On the Ritt property and weak type maximal inequalities for convolution powers on L1 (Z)

Christophe Cuny

ON THE RITT PROPERTY AND WEAK TYPE MAXIMAL INEQUALITIES FOR CONVOLUTION POWERS ON $\ell^1(\mathbb{Z})$

CHRISTOPHE CUNY

Abstract. In this paper we study the behaviour of convolution powers of probability measures μ on $\mathbb{Z}$, such that $(\mu(n))_{n\in\mathbb{N}}$ is completely monotone or such that ν is centered with a second moment. In particular we exhibit many new examples of probability measures on $\mathbb{Z}$ having the so called Ritt property and whose convolution powers satisfy weak type maximal inequalities in $\ell^1(\mathbb{Z})$.

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1. Introduction

Let μ be a probability on $\mathbb{Z}$. Given an invertible bi-measurable transformation τ on a measure space $(\mathbb{S}, \mathcal{S}, \lambda)$ we define a *positive contraction* of every $L^p(\lambda)$, $1 \leq p \leq \infty$, by setting

$$P_\mu(\tau)(f) := \sum_{k\in\mathbb{Z}} \mu(k) f \circ \tau^k \qquad \forall f \in L^p(m)\,.$$

Several authors, see for instance [4], [5], [22], [6], [16], [3], [20], [26], [24], [23] or [7], studied the almost everywhere behaviour of the iterates of $\mu(\tau)$, i.e. of $(\mu^{*n}(\tau))_{n\geq 1}$, acting on $L^p(\lambda)$, $1 \leq p < \infty$.

When $p > 1$, the almost everywhere behaviour has been characterized thanks to the so called *bounded angular ratio* property, introduced in [6] and which is equivalent to the so called *Ritt* property on $\ell^p(\mathbb{Z})$, $p > 1$. Let us recall the definition of those properties.

Definition 1.1. *Let μ be a probability measure on $\mathbb{Z}$. We say that μ is strictly aperiodic if $|\hat{\mu}(\theta)| < 1$ for every $\theta \in (0, 2\pi)$. We say that μ has bounded angular ratio (BAR) if moreover*

$$\sup_{\theta\in(0,2\pi)} \frac{|1-\hat{\mu}(\theta)|}{1-|\hat{\mu}(\theta)|} < \infty\,. \tag{1}$$

The strict aperiodicity is equivalent to the fact that the support of μ is not contained in a coset of a proper subgroup of $\mathbb{Z}$. In particular, it holds whenever the support of μ contains two consecutive integers.

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Definition 1.2. *We say that a probability measure μ on $\mathbb{Z}$ is Ritt on $\ell^p(\mathbb{Z})$, for some $p \geq 1$, if*

$$\sup_{n\geq 1} n\|\mu^{*n} - \mu^{*(n+1)}\|_{\ell^p(\mathbb{Z})} < \infty\,.$$

When $p = 1$ we say simply that μ is Ritt, because then, it is Ritt on all $\ell^r(\mathbb{Z})$, $r \geq 1$. Denote by $\mathcal{R}$ the set of Ritt probability measures on $\mathbb{Z}$.

A version of the next Theorem may be found for instance in Cohen, Cuny and Lin [7, Theorem 4.3]. Their Theorem 4.3 is not formulated exactly as below but the proof of Theorem 1.1 may be done similarly. The equivalence the item (vi) with the other items follow from their Proposition 6.4. In all the paper we use the notation $\mathbb{N} := \{0, 1, 2 \ldots\}$.

Theorem 1.1. *Let μ be a strictly aperiodic probability on $\mathbb{Z}$. The following are equivalent:*

i) μ has BAR;

ii) There exist $p > 1$ and $C_p > 0$ such that for every invertible bi-measurable transformation τ on a measure space $(S, \mathcal{S}, \lambda)$,

$$\|\sup_{n\geq 1} |(P_\mu(\tau))^n f|\,\|_{p,\lambda} \leq C_p \|f\|_{p,\lambda} \qquad \forall f \in L^p(\lambda)\,; \tag{2}$$

(iii) There exists $p > 1$ such that for every invertible bi-measurable transformation τ on a probability space $(S, \mathcal{S}, \lambda)$ and every $f \in L^p(\lambda)$, $((P_\mu(\tau))^n f)_{n\in\mathbb{N}}$ converges λ-a.e.

(iv) There exist $p > 1$ and $C_p > 0$ such that

$$\|\sup_{n\geq 1} |(P_\mu(R))^n f|\,\|_{\ell^p(\mathbb{Z})} \leq C_p \|f\|_{\ell^p(\mathbb{Z})} \qquad f \in \ell^p(\mathbb{Z})\,, \tag{3}$$

where R is the right shift on $\mathbb{Z}$;

v) There exists $p > 1$ such that μ is Ritt on $\ell^p(\mathbb{Z})$.

(vi) There exists $p > 1$ such that for every $m \in \mathbb{N}$, there exists $C_{m,p} > 0$ such that

$$\|\sup_{n\geq 1} n^m |(I - P_\mu)^m (P_\mu(R))^n f|\,\|_{\ell^p(\mathbb{Z})} \leq C_{m,p} \|f\|_{\ell^p(\mathbb{Z})} \qquad f \in \ell^p(\mathbb{Z})\,,$$

Actually, if any of the above properties holds, then the conclusion of (ii), (iii), (iv) or (v) holds for all $p > 1$.

The proof of the above theorem follows from recent works of Le Merdy and Xu, [17] and [18], who studied positive Ritt contractions T of $L^p(\mathbb{S}, \mathcal{S}, m)$ (p being fixed). Recall that a contraction T on a Banach space X is Ritt if $\sup_{n\in\mathbb{N}} n\|T^n - T^{n+1}\|_X < \infty$, which is compatible with our definition 1.2 which just says that the operator of convolution by μ is Ritt on $X = \ell^p(\mathbb{Z})$.

Le Merdy and Xu proved that any positive Ritt contraction satisfies maximal inequalities in spirit of (2). They also obtained square function estimates, oscillation inequalities and variation inequalities. See also [7] for related results.

In this paper we are concerned with the case when $p = 1$, and we address the following two questions.

- Question 1: For what probability measures μ on $\mathbb{Z}$ does one have a weak type $(1,1)$-maximal inequality:

$$\#\{k \in \mathbb{Z} : \sup_{n\geq 1} |\mu^{*n} * f| > \lambda\} \leq \frac{C}{\lambda}\|f\|_{\ell^1(\mathbb{Z})} \qquad \forall \lambda \geq 0\,. \tag{4}$$

More generally, given $m \in \mathbb{N}$, does there exist $C_m > 0$ such that (with the convention $(\delta_0 - \mu)^{*0} = \delta_0$)

$$\sup_{\lambda>0} \lambda \#\{k \in \mathbb{Z} : \sup_{n\geq 1} n^m |\mu^{*n} * (\delta_0 - \mu)^{*m} * f(k)| \geq \lambda\} \leq C_m \|f\|_{\ell^1(\mathbb{Z})} \qquad \forall f \in \ell^1(\mathbb{Z})\,. \tag{5}$$

- Question 2: For what probability measures μ on $\mathbb{Z}$ does one have the Ritt property in $\ell^1(\mathbb{Z})$:

$$\sup_{n\geq 1} n\|\mu^{*n} - \mu^{*(n+1)}\|_{\ell^1(\mathbb{Z})} < \infty\,. \tag{6}$$

Notice that if μ satisfies (4) then, by the Marcinkiewicz interpolation theorem (between weak L^1 and L^∞), it does satisfy (2), hence μ has BAR. Notice also that if μ satisfies (6) then, by Theorem 1.1 has also BAR. Hence, the questions we intend to answer are: what extra conditions, in addition to the BAR property, are sufficient to have (4), (5) or (6) ?

Let us discuss the known results concerning those questions, before presenting our results. As far as we know, when $m \geq 1$, (5) has not been investigated before.

The simplest examples of probability measures having BAR are the symmetric ones. Bellow, Jones and Rosenblatt [6] proved that if μ is symmetric such that $(\mu(n))_{n\geq 0}$ is non increasing then (4) holds. We do not know whether (6) holds as well, in this case, but we provide sufficient conditions in Section 5.

Another case where (4) holds is when $\sum_{k\in\mathbb{Z}} k^2\mu(k) < \infty$ (μ has a second moment) and $\sum_{k\in\mathbb{Z}} k\mu(k) = 0$. This has been proved by Bellow and Calderón [3]. Again the Ritt property is not known in that case. The proof of Bellow and Calderón is based on general intermediary results that have been extended recently by Wedrychowicz [26]. Wedrychowicz proved that (4) holds for centered probability measures (hence with a first moment) having BAR and satisfying some extra conditions. Examples without second moment are presented in [26].

Several examples of probabilities having the Ritt property in $\ell^1(\mathbb{Z})$ may be found in Dungey [10], see sections 4 and 5 there.

Let us now present our results. As mentionned above, the method of Bellow and Calderón is fairly general. Actually, if one follows carefully their paper, one realizes that the following definition comes somewhat naturally into play.

Definition 1.3. *We say that a probability measure μ on $\mathbb{Z}$ satisfies the hypothesis* (**H**) *if $\hat{\mu}$ is twice continuously differentiable on $[-\pi,\pi]-\{0\}$ and if there exists an even and continuous function ψ on $[-\pi,\pi]$, vanishing at 0 and continuously differentiable on $[-\pi,\pi]-\{0\}$, and some constants $c, C > 0$ such that for every $\theta \in (0,\pi]$*

(i) $|\hat{\mu}(\theta)| \leq 1 - c\psi(\theta)$;

(*ii*) $|\theta\hat{\mu}'(\theta)| \leq C\psi(\theta)$;
(*iii*) $|\hat{\mu}'(\theta)| \leq C\psi'(\theta)$;
(*iv*) $|\theta\hat{\mu}''(\theta)| \leq C\psi'(\theta)$.

Let us denote by $\mathcal{H}$ the set of probability measures satisfying hypothesis (**H**).

The relevance of the hypothesis (**H**) lies in the following, where we also give stability properties of $\mathcal{H}$ as well as of $\mathcal{R}$. We say that a set of probability measures on $\mathbb{Z}$ is stable by symmetrization if whenever $\mu = (\mu(n))_{n\in\mathbb{Z}}$ belongs to that set so does $\check{\mu} = (\mu(-n))_{n\in\mathbb{Z}}$.

Theorem 1.2.

(*i*) *The set $\mathcal{H}$ is convex and stable by convolution and by symmetrization.*
(*ii*) *The set $\mathcal{R}$ is convex and stable by convolution and by symmetrization.*
(*iii*) *Let $\mu \in \mathcal{H}$. Then, μ satisfies* (4).
(*iv*) *Let $\mu \in \mathcal{H} \cap \mathcal{R}$. Then, for every $m \in \mathbb{N}$, there exists $C_m > 0$ such that μ satisfies* (5).

Theorem 1.2 follows from several results: item (*i*) follows from Proposition 2.4, item (*ii*) may be proved as Proposition 3.10, items (*iii*) and (*iv*) follow from Proposition 2.3.

Then, our goal is to provide many examples of elements of $\mathcal{H} \cap \mathcal{R}$.

Our first examples are the ones already considered by Bellow and Calderón, in particular the fact that μ as in the next theorem satisfies (4) is not new, while the Ritt property is new. The proof of Theorem 1.3 is done in Section 2.3.

Theorem 1.3. *Let μ be a centered and strictly aperiodic probability measure on $\mathbb{Z}$ with finite second moment. Then $\mu \in \mathcal{H} \cap \mathcal{R}$.*

Then, we shall consider probability measures μ, such that $(\mu(n))_{n\geq 0}$ is completely monotone (see the next section for the definition). In this context we are able to characterize the BAR property. The idea of considering completely monotone sequences was motivated by Gomilko-Haase-Tomilov [12] and Cohen-Cuny-Lin [7].

Theorem 1.4. *Let μ be a probability measure on $\mathbb{Z}$ supported on $\mathbb{N}$, such that $(\mu(n))_{n\in\mathbb{N}}$ is completely monotone. Then*

(*i*) *μ has BAR if and only if there exists $C > 0$ such that*

$$\sum_{k=1}^{n} k\mu(k) \leq Cn \sum_{k\geq n} \mu(k) \qquad \forall n \geq 1\,. \tag{7}$$

(*ii*) *Assume that μ has BAR. Let σ be a probability measure on $\mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}} n^2\sigma(n) < \infty$. Then, $\mu * \sigma \in \mathcal{H} \cap \mathcal{R}$ and for every $\alpha \in (0,1]$ $\alpha\mu + (1-\alpha)\sigma \in \mathcal{H} \cap \mathcal{R}$. In particular (take $\sigma = \delta_0$), $\mu \in \mathcal{H} \cap \mathcal{R}$.*

Remarks. Notice that we do not assume σ to be centered. The conclusion of item (*ii*) actually holds for σ such that $\hat{\sigma}$ is twice continuously differentiable on $[-\pi,\pi] - \{0\}$ with $\hat{\sigma}'$ and $\theta \mapsto \theta\hat{\sigma}''(\theta)$ bounded. Moreover, see Proposition 3.10, it is possible to relax the

conditions on $\hat{\sigma}$ if one is only concerned with the Ritt Property. We were not able to provide a perturbation result in the spirit of Theorem 1.5.

Item (i) follows from Propositions 3.3 and 3.5. Item (ii) is proved in sections 3.2 and 3.3.

The proof of the Ritt property in Theorem 1.4 is based on a recent of Gomilko and Tomilov [14]. The fact that when μ has BAR $\delta_1 * \mu$ is Ritt has been proven by Gomilko and Tomilov [15], see their Theorem 7.1. Their proof is also based on [14].

