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Well-posedness of the Cauchy problem for a space-dependent anyon Boltzmann equation.

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Abstract. A fully non-linear kinetic Boltzmann equation for anyons is studied in a periodic 1d setting with large initial data. Strong $L^1$ solutions are obtained for the Cauchy problem. The main results concern global existence, uniqueness and stability. We use the Bony functional, the two-dimensional velocity frame specific for anyons, and an initial layer analysis that moves the solution away from a critical value.

1 Anyons and the Boltzmann equation.

Let us first recall the definition of anyon. Consider the wave function $\psi(R, \theta, r, \varphi)$ for two identical particles with center of mass coordinates $(R, \theta)$ and relative coordinates $(r, \varphi)$. Exchanging them, $\varphi \rightarrow \varphi + \pi$, gives a phase factor $e^{2\pi i}$ for bosons and $e^{\pi i}$ for fermions. In three or more dimensions those are all possibilities. Leinaas and Myrheim proved in 1977 [10], that in one and two dimensions any phase factor is possible in the particle exchange. This became an important topic after the first experimental confirmations in the early 1980-ies, and Frank Wilczek in analogy with the terms bos(e)-ons and fermi-ons coined the name any-ons for the new quasi-particles with any phase. Anyon quasi-particles with e.g. fractional electric charge, have since been observed in various types of experiments.

By moving to a definition in terms of a generalized Pauli exclusion principle, Haldane [9] extended this to a fractional exclusion statistics valid for any dimension, and coinciding with the anyon definition in the one and two dimensional cases. Haldane statistics has also been realized for neutral fermionic atoms at ultra-low temperatures in three dimensions [3]. Wu later derived [17] occupation-number distributions for ideal gases under Haldane statistics by counting states under the new fractional exclusion principle. From the number of quantum states of $N$ identical particles occupying $G$ states being

$$\frac{(G + N - 1)!}{N!(G - 1)!} \quad \text{and} \quad \frac{G!}{N!(G - N)!}$$

in the boson resp. fermion cases, he derived the interpolated number of quantum states for the fractional exclusions to be

$$\frac{(G + (N - 1)(1 - \alpha))!}{N!(G - \alpha N - (1 - \alpha))!}, \quad 0 < \alpha < 1.$$  \hspace{1cm} (1.1)
He then obtained for ideal gases the equilibrium statistical distribution

\[
\frac{1}{w(e^{(\epsilon-\mu)/T}) + \alpha},
\]

(1.2)

where \(\epsilon\) denotes particle energy, \(\mu\) chemical potential, \(T\) temperature, and the function \(w(\zeta)\) satisfies

\[
w(\zeta)^\alpha(1 + w(\zeta))^{1-\alpha} = \zeta \equiv e^{(\epsilon-\mu)/T}.
\]

In particular \(w(\zeta) = \zeta - 1\) for \(\alpha = 0\) (bosons) and \(w(\zeta) = \zeta\) for \(\alpha = 1\) (fermions).

In elastic pair collisions, the velocities \((v, v_s)\) before and \((v', v'_s)\) after a collision are related by

\[
v' = v - n[(v - v_s) \cdot n], \quad v'_s = v_s + n[(v - v_s) \cdot n], \quad n \in S^{d-1}.
\]

This preserves mass, linear momentum, and energy in Boltzmann type collision operators. We shall write \(f = f(v), \quad f_s = f_s(v_s), \quad f' = f(v'), \quad f'_s = f'_s(v'_s)\). An important question for gases with fractional exclusion statistics, is how to calculate their transport properties, in particular how the Boltzmann equation

\[
\partial_t f + v \cdot \nabla_x f = Q(f)
\]

gets modified. An answer was given by Bhaduri, Bhalerao, and Murthy [2] by generalizing to anyons the filling factors \(F(f)\) from the fermion and boson cases, \(F(f) = (1 + \eta f), \quad \eta = \mp 1\), and by inductive reasoning obtaining as anyon filling factors

\[
F(f) = (1 - \alpha f)^\alpha(1 + (1 - \alpha)f)^{1-\alpha}, \quad 0 < \alpha < 1.
\]

Namely, with a filling factor \(F(f)\) in the collision operator \(Q\), the entropy production term becomes

\[
\int Q(f) \log \frac{f}{F(f)} dv,
\]

which for equilibrium implies

\[
\frac{f'}{F(f')} \frac{f_s'}{F(f'_s)} = \frac{f}{F(f)} \frac{f_s}{F(f_s)}.
\]

Using conservation laws and properties of the Cauchy equation, one concludes that in equilibrium

\[
\frac{f}{F(f)}
\]

is a Maxwellian. Inserting Wu’s equilibrium (1.2) for \(f\) and taking the quotient Maxwellian as \(e^{-(\epsilon-\mu)/T}\) with \(\epsilon = |v - v_0|^2\) when the bulk velocity is \(v_0\), this gives

\[
f = \frac{1}{w(e^{(\epsilon-\mu)/T}) + \alpha}, \quad F(f) = f e^{(\epsilon-\mu)/T} = \frac{e^{(\epsilon-\mu)/T}}{w(e^{(\epsilon-\mu)/T}) + \alpha}.
\]

In particular in the fermion and boson cases,

\[
f = \frac{1}{e^{(\epsilon-\mu)/T} - \eta}, \quad F(f) = \frac{e^{(\epsilon-\mu)/T}}{e^{(\epsilon-\mu)/T} - \eta}, \quad \eta = \mp 1.
\]

