Logical Families of Nonlocal Boxes

Zeno Toffano, Cassandre Chatard, Kaicheng Ding, Alice Portalier, Gregoire Vilde

Logical Families of Nonlocal Boxes

Zeno Toffano[1,2], Cassandre Chatard[2], Kaicheng Ding[2], Alice Portalier[2] and Gregoire Vilde[2*]
[1]*Laboratoire de Signaux et Systemes (L2S - UMR8506) - CNRS - Universite Paris Saclay, France*
[2]*CentraleSupelec, Gif-sur-Yvette, France*

The well known nonlocal PR Box [1] correlates outputs (a, b) to inputs (x, y) in a two-party correlation by means of the logical constraint equation $a \oplus b = x \wedge y$. It violates the CHSH Bell Inequality (BI) [2] maximmaly : giving the number 4 beyond the quantum limit $2\sqrt{2}$, this last being obtained for a maximally nonlocal entangled quantum state (Bell state). The PR box is also no-signaling.

Here we consider families of boxes with all possible combinations of logical functions f correlated by the equation : $f_o(a, b) = f_i(x, y)$. The indexes o (output) and i (input) correspond to different Boolean functions and the varaibles $(x, y, a, b)\epsilon\{0, 1\}$ are Boolean. There are 16 different Boolean functions for 2 binary variables ($n = 2$), these are ordered with increasing binary number in the truth table : f_0 has the truth table $(0, 0, 0, 0)$, f_1 is NOR $(1, 0, 0, 0)$ and so on... For example OR ($\vee$) is f_{14} $(0, 1, 1, 1)$ and in the PR Box, XOR (exclusive or :$\oplus$) is $f_{o=6}(a, b)$ $(0, 1, 1, 0)$ and AND ($\wedge$) is $f_{i=8}(x, y)$ $(0, 0, 0, 1)$.

We define the joint mean value for the possible outcomes of the box as a function of the marginal probabilities : $C_{x,y}^{(i,o)} = \sum_{a,b}[P(a, b \mid f_o(a, b) = f_i(x, y)) \cdot A \cdot B]$ [3]. The measurement outcomes $A = 2a - 1$ (Alice) and $B = 2b - 1$ (Bob) give the values ± 1. The Bell parameter considering the four input possibilities is : $S^{(i,o)} = C_{00}^{(i,o)} + C_{01}^{(i,o)} + C_{10}^{(i,o)} - C_{11}^{(i,o)}$. If $S > 2$ we have nonlocality. There are actually four options by changing the place of the "$-$" in the expression of $S^{(i,o)}$. We calculated every case and kept the value of S with the maximal absolute value. We also tested no-signaling by the condition : $P(a \mid x, y) = P(a \mid x)$ and $P(b \mid x, y) = P(b \mid y)$.

For $n = 2$, there are $16 \times 16 = 256$ equations. A family of 16 equations giving maximal violation of the BI ($|S| = 4$) and no-signaling, corresponds to the following combinations of logical functions : output $o : \{6, 9\}$ and input $i : \{1, 2, 4, 8\}$ (8 cases) and also $o : \{6, 9\}$ and $i : \{7, 11, 13, 14\}$ (8 cases). The PR Box is part of this family ($o = 6$, $i = 8$). Another family of 32 equations giving a BI violation $|S| = \frac{10}{3} \approx 3.33 > 2\sqrt{2} > 2$, exceeding the quantum limit, but now signaling, corresponds to : $o : \{7, 11, 13, 14\}$ and $i : \{1, 2, 4, 8\}$, and the converse $o : \{1, 2, 4, 8\}$ and $i : \{7, 11, 13, 14\}$. An example of this kind of nonlocal logical box is : $a \vee b = x \wedge y$ ($o = 14$, $i = 8$).

For three-party correlations, $n = 3$, we have 256 logical functions. The expression of the joint outcome mean value $C_{x,y,z}^{(i,o)}$ is similar as before. We use the Svetlichny BI [4], which is written as follows : $M^{(i,o)} = C_{000}^{(i,o)} + C_{001}^{(i,o)} + C_{010}^{(i,o)} + C_{100}^{(i,o)} - C_{011}^{(i,o)} - C_{101}^{(i,o)} - C_{110}^{(i,o)} - C_{111}^{(i,o)}$, if $M > 4$ we have non-locality.

$n = 3$ corresponds to $256 \times 256 = 65536$ equations. Non-locality is obtained for six different values: $M_{NL} = 8\,; 6.4\,; 6\,; 5.33\,; 4.8\,; 4.57$ with the corresponding number of cases $N_{case}(M_{NL}) = 71\,; 560\,; 112\,; 840\,; 896\,; 560$. The cases of non-locality and non-signaling are less and occur only for $M_{NL+NS} = 8\,; 6\,; 5.33$.