Theorem 1.5. *Let μ be a* centered *probability measure on $\mathbb{Z}$ supported on $\{-1\} \cup \mathbb{Z}$, such that $(\mu(n))_{n\in\mathbb{N}}$ is completely monotone. Then*

(i) *μ has BAR if and only if there exists $C > 0$ such that*

$$n \sum_{k\geq n} k\mu(k) \leq C \sum_{k=1}^{n} k^2 \mu(k) \qquad \forall n \geq 1\,. \tag{8}$$

(ii) *Assume that μ has BAR. Let σ be a centered probability measure on $\mathbb{Z}$, such that there exists $c > 0$ such that $\sum_{n\in\mathbb{Z}} n^2 |\sigma(n) - c\mu(n)| < \infty$. Then, $\sigma \in \mathcal{H} \cap \mathcal{R}$. In particular (take $c = 1$ and $\sigma = \mu$), $\mu \in \mathcal{H} \cap \mathcal{R}$.*

Moreover (see section 5), we also study symmetric probability measures with completely monotone coefficients.

In the above theorems, we obtain weak type maximal inequalities in $\ell^1(\mathbb{Z})$. Of course, by mean of transference principles (see e.g. [2] or [25, page 164], one may derive similar results for the operator $P_\mu(\tau)$ in the spirit of (2) as well as some almost everywhere convergence results. We leave that "standard" task to the reader.

The paper is organized as follows. In Section 2, we prove Theorem 1.2 and prove the Ritt property under a slightly stronger assumption than hypothesis (**H**). In section 3, we consider probability measures as in Theorem 1.4 and prove Theorem 1.4 In section 4, we consider probability measures as in Theorem 1.5 and prove Theorem 1.5. In section 5 we consider symmetric probability measures. Finally, in section 6 we discuss several open questions on the topic.

Before going to the proofs, we would like to mention that the above theorems provide new situations to which the results of Cuny and Lin [9] apply, see examples 1 and 2 there.

2. General criteria for maximal inequalities and for the Ritt property

In this section we give general conditions ensuring weak type maximal inequalities associated with sequences of probabilities on $\mathbb{Z}$ and conditions ensuring the Ritt property.

In the case of weak type maximal inequalities, the obtained conditions are derived from slight modifications of known results, see e.g. [3] and [26].

2.1. **Sufficient conditions for weak type maximal inequalities.** We start with the following result of Bellow and Calderón [3], see also Zo [28] for a related result. Actually, Bellow and Calderón considered only the case of probability measures, but their proof extends to the situation below.

Theorem 2.1 (Bellow-Calderón)**.** *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite* signed *measures on $\mathbb{Z}$ such that $\sup_{n\in\mathbb{N}}\|\sigma_n\|_{\ell^1}<\infty$. Assume that there exists $C>0$ such that $k,\ell\in\mathbb{Z}$ with $0<2|k|\le\ell$,*

$$|\sigma_n(k+\ell)-\sigma_n(\ell)|\le C\frac{k}{\ell^2}\qquad\forall n\in\mathbb{N}\,. \tag{9}$$

Then, there exists $C'>0$ such that for every $f\in\ell^1(\mathbb{Z})$,

$$\sup_{\lambda>0}\#\{k\in\mathbb{Z}\,:\,\sup_{n\in\mathbb{N}}|\sigma_n*f(k)|\ge\lambda\}\le\frac{C'}{\lambda}\|f\|_{\ell^1}\,.$$

In order to apply Theorem 2.1 we shall need the following version of Corollary 3.4 of [3].

Lemma 2.2. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite signed measures, such that for every $n\in\mathbb{N}$, $\hat\sigma_n$ is twice continuously differentiable on $\mathbb{R}-2\pi\mathbb{Z}$. If moreover*

$$\sup_{n\in\mathbb{N}}\int_{-\pi}^{\pi}|\theta\hat\sigma_n''(\theta)|d\theta<\infty\,, \tag{10}$$

and

$$\lim_{\theta\to0,\theta\neq0}\theta\hat\sigma_n'(\theta)=0\qquad\forall n\in\mathbb{N}\,, \tag{11}$$

then (9) *holds.*

Remark. It follows from (10) (and the continuity of $\hat\sigma_n$ at 0) that the limit in (11) exists, hence condition (11) is just that the limit is 0.

Proof. For every $k\in\mathbb{Z}-\{0\}$, we have $\sigma_n(k)=\int_{-\pi}^{\pi}\hat\sigma_n(\theta)\mathrm{e}^{-ik\theta}d\theta$. Let $\pi>\varepsilon>0$. Performing two integration by parts as in [3] to evaluate $\int_{\varepsilon}^{\pi}\hat\sigma_n(\theta)\mathrm{e}^{-ik\theta}d\theta$ and $\int_{-\pi}^{-\varepsilon}\hat\sigma_n(\theta)\mathrm{e}^{-ik\theta}d\theta$, using our assumptions and letting $\varepsilon\to0$, we see that

$$\sigma_n(k)=\int_{-\pi}^{\pi}\hat\sigma_n''(\theta)\frac{1-\mathrm{e}^{-ik\theta}}{k^2}\,d\theta\,.$$

Then, we conclude as in [3]. □

Proposition 2.3. *Let μ be a probability measure on $\mathbb{Z}$ satisfying hypothesis* (**H**)*. Then, for every $m\in\mathbb{N}$,*

$$\sup_{n\ge1}n^m\int_{-\pi}^{\pi}|\theta||(\hat\mu^n(1-\hat\mu)^m)''(\theta)|d\theta<\infty\,, \tag{12}$$

and

$$\lim_{\theta\to0,\theta\neq0}\theta(\hat\mu^n(1-\hat\mu)^m)'(\theta)=0\qquad\forall n\in\mathbb{N}\,,$$

In particular, there exists $C > 0$ such that for every $f \in \ell^1(\mathbb{Z})$,

$$\sup_{\lambda>0} \lambda \#\{k \in \mathbb{Z} : \sup_{n\geq 1} |\mu^{*n} * f(k)| \geq \lambda\} \leq C\|f\|_{\ell^1} . \tag{13}$$

If moreover μ is Ritt then, for every $m \geq 1$, there exists $C_m > 0$ such that for every $f \in \ell^1(\mathbb{Z})$,

$$\sup_{\lambda>0} \lambda \#\{k \in \mathbb{Z} : \sup_{n\geq 1} n^m |\mu^{*n} * (\delta_0 - \mu)^{*m} * f(k)| \geq \lambda\} \leq C_m\|f\|_{\ell^1} . \tag{14}$$

Remarks. The proposition is related to Theorem 2.10 of Wedrychowicz [26]. Notice that, by (ii), ψ is non-negative and by (iii) it is non-decreasing. We shall see in proposition 2.5 that if there exists $C > 0$ such that for every $\theta \in (0, \pi]$, $\psi(\theta) \leq C\theta\psi'(\theta)$, then μ is automatically Ritt.

Proof. If $\mu = \delta_0$ the result is trivial. Hence we assume that $\mu \neq \delta_0$. In particular, by (ii), ψ cannot vanish in a neighbourghood of 0, hence is positive on $(0, \pi]$. Then $|\hat{\mu}| < 1$ on $(0, \pi]$ (hence μ is strictly aperiodic).

We have, on $(0, \pi]$.

$$\begin{aligned} n^m|(\hat{\mu}^n(1-\hat{\mu})^m)''| &\leq n^{m+2}|\hat{\mu}|^{n-2}|\hat{\mu}'|^2|1-\hat{\mu}|^m + 2mn^{m+1}|\hat{\mu}|^{n-1}|\hat{\mu}'|^2|1-\hat{\mu}|^{m-1} \\ &+n^{m+1}|\hat{\mu}|^{n-1}|\hat{\mu}''||1-\hat{\mu}|^{m-1} + mn^m|\hat{\mu}|^n|\hat{\mu}''|\,|1-\hat{\mu}|^{m-1} + m(m-1)n^m|\hat{\mu}|^n|\hat{\mu}'|^2|1-\hat{\mu}|^{m-2} . \end{aligned} \tag{15}$$

Using (i) and the fact that ψ is continuous with $\psi(0) = 0$, there exist $\eta > 0$ and $c > 0$, such that for every $\theta \in [0, \eta]$,

$$|\hat{\mu}(\theta)| \leq \mathrm{e}^{-c\psi(\theta)} . \tag{16}$$

Since, $\sup_{\theta\in[\eta,\pi]} |\hat{\mu}(\theta)| < 1$, taking c smaller if necessary, we may assume that (16) holds for every $\theta \in [0, \pi]$.

Using (iii) and that ψ is continuous at 0, we see that ψ' and $\hat{\mu}'$ are in L^1, hence that

$$|1 - \hat{\mu}(\theta)| \leq \psi(\theta) \qquad \forall\theta \in (0, \pi] . \tag{17}$$

Combining (16) and (26) with (ii), (iii) and (iv) (taking care with the cases $m = 0$ and $m = 1$), we see that there exists $C_m > 0$ such that for every $\theta \in (0, \pi]$ and every $n \in \mathbb{N}$,

$$|\theta(\hat{\mu}^n)''(\theta)| \leq C_m\big(n^2\psi^{m+1}(\theta) + n\psi^m(\theta) + (m-1)\psi^{m-1}(\theta)\big)\mathrm{e}^{-c(n-2)\psi(\theta)}\, n^m\psi'(\theta) .$$

Using that the integrand below is even and the change of variable $u = (n-2)\psi(\theta)$, we see that

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} |\theta||(\hat{\mu}^n(1-\hat{\mu})^m)''(\theta)|d\theta \\ &\leq \tilde{C}_m \int_0^{+\infty} (u^{m+1} + u^m + (m-1)u^{m-1})\mathrm{e}^{-cu}\, du \, < \infty, \end{aligned}$$

and (12) holds.

The fact that (11) holds follows from item (ii), using that ψ is continuous at 0, with $\psi(0) = 0$, and that $\hat{\mu}$ is bounded.

Let $m \in \mathbb{N}$ and set $\sigma_n = \sigma_{n,m} := n^m \mu^{*n} (\delta_0 - \mu)^{*m}$ for every $n \in \mathbb{N}$.

It follows from Theorem 2.1 and Lemma 2.2, that (13) holds.

When $m \geq 1$, it follows from Theorem 2.1 and Lemma 2.2, that (14) holds provided that

$$\sup_{n \in \mathbb{N}} \|\sigma_{n,m}\|_{\ell^1(\mathbb{Z})} < \infty \,. \tag{18}$$

When $m = 1$, (18) is just the definition of the Ritt property.

Let $m \geq 2$, write $n = m\ell + k$, with $\ell \in \mathbb{N}$ and $0 \leq k \leq m-1$. We have

$$\|n^m \mu^{*n} * (\delta_0 - \mu)^{*m}\|_{\ell^1(\mathbb{Z})} \leq m^m \|(\ell+1)\mu^{*\ell} * (\delta_0 - \mu)\|^m_{\ell^1(\mathbb{Z})} \,,$$

and the latter is bounded uniformly with respect to $\ell \in \mathbb{N}$, since μ is Ritt. □

To conclude this subsection we shall study stability properties of set of probabilities satisfying the weak type maximal inequalities.

It is well-known, see e.g. Proposition 3.2 of [10], that the set of Ritt probability measures on $\mathbb{Z}$ is convex and stable by convolution. Actually, [10] deals with probability measures supported by $\mathbb{N}$, but the proof is the same.

Let $p > 1$. It is not difficult to see that the set of probability measures μ on $\mathbb{Z}$, such that there exists $C_p > 0$ such that (3) holds is also convex and stable by convolution.

However it is unclear (and probably not true) whether the set of probability measures μ on $\mathbb{Z}$ satisfying (14) for every $m \in \mathbb{N}$ (or for some $m \in \mathbb{N}$) is also convex and stable by convolution. Nevertheless, we have the following.

Proposition 2.4. *Let μ_1 and μ_2 be probability measures satisfying hypothesis* (**H**). *Let $\alpha \in (0,1)$. Then, $\check{\mu}_1$, $\mu_1 * \mu_2$ and $\alpha\mu_1 + (1-\alpha)\mu_2$ satisfy hypothesis* (**H**).

Remark. Recall that $\check{\mu}_1$ is the probability measure defined by $\check{\mu}_1(n) = \mu_1(-n)$ for every $n \in \mathbb{Z}$.

Proof. The fact that $\check{\mu}_1$ satisfies hypothesis (**H**) is obvious.

Let ψ_i, c_i, C_i be the terms associated with μ_i ($i \in \{0,1\}$) such that the items $(i) - (iv)$ of hypothesis (**H**) be satisfied.

Define $\mu := \mu_1 * \mu_2$ and $\psi := c_1\psi_1 + c_2\psi_2$. Let $\theta \in (0, \pi]$. We have

$$|\hat{\mu}(\theta)| = |\hat{\mu}_1(\theta)|\,|\hat{\mu}_2(\theta)| \leq 1 - \psi(\theta) + c_1c_2\psi_1(\theta)\psi_2(\theta).$$

Since ψ_1 and ψ_2 are continuous with $\psi_1(0) = \psi_2(0) = 0$, there exist $c \in (0,1)$ and $\eta \in (0,\pi)$ such that $c_1c_2\psi_1(\theta)\psi_2(\theta) \leq (1-c)\psi(\theta)$ for every $\theta \in (0,\eta)$.

Hence, $|\hat{\mu}| \leq 1 - c\psi$ on $(0,\eta)$. Arguing as in the previous proof, we see that taking c smaller if necessary, the inequality holds on $(0,\pi]$ either.

