integer \(q\), we have

\[
\lambda(E_q(\psi) \cap [0, 1]^2) = \frac{2\varphi(q)\psi(q)}{q}.
\]

(ii) Let \(q\) and \(s\) be distinct positive integers. Then, we have the upper bound

\[
\lambda(E_q(\psi) \cap E_s(\psi) \cap [0, 1]^2) \leq 4\psi(q)\psi(s).
\]
**Proof.** Denote by \( \chi_q \) the characteristic function of the interval \([-\psi(q), \psi(q)]\). Then the characteristic function \( \chi_{E_q(\psi)} \) of the subset \( E_q(\psi) \subset \mathbb{R}^2 \) is equal to

\[
\chi_{E_q(\psi)}(\xi, y) = \sum_{p \in \mathbb{Z}} \chi_q(q\xi + p - y) = \sum_{\gcd(p, q) = 1} \chi_q(q\xi - p - y).
\]

Observe that if \( (\xi, y) \) belongs to \([0, 1]^2\), the indices \( p \) of non-vanishing terms occurring in the last sum are located in the interval \(-1 \leq p \leq q\). Integrating first with respect to \( x \), we find

\[
\lambda(E_q(\psi) \cap [0, 1]^2) = \int_0^1 \int_0^1 \chi_{E_q(\psi)}(x, y) \, dx \, dy
\]

\[
= \sum_{p \in \mathbb{Z}} \int_0^1 \int_0^1 \chi_q(qx - p - y) \, dx \, dy
\]

\[
= \int_{-1}^1 -1 + y + \psi(q) \frac{dy}{q} + \sum_{1 \leq p \leq q - 2} \int_0^1 \frac{2\psi(q)}{q} \frac{dy}{q}
\]

\[
= \int_{-1}^1 \frac{2\psi(q)}{q} \frac{dy}{q} + \sum_{1 \leq p \leq q - 2} \int_{1 - \psi(q)}^1 \frac{1 - y + \psi(q)}{q} \frac{dy}{q}
\]

\[
= 2 \varphi(q) \psi(q)\frac{q}{q}.
\]

The first term appearing in the third equality of the above formula corresponds to the summation index \( p = -1 \) and the two last ones to \( p = q - 1 \). We have thus proved (i).

For the second assertion, we majorize

\[
\lambda(E_q(\psi) \cap E_s(\psi) \cap [0, 1]^2) = \int_0^1 \int_0^1 \chi_{E_q(\psi)}(x, y) \chi_{E_s(\psi)}(x, y) \, dx \, dy
\]

\[
\leq \int_0^1 \int_0^1 \left( \sum_{p \in \mathbb{Z}} \chi_q(qx + p - y) \right) \left( \sum_{r \in \mathbb{Z}} \chi_s(sx + r - y) \right) \, dx \, dy
\]

\[
= \int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) \, dx \, dy,
\]

where \( \|\cdot\| \) stands as usual for the distance to the nearest integer. Now, (ii) follows from the probabilistic independence formula

\[
\int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) \, dx \, dy = 4\psi(q)\psi(s),
\]

obtained by Cassels on page 124 of [4] (see Proof (ii)). \( \square \)
3.2 A zero-one law

We say that a subset of \( \mathbb{R}^2 \) is a \textit{null} set if it has Lebesgue measure 0. A set whose complementary is a null set is called a \textit{full} set. The goal of this section is to prove the

**Proposition.** Let \( \psi \) be an approximating function as in Theorem 2. Then the subset \( \mathcal{E}(\psi) \) is either a null set or a full set.

For proving the proposition, it is convenient to introduce the larger subset

\[
\mathcal{E}'(\psi) = \bigcup_{k \geq 1} \mathcal{E}(k\psi).
\]

In other words, \( \mathcal{E}'(\psi) \) is the set of all points \( \left( \frac{\xi}{y} \right) \) in \( \mathbb{R}^2 \) for which there exist a positive real number \( \kappa \), depending possibly on \( \left( \frac{\xi}{y} \right) \), and infinitely many primitive points \( (p,q) \) satisfying

\[
q \geq 1 \quad \text{and} \quad \left| q\xi + p - y \right| \leq \kappa \psi(q).
\]

Observe that \( \mathcal{E}(k\psi) \subseteq \mathcal{E}(k'\psi) \) if \( 1 \leq k \leq k' \). In particular, \( \mathcal{E}(\psi) \) is contained in \( \mathcal{E}'(\psi) \).