In this work we show that in the case of two-party correlations, we have two families of nonlocal boxes : one with 16 boxes, comprising the PR box, and the other one, to our knowledge unknown up to now, giving 32 nonlocal boxes beyond the quantum limit but signaling. For three-party correlations the number increases and the picture is much more complex.

[1] S. Popescu, D. Rohrlich, "Quantum nonlocality as an axiom", Foundations of Physics, 24, p.379-385 (1994).
[2] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, "Proposed experiment to test local hidden-variable theories", Phys. Rev. Lett. 23 (15) : 880–4 (1969).
[3] Z. Toffano, "Intrication quantique : mythe ou realite ? / Quantum Entanglement : Myth or Reality", Res-Systemica, Revue Francaise de Systemique ; 12 (2014).
[4] G. Svetlichny, "Distinguishing three-body from two-body nonseparability by a Bell-type inequality", Phys. Rev. D 35, 3066, (1987).

* zeno.toffano@centralesupelec.fr

Logical Families of Nonlocal Boxes

Zeno Toffano[(1,2)], Cassandre Chatard[(2)], Kaicheng Ding[(2)], Alice Portalier[(2)], Grégoire Vildé[(2)]

(1) Laboratoire de Signaux et Systèmes (L2S, UMR8506) - CNRS - Université Paris-Saclay
(2) CentraleSupélec, Gif-sur-Yvette, FRANCE

Bell Inequality tests on logical boxes for two and three-party correlations

The well known nonlocal PR box [1] correlates outputs (a,b) to inputs (x,y) in a two-party correlation by means of the logical constraint correlation equation $a \oplus b = x \wedge y$. This box violates the CHSH Bell Inequality (BI) [2] maximally: giving the number 4 beyond the quantum limit $2\sqrt{2}$ (Tsirelson bound) obtained for a Bell state which is a maximally nonlocal entangled quantum state, this box is also no-signaling.

Here we consider families of <u>boxes with all possible combinations of logical functions</u> correlated by the equation:

$$f_n^{(o)}(a,b) = f_m^{(i)}(x,y) \qquad (1)$$

$f_n^{(o)}(a,b)$ (output) and $f_m^{(i)}(x,y)$ (input) stand for the 16 possible different Boolean functions (see the list below) and $(x,y,a,b) \in \{0,1\}$ are the Boolean variables of these functions.

We define the joint mean value for the possible outcomes of the box as a function of the marginal probabilities [3]:

$$C_{x,y}^{(i,o)} = \sum_{a,b} P(a,b|f_n^{(o)}(a,b) = f_m^{(i)}(x,y)) \cdot A(a) \cdot B(b) \quad \text{where } A(a) = 2a-1 \text{ and } B(b) = 2b-1 \qquad (2)$$

The measurement outcomes (A,B) , Alice and Bob, give the values ± 1 . The Bell parameter considering the four input possibilities is:

$$S^{(i,o)} = C_{00}^{(i,o)} + C_{01}^{(i,o)} + C_{10}^{(i,o)} - C_{11}^{(i,o)} \quad \text{If } |S| > 2 \ ^{(*)} \text{ we have nonlocality} \qquad (3)$$

We also tested no-signaling by the condition: $P(a|x,y) = P(a|x)$ and $P(b|x,y) = P(b|y)$ (4)

For two party correlations we have to consider $16 \times 16 = 256$ equations. We find two families of nonlocal boxes: one with 16 no-signaling boxes, comprising the PR box, giving maximal violation ($|S| = 4$) and the other one, to our knowledge unknown up to now, giving 32 nonlocal boxes beyond the quantum limit but signaling with $|S| = \frac{10}{3} \approx 3.33 > 2\sqrt{2} > 2$ which also violates the BI above the quantum limit.

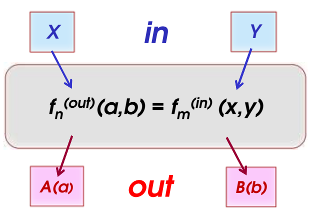
We also tested three party correlations ($256 \times 256 = 65536$ equations) using the Svetlichny Bell Inequality [4], which is written as follows:

$$M^{(i,o)} = C_{000}^{(i,o)} + C_{001}^{(i,o)} + C_{010}^{(i,o)} + C_{100}^{(i,o)} - C_{011}^{(i,o)} - C_{101}^{(i,o)} - C_{110}^{(i,o)} - C_{111}^{(i,o)} \quad \text{if } |M| > 4 \ ^{(*)} \text{ we have nonlocality} \qquad (5)$$

Non locality is obtained for six families corresponding to different values of $|M| > 4$. Among these three are also no-signaling.