Using that $\hat{\mu}' = \hat{\mu}_1'\hat{\mu}_2 + \hat{\mu}_1\hat{\mu}_2'$, we infer that items (ii) and (iii) of hypothesis (**H**) hold.

We have $\hat{\mu}'' = \hat{\mu}_1''\hat{\mu}_2 + 2\hat{\mu}_1'\hat{\mu}_2' + \hat{\mu}_1\hat{\mu}_2''$. Hence, for every $\theta \in (0,\pi]$

$$|\theta\hat{\mu}''(\theta)| \leq C_1\psi_1'(\theta) + 2C_1\psi_1(\theta)C_2\psi_2'(\theta) + C_2\psi_2'(\theta)\,,$$

and we see that item (iv) holds, since ψ_1 is bounded.

Let $\alpha \in (0,1)$. Let $\mu := \alpha\mu_1 + (1-\alpha)\mu_2$. One can see that items $(i)-(iv)$ of hypothesis $(\mathbf{H})$ hold with $\psi := \alpha\psi_1 + (1-\alpha)\psi_2$. $\square$

2.2. **A sufficient condition for the Ritt property.** In this subsection we derive a condition ensuring that a probability measure is Ritt. This condition will be used for centered probability measure with either a second moment, or a first moment and completly monotone coefficients. For non centered probability measure another argument will be needed.

We start with a general result.

Proposition 2.5. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite signed measures on $\mathbb{Z}$, such that for every $n \in \mathbb{N}$, $\hat{\sigma}_n$ is twice differentiable on $\mathbb{R} - 2\pi\mathbb{Z}$. Assume moreover the following*

(i) $\sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} \frac{|\hat{\sigma}_n(\theta)|}{|\theta|} d\theta < \infty$;

(ii) $\sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} |\hat{\sigma}_n'(\theta)| d\theta < \infty$;

(iii) $\sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} |\theta|\, |\hat{\sigma}_n''(\theta)| d\theta < \infty$.

Then, $\sup_{n\in\mathbb{N}} \|\sigma_n\|_{\ell^1(\mathbb{Z})} < \infty$.

Proof. We first notice that, by (i),

$$\sup_{n\in\mathbb{N}} |\sigma_n(0)| \leq \sup_{n\in\mathbb{N}} \int_{-\pi}^{\pi} |\hat{\sigma}_n(\theta)| d\theta < \infty$$

Let $k \in \mathbb{Z} - \{0\}$. We have

$$\begin{aligned}\sigma_n(k) &= \int_{-\pi}^{\pi} \hat{\sigma}_n(\theta) \mathrm{e}^{-ik\theta} d\theta \\ &= \int_{-\pi/|k|}^{\pi/|k|} \hat{\sigma}_n(\theta) \mathrm{e}^{-ik\theta} d\theta + \int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n(\theta) \mathrm{e}^{-ik\theta}\, d\theta\,.\end{aligned}$$

Integrating by part and using that $\hat{\sigma}_n$ is 2π-periodic, we have

$$\int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n(\theta) \mathrm{e}^{-ik\theta}\, d\theta = -\int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n'(\theta) \frac{\mathrm{e}^{-ik\theta}}{-ik}\, d\theta + \frac{\sigma_n(-\pi/|k|) - \sigma_n(\pi/|k|)}{-ik}\,,$$

and

$$\int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n'(\theta) \frac{\mathrm{e}^{-ik\theta}}{-ik}\, d\theta = -\int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n''(\theta) \frac{\mathrm{e}^{-ik\theta}}{-k^2}\, d\theta + \frac{\sigma_n'(-\pi/|k|) - \sigma_n'(\pi/|k|)}{-k^2}$$

Now,

$$\sum_{|k|\geq 1} |\int_{-\pi/|k|}^{\pi/|k|} \hat{\sigma}_n(\theta) \mathrm{e}^{-ik\theta}\, d\theta| \leq \int_{-\pi}^{\pi} |\hat{\sigma}_n(\theta)| \sum_{1\leq|k|\leq\pi/|\theta|} 1\, d\theta \leq 2\pi \int_{-\pi}^{\pi} \frac{|\hat{\sigma}_n(\theta)|}{|\theta|} d\theta\,,$$

and

$$\sum_{|k|\geq 1} |\int_{[-\pi,\pi]-[-\pi/|k|,\pi/|k|]} \hat{\sigma}_n''(\theta) \frac{\mathrm{e}^{-ik\theta}}{-k^2}\, d\theta| \leq \int_{-\pi}^{\pi} |\hat{\sigma}_n''(\theta)| \sum_{|k|\geq\pi/|\theta|} \frac{1}{k^2} \leq C \int_{-\pi}^{\pi} |\theta|\, |\hat{\sigma}_n''(\theta)| d\theta\,.$$

Hence, it remains to show that $\sup_{n\in\mathbb{N}}\sum_{|k|\geq 1}\frac{|\hat{\sigma}_n(\pi/k)|}{|k|}<\infty$ and $\sup_{n\in\mathbb{N}}\sum_{|k|\geq 1}\frac{|\sigma_n'(\pi/k)|}{k^2}<\infty$.

Let $f_n(\theta):=\theta\hat{\sigma}_n(\theta)$, for every $\theta\in\mathbb{R}-2\pi\mathbb{Z}$. Then f_n is differentiable on $\mathbb{R}-2\pi\mathbb{Z}$ and, by (i) and (ii), $\hat{\sigma}_n'\in L^1([0,2\pi])$, $f_n'\in L^1([0,2\pi])$. Hence, $\hat{\sigma}_n$ and f_n can be continuously extended to $\mathbb{R}$ with $f_n(0)=0$. Then, for every $k\geq 1$,

$$\frac{\pi}{k}|\hat{\sigma}_n(\pi/k)|=\Big|\int_0^{\pi/k}f_n'(\theta)d\theta\Big|\leq\int_0^{\pi/k}|\hat{\sigma}_n(\theta)|d\theta+\int_0^{\pi/k}\theta|\hat{\sigma}_n'(\theta)|d\theta\,.$$

Dealing similarly with $k\leq -1$ we infer that

$$\begin{aligned}\sum_{|k|\geq 1}\frac{|\hat{\sigma}_n(\pi/k)|}{|k|}&\leq\sum_{k\geq 1}\Big(\int_{-\pi/k}^{\pi/k}|\hat{\sigma}_n(\theta)|d\theta+\int_{-\pi/k}^{\pi/k}\theta|\hat{\sigma}_n'(\theta)|d\theta\Big)\\&\leq\pi\int_{-\pi}^{\pi}\frac{|\hat{\sigma}_n(\theta)|}{|\theta|}d\theta+\pi\int_{-\pi}^{\pi}|\hat{\sigma}_n'(\theta)|d\theta\,,\end{aligned}$$

which is bounded uniformly with respect to n.

Proceeding as above with $g_n(\theta):=\theta^2\hat{\sigma}_n'(\theta)$ in place of $f_n(\theta)$ we see that, by (ii) and (iii), $\sup_{n\in\mathbb{N}}\sum_{|k|\geq 1}\frac{|\sigma_n'(\pi/k)|}{k^2}<\infty$. □

Let μ be a probability measure on $\mathbb{Z}$. We say that μ satisfies hypothesis $(\tilde{\mathbf{H}})$ if it satisfies hypothesis $(\mathbf{H})$ with a function ψ such that there exists $D>0$ such that for every $\theta\in(0,\pi]$,

$$\psi(\theta)\leq D\theta\psi'(\theta)\,. \tag{19}$$

Proposition 2.6. *Let μ be a probability measure on $\mathbb{Z}$ satisfying hypothesis $(\tilde{\mathbf{H}})$. Then $(\sigma_n)_{n\in\mathbb{N}}:=(n(\mu^{*n}-\mu^{*(n+1)}))_{n\in\mathbb{N}}$ satisfies to items (i), (ii) and (iii) of Proposition 2.5. In particular, $\sup_{n\in\mathbb{N}}n\|\mu^{*n}-\mu^{*(n+1)}\|_{\ell^1(\mathbb{Z})}<\infty$, i.e. μ is Ritt.*

Proof. By Proposition 2.3 we already know that (iii) holds. It follows from the proof of Proposition 2.3 and from (19) that there exist $C,c>0$ such that for every $\theta\in(0,\pi]$,

$$\begin{aligned}|\hat{\sigma}_n(\theta)|/\theta&\leq Cne^{-cn\psi(\theta)}\psi(\theta)/\theta\leq CDne^{-cn\psi(\theta)}\psi'(\theta)\\|\hat{\sigma}_n'(\theta)|&\leq Cne^{-cn\psi(\theta)}\psi'(\theta)(n\psi(\theta)+1)\,.\end{aligned}$$

then, we conclude as in the proof of Proposition 2.3. □

We now provide a sufficient condition on sequence of finite signed measure on $\mathbb{Z}$ to be bounded in $\ell^1(\mathbb{Z})$, that will be needed in the sequel.

Proposition 2.7. *Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of finite signed measures on $\mathbb{Z}$ such that for every $n\in\mathbb{N}$, $\hat{\sigma}_n$ is continuously differentiable on $[-\pi,\pi]-\{0\}$. Assume that*

- (i) $\sup_{n\in\mathbb{N}}n\int_{-\pi}^{\pi}|\hat{\sigma}_n(\theta)|d\theta<\infty$;
- (ii) $\sup_{n\in\mathbb{N}}\int_{-\pi}^{\pi}\frac{|\hat{\sigma}_n'(\theta)|^2}{n+1}d\theta<\infty$.

Then, $\sup_{n\in\mathbb{N}}\|\sigma_n\|_{\ell^1(\mathbb{Z})}<\infty$.

Proof. Let $n \geq 1$. Let $k \in \mathbb{Z}$. We have

$$\sigma_n(k) = \int_{-\pi}^{\pi} \hat{\sigma}_n(\theta) \mathrm{e}^{-ik\theta} d\theta \,, \tag{20}$$

and if $k \neq 0$,

$$\sigma_n(k) = \int_{-\pi}^{\pi} \hat{\sigma}_n'(\theta) \frac{\mathrm{e}^{-ik\theta}}{ik} d\theta \,, \tag{21}$$

Using (20), we infer that $\sum_{0 \leq |k| \leq n} |\sigma_n(k)| \leq (2n+1) \int_{-\pi}^{\pi} |\hat{\sigma}_n(\theta)| d\theta$. Using (21), Cauchy-Schwarz and Parseval, we infer that

$$(\sum_{|k|>n} |\sigma_n(k)|)^2 \leq \Big(\int_{-\pi}^{\pi} |\hat{\sigma}_n'(\theta)|^2 d\theta \Big) \sum_{|k|>n} \frac{1}{k^2} \leq \frac{C}{n} \int_{-\pi}^{\pi} |\hat{\sigma}_n'(\theta)|^2 d\theta \,.$$

Then, we conclude thanks to (i) and (ii). □

2.3. **Centered probability measures with a second moment.** It is known, see [3], that a centered and strictly aperiodic probability measure μ on $\mathbb{Z}$ with a second moment satisfies (13). As an application of the previous subsections we add here that μ is moreover Ritt and satisfies (14). Indeed, we shall prove Theorem 1.3.

By Proposition 2.3 and Proposition 2.6, it suffices to prove that a centered and strictly aperiodic probability measure μ with a second moment satisfies condition $(\tilde{\mathbf{H}})$ for some function ψ.

We shall take $\psi(\theta) = \theta^2$, for every $\theta \in [-\pi, \pi]$. Then ψ satisfies (19) hence we just have to prove that μ satisfies $(\mathbf{H})$.

Since μ has a second moment and is centered, it is twice continuously differentiable on $[-\pi, \pi]$ and we have

$$\lim_{\theta \to 0, \theta \neq 0} (1 - \mathrm{Re}\, \hat{\mu}(\theta))/\theta^2 = \hat{\mu}''(0)/2 > 0 \,,$$

and

$$\lim_{\theta \to 0, \theta \neq 0} \mathrm{Im}\, \hat{\mu}(\theta))/\theta^2 = 0 \,.$$

It follows that item (i) of hypothesis $(\mathbf{H})$ is satisfied for θ close enough to 0. Then, taking c smaller if necessary, it holds on $(0, \pi]$ by strict aperiodicity.

Using again that μ has a second moment and is centered we see that for every $\theta \in [-\pi, \pi]$, $|\hat{\mu}'(\theta)| \leq \|\hat{\mu}''\|_\infty |\theta|$. Hence items (ii) and (iii) of hypothesis $(\mathbf{H})$ hold. Similarly, item (iv) holds.

3. Probability measures without first moment

In this section as well as in sections 4 and 5, we shall consider probability measures μ on $\mathbb{Z}$ such that $(\mu(n))_{n \in \mathbb{N}}$ is *completely monotone sequence*. Let us recall some definition and facts.

Definition 3.1. *Let Δ be the operator defined for every sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers, by $(\Delta t_n)_{n\in\mathbb{N}} = (t_n - t_{n+1})_{n\in\mathbb{N}}$. We say that a sequence $(t_n)_{n\geq 0}$ is completely monotone if for every $m \geq 0$ (with the convention $\Delta^0 = Id$, $(\Delta^m t_n)_{n\geq 0}$ is non-negative.*

Definition 3.2. *We say that an infinitely differentiable function $f : [s, +\infty) \to [0, +\infty)$ is completely monotone, if for every $m \geq 0$, $(-1)^m f^{(m)} \geq 0$.*

The following characterization of completely monotone sequences is due to Hausdorff and may be found in Widder [27], p.108.