**Lemma 3.** Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) be a function tending to zero at infinity. Then the difference \( \mathcal{E}'(\psi) \setminus \mathcal{E}(\psi) \) is a set of null Lebesgue measure.

**Proof.** We show that all sets \( \mathcal{E}(k\psi), k \geq 1 \), have the same Lebesgue measure. For every real number \( y \), denote by \( \mathcal{E}(\psi, y) \subseteq \mathbb{R} \) the section of \( \mathcal{E}(\psi) \) on the horizontal line \( \mathbb{R} \times \{ y \} \), i.e.

\[
\mathcal{E}(\psi, y) = \left\{ \xi \in \mathbb{R} ; \left( \frac{\xi}{y} \right) \in \mathcal{E}(\psi) \right\}.
\]

Then, using (8), we can express

\[
\mathcal{E}(\psi, y) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \bigcup_{p \in \mathbb{Z} \text{ s.t.}(p,q)=1} \left[ -p + y - \psi(q), -p + y + \psi(q) \right] \bigg/ q
\]

as a limsup set of intervals. If we restrict to a bounded part of \( \mathcal{E}(\psi, y) \), the above union over \( p \) reduces to a finite one. Observe that the centers \( -\frac{p+y}{q} \) of these intervals do not depend on \( \psi \), and that their length is multiplied by the constant factor \( k \) when replacing \( \psi \) by \( k\psi \). Appealing now to a result due to Cassels [5], we infer that all limsup sets \( \mathcal{E}(k\psi, y), k \geq 1 \), have the same Lebesgue measure. See also Corollary of Lemma 2.1 on page 30 of [8]. Notice that for fixed \( k \), the length \( \frac{2k\psi(q)}{q} \) of the intervals
\[
\begin{pmatrix}
\frac{-p+y-k\psi(q)}{q} & \frac{-p+y+k\psi(q)}{q}
\end{pmatrix}
\]
tend to 0 as \(q\) tends to infinity, as required by Lemma 2.1. By Fubini, the fibered sets

\[
\mathcal{E}(k\psi) = \prod_{y \in \mathbb{R}} \left( \mathcal{E}(k\psi, y) \times \{y\} \right), \quad k \geq 1,
\]

have as well the same Lebesgue measure in \(\mathbb{R}^2\).

**Lemma 4.** Let \(\psi : \mathbb{N} \to \mathbb{R}^+\) be a non-increasing function satisfying (3). Then \(\mathcal{E}'(\psi)\) is either a null or a full set.

**Proof.** It is based on the following observation. Let \(\begin{pmatrix} \xi \\ y \end{pmatrix}\) belong to \(\mathcal{E}'(\psi)\) and let \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) be a matrix in \(SL(2, \mathbb{Z})\) such that \(c\xi + d > 0\). Then the point \(\begin{pmatrix} \xi' \\ y' \end{pmatrix}\) with coordinates

\[
\xi' = \frac{a\xi + b}{c\xi + d} \quad \text{and} \quad y' = \frac{y}{c\xi + d}
\]

belongs to \(\mathcal{E}'(\psi)\). Indeed, substituting

\[
q = aq' + cp', \quad p = bq' + dp'
\]

in (9) and dividing by \(c\xi + d\), we obtain the inequalities

\[
q' \geq 1 \quad \text{and} \quad |q'\xi' + p' - y'| \leq \frac{\kappa}{c\xi + d}\psi(q) \leq \kappa'\psi(q'),
\]

for some \(\kappa' > 0\) independent of \(q'\). The positivity of \(q'\) is proved as follows. Note that (9) implies the estimate

\[
p = -q\xi + \mathcal{O}_{\xi,y}(1).
\]

Then, inverting the linear substitution (10), we find

\[
q' = dq - cp = q(c\xi + d) + \mathcal{O}_{\gamma,\xi,y}(1).
\]

Since we have assumed that \(c\xi + d > 0\), the term \(q(c\xi + d)\) is arbitrarily large when \(q\) is large enough. The condition (3) now shows that \(\psi(q) \asymp \psi(q')\). Thus (11) is satisfied for infinitely many primitive points \((p', q')\), since the linear substitution (10) is unimodular. We have shown that \(\begin{pmatrix} \xi' \\ y' \end{pmatrix}\) belongs to \(\mathcal{E}'(\psi)\).