(*)There are actually four options to set S by changing the place of the " – " sign in Eq.3. We have therefore chosen to calculate in each case the value of S and keep the one whose absolute value is maximum. Same procedure in Eq.5 for M for the three-party correlations.

two-party correlation box logical equation



two-party correlation the PR box : $a \oplus b = x \wedge y$

x	0		0		1		1	
y	0		1		0		1	
f8 : x ∧ y	0		0		0		1	
C_{xy}	+1		+1		+1		-1	
f6 : a ⊕ b	0		1		1		0	
a , A	0	-1	0	-1	1	+1	1	+1
b , B	0	-1	1	+1	0	-1	1	+1

C_{xy}= P(0,0|x,y)·(-1)·(-1) + P(0,1|x,y)·(-1)·(+1) + P(1,0|x,y)·(+1)·(-1) + P(1,1|x,y)·(+1)·(+1) *(Eq 2)*

$C_{00}=C_{01}=C_{10}=1/2+0+0+1/2=+1 \quad C_{11}= 0-1/2-1/2+0=-1$

$S=C_{00}+C_{01}+C_{10}-C_{11}=$ **+4** *(Eq 3)*

Boolean function truth tables

We consider the 16 possible different 2-input Boolean functions

Logical function for p, q $f_n(p,q)$	Truth table for 2 inputs (p,q) (1,1) (1,0) (0,1) (0,0)	Logical operator	
f_0	(0 0 0 0)	contradiction :	False ; ⊥
f_1	(0 0 0 1)	not or :	NOR ; ¬P ∧ ¬Q
f_2	(0 0 1 0)	non-Implication :	P ⊇ Q
f_3	(0 0 1 1)	negation of p :	¬P
f_4	(0 1 0 0)	converse non-implication :	P ⊆ Q
f_5	(0 1 0 1)	negation of q :	¬Q
f_6	(0 1 1 0)	exclusive or :	XOR ; P ⊕ Q
f_7	(0 1 1 1)	not and :	NAND ; ¬P ∨ ¬Q
f_8	(1 0 0 0)	conjunction (and) :	AND ; P ∧ Q
f_9	(1 0 0 1)	equivalence :	XNOR ; P ≡ Q
f_{10}	(1 0 1 0)	right projection of q :	Q
f_{11}	(1 0 1 1)	converse implication	P ⊂ Q
f_{12}	(1 1 0 0)	left projection of p :	P
f_{13}	(1 1 0 1)	implication :	P ⊃ Q
f_{14}	(1 1 1 0)	disjunction (or) :	OR ; P ∨ Q
f_{15}	(1 1 1 1)	tautology :	True ; T

According to standard numeration, logical functions $\boldsymbol{f_n(p,q)}$ are ordered with increasing binary number in the truth table.

Bell Inequality for the 256 combinations of logical functions in the two-party correlation logical boxes

OUTPUT / INPUT	f0	f1	f2	f4	f8	f3	f5	f12	f10	f6	f9	f7	f11	f13	f14	f15
f0																
f1										4	4	3,33	3,33	3,33	3,33	
f2										4	4	3,33	3,33	3,33	3,33	
f4										4	4	3,33	3,33	3,33	3,33	
f8										4	4	3,33	3,33	3,33	3,33	
f3																
f5																
f12																
f10																
f6																
f9																
f7		3,33	3,33	3,33	3,33					4	4					
f11		3,33	3,33	3,33	3,33					4	4					
f13		3,33	3,33	3,33	3,33					4	4					
f14		3,33	3,33	3,33	3,33					4	4					
f15																

The PR box [1] corresponds to the AND function, $f_8(x,y)$, at the input and to the XOR function, $f_6(a,b)$, at the output. Bell parameter is maximal $|S| = 4$. It is part of the no-signaling family of 16 nonlocal boxes (**orange cases**).

The other family of 32 nonlocal boxes (**green cases**) is signaling and giving a BI violation $|S| = \frac{10}{3} \approx 3.33 > 2\sqrt{2} > 2$, exceeding the quantum limit. An example of this kind of nonlocal logical box has the logical constraint $a \vee b = x \wedge y$.

Bell Inequality for the 65536 combinations of logical functions in the three-party correlation logical boxes

Calculation (computer) of the Svetlichny [4] parameter M (Eq. 5) for $256 \times 256 = 65536$ equations.
Non-locality $|M| > 4$ is obtained for six different values of $|M|$.

$\|M\|$	8	6.4	6	5.33	4.8	4.57
Number of boxes	71	560	112	840	896	560
Signaling :	NO	YES	NO	NO	YES	YES

[1] S. Popescu, D. Rohrlich, "Quantum nonlocality as an axiom", Foundations of Physics, 24, p.379-385 (1994).
[2] J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, "Proposed experiment to test local hidden-variable theories", Phys. Rev. Lett. 23 (15): 880–4 (1969).
[3] Z. Toffano, "Intrication quantique : mythe ou réalité ? / Quantum Entanglement: Myth or Reality", Res-Systemica, Revue Française de Systémique; 12 (2014).
[4] G. Svetlichny, "Distinguishing three-body from two-body non-separability by a Bell-type inequality", Phys. Rev. D 35, 3066, (1987).