Proposition 3.1 (Hausdorff)**.** *A sequence $(\mu_n)_{n\in\mathbb{N}}$ is completely monotone if and only if there exists a finite positive measure ν on $[0, 1]$, such that $\mu_n = \int_0^1 t^n \nu(dt$ for every $n \in \mathbb{N}$.*

A way to generate completely monotone sequences is the following, see [27], Theorem 11d, p. 158.

Proposition 3.2. *Let f be a completely monotone function. Then $(f(n + 1))_{n\in\mathbb{N}}$ is a completely monotone sequence.*

Definition 3.3. *We say that a probability measure μ on $\mathbb{Z}$ is CM if it is supported on $\mathbb{N}$ and if there exists a finite (positive) measure ν on $[0, 1]$, such that*

$$\int_0^1 \frac{\nu(dt)}{1-t} = 1\,.^1 \tag{22}$$

and

$$\mu(n) = \int_0^1 t^n \nu(dt) \qquad \forall n \in \mathbb{N}\,. \tag{23}$$

To emphasize the measure ν we shall say that μ is a CM probability measure on $\mathbb{Z}$ with representative measure ν.

Notice that for μ as above, $\mu(n) > 0$ for every $n \in \mathbb{N}$, hence μ is strictly aperiodic.

3.1. **Characterization of the BAR property.** We first give an equivalent formulation of the BAR property that will be more convenient in the sequel.

Definition 3.4. *We say that a subset of $\mathbb{C}$ is a Stolz region if it is the convex hull of 1 and a circle centered at 0, with radius $0 < r < 1$.*

It is known that μ is strictly aperiodic and has BAR if and only if the range of $\hat{\mu}$ is included in a Stolz region.

[1]All along the paper (for esthetical reasons) we shall adopt the convention $\int_a^b \varphi d\nu = \int_{[a,b]} \varphi d\nu$. Hence, for non-negative φ we will have $\int_a^c \varphi d\nu \leq \int_a^b \varphi d\nu + \int_b^c \varphi d\nu$ with equality if $\nu(\{b\}) = 0$

If μ is strictly aperiodic, for every $\varepsilon \in (0,\pi)$, $\hat{\mu}([\varepsilon, 2\pi-\varepsilon])$ is included in a disk centered at 0 with radius strictly smaller than 1. Hence, a strictly aperiodic μ has BAR if and only if

$$\sup_{\theta\in(0,2\pi)} \frac{|\mathrm{Im}\,(\hat{\mu}(\theta))|}{1-\mathrm{Re}\,(\hat{\mu}(\theta))|} < \infty\,. \tag{24}$$

We shall consider the following condition on ν: there exists $L>0$ such that for every $x\in[0,1)$,

$$\int_0^x \frac{t}{(1-t)^2}\,\nu(dt) \le \frac{L}{1-x}\int_x^1 \frac{t}{1-t}\,\nu(dt)\,. \tag{25}$$

Notice that this condition implies that $\int_0^1 \frac{\nu(dt}{(1-t)^2} = +\infty$ or, equivalently, that $\sum_{n\in\mathbb{N}} na_n = +\infty$, i.e. μ does not have first moment.

Proposition 3.3. *Let μ be a CM probability measure on $\mathbb{Z}$ with representative measure ν. Then, μ has BAR if and only if there exists $L>0$ such that ν satisfies* (25)*. Moreover, then*

$$\frac{1-\mathrm{Re}\,\hat{\mu}(\theta)}{|\theta|} \underset{\theta\to 0}{\longrightarrow} +\infty\,. \tag{26}$$

We deduce the following corollary, in the spirit of Theorem 4.1 of Dungey [10].

Corollary 3.4. *Let μ be a CM probability measure on $\mathbb{Z}$ with representative measure ν satisfying* (25) *for some $L>0$. Let τ be a probability measure on $\mathbb{Z}$ such that there exists $c>0$ such that*

$$\sum_{n\in\mathbb{Z}} n|\tau(n)-a\mu(n)| < \infty\,.$$

Then, τ has BAR.

Throughout the paper we will make use of the following easy inequalities.

$$|\sin\theta| \le |\theta| \quad , \quad 1-\cos\theta \le \frac{\theta^2}{2} \qquad \forall\theta\in\mathbb{R}\,,$$

$$|\sin\theta| \ge 2|\theta|/\pi \quad , \quad 1-\cos\theta \ge \frac{\theta^2}{4} \qquad \forall\theta\in[-1,1]\,.$$

Proof of Proposition 3.3. Assume first that ν satisfies (25). Since ν is not null, the support of μ is $\mathbb{N}$ and μ is strictly aperiodic.

Hence, we just have to prove that there exists $K>0$, such that

$$|\mathrm{Im}\,\hat{\mu}(\theta)| \le K(1-\mathrm{Re}\,\hat{\mu}(\theta)) \qquad \forall\theta\in[-\pi,\pi]\,. \tag{27}$$

We have, for every $\theta\in[-\pi,\pi]$, $\hat{\mu}(\theta) = \int_0^1 \frac{\nu(dt)}{1-te^{i\theta}}$.

Notice that $|1-te^{i\theta}|^2 = 1+t^2-2t\cos\theta = (1-t)^2+2t(1-\cos\theta)$ and that

$$\mathrm{Re}\,\big(\frac{1}{1-t}-\frac{(1-te^{-i\theta})}{|1-te^{i\theta}|^2}\big)=\frac{(1-t)^2+2t(1-\cos\theta)-(1-t)(1-t\cos\theta)}{(1-t)|1-te^{i\theta}|^2}$$
$$=\frac{t(1-\cos\theta)}{(1-t)|1-te^{i\theta}|^2}\,.$$

Hence, using (22), we have

$$1-\mathrm{Re}\,\hat{\mu}(\theta)=\int_0^1\frac{t(1-\cos\theta)}{(1-t)((1-t)^2+2t(1-\cos\theta))}\,\nu(dt)\,. \tag{28}$$

Moreover,

$$\mathrm{Im}\,\hat{\mu}(\theta)=\int_0^1\frac{t\sin\theta}{|1-te^{i\theta}|^2}\,\nu(dt)=\int_0^1\frac{t\sin\theta}{(1-t)^2+2t(1-\cos\theta)}\,\nu(dt)\,. \tag{29}$$

Since, $\hat{\mu}$ is continuous and $1-\mathrm{Re}\,\hat{\mu}$ vanishes only at 0, on $[-\pi,\pi]$, it is enough to prove (27) for $\theta\in[-1/2,1/2]$. Moreover, (27) is clear for $\theta=0$. So, let $\theta\in[-1/2,1/2]-\{0\}$.

Let us first estimate $1-\mathrm{Re}\,\hat{\mu}(\theta)$. Using that $(1-t)^2+2t(1-\cos\theta)\le(1-t)^2+\theta^2\le 2\max((1-t)^2,\theta^2)$, we obtain

$$1-\mathrm{Re}\,\hat{\mu}(\theta)\ge\frac{1}{2}\int_0^{1-|\theta|}\frac{t(1-\cos\theta)}{(1-t)^3}\nu(dt)\;+\;\frac{1}{8}\int_{1-|\theta|}^1\frac{t}{1-t}\nu(dt) \tag{30}$$

Now, we estimate $\mathrm{Im}\,\hat{\mu}$. We have,

$$\int_{1-|\theta|}^1\frac{t|\sin\theta|}{(1-t)^2+2t(1-\cos\theta)}\,\nu(dt)\le\int_{1-|\theta|}^1\frac{t\theta^2}{(1-t)((1-t)^2+2t(1-\cos\theta))}\,\nu(dt)$$
$$\le 4\int_{1-|\theta|}^1\frac{t(1-\cos\theta)}{(1-t)((1-t)^2+2t(1-\cos\theta))}\,\nu(dt)\le 4(1-\mathrm{Re}\,\hat{\mu}(\theta))\,.$$

Now, using our assumption on ν and (30), we obtain

$$\int_0^{1-|\theta|}\frac{t|\sin\theta|}{(1-t)^2+2t(1-\cos\theta)}\,\nu(dt)\le\int_0^{1-|\theta|}\frac{t|\sin\theta|}{(1-t)^2}\,\nu(dt)$$
$$\le\frac{L}{|\theta|}\int_{1-|\theta|}^1\frac{t}{1-t}\,\nu(dt)\le 8L(1-\mathrm{Re}\,\hat{\mu}(\theta))\,.$$

and we see that (27) holds.

Let us prove the converse. Assume that (27) holds.
Let $S\ge 1$ be fixed for the moment. Let $\theta\in[-1/2S,1/2S]-\{0\}$.
Using that $|1-te^{i\theta}|^2\le(1+1/S^2)(1-t)^2$, whenever $0\le t\le 1-S|\theta|$, we see that

$$\int_0^{1-S|\theta|}\frac{\nu(dt)}{(1-t)^2}\le\frac{1+1/S^2}{2|\theta|/\pi}|\mathrm{Im}\,\hat{\mu}(\theta)|\le C\frac{1+1/S^2}{2|\theta|/\pi}(1-\mathrm{Re}\,\hat{\mu}(\theta))\,. \tag{31}$$

Now, we see that

$$1-\mathrm{Re}\,\hat{\mu}(\theta)) \le \frac{1-\cos\theta}{S|\theta|}\int_0^{1-S|\theta|}\frac{t\nu(dt)}{(1-t)^2}+\int_{1-S|\theta|}^1\frac{t\nu(dt)}{1-t}\,.$$

Hence, taking S large enough and using (31), we infer that there exists $D>0$ such that

$$\int_0^{1-S|\theta|}\frac{\nu(dt)}{(1-t)^2}\le\frac{D}{|\theta|}\int_{1-S|\theta|}^1\frac{t\nu(dt)}{1-t}\,,$$

which prove that (25) holds □

It remains to prove (26). Using (25), we see that

$$\frac{1-\mathrm{Re}\,\hat{\mu}(\theta)}{|\theta|}\ge\frac{1-\cos\theta}{2|\theta|^3}\int_{1-|\theta|}^1\frac{t}{(1-t)}\,\nu(dt)\ge\frac{1-\cos\theta}{2|\theta|^2}\int_0^{1-|\theta|}\frac{t}{(1-t)^2}\,\nu(dt)\underset{\theta\to0}{\longrightarrow}+\infty\,,$$

hence the result.

□

Proof of Corollary 3.4. By assumption and Proposition 3.3, there exists $K>0$ such that $\sum_{n\ge1}n|\tau(n)-a\mu(n)|\le K$ and for every $\theta\in[-\pi,\pi]$, $|\mathrm{Im}\,(\hat{\mu}(\theta))|\le K(1-\mathrm{Re}\,(\hat{\mu}(\theta))$.

Let us prove that τ is strictly aperiodic. If τ were not strictly aperiodic, there would exists $\ell\ge2$ and $0\le k\le\ell-1$, such that the support of τ would be contained in $k+\ell\mathbb{Z}$. In particular $\tau(k+1+\ell m)=0$ for every $m\in\mathbb{Z}$. Hence, $\sum_{m\in\mathbb{Z}}|m|\mu(k+1+\ell m)<\infty$ and (using that $(\mu(n))_{n\ge1}$ is non increasing) μ must have a first moment, contradicting (25) (see the remark after (25)).

We first prove that there exists $C>0$ such that for every $\theta\in[-\pi,\pi]$,

$$\mathrm{Re}\,(1-\hat{\tau}(\theta))\ge C|\theta|\,. \tag{32}$$

Since τ is strictly aperiodic, it is enough to prove the result for small enough θ's. By Proposition 3.3, there exists $\delta\in(0,\pi)$, such that for every $\theta\in[-\delta,\delta]$, $|\theta|\le a(1-\mathrm{Re}\,(\hat{\mu}(\theta))/2K$. Then, using that $1-\cos u\le|u|$ for every $u\in\mathbb{R}$,

$$\begin{aligned}|\theta|&\le(1-\mathrm{Re}\,(\hat{\tau}(\theta))/2K+\frac{1}{2K}\sum_{n\ge1}|\tau(n)-a\mu(n)|(1-\cos(n\theta))\\&\le(1-\mathrm{Re}\,(\hat{\tau}(\theta))/2K+|\theta|/2\,,\end{aligned}$$

and (32) follows.

Let $\theta\in[-\pi,\pi]$. We have, using that $|\sin u|\le u$ for every $u\in\mathbb{R}$,

$$\begin{aligned}|\mathrm{Im}\,(\hat{\tau}(\theta))|&\le a|\mathrm{Im}\,(\hat{\mu}(\theta))|+\sum_{n\ge1}|\tau(n)-a\mu(n)|\,|\sin(n\theta)|\le aK(1-\mathrm{Re}\,(\hat{\mu}(\theta))+K|\theta|\\&\le K(1-\mathrm{Re}\,(\hat{\tau}(\theta))+\sum_{n\ge1}|\tau(n)-a\mu(n)|\,|(1-\cos(n\theta)|+K|\theta|\\&\le K(1-\mathrm{Re}\,(\hat{\tau}(\theta))+2K|\theta|\le K(1+2C)(1-\mathrm{Re}\,(\hat{\tau}(\theta))\,,\end{aligned}$$

and the corollary is proved. □

From a pratical point of view it is better to have a condition on $(\mu(n))_{n\in\mathbb{Z}}$. Indeed we may consider completely monotone sequences given thanks to Proposition 3.2, in which case, we do not know ν.