We now prove that the intersection \(\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)\) is either a full or a null subset of the half plane \(\mathbb{R} \times \mathbb{R}^+\). To that purpose, we consider the map

\[
\Phi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+, \quad \text{defined by} \quad \Phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x/y \\ 1/y \end{pmatrix}.
\]
Clearly $\Phi$ is a continuous involution of $\mathbb{R} \times \mathbb{R}^+$. The image
\[ \Omega := \Phi\left(\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)\right) \]
is formed by all points of the type
\[ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ y \end{pmatrix}, \]
where \( \begin{pmatrix} \xi \\ y \end{pmatrix} \) ranges over $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$. Now, the above condition $c\xi + d > 0$ is obviously equivalent to $cu + dv > 0$ since $y$ is positive. Then, the point
\[ \Phi\left( \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} \right) = \begin{pmatrix} \frac{au + bv}{cu + dv} \\ \frac{1}{cu + dv} \end{pmatrix} = \begin{pmatrix} \frac{a\xi + b}{c\xi + d} \\ \frac{y}{c\xi + d} \end{pmatrix} \]
belongs to $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$, by the preceding observation. Applying the involution $\Phi$, we find that
\[ \Phi\left( \begin{pmatrix} \frac{a\xi + b}{c\xi + d} \\ \frac{y}{c\xi + d} \end{pmatrix} \right) = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \]
belongs to $\Omega$. In other words, setting $\Gamma = SL(2, \mathbb{Z})$, we have established the inclusion
\[ (\Gamma \Omega) \cap (\mathbb{R} \times \mathbb{R}^+) \subseteq \Omega. \]
Since the reversed inclusion is obvious, the equality $\Omega = (\Gamma \Omega) \cap (\mathbb{R} \times \mathbb{R}^+)$ holds in fact. Assuming that $\Omega$ is not a null set, the ergodicity of the linear action of $\Gamma$ on $\mathbb{R}^2$ [13] shows that $\Gamma \Omega$ is a full set in $\mathbb{R}^2$. Hence $\Omega$ is a full set in the half plane $\mathbb{R} \times \mathbb{R}^+$. Transforming now $\Omega$ by $\Phi$, we find that
\[ \Phi(\Omega) = \mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+), \]
is as well a full set in $\mathbb{R} \times \mathbb{R}^+$, thus proving the claim.

We finally use another transformation to carry the zero-one law from the positive half plane $\mathbb{R} \times \mathbb{R}^+$ to the negative one $\mathbb{R} \times \mathbb{R}^-$. Writing (9) in the equivalent form
\[ q \geq 1 \quad \text{and} \quad |q(-\xi) + (-p) - (-y)| \leq \kappa \psi(q), \]
shows that $\mathcal{E}'(\psi)$ is invariant under the symmetry \( \begin{pmatrix} \xi \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\xi \\ -y \end{pmatrix} \) which maps $\mathbb{R} \times \mathbb{R}^+$ onto $\mathbb{R} \times \mathbb{R}^-$. Therefore $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^-)$ is a null or a full set in $\mathbb{R} \times \mathbb{R}^-$ when $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ is accordingly a null or a full set in $\mathbb{R} \times \mathbb{R}^+$.

Now, the combination of Lemma 3 and Lemma 4 obviously yields our proposition.
3.3 Concluding the proof of Theorem 2

Assume first that $\sum \psi(\ell)$ converges. We have to show that the set

$$E(\psi) = \limsup_{q \to +\infty} E_q(\psi)$$

has null Lebesgue measure. Lemma 2 shows that the partial sums

$$\sum_{q=1}^{Q} \lambda(E_q(\psi) \cap [0,1]^2) = 2 \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \leq 2 \sum_{q=1}^{Q} \psi(q)$$

converge (*). Then, Borel-Cantelli Lemma ensures that the lim sup set $E(\psi) \cap [0,1]^2$ is a null set. Thus $E(\psi)$ cannot be a full set. Now, the above proposition tells us that $E(\psi)$ is a null set.

We now consider the case of a divergent series $\sum \psi(\ell)$. Observe that the estimate

$$\frac{1}{2} \sum_{q=1}^{Q} \psi(q) \leq \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \leq \sum_{q=1}^{Q} \psi(q)$$

holds true for any large integer $Q$, since the sequence $\psi(\ell)_{\ell \geq 1}$ is non-increasing. The right inequality is obvious, while the left one easily follows from Abel summation process. See for instance Chapter 2 of [8], where full details are provided. By Lemma 2 and (12), the sums

$$\sum_{q=1}^{Q} \lambda(E_q(\psi) \cap [0,1]^2) = 2 \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \geq \sum_{q=1}^{Q} \psi(q)$$

are then unbounded. Then, using a classical converse to Borel-Cantelli Lemma, we have the lower bound

$$\lambda \left( E(\psi) \cap [0,1]^2 \right) = \lambda \left( \limsup_{q \to +\infty} (E_q(\psi) \cap [0,1]^2) \right) \geq \limsup_{Q \to +\infty} \frac{\left(\sum_{q=1}^{Q} \lambda(E_q(\psi) \cap [0,1]^2)\right)^2}{\sum_{q=1}^{Q} \sum_{s=1}^{Q} \lambda(E_q(\psi) \cap E_s(\psi) \cap [0,1]^2)}.$$ 