Proposition 3.5. *Let μ be a CM probability measure on $\mathbb{Z}$ with representative measure ν. Then, ν satisfies* (25) *if and only if there exists $D>0$, such that for every $n\geq 1$,*

$$\sum_{k=1}^{n} k\mu(k) \leq Dn\sum_{k\geq n}\mu(k)\,. \tag{33}$$

Proof. Assume (25). Let $n\geq 1$. We have

$$\sum_{k=1}^{n} k\mu(k) \leq \int_0^{1-1/n}\frac{t}{(1-t)^2}\nu(dt) + n\int_{1-1/n}^{1}\frac{t}{1-t}\nu(dt) \leq (1+L)n\int_{1-1/n}^{1}\frac{t}{1-t}\nu(dt)\,.$$

Using that $\sum_{k\geq n}\mu(k) = \int_0^1 \frac{t^n}{1-t}\nu(dt)$ and that $(1-1/n)^n \underset{n\to\infty}{\longrightarrow} \mathrm{e}^{-1}$, we see that (33) holds.

Assume now that (33) holds.

Let $A\geq 1$ be a positive integer fixed for the moment. Let $n\geq 2$.

Let $1\leq m\leq n-1$ be an integer and let $t\in[1-1/m, 1-1/(m+1)]$. Using that the sequence $((1-1/k)^{k-1})_{k\geq 1}$ decreases to $1/\mathrm{e}$, we obtain that (with the convention $0^0=1$)

$$\sum_{k=1}^{An} kt^k \geq t\sum_{k=0}^{m-1}(k+1)(1-1/m)^{m-1} \geq \frac{tm(m+1)}{2\mathrm{e}} \geq \frac{t}{\mathrm{e}(1-t)^2}\,.$$

Hence,

$$\int_0^{1-1/n}\frac{t}{(1-t)^2}\nu(dt) \leq \mathrm{e}\sum_{k=1}^{An} k\mu(k) \leq \mathrm{e}DAn\sum_{k\geq An}\mu(k) = \mathrm{e}DAn\int_0^1\sum_{k\geq An} t^k\nu(dt)\,.$$

Now notice that for $t\in[0,1-1/n]$,

$$\sum_{k\geq An} t^{k-1} \leq \frac{1}{A^2n^2}\sum_{k\geq An} k(k+1)t^{k-1} \leq \frac{1}{A^2n^2(1-t)^3} \leq \frac{1}{A^2n(1-t)^2}\,,$$

and that for $t\in[1-1/n,1]$, $\sum_{k\geq An} t^{k-1}\leq 1/(1-t)$.

Hence, taking A large enough we infer that (25) holds. □

3.2. **Hypothesis (H) for CM probability measures.** We shall prove that the conditions imposed in the previous subsection guarantee hypothesis (**H**).

Proposition 3.6. *Let μ be a CM probability measure on $\mathbb{Z}$ satisfying* (33)*. Then, μ satisfies hypothesis* (**H**)*.*

Proof. To check the conditions $(i)-(iv)$ of hypothesis $(\mathbf{H})$ with a suitable function ψ we must first estimate $\hat{\mu}$ and its derivatives.

Let us first compute the derivatives of $\hat{\mu}$. Recall that for every $\theta \in [-\pi, \pi]$,

$$\begin{aligned} 1 - \operatorname{Re}\hat{\mu}(\theta) &= \int_0^1 \frac{t(1-\cos\theta)}{(1-t)\big((1-t)^2 + 2t(1-\cos\theta)\big)}\,\nu(dt) \\ &= \frac{1}{2}\int_0^1 \frac{\nu(dt)}{1-t} - \frac{1}{2}\int_0^1 \frac{(1-t)}{(1-t)^2 + 2t(1-\cos\theta)}\,\nu(dt)\,, \end{aligned}$$

and

$$\operatorname{Im}\hat{\mu}(\theta) = \int_0^1 \frac{t\sin\theta}{|1-te^{i\theta}|^2}\,\nu(dt) = \int_0^1 \frac{t\sin\theta}{(1-t)^2 + 2t(1-\cos\theta)}\,\nu(dt)\,.$$

Hence, for every $\theta \in [-\pi, \pi] - \{0\}$,

$$\operatorname{Re}\hat{\mu}'(\theta) = -\sin\theta \int_0^1 \frac{t(1-t)}{\big((1-t)^2 + 2t(1-\cos\theta)\big)^2}\,\nu(dt)\,, \tag{34}$$

$$\begin{aligned} \operatorname{Re}\hat{\mu}''(\theta) = &-\cos\theta \int_0^1 \frac{t(1-t)}{\big((1-t)^2 + 2t(1-\cos\theta)\big)^2}\,\nu(dt) \\ &+2\sin^2\theta \int_0^1 \frac{t(1-t)}{\big((1-t)^2 + 2t(1-\cos\theta)\big)^3}\,\nu(dt) \end{aligned} \tag{35}$$

$$\begin{aligned} \operatorname{Im}\hat{\mu}'(\theta) = &\int_0^1 \frac{t\cos\theta}{(1-t)^2 + 2t(1-\cos\theta)}\,\nu(dt) \\ &- \int_0^1 \frac{2t^2\sin^2\theta}{\big((1-t)^2 + 2t(1-\cos\theta)\big)^2}\,\nu(dt)\,, \end{aligned} \tag{36}$$

$$\begin{aligned} \operatorname{Im}\hat{\mu}''(\theta) = &\int_0^1 \frac{-t\sin\theta}{(1-t)^2 + 2t(1-\cos\theta)}\,\nu(dt) \\ &- \int_0^1 \frac{8t^2\sin\theta\cos\theta}{\big((1-t)^2 + 2t(1-\cos\theta)\big)^2}\,\nu(dt) + \int_0^1 \frac{4t^3\sin^3\theta}{\big((1-t)^2 + 2t(1-\cos\theta)\big)^3}\,\nu(dt)\,. \end{aligned} \tag{37}$$

Define, for $\theta \in [-\pi, \pi]$,

$$\psi(\theta) = \int_0^1 \frac{t|\theta|}{(1-t)(1-t+t|\theta|)}\,\nu(dt) = 1 - \int_0^1 \frac{\nu(dt)}{(1-t+t|\theta|)}\,. \tag{38}$$

Hence, for every $\theta \in (0, \pi]$,

$$\psi'(\theta) = \int_0^1 \frac{t\nu(dt)}{(1-t+t\theta)^2}\,. \tag{39}$$

Notice that, for every $\theta \in (0, 1/2]$,

$$\frac{\theta}{2}\int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^2} + \frac{1}{2}\int_{1-\theta}^1 \frac{t\nu(dt)}{1-t} \leq \psi(\theta) \leq \theta \int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^2} + \int_{1-\theta}^1 \frac{\nu(dt)}{1-t}\,; \tag{40}$$

$$\frac{\theta}{4}\int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^2} + \frac{1}{4\theta}\int_{1-\theta}^1 t\nu(dt) \leq \theta\psi'(\theta) \leq \theta \int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^2} + \frac{2}{|\theta|}\int_{1-\theta}^1 \nu(dt)\,. \tag{41}$$

Claim 1. *There exists $C > 0$, such that for every $\theta \in [0, \pi]$,*

$$1 - \operatorname{Re}\hat{\mu}(\theta) \geq C\psi(\theta)\,.$$

Proof. It suffices to prove the claim for $\theta \in (0, 1/2]$. Using (30), (25) and (40) we have

$$1 - \operatorname{Re}\hat{\mu}(\theta) \geq \int_{1-|\theta|}^1 \frac{t}{8(1-t)}\nu(dt) \geq \frac{1}{8(L+2)}\psi(\theta)\,,$$

and the claim follows. □

Claim 2. *There exists $C > 0$, such that for every $\theta \in (0, \pi]$, $|\hat{\mu}'(\theta)| \leq C\psi'(\theta)$.*

Proof. Again, we only consider the case when $\theta \in (0, 1/2]$. We deal separately with the real and imaginary part of μ'. We have, using (34) and (41)

$$|\operatorname{Re}\hat{\mu}'(\theta)| \leq 2\theta \int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^3} + \frac{2}{\theta^3}\int_{1-\theta}^1 t(1-t)\nu(dt) \leq 8\psi'(\theta)\,,$$

Similarly,

$$|\operatorname{Im}\hat{\mu}'(\theta)| \leq \int_0^{1-\theta} \Big(\frac{t}{(1-t)^2} + \frac{t\theta^2}{(1-t)^4}\Big)\nu(dt) + \Big(\frac{1}{1-\cos\theta} + \frac{2\theta^2}{(1-\cos\theta)^2}\Big)\int_{1-\theta}^1 t\nu(dt)$$
$$\leq C\psi'(\theta)\,.$$

Claim 3. *There exists $C > 0$, such that for every $\theta \in (0, \pi]$, $|\theta\hat{\mu}'(\theta)| \leq C\psi(\theta)$.*

Proof. Combine Claim 2 and (41). □

Claim 4. *There exists $C > 0$, such that for every $\theta \in (0, \pi]$, $|\theta\hat{\mu}''(\theta)| \leq C\psi'(\theta)$.*

Proof. We assume that $\theta \in (0, 1/2]$. By (35) and (41), we have

$$|\operatorname{Re}\hat{\mu}''(\theta)| \leq \int_0^{1-\theta} \Big(\frac{2t}{(1-t)^3} + \frac{4t\theta^2}{(1-t)^5}\Big)\nu(dt)$$
$$+ \frac{2}{(1-\cos\theta)^2}\int_{1-\theta}^1 \frac{t}{1-t}\nu(dt) + \frac{4\theta^2}{(1-\cos\theta)^3}\int_{1-\theta}^1 t(1-t)\nu(dt) \leq C\psi'(\theta)/\theta.$$

Similar computations based on (34) and (41) yields

$$|\operatorname{Im}\hat{\mu}''(\theta)| \leq C\psi'(\theta)/\theta\,.$$

Then, items *ii*), (*iii*) and (*iv*) of Proposition 2.3 follows from the combination of the claims 2, 3 and 4.

Let us prove item (*i*) of Proposition 2.3. Let $\theta \in [0, 1/2]$. Recall that $|\mathrm{Im}\,\hat{\mu}(\theta)| \leq C(1 - \mathrm{Re}\,\hat{\mu}(\theta))$. Hence,

$$|\hat{\mu}(\theta)|^2 = |\mathrm{Im}\,\hat{\mu}(\theta)|^2 + 1 - 2(1 - \mathrm{Re}\,\hat{\mu}(\theta)) + (1 - \mathrm{Re}\,\hat{\mu}(\theta))^2$$
$$1 - (1 - \mathrm{Re}\,\hat{\mu}(\theta))(2 - (C+1)(1 - \mathrm{Re}\,\hat{\mu}(\theta)))$$

Since $(1 - \mathrm{Re}\,\hat{\mu}(\theta)) \underset{\theta\to 0}{\longrightarrow} 0$, using Claim 1, we infer that for θ small enough

$$|\hat{\mu}(\theta)|^2 \leq 1 - \tilde{C}\psi(\theta)\,.$$

Hence (*i*) holds for, say, $\theta \in [0, \eta]$, with η small enough. Since $\sup_{\theta\in[\eta,\pi]} |\hat{\mu}(\theta)| < 1$, wee see that (*i*) holds for every $\theta \in [0, \pi]$, taking c smaller if necessary.

□

Corollary 3.7. *Let μ be a CM probability measure on $\mathbb{Z}$. Let σ be a probability measure on $\mathbb{Z}$ such that $\hat{\sigma}$ is twice continuously differentiable on $[-\pi, \pi] - \{0\}$ and such that $\hat{\sigma}'$ and $\theta \mapsto \theta\hat{\sigma}''(\theta)$ are bounded. Then, $\sigma * \mu$ satisfies hypothesis* (**H**)*. Moreover, if μ satisfies hypothesis* ($\tilde{\mathbf{H}}$)*, so does $\sigma * \mu$.*

Remark. The assumptions on ν holds, for instance, as soon as $\sum_{n\in\mathbb{Z}} n^2\sigma(n) < \infty$.
Proof. Let ψ be the function defined in (38). Since $\int_0^1 \frac{\nu(dt)}{(1-t)^2} = +\infty$, one easily infers from (39) that $\liminf_{\theta\to 0,\theta>0} \psi'(\theta) = +\infty$. In particular, there exists $K > 0$ such that for every $\theta \in (0, \pi]$, $\psi'(\theta) \geq K$ and, consequently, $\psi(\theta) \geq K\theta$. Then, the fact that $\sigma * \mu$ satisfies hypothesis (**H**), with the same function ψ as μ, may be proved exactly as Proposition 2.4. Since we use the same function ψ for $\sigma * \mu$ and μ, then $\sigma * \mu$ satisfies hypothesis ($\tilde{\mathbf{H}}$) as soon as μ does. □

Corollary 3.8. *Let τ be a probability measure on $\mathbb{Z}$. Assume that there exists a CM probability measure μ and $c > 0$ such that $\sum_{n\in\mathbb{Z}} n^2|\tau(n) - c\mu(n)| < \infty$. Then, τ satisfies hypothesis* (**H**)*. If moreover μ satisfies hypothesis* ($\tilde{\mathbf{H}}$)*, so does τ.*

Remark. It follows from the proof that we only need that $\hat{\sigma}$ be twice continuously differentiable on $[-\pi, \pi] - \{0\}$ and that $\hat{\tau}'$ and $\theta \mapsto \theta\hat{\tau}''(\theta)$ be bounded.
Proof. Define a *signed* measure by setting $\sigma := \tau - c\mu$. Then, $\hat{\sigma}$ is twice continuously differentiable on $[-\pi, \pi]$, $\hat{\sigma}(0) = 1 - c$ and there exists $C > 0$ such that for every $\theta \in [0, \pi]$, $|\hat{\sigma}(\theta) - (1 - c)| \leq C\theta$. Then, the proof may be finished using the same arguments as in the proof of Corollary 3.7. □

3.3. **The Ritt property on $\ell^1(\mathbb{Z})$.** In this section, we finish the proof of Theorem 1.4. We first prove the Ritt property of CM probability measures, which corresponds to the case where $\sigma = \delta_0$.

Let μ be a probability measure on $\mathbb{Z}$. Notice that the fact μ is Ritt is equivalent to the fact that

$$\sup_{n\geq 1} n\|\pi_\mu^n - \pi_\mu^{n+1}\|_{\ell^1(\mathbb{Z})} < \infty\,,$$

where π_μ stands for the operator of convolution by μ.

Let Γ be the open unit disk in the complex plane. By Theorem 1.5 of Dungey, μ is Ritt if and only if the spectrum $\sigma(\pi_\mu)$ of π_μ is contained in $\Gamma\cup\{1\}$ and the semi-group $(\mathrm{e}^{-t(I-\pi_\mu)})_{t\geq 0}$ is bounded analytic. The fact that $(\mathrm{e}^{-t(I-\pi_\mu)})_{t\geq 0}$ is bounded analytic means that

$$\sup_{t>0}\Big(\|\mathrm{e}^{-t(\delta_0-\mu)}\|_{\ell^1(\mathbb{Z})} + t\|(I-T)\mathrm{e}^{-t(\delta_0-\pi_\mu)}\|_{\ell^1(\mathbb{Z})}\Big) < \infty\,.$$

Remark. Notice that Theorem 1.5 of Dungey is valid for probabilities supported on $\mathbb{N}$.

Proposition 3.9. *Let μ be a CM probability measure on $\mathbb{Z}$ with representative measure ν satisfying* (25)*. Then, μ is Ritt.*

We already saw that ν satisfies (25) if and only if μ has BAR. The fact that a CM probability measure on $\mathbb{Z}$ having BAR is Ritt has been proved very recently (see their Theorem 7.1) by Gomilko and Tomilov [15] as a consequence of another very recent result of their own [14].

The latter paper deals with subordination semi-groups hence is written in a continuous setting.

For reader's convenience we explain below how to derive Proposition 3.9 from the work [14].

First of all, by Theorem 2.1 of Dungey [10], we have $\sigma(\pi_\mu)\subset\hat{\mu}([-\pi,\pi])\subset\Gamma\cup\{1\}$, where the latter inclusion follows from the fact that μ has BAR. Hence, Proposition 3.9 will be proved if we can prove that $(\mathrm{e}^{-t(I-\pi_\mu)})_{t\geq 0}$ is bounded analytic.

Definition 3.5. *An infinitely differentiable function $f:(0,+\infty)\to[0,+\infty)$ is called a Bernstein function if f' is completely monotone. If $\lim_{x\to 0^+} f(x)$ exists and if f admits an holomorphic extension to $\{z\in\mathbb{C}:\operatorname{Im} z>0\}$, such that $\operatorname{Im} f(z)\geq 0$, then f is called* complete Bernstein*.*

For every $x\geq 0$, define $\chi(x) := 1-\int_0^1 \frac{\nu(dt)}{1-t+tx} = \int_0^1 \frac{\nu(dt)}{1-t} - \int_0^1 \frac{\nu(dt)}{1-t+tx}$. Then χ is non-decreasing, with $\chi(0)=0$, hence it is non-negative. It is not hard to see that it is infinitely differentiable and that χ' is completely monotone, hence χ is a Bernstein function and one can easily see that it is actually a complete Bernstein function.

Since χ is Bernstein, it is well known (see e.g. Theorem 1.2.4 of [11]) that there exists a convolution semi-group $(\sigma_t)_{t\geq 0}$ (of probability measures on $[0,\infty)$), such that for every $x\geq 0$, and every $t\geq 0$,

$$\int_0^\infty \mathrm{e}^{-xy}\sigma_t(dy) = \mathrm{e}^{-t\chi(x)}\,.$$

Following Dungey [10, p. 1734], we consider the Poisson semi-group $(P_s)_{s\geq 0}$ acting by convolution on $\ell^1(\mathbb{N})$, and defined by

$$P_s := \mathrm{e}^{-s(\delta_0-\delta_1)} = \mathrm{e}^{-s}\sum_{k\geq 0}\frac{s^k}{k!}\delta_k \qquad \forall s\geq 0\,.$$

Consider now the associated subordinated semi-group $(Q_t)_{s\geq 0}$ defined by

$$Q_t := \int_0^\infty P_s\,\sigma_t(ds) \qquad \forall t\geq 0\,.$$

Let $t\geq 0$. Then, Q_t is a probability measure on $\mathbb{N}$, whose generating function is given (on $[0,1]$) by

$$x\mapsto \int_0^\infty \mathrm{e}^{-s(1-x)}\sigma_t(ds) = \mathrm{e}^{-t\chi(1-x)}\,.$$

Let G_μ denote the generating function of μ, i.e.

$$G_\mu(x) = \sum_{n\geq 0}\mu(n)x^n = \int_0^1 \frac{\nu(dt}{1-tx} = 1-\chi(1-x)\,,$$

for every $x\in[0,1]$. Then, for every $t\geq 0$, the generating function of the probability $\mathrm{e}^{-t(I-\mu)} = \mathrm{e}^{-t}\sum_{k\geq 0}\frac{t^k\mu^{*k}}{k!}$ is given by

$$\mathrm{e}^{-t}\sum_{k\geq 0}\frac{t^kG_\mu^k}{k!} = \mathrm{e}^{-t(1-G_\mu)}\,.$$

In particular, we see that the semi-groups $(\mathrm{e}^{-t(I-\pi_\mu)})_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ coïncide. Hence, to prove that $(\mathrm{e}^{-t(I-\pi_\mu)})_{t\geq 0}$ is bounded analytic, it is enough to prove that any subordinated semi-group associated with $(\sigma_t)_{t\geq 0}$ is bounded analytic (see the introduction of [14] for more details). To prove the latter point, since χ is complete Bernstein, by Corollary 7.10 of [14], it is enough to prove that χ sends the half-plane $\{z\in\mathbb{C} : \mathrm{Re}\,z\geq 0\}$ to a sector $\{z\in\mathbb{C} : |\mathrm{Im}\,z|\leq C\mathrm{Re}\,z\}$, for some $C>0$.

Let $z=a+ib$ such that $a\geq 0$ and $|z|^2=a^2+b^2\leq 1/4$. We have, using (25)

$$|\mathrm{Im}\,\chi(z)| = |b|\int_0^1 \frac{t}{(1-t+at)^2+t^2b^2}\nu(dt)$$
$$\leq |b|\int_0^{1-|z|}\frac{t}{(1-t)^2}\nu(dt) + \frac{4b}{|z|^2}\int_{1-|z|}^1 t\nu(dt) \leq K\int_{1-|z|}^1\frac{t}{1-t}\nu(dt)\,.$$

On the other hand,

$$\mathrm{Re}\,\chi(z) = \int_0^1 \frac{at(1-t)+|z|^2t^2}{(1-t)\big((1-t+at)^2+t^2b^2\big)}\,\nu(dt) \geq \int_{1-|z|}^1 \frac{|z|^2t^2}{5|z|^2(1-t)}\,\nu(dt)$$
$$\geq \frac{1}{10}\int_{1-|z|}^1\frac{t}{1-t}\nu(dt)\,.$$

This gives the desired bound when $|z|^2 \leq 1/4$.

Assume now that $|z|^2 \geq 1/4$. In particular, we have $4|z| \geq 2$. Hence,

$$\begin{aligned}|\mathrm{Im}\,\chi(z)| &\leq \int_0^{(4|z|)^{-1}} \frac{t|z|}{(1-t)^2}\nu(dt) + \frac{|z|}{|z|^2}\int_{(4|z|)^{-1}}^1 t^{-1}\nu(dt)\\ &\leq \frac{1}{4}\int_0^{1/2}\frac{\nu(dt)}{(1-t)^2} + 4\int_0^1 \nu(dt) < \infty\,.\end{aligned}$$

Moreover, using that the integrand below is non decreasing with respect to $|z|$, we have

$$\begin{aligned}\mathrm{Re}\,\chi(z) &\geq \int_0^1 \frac{|z|^2t^2}{2(1-t)\big((1-t)^2+t^2|z|^2\big)}\,\nu(dt)\\ &\geq \frac{1}{8}\int_0^1 \frac{t^2}{(1-t)^2+t^2/4}\nu(dt) > 0\,,\end{aligned}$$

which finishes the proof. □

Proposition 3.10. *Let μ be a CM probability measure on $\mathbb{Z}$ with representative measure ν satisfying* (25)*. Let σ be a probability measure on $\mathbb{Z}$ such that $\hat{\sigma}$ is continuously differentiable on $[-\pi,\pi]-\{0\}$ and such that $\hat{\sigma}'$ is bounded on $[-\pi,\pi]-\{0\}$. Then,*

$$\sup_{n\in\mathbb{N}} n\|(\delta_0-\sigma)*\mu^{*n}\|_{\ell^1} < \infty\,. \tag{42}$$

In particular, $\sigma*\mu$ is Ritt and for every $\alpha\in(0,1]$, $\alpha\mu+(1-\alpha)\sigma$ is Ritt.

Proof. To prove (42), we check that $(\sigma_n)_{n\in\mathbb{N}} := (n(\delta_0-\sigma)*\mu^{*n})_{n\in\mathbb{N}}$ satisfies items (i) and (ii) of Proposition 2.7.

By assumption there exists $L>0$ such that $|\hat{\sigma}'|\leq L$ and it follows that $|1-\hat{\sigma}(\theta)|\leq L|\theta|$ for every $\theta\in[-\pi,\pi]$.

Let ψ be the function given in (38). Recall that there exists $K>0$ such that for every $\theta\in[-\pi,\pi]-\{0\}$, $\psi(\theta)\geq K\theta$ and $\psi'(\theta)\geq K$. Hence, for every $n\in\mathbb{N}$ and every $\theta\in[-\pi,\pi]-\{0\}$,

$$n|\hat{\sigma}_n(\theta)| \leq \frac{Ln^2}{K^2}\psi(\theta)\psi'(\theta)(\hat{\mu}(\theta))^n\,.$$

Hence, arguing as in the proof of Proposition 2.6, we see that $(\sigma_n)_{n\in\mathbb{N}}$ satisfies item (i) in Proposition 2.7.

For every $\theta\in[-\pi,\pi]-\{0\}$, we have

$$\hat{\sigma}_n'(\theta) = -n\hat{\sigma}'(\theta)\hat{\mu}^n(\theta) + n^2(1-\hat{\sigma}(\theta))\hat{\mu}'(\theta)\hat{\mu}^{n-1}(\theta)\,.$$

Then, we infer that

$$|\hat{\sigma}_n'(\theta)|^2 \leq \frac{2n^2L^2}{K}\psi'(\theta)|\hat{\mu}^{2n}|(\theta) + \frac{2n^4L^2}{K^3}\psi^2(\theta)\psi'(\theta)|\hat{\mu}^{n-1}|(\theta)\,,$$

Hence, arguing as in the proof of Proposition 2.6, we see that $(\sigma_n)_{n\in\mathbb{N}}$ satisfies item (ii) in Proposition 2.7.

It remains to prove the second part of the Proposition.

Let $n \geq 1$. We have

$$(\delta_0 - \sigma * \mu) * (\sigma * \mu)^{*n} = \sigma^{*n} * \left[(\delta_0 - \mu) * \mu^{*n}\right] + \left[(\delta_0 - \sigma) * \mu^{*(n+1)}\right] * \sigma^{*n} ,$$

which proves that $\sigma * \mu$ is Ritt.

Let $\alpha \in (0, 1]$ and $n \geq 1$, and $\tau := \alpha\mu + (1 - \alpha)\sigma$. We have

$$(\delta_0 - \tau) * \tau^{*n} =$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\alpha^k (1-\alpha)^{n-k}}{k+1} \left[\alpha(k+1)(\delta_0 - \mu) * \mu^{*k} + (1-\alpha)(k+1)(\delta_0 - \sigma) * \mu^{*k}\right] * \sigma^{*(n-k)} .$$

Hence,

$$(n+1)\|(\delta_0 - \tau) * \tau^{*n}\|_{\ell^1} \leq \frac{C}{\alpha} \sum_{k=0}^{n} \binom{n+1}{k+1} \alpha^{k+1} (1-\alpha)^{(n+1)-(k+1)} \leq C ,$$

and we see that τ is Ritt. □

3.4. **Examples.** To exhibit examples we will make use of Proposition 3.2. Hence we shall first exhibit completely monotone functions.

Lemma 3.11 (Miller-Samko [21])**.** *Let $f, g : (0, +\infty) \to (0, +\infty)$ be infinitely differentiable functions functions such that g' is completely monotone.*

(i) If f is completely monotone then $f \circ g$ is completely monotone either;
(ii) If f' is completely monotone then $(f \circ g)'$ is completely monotone either.

Proof. Item (i) is just Theorem 2 of [21]. Let us prove item (ii). We have $(f \circ g)' = f' \circ g \times g'$. By (i), $f' \circ g$ is completely monotone. Then, $(f \circ g)'$ is completely monotone by Theorem 1 of [21]. □

Define by induction $L_1(x) = L(x) := \log(1 + x)$ and $L_{k+1}(x) = L(L_k(x))$ for every $x > 0$.

Corollary 3.12. *For every integer $k \geq 1$ and every real numbers $\alpha_1, \ldots, \alpha_k \in [0, +\infty)$ and $\alpha \in [0, +\infty)$ the function given by*

$$f_{\alpha,\alpha_1,\ldots,\alpha_k}(x) = \frac{1}{x^\alpha L_1(x)^{\alpha_1} \ldots L_k^{\alpha_k}(x)} \qquad \forall x \geq 0 ,$$

is completely monotone.