See for instance Lemma 2.3 in [8]. Lemma 2 and (12) now show that the numerator on the right hand side of (13) equals

$$4 \left( \sum_{q=1}^{Q} \frac{\varphi(q)\psi(q)}{q} \right)^2 \geq \left( \sum_{q=1}^{Q} \psi(q) \right)^2,$$

(* Here again we assume without loss of generality that $\psi(q) \leq 1/2$ for every $q \geq 1$, so that Lemma 2 may be applied.)
when $Q$ is large, while the denominator is bounded from above by
\[
4 \sum_{q=1}^{Q} \psi(q)\psi(s) + 2 \sum_{q=1}^{Q} \psi(q) \leq 4 \left( \sum_{q=1}^{Q} \psi(q) \right)^2 + 2 \sum_{q=1}^{Q} \psi(q).
\]
Thus (13) yields the lower bound
\[
\lambda \left( \mathcal{E}(\psi) \cap [0,1]^2 \right) \geq \frac{1}{4}.
\]
Hence $\mathcal{E}(\psi)$ is not a null set; it is thus a full set according to our proposition.

4 An approach to our problem

In this section, we apply a transference principle between homogeneous and inhomogeneous approximation, as displayed in Chapter V of [4] and in [3], for constructing explicit integer solutions of the inequality
\[
|q\xi + p - y| \leq \frac{2}{|q|}.
\]

Let $(p_k/q_k)_{k \geq 0}$ be the sequence of convergents to the irrational number $\xi$. The theory of continued fractions, see for instance the monograph [9], tells us that
\[
|q_k\xi - p_k| \leq \frac{1}{q_{k+1}} \quad \text{and} \quad p_kq_{k+1} - p_{k+1}q_k = (-1)^{k+1},
\]
for any $k \geq 0$. Setting $\nu_k = (-1)^{k+1}q_ky$, we thus have the relations
\[
\nu_k q_{k+1} + \nu_{k+1} q_k = 0 \quad \text{and} \quad \nu_k (q_{k+1} \xi - p_{k+1}) + \nu_{k+1} (q_k \xi - p_k) = y.
\]
Now, let $n_k$ be anyone of the two integers $\lfloor \nu_k \rfloor$ and $\lceil \nu_k \rceil$ (†). Then,
\[
|\nu_k - n_k| < 1,
\]
and $n_k$ is either equal to $(-1)^{k+1}[yq_k]$ or to $(-1)^{k+1}[yq_k]$. Setting
\[
p = -n_k p_{k+1} - n_{k+1} p_k \quad \text{and} \quad q = n_k q_{k+1} + n_{k+1} q_k,
\]
(†) As usual $[x]$ and $\lfloor x \rfloor$ stand respectively for the floor and the ceiling of the real number $x$. Then $\lfloor x \rfloor = [x] + 1$, unless $x$ is an integer in which case $\lfloor x \rfloor = [x] = x$. 12
we deduce from (16) the expressions

\begin{align}
q\xi + p - y &= n_k(q_{k+1}\xi - p_{k+1}) + n_{k+1}(q_k\xi - p_k) - y \\
&= (n_k - \nu_k)(q_{k+1}\xi - p_{k+1}) + (n_{k+1} - \nu_{k+1})(q_k\xi - p_k)
\end{align}

and

\begin{align}
q &= (n_k - \nu_k)q_{k+1} + (n_{k+1} - \nu_{k+1})q_k.
\end{align}

Recall that \(q_k\xi - p_k\) and \(q_{k+1}\xi - p_{k+1}\) have opposite signs. Assuming that \(n_k - \nu_k\) and \(n_{k+1} - \nu_{k+1}\) have the same sign, we infer from the formulas (19), (20) and from (15), (17) that

\begin{align}
|q\xi + p - y| < \frac{1}{q_{k+1}} \quad \text{and} \quad |q| < 2q_{k+1}.
\end{align}

Otherwise, we have

\begin{align}
|q\xi + p - y| < \frac{2}{q_{k+1}} \quad \text{and} \quad |q| < q_{k+1}.
\end{align}

The inequalities (21) and (22) obviously imply (14).