Proof. Obviously, $x \mapsto x^{-\alpha}$ is completely monotone. By (ii) of the previous lemma L_k admits a completely monotone derivative and then $L_k^{-\alpha_k}$ is completely monotone by (i). The fact that $f_{\alpha,\alpha_1,\ldots,\alpha_k}$ is also completely monotone then follows from Theorem 1 of [21]. □

Example 1. Let μ be a probability measure supported on $\mathbb{N}$ such that $\mu(n) = c f_{\alpha,\alpha_1,\ldots,\alpha_k}(n+1)$ for every $n \in \mathbb{N}$, where $\alpha_1, \ldots, \alpha_k \in [0, +\infty)$, $\alpha \in (1, 2)$ and c is a normalizing constant

ensuring that we have a probability. Then, $\mu \in \mathcal{H} \cap \mathcal{R}$ Of course one may take $\alpha = 1$ and $\alpha_1 > 1$, and so on... But for $\alpha = 2$, μ does not even have BAR.

It is more difficult to produce examples allowing negative $\alpha_k' s$. One way to handle the difficulty is to proceed as in the proof of Proposition 5.11 of [8].

Example 2. Our next example is a basic example of Ritt probability measures already considered by Dungey [10] and Gomilko and Tomilov [15]. Let $\gamma \in (0,1)$. We have a power series expansion $1-(1-t)^\gamma = \sum_{n\geq 1} a_n(\gamma) t^n$, $0 \leq t \leq 1$. Notice that $\sum_{n\geq 1} a_n(\gamma) = 1$ and $a_n(\gamma) \geq 0$ for every $n \geq 1$. Define two probability measures τ and μ by setting for every $n \in \mathbb{N}$, $\mu(n) = a_{n+1}(\gamma) = \tau(n+1)$. so that $\tau = \delta_1 * \mu$. Then, see for instance example 3.10a of [15], τ is a CM probability measure which has BAR. In particular, $\tau \in \mathcal{H} \cap \mathcal{R}$ and $\mu \in \mathcal{H} \cap \mathcal{R}$.

4. Probability measures with a first moment

When μ has a first moment, a necessary condition for the BAR property is that μ be centered, i.e. $\sum_{n\in\mathbb{Z}} n\mu(n) = 0$, see Proposition 1.9 of [6].

Hence we cannot consider probability measures μ supported by $\mathbb{N}$ anymore. We shall consider the following situation.

Definition 4.1. *We say that a probability measure μ on $\mathbb{Z}$ is CCM if it is supported on $\{-1\} \cup \mathbb{N}$ and if there exists a finite positive measure ν on $[0,1]$, such that*

$$\int_0^1 \frac{\nu(dt)}{(1-t)^2} = 1\,.$$

and

$$\mu(n) := \int_0^1 t^n \nu(dt) \qquad \forall n \in \mathbb{N};$$
$$\mu(-1) = 1 - \int_0^1 \frac{\nu(dt)}{1-t} = \int_0^1 \frac{t\nu(dt)}{(1-t)^2}\,.$$

It is not hard to see that μ is indeed a probability measure and that it is centered.

4.1. Characterization of the BAR property.

Let μ be a CCM probability measure on $\mathbb{Z}$ with representative measure ν.

For every $\theta \in [-\pi, \pi]$, we have

$$\hat{\mu}(\theta) = \int_0^1 \frac{1 - 2t + 2t^2 \mathrm{e}^{-i\theta} - t^2 \mathrm{e}^{-2i\theta}}{(1-t)^2((1-t)^2 + 2t(1-\cos\theta))}\, \nu(dt)\,.$$

In particular,

$$1 - \operatorname{Re}\hat{\mu}(\theta) = (1-\cos\theta) \int_0^1 \frac{2t(1-t\cos\theta)}{(1-t)^2((1-t)^2 + 2t(1-\cos\theta))}\, \nu(dt)\,, \tag{43}$$

and

$$\operatorname{Im}\hat{\mu}(\theta) = 2\sin\theta\,(1-\cos\theta)\int_0^1 \frac{t^2}{(1-t)^2((1-t)^2+2t(1-\cos\theta))}\,\nu(dt) \tag{44}$$

Consider the following condition on ν: there exists $L>0$, such that for every $x\in[0,1)$,

$$\frac{1}{1-x}\int_x^1 \frac{t\nu(dt)}{(1-t)^2} \le L\int_0^x \frac{t\nu(dt)}{(1-t)^3}\,. \tag{45}$$

Notice that if $\int_0^1 \frac{\nu(dt)}{(1-t)^3}<\infty$ (i.e. μ has a moment of order 2), condition (45) is automatically satisfied.

Proposition 4.1. *Let μ be a CCM probability measure on $\mathbb{Z}$ with representative measure ν. Then, μ has BAR if and only if there exists $L>0$ such that ν satisfies* (45).

Proof. Assume (45). Let us prove that μ satisfies (24). As noticed previously, it is enough to consider $\theta\in[-1/2,1/2]$. We have

$$|\operatorname{Im}\hat{\mu}(\theta)| \le C\Big(|\theta|^3\int_0^{1-|\theta|}\frac{t}{(1-t)^4}\nu(dt) + |\theta|\int_{1-|\theta|}^1 \frac{t\nu(dt)}{(1-t)^2}\Big)\,.$$

Using that $1-t\cos\theta\ge 1-t$, we see that

$$1-\operatorname{Re}\hat{\mu}(\theta) \ge \tilde{C}\theta^2\int_0^{1-|\theta|}\frac{t\nu(dt)}{(1-t)^3}\,,$$

and (24) holds, by (45).

Let us prove that if μ has BAR, then (45) holds. There exists $C>0$ such that for every $\theta\in[-1/2,1/2]$,

$$|\operatorname{Im}\hat{\mu}(\theta)| \le C(1-\operatorname{Re}\hat{\mu}(\theta))\,. \tag{46}$$

Let $\theta\in[-1/2,1/2]$ and $\alpha\in(0,1]$. We have

$$|\operatorname{Im}\hat{\mu}(\theta)| \ge \frac{|\theta|}{4(1+\alpha^2)}\int_{1-\alpha|\theta|}^1 \frac{t\nu(dt)}{(1-t)^2}\,.$$

It is not hard to prove that there exists $C_\alpha, D>0$ such that

$$1-t\cos\theta \le C_\alpha(1-t) \qquad \forall t\in[0,1-\alpha|\theta|]\,;$$
$$1-t\cos\theta \le D\alpha|\theta| \qquad \forall t\in(1-\alpha|\theta|,1]\,.$$

Hence, using (46), we infer that

$$\frac{|\theta|}{4(1+\alpha^2)}\int_{1-\alpha|\theta|}^1 \frac{t\nu(dt)}{(1-t)^2} \le C\Big(C_\alpha|\theta|^2\int_0^{1-|\theta|}\frac{t\nu(dt)}{(1-t)^3} + \alpha|\theta|\int_{1-\alpha|\theta|}^1 \frac{t\nu(dt)}{(1-t)^2}\Big)\,.$$

Taking $\alpha = 1/(8C)$ gives the desired result. □

As before, we shall now characterize the BAR property in terms of the coefficients of μ.

Proposition 4.2. *Let μ be a CCM probability measure on $\mathbb{Z}$ with representative measure ν. Then, ν satisfies* (45) *if and only if there exists $L>0$ such that*

$$n\sum_{k\geq n}k\mu(k)\leq L\sum_{k=1}^{n}k^2\mu(k)\qquad\forall n\in\mathbb{N}\,. \tag{47}$$

Proof. Assume (45). Let $n\geq 2$, we have

$$\begin{aligned}n\sum_{k\geq n}k\mu(k)&\leq\int_0^{1-1/n}\sum_{k\geq n}k^2t^k\nu(dt)+n\int_{1-1/n}^1\frac{t\nu(dt)}{(1-t)^2}\\&\leq(1+L)\int_0^{1-1/n}\frac{t\nu(dt)}{(1-t)^3}\,.\end{aligned}$$

Now, for every $1\leq\ell\leq n-1$ and every $t\in[1-1/\ell,1-1/(\ell+1)]$, we have $\sum_{k=1}^n k^2t^k\geq t\sum_{k=1}^{\ell}k^2\mathrm{e}^{-1}\geq Ct/(1-t)^3$, where we used that $(1-1/m)^{m-1}$ decreases to e^{-1}. Hence, (47) holds.

Assume that (47) holds. Let $\gamma\in(0,1]$ and $n\geq 2$. For every $t\in[1-1/n,1]$, since $\gamma\leq 1$, we have

$$\sum_{k\geq\gamma n}kt^k=\sum_{k\geq 0}(k+n)t^{k+n}\geq\frac{t(1-1/n)^n}{(1-t)^2}\geq\frac{t(2\mathrm{e})^{-1}}{(1-t)^2}\,.$$

Hence

$$\begin{aligned}n\int_{1-1/n}^1\frac{t\nu(dt)}{(1-t)^2}&\leq 2\mathrm{e}n\sum_{k\geq\gamma n}k\mu(k)\leq\frac{2nL\mathrm{e}}{[\gamma n]}\sum_{k=1}^{[\gamma n]}k^2\mu(k)\\&\leq\frac{2nL\mathrm{e}}{[\gamma n]}\Big(\int_0^{1-1/n}\frac{t\nu(dt)}{(1-t)^3}+[\gamma n]^2\int_{1-1/n}^1\frac{t\nu(dt)}{1-t}\Big)\,,\end{aligned}$$

and we conclude by taking γ small enough □

Theorem 4.3. *Let μ be a CCM probability measure on $\mathbb{Z}$ with representative measure ν. Assume that $(\mu(n))_{n\in\mathbb{N}}$ satisfies* (47)*. Then, μ is Ritt and for every $m\in\mathbb{N}$, there exists $C_m>0$ such that for every $f\in\ell^1(\mathbb{Z})$,*

$$\sup_{\lambda>0}\lambda\#\{k\in\mathbb{Z}\,:\,\sup_{n\geq 1}n^m|\mu^{*n}*(\delta_0-\mu)^{*m}*f(k)|\geq\lambda\}\leq C_m\|f\|_{\ell^1}\,.$$

Proof. It suffices to check that μ satisfies hypothesis (**H**) and to apply Propositions 2.3 and 2.5.

To check the conditions we must estimate $\hat{\mu}$ and its derivatives.

Define

$$\psi(\theta)=\theta^2\int_0^1\frac{t\nu(dt)}{(1-t)((1-t)^2+\theta^2)}=\frac{t\nu(dt)}{1-t}-\int_0^1\frac{t(1-t)\nu(dt)}{(1-t)^2+\theta^2}\,. \tag{48}$$

Then,

$$\psi'(\theta) = 2\theta \int_0^1 \frac{t(1-t)\nu(dt)}{((1-t)^2+\theta^2)^2}\,. \tag{49}$$

Hence for every $\theta \in [0, 1/2]$, we have

$$\frac{\theta^2}{2}\int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^3} + \frac{1}{2}\int_{1-\theta}^1 \frac{t\nu(dt)}{1-t} \le \psi(\theta) \le \theta^2 \int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^3} + \int_{1-\theta}^1 \frac{t\nu(dt)}{1-t} \tag{50}$$

$$\theta \int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^3} + \frac{1}{\theta^3}\int_{1-\theta}^1 t(1-t)\nu(dt) \le \psi'(\theta) \le 2\theta \int_0^{1-\theta} \frac{t\nu(dt)}{(1-t)^3} + \frac{2}{\theta^3}\int_{1-\theta}^1 t(1-t)\nu(dt)\,.$$

In particular, using (45), we see that (19) holds.

Let us compute the derivatives of $\hat{\mu}$. We shall not give the full details here. Using (43), we infer that

$$1 - \operatorname{Re}\hat{\mu}(\theta) = \int_0^1 \frac{1-t\cos\theta}{(1-t)^2}\nu(dt) - \int_0^1 \frac{1-t\cos\theta}{(1-t)^2+2t(1-\cos\theta)}\nu(dt)\,;$$

$$\begin{aligned} \operatorname{Re}\hat{\mu}'(\theta) &= 2\sin\theta(1-\cos\theta)\int_0^1 \frac{t^2}{(1-t)^2((1-t)^2+2t(1-\cos\theta))}\nu(dt) \\ &\qquad + \sin\theta \int_0^1 \frac{2t(1-t\cos\theta)}{((1-t)^2+2t(1-\cos\theta))^2}\nu(dt) \\ &= -\sin\theta\int_0^1 \frac{t}{(1-t)^2}\nu(dt) - \sin\theta\int_0^1 \frac{t(1-t^2)}{((1-t)^2+2t(1-\cos\theta))^2}\nu(dt)\,; \end{aligned} \tag{51}$$

and

$$\begin{aligned} \operatorname{Re}\hat{\mu}''(\theta) &= -\cos\theta\int_0^1 \frac{t}{(1-t)^2}\nu(dt) - \cos\theta\int_0^1 \frac{t(1-t^2)}{((1-t)^2+2t(1-\cos\theta))^2}\nu(dt) \\ &\qquad + 4\sin^2\theta\int_0^1 \frac{t^2(1-t^2)}{((1-t)^2+2t(1-\cos\theta))^3}\nu(dt)\,. \end{aligned} \tag{52}$$

Using (44), we infer that

$$\operatorname{Im}\hat{\mu}(\theta) = \sin\theta\int_0^1 \frac{t}{(1-t)^2}\nu(dt) - \sin\theta\int_0^1 \frac{t}{(1-t)^2+2t(1-\cos\theta)}\nu(dt)\,;$$

$$\begin{aligned} \operatorname{Im}\hat{\mu}'(\theta) &= 2\cos\theta(1-\cos\theta)\int_0^1 \frac{t^2}{(1-t)^2((1-t)^2+2t(1-\cos\theta))}\nu(dt) \\ &= \cos\theta\int_0^1 \frac{t}{(1-t)^2}\nu(dt) + \int_0^1 \frac{2t^2 - t(1+t^2)\cos\theta}{((1-t)^2+2t(1-\cos\theta))^2}\nu(dt)\,; \end{aligned} \tag{53}$$

and

$$\operatorname{Im}\hat{\mu}''(\theta) = -\sin\theta\int_0^1 \frac{t}{(1-t)^2}\nu(dt) + \sin\theta\int_0^1 \frac{t(1+t^2)}{((1-t)^2+2t(1-\cos\theta))^2}\nu(dt) \\ -4\sin\theta\int_0^1 \frac{t(2t^2-t(1+t^2)\cos\theta)}{((1-t)^2+2t(1-\cos\theta))^3}\nu(dt) \tag{54}$$

We now derive the necessary estimates on $\hat{\mu}$ and its derivatives.