Since the linear substitution (18) is unimodular, the integers \(p\) and \(q\) are coprime if and only if \(n_k\) and \(n_{k+1}\) are coprime. Recall that the two choices \(n_k = \lfloor \nu_k \rfloor\) and \(n_k = \lceil \nu_k \rceil\) are admissible, both for \(n_k\) and \(n_{k+1}\). It thus remains to find indices \(k\) for which at least one of the coprimality conditions

\begin{align}
\gcd(\lfloor yq_k \rfloor, \lfloor yq_{k+1} \rfloor) = 1 \quad \text{or} \quad \gcd(\lceil yq_k \rceil, \lceil yq_{k+1} \rceil) = 1
\end{align}

is verified. Note that (23) obviously fails for all \(k \geq 0\) when \(y\) is an integer not equal to 1 or to \(-1\). Otherwise, the contingent existence of infinitely many indices \(k\) satisfying (23) is a non-trivial problem that we leave hanging.

Let us mention that the proof of (1) in [6] follows the same idea, finding a primitive integer point inside the square centered at the point \((\nu_k, \nu_{k+1}) \in \mathbb{R}^2\) with side \(C \log |\nu_k| / \log \log |\nu_k|\) for some suitable large absolute constant \(C\).

### 4.1 Proof of Theorem 3

We quote the following metrical result due to Harman (Theorem 8.3 in [8]). Assume that the series (5) diverges. Then for almost all positive real numbers \(y\), there exist infinitely many indices \(k\) such that the integer part \([yq_k]\) is a prime number. These
indices $k$ fulfill (23) since, assuming for simplicity that $y$ is irrational, either $\lfloor yq_{k+1} \rfloor$ or $\lceil yq_{k+1} \rceil + 1$ is not divisible by $\lfloor yq_k \rfloor$ and is thus relatively prime with $\lfloor yq_k \rfloor$. Hence (14) has infinitely many coprime solutions $(p, q)$ for almost every positive real number $y$. Writing now (14) in the equivalent form

$$\left| (-q)\xi + (-p) - (-y) \right| \leq \frac{2}{\lfloor q \rfloor}$$

shows that, $\xi$ being given, the set of all real numbers $y$ for which (14) has infinitely many coprime solutions is invariant by the symmetry $y \mapsto -y$. The first assertion is thus established. To complete the proof, note that

$$\lim_{k \to +\infty} \frac{\log q_k}{k} = \frac{\pi^2}{12 \log 2}$$

for almost every $\xi$ by Khintchine-Levy Theorem (see equation (4.18) in [2]). Thus the series (5) diverges for almost every $\xi$.

5 Generic density exponents

We prove in this section Theorem 4, as a consequence of Borel-Cantelli Lemma combined with the following counting result.

**Lemma 5.** Let $x$ be a point in $\mathbb{R}^2$ whose orbit $\Gamma x$ is dense in $\mathbb{R}^2$. For every symmetric compact set $\Omega$ in $\mathbb{R}^2 \setminus \{0\}$ there exists $c > 0$ such that

$$\text{Card}\{\gamma \in \Gamma; \gamma x \in \Omega, |\gamma| \leq T\} \leq cT$$

for any real number $T \geq 1$.

**Proof.** Ledrappier [11] has shown that the limit formula

$$\lim_{T \to +\infty} \frac{1}{T} \sum_{\gamma \in \Gamma, |\gamma| \leq T} f(\gamma x) = \frac{4}{|x| \text{vol}(\Gamma \setminus SL(2, \mathbb{R}))} \int f(y) \frac{dy}{|y|}$$

holds for any even continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ having compact support on $\mathbb{R}^2 \setminus \{0\}$, with a suitable normalisation of Haar measure on $SL(2, \mathbb{R})$. Approximating uniformly from above and from below the characteristic function of $\Omega$ by even continuous functions, we deduce that

$$\lim_{T \to +\infty} \frac{\text{Card}\{\gamma \in \Gamma; \gamma x \in \Omega, |\gamma| \leq T\}}{T} = \frac{4}{|x| \text{vol}(\Gamma \setminus SL(2, \mathbb{R}))} \int_{\Omega} \frac{dy}{|y|}.$$
Lemma 5 immediately follows.

For any point \( y \in \mathbb{R}^2 \) and any positive real number \( r \), we denote by
\[
B(y, r) = \{ z \in \mathbb{R}^2 ; |z - y| \leq r \}
\]
the closed disc centered at \( y \) with radius \( r \).

**Lemma 6.** Let \( x \) be a point in \( \mathbb{R}^2 \) whose orbit \( \Gamma x \) is dense, \( \Omega \) a symmetric compact set in \( \mathbb{R}^2 \setminus \{0\} \) and \( \mu \) a real number \( > 1/2 \). For every integer \( n \geq 1 \), put
\[
B_n = \bigcup_{|\gamma| = n, \gamma x \in \Omega} B(\gamma x, n^{-\mu}).
\]
Then the set
\[
B := \limsup_{n \to +\infty} B_n = \bigcap_{N \geq 1} \bigcup_{n \geq N} B_n = \bigcap_{N \geq 1} \bigcup_{|\gamma| \geq N, \gamma x \in \Omega} B(\gamma x, |\gamma|^{-\mu})
\]
has null Lebesgue measure.

**Proof.** We apply Borel-Cantelli Lemma and we prove that the series \( \sum_{n \geq 1} \lambda(B_n) \) converges if \( \mu > 1/2 \).

For every positive integer \( n \), set
\[
M_n = \text{Card}\{ \gamma \in \Gamma ; \gamma x \in \Omega, |\gamma| = n \}.
\]
Lemma 5 gives us the upper bound
(24)
\[
M_1 + \cdots + M_n = \text{Card}\{ \gamma \in \Gamma ; \gamma x \in \Omega, |\gamma| \leq n \} \leq cn,
\]
for some \( c > 0 \) independent of \( n \geq 1 \). Since a ball of radius \( r \) has Lebesgue measure \( 4r^2 \), we trivially bound from above
\[
\lambda(B_n) \leq \sum_{|\gamma| = n, \gamma x \in \Omega} 4n^{-2\mu} = 4M_n n^{-2\mu}.
\]

Summing by parts, we deduce from (24) that
\[
\sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} = \sum_{n=1}^{N-1} (M_1 + \cdots + M_n) \left( \frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{M_1 + \cdots + M_N}{N^{2\mu}} \\
\leq c \sum_{n=1}^{N-1} n \left( \frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{cN}{N^{2\mu}} = c \sum_{n=1}^{N} \frac{1}{n^{2\mu}}.
\]

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The partial sums
\[ \sum_{n=1}^{N} \lambda(B_n) \leq 4 \sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} \leq 4c \sum_{n=1}^{N} \frac{1}{n^{2\mu}} \]
thus converge if \( \mu > 1/2 \). \( \square \)

5.1 Proof of Theorem 4

We argue by contradiction and suppose on the contrary that \( \mu_\Gamma(x) > 1/2 \). Fix a real number \( \mu \) with \( 1/2 < \mu < \mu_\Gamma(x) \). Then for almost all points \( y \in \mathbb{R}^2 \), we have \( \mu(x,y) > \mu \). This means that there exist infinitely many \( \gamma \in \Gamma \) satisfying (6), or equivalently that \( y \) belongs to infinitely many balls of the form \( B(\gamma x, |\gamma|^{-\mu}) \). We now restrict our attention to points \( y \) with \( \mu(x,y) > \mu \) lying in an annulus
\[ \Omega' = \{ z \in \mathbb{R}^2; a' \leq |z| \leq b' \}, \]
where \( b' > a' > 0 \) are arbitrarily fixed. Since \( y \) belongs to the intersection \( \Omega' \cap B(\gamma x, |\gamma|^{-\mu}) \), we deduce from the triangle inequality the estimate
\[ a' - |\gamma|^{-\mu} \leq |\gamma x| \leq b' + |\gamma|^{-\mu}. \]
Fixing \( a < a' \) and \( b > b' \), the center \( \gamma x \) then lies in the larger annulus
\[ \Omega = \{ z \in \mathbb{R}^2; a \leq |z| \leq b \}, \]
provided that \( |\gamma| \) is large enough. It follows that \( y \) falls inside the union of balls
\[ \bigcup_{|\gamma| \geq N, \gamma \in \Omega} B(\gamma x, |\gamma|^{-\mu}) \]
considered in Lemma 6 for every integer \( N \) large enough, and thus \( y \) belongs to \( B \). However, Lemma 6 asserts that \( B \) is a null set which is a contradiction.

References


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