Using (43), we infer that

Claim 5. *There exists $C>0$ such that $1-\operatorname{Re}\hat{\mu}(\theta) \geq C\theta^2\int_0^{1-\theta}\frac{t}{(1-t)^3}\nu(dt)$, for every $\theta\in(0,1/2]$.*

Using (51), we infer that

Claim 6. *There exists $C>0$ such that $|\operatorname{Re}\hat{\mu}'(\theta)| \leq C\theta\int_0^{1-\theta}\frac{t}{(1-t)^3}\nu(dt) + \frac{C}{\theta^2}\int_{1-\theta}^1 t\nu(dt)$, for every $\theta\in(0,1/2]$.*

Using (53), we infer that

Claim 7. *There exists $C>0$ such that $|\operatorname{Im}\hat{\mu}'(\theta)| \leq C\theta^2\int_0^{1-\theta}\frac{t}{(1-t)^4}\nu(dt)+C\int_{1-\theta}^1\frac{t}{(1-t)^2}\nu(dt)$, for every $\theta\in(0,1/2]$.*

Using (52), we infer that

Claim 8. *There exists $C>0$ such that $|\operatorname{Re}\hat{\mu}''(\theta)| \leq C\int_0^{1-\theta}\frac{t}{(1-t)^3}\nu(dt) + \frac{C}{\theta^4}\int_{1-\theta}^1 t(1-t)\nu(dt$, for every $\theta\in(0,1/2]$.*

Notice that there exists $\alpha>0$ such that for every $t\in[0,1]$ and every $\theta\in(0,1/2]$,

$$|2t^2-t(1+t^2)\cos\theta)| = t|(1+t^2)(1-\cos\theta)-(1-t)^2| \leq \alpha\max(\theta^2,(1-t)^2)\,.$$

Combining this estimate with (54), we infer that

Claim 9. *There exists $C>0$ such that $|\operatorname{Im}\hat{\mu}''(\theta)| \leq C\theta\int_0^{1-\theta}\frac{t}{(1-t)^4}\nu(dt) + \frac{C}{\theta^3}\int_{1-\theta}^1 t\nu(dt)$, for every $\theta\in(0,1/2]$.*

We already saw that (19) holds. Let us prove that items $(i)-(iv)$ of Proposition 2.3 hold.

Item (i) follows from Claim 5 and (45) (see the proof of Theorem 3.6.

Item (ii) follows from Claims 6 and 7 combined with (45) and (50).

Item (iii) follows from item (ii) combined with (19).

Item (iv) follows from Claims 8 and 9 combined with (45) and (50). □

Proposition 4.4. *Let τ be a centered probability measure on $\mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}}|n|\tau(n)<\infty$. Assume moreover that there exists a CCM probability measure μ satisfying* (47) *and that there exists $a>0$ such that $\sum_{n\in\mathbb{Z}}n^2|\tau(n)-a\mu(n)|<\infty$. Then the conclusion of Theorem 4.3 holds for τ.*

Proof. We shall assume that $\sum_{n\in\mathbb{Z}} n^2\tau(n) = \infty$, otherwise, the result holds by Theorem 1.3. In particular we must have $\sum_{n\in\mathbb{Z}} n^2\mu(n) = \infty$ and by (48) and (49)

$$\liminf_{\theta\in(0,\pi]} \psi(\theta)/\theta^2 = +\infty \quad \text{and} \quad \liminf_{\theta\in(0,\pi]} \psi'(\theta)/\theta = +\infty \qquad (\theta \to 0)\,. \tag{55}$$

It follows from the proof of Theorem 4.3 that there exists an even function ψ continuous on $[-\pi,\pi]$ and continuously differentiable on $(0,\pi]$, with $\psi(0) = 0$ such that μ and ψ satisfy the item $(i)-(iv)$ of hypothesis $(\mathbf{H})$, for some $C, c > 0$.

Since $\hat{\tau} = (\hat{\tau} - a\hat{\nu}) + a\hat{\mu}$ is clearly twice differentiable on $(0,\pi]$, the proposition will be proved if we can show that the items $(i)-(iv)$ of hypothesis $(\mathbf{H})$ hold with τ in place of μ with the same ψ, but for possibly different $C, c > 0$.

We already saw that τ must be strictly aperiodic. Hence $|\hat{\mu}| < 1$ on $(0,\pi]$. In particular, to prove item (i) it suffices to consider $\theta \in (0,\eta]$ for some small enough $\eta > 0$.

For every $\theta \in (0,\pi]$, we have

$$\hat{\tau}(\theta) = \sum_{n\in\mathbb{Z}} (\tau(n) - a\mu(n))(\mathrm{e}^{in\theta} - 1) + \big[1 - a + a\sum_{n\in\mathbb{Z}} \mu(n)\mathrm{e}^{in\theta}\big] := \chi(\theta) + \phi(\theta)\,.$$

Using that $\sum_{n\in\mathbb{Z}} n^2|\tau(n) - a\mu(n)| < \infty$ and that $\sum_{n\in\mathbb{Z}} n(\tau(n) - a\mu(n)) = 0$, we see that $\lim_{\theta\to 0,\theta\neq 0} \chi(\theta)/\theta^2 = \chi''(0)$ exists. In particular, $\lim_{\theta\to 0,\theta\neq 0} \chi(\theta)/\psi(\theta) = 0$.

Now, we have

$$|\phi(\theta)|^2 = (1-a+a\mathrm{Re}\,\hat{\mu}(\theta))^2 + a^2(\mathrm{Im}\,\hat{\mu}(\theta))^2 = 1 - 2a(1-\mathrm{Re}\,\hat{\mu}(\theta)) + a^2(1-\mathrm{Re}\,\hat{\mu}(\theta))^2 + a^2(\mathrm{Im}\,\hat{\mu}(\theta))^2.$$

Hence, using Claim 5, we infer that there exists $\eta > 0$ such that for every $\theta \in (0,\eta]$, $|\hat{\tau}(\theta)| \le 1 - \delta\psi(\theta)$, for some $\delta > 0$.

The proofs of item $(ii)-(iii)$ are similar (but simpler) hence we leave them to the reader. □

Example 3. Let $\alpha \in (2,+\infty)$ and $\alpha_1,\dots,\alpha_k \ge 0$. Let μ be a probability on $\mathbb{Z}$ such that $\sum_{n\in\mathbb{Z}} |n|\mu(n) < \infty$, $\sum_{n\in\mathbb{Z}} n\mu(n) = 0$ and $\sum_{n\in\mathbb{Z}} n^2|\mu(n) - af_{\alpha,\alpha_1,\dots,\alpha_k}(n+1)| < \infty$, for some $a > 0$, where we extended $f_{\alpha,\alpha_1,\dots,\alpha_k}$ to $\mathbb{Z}^-$ by setting, $f_{\alpha,\alpha_1,\dots,\alpha_k}(-n) = 0$ for every $n \in \mathbb{N}$.

5. Symmetric probability measures

In this section we consider symmetric probability measures. If μ is symmetric (i.e. $\check{\mu} = \mu$), then $\hat{\mu}$ is real valued, hence has BAR. It is known that if moreover $(\mu(n))_{n\in\mathbb{N}}$ is non-increasing then (13) holds. However we are not aware of any result concerning the Ritt property or (14) with $m \ge 1$.

We shall again investigate the situation where we have completely monotone coefficients. To be more precise we consider the following situation.

Definition 5.1. *We say that a probability measure μ on $\mathbb{Z}$ is SCM if it is symmetric and if there exists a finite positive measure on $[0,1]$ such that*

$$\int_0^1 \frac{1}{1-t}\nu(dt) = 1/2 \quad , \quad \mu(0) = 2\int_0^1 \nu(dt);$$

$$\mu(n) = \int_0^1 t^n \nu(dt) \qquad \forall n \geq 1\,.$$

Let μ be an SCM probability measure on $\mathbb{Z}$ with repesentative measure ν. Define another measure on $\mathbb{Z}$, supported on $\mathbb{N}$, by setting

$$\mu_1(0) = 2\int_0^1 \nu(dt);$$

$$\mu_1(n) = 2\int_0^1 t^n \nu(dt) \qquad \forall n \geq 1\,.$$

Then μ_1 is a probability measure, $(\mu_1(n))_{n\in\mathbb{N}}$ is completely monotone and $\mu = \frac{1}{2}(\check{\mu}_1 + \mu_1)$. In particular, it follows from Proposition 2.4 and Theorem 3.6, that μ satisfies hypothesis (**H**) as soon as μ satisfies (33). The fact that μ is Ritt when it satisfies (33) may be proved similarly (but more easily).

We could use a similar argument based on Theorem 4.3. However, doing so, we would miss some symmetric probability measures satisfying hypothesis (**H**).

Let us explain how to be more precise. Let μ be a SCM probability measure.

It follows from previous computations that, for every $\theta \in \mathbb{R}$,

$$1 - \hat{\mu}(\theta) = 1 - \operatorname{Re}\hat{\mu}(\theta) = \int_0^1 \frac{t(1-\cos\theta)}{(1-t)((1-t)^2 + 2t(1-\cos\theta))}\,\nu(dt)\,.$$

Consider the following condition on ν: there exists $L > 0$ such that for every $x \in [0,1)$,

$$\int_x^1 \frac{t}{1-t}\nu(dt) \leq L(1-x)^2 \int_0^x \frac{t}{(1-t)^3}\nu(dt)\,. \tag{56}$$

This condition can be proved to be equivalent to the following one: there exists $D > 0$ such that for every $n \geq 1$,

$$n^2 \sum_{k\geq n} \mu(k) \leq L \sum_{k=1}^{n} k^2 \mu(k)\,. \tag{57}$$

One can prove that if (56) holds, then μ satisfies hypothesis ($\tilde{\mathbf{H}}$) with ψ given by

$$\psi(\theta) = \theta^2 \int_0^1 \frac{t}{(1-t)(1-t+|\theta|)^2}\nu(dt) \qquad \forall \theta \in [-\pi,\pi] - \{0\}\,.$$

Notice that $\psi'(\theta) = 2\theta \int_0^1 \frac{t}{(1-t+\theta)^3}\nu(dt)$, for every $\theta \in (0,\pi]$.

Then, one can prove that a SCM probability measure satisfying (57) is Ritt and satisfies (14) for every $m \in \mathbb{N}$ and some $C_m > 0$.

In particular, we have the following.

Theorem 5.1. *Let μ be a SCM probability measure such that $(\mu(n))_{n\in\mathbb{N}}$ satisfies either* (33) *or* (57)*. Then, μ is Ritt and satisfies* (14) *for every $m \in \mathbb{N}$ and some $C_m > 0$.*

Example 4. Let $\alpha > 1$ and $\alpha_1, \dots, \alpha_k \geq 0$. Let μ be a symmetric probability measure defined by $\mu(0) = 2cf_{\alpha,\alpha_1,\dots,\alpha_k}(1)$ and for every $n \geq 1$ $\mu(n) = cf_{\alpha,\alpha_1,\dots,\alpha_k}(n+1)$, where c is a normalizing sequence ensuring that μ is a probability. Then, μ is a SCM probability measure for which the above theorem apply.

6. Discussion and open questions

- Most of the examples of (strictly aperiodic) probability measures on $\mathbb{Z}$ that have BAR are known to be Ritt. We do not believe that the BAR property and the Ritt property are equivalent, but one has to find a counterexample. This problem was also formulated by Dungey [10] (see his remarks page 1729).

- One may wonder whether, in the symmetric case, the condition "$(\mu(n))_{n\in\mathbb{N}}$ is non-increasing" is sufficient for the Ritt property or for weak type maximal inequalities (5), since it is sufficient for the weak type maximal inequality (4). At least, for a SCM probability measure on $\mathbb{Z}$, can one remove the conditions (33) and (57) from Theorem 5.1 ?

- Let μ be a probability measure on $\mathbb{Z}$. Let $f \in \ell^p(\mathbb{Z})$, $p \geq 1$. Consider the square function defined by $s_\mu(f)(k) := \Big(\sum_{n\geq 1} n\big((\mu^{*n} - \mu^{*(n+1)}) * f(k)\big)^2\Big)^{1/2}$. Assume that μ has BAR. When $p > 1$, it follows from the work of Le Merdy and Xu [17] that there exists $C_p > 0$ such that for every $f \in \ell^p(\mathbb{Z})$, $\|s_\mu(f)\|_p \leq C_p\|f\|_p$, i.e. $s(f)$ satifies a strong $p-p$ inequality. A natural question is whether $s(f)$ satisfies a weak $1-1$ inequality.

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Laboratoire MICS, Centralesupelec, Grande Voie des Vignes, 92295 Chatenay-Malabry cedex, FRANCE.
E-mail address: christophe.cuny@ecp.fr