This is consistent with taking an interpolation between the fermion and boson factors as general filling factor, \(F(f) = (1 - \alpha f)^\alpha(1 + (1 - \alpha)f)^{1-\alpha}, \quad 0 < \alpha < 1\). It gives the collision operator \(Q\) of [2] for Haldane statistics,

\[
Q(f)(v) = \int_{\mathbb{R}^d \times S^{d-1}} B(|v - v_s|, \omega)[f' f'_s F(f) F(f_s) - f f_s F(f') F(f'_s)] dv_s d\omega.
\]

(1.3)
Here $d\omega$ corresponds to the Lebesgue probability measure on the $(d-1)$-sphere. The collision kernel $B(z,\omega)$ in the variables $(z,\omega) \in \mathbb{R}^d \times S^{d-1}$ is positive, locally integrable, and only depends on $|z|$ and $|(z,\omega)|$. It is discussed in [2] but, as common in quantum kinetic theory, without explicit bounds on the kernel. We restrict to a bounded collision kernel truncated for small relative velocities and grazing collisions. The precise assumptions on $B$ are given in the beginning of Section 2.

The anyon Boltzmann equation for $0 < \alpha < 1$ retains important properties from the Fermi-Dirac case. In the filling factor $F(f) = (1-\alpha f)^\alpha(1+(1-\alpha)f)^{1-\alpha}$, $0 < \alpha < 1$, the factor $(1-\alpha f)^\alpha$ requires the value of $f$ not to exceed $\frac{1}{\alpha}$. This is formally preserved by the equation, since the gain term vanishes for $f = \frac{1}{\alpha}$, making the $Q$-term (1.3) and the derivative left hand side of the Boltzmann equation negative there. Positivity is formally preserved, since the derivative equals the positive gain term for $f = 0$, where the loss term vanishes. $F$ is concave with maximum value one at $f = 0$ for $\alpha \geq \frac{1}{2}$, and maximum value $(\frac{1}{\alpha} - 1)^{1-2\alpha} > 1$ at $f = \frac{1}{\alpha(1-\alpha)}$ for $\alpha < \frac{1}{2}$. The collision operator vanishes identically for the equilibrium distribution functions obtained by Wu.

The Boltzmann equation for the limiting cases, representing boson statistics ($\alpha = 0$) and fermion statistics ($\alpha = 1$), was introduced by Nordheim [15] in 1928. Here the quartic terms in the collision integral cancel, which is used in the analysis. General existence results for the space-homogeneous isotropic boson large data case were obtained in [12], followed by a number of other papers, e.g. [7], [13], [8], [14], and for the space-dependent case near equilibrium in [16]. In the space-dependent fermion case general existence results were obtained in [6], [11] and [14].

For $0 < \alpha < 1$ there are no cancellations in the collision term. Moreover, the Lipschitz continuity of the collision term for $\alpha \in [0,1]$, is for $0 < \alpha < 1$ replaced by a weaker Hölder continuity near $f = \frac{1}{\alpha}$. The space-homogeneous initial value problem for the Boltzmann equation with Haldane statistics is

$$\frac{df}{dt} = Q(f), \quad f(0,v) = f_0(v). \quad (1.4)$$

Because of the filling factor $F$, the range for the initial value $f_0$ should belong to $[0,\frac{1}{\alpha}]$, which is also formally preserved by the equation. A good control of $\int f(t,x,v)dv$, which in the space-homogeneous case is given by the mass conservation, can be used to keep $f$ uniformly away from $\frac{1}{\alpha}$, and $F(f)$ Lipschitz continuous. That was a basic observation behind the existence result for the space-homogeneous anyon Boltzmann equation.

**Proposition 1.1** [1] Consider the space-homogeneous equation (1.4) with velocities in $\mathbb{R}^d$, $d \geq 2$ and for hard potential kernels with

$$0 < B(z,\theta) \leq C|z|^\beta|\sin \theta \cos \theta|^{d-1}, \quad (z,\theta) \in \mathbb{R}_+ \times [-\frac{\pi}{2}, \frac{\pi}{2}], \quad (1.5)$$

where $0 < \beta \leq 1$, $d > 2$ or $0 < \beta < 1$, $d = 2$. Let the initial value $f_0$ have finite mass and energy. If $0 < f_0 \leq \frac{1}{\alpha}$ and $\text{ess sup}(1 + |v|^s)f_0 < \infty$ for $s = d - 1 + \beta$, then the initial value problem for (1.4) has a strong solution in the space of functions continuous from $t \geq 0$ into $L^1 \cap L^\infty$, which conserves mass and energy, and for $t_0 > 0$ given, has

$$\text{ess sup}_{v \in \mathbb{R}^d, t \leq t_0} |v|^{s'}f(t,v) \text{ bounded, where } s' = \min\{s, \frac{2\beta(d + 1) + 2}{d}\}.$$
and converging in $L^1$ to $f_0$, there is a subsequence of the solutions converging in $L^1$ to a solution with initial value $f_0$.

**2 The main results.**

The present paper considers the space-dependent anyon Boltzmann equation in a slab. With

$$\cos \theta = n \cdot \frac{v - v_*}{|v - v_*|},$$

the kernel $B(|v - v_*|, \omega)$ will from now on be written $B(|v - v_*|, \theta)$ and be assumed measurable with

$$0 \leq B \leq B_0,$$

for some $B_0 > 0$. It is also assumed for some $\gamma, \gamma', c_B > 0$, that

$$B(|v - v_*|, \theta) = 0 \text{ for } |\cos \theta| < \gamma', \quad \text{for } 1 - |\cos \theta| < \gamma', \quad \text{and for } |v - v_*| < \gamma,$$

and that

$$\int B(|v - v_*|, \theta) d\theta \geq c_B > 0 \quad \text{for } |v - v_*| \geq \gamma.$$ (2.3)

The initial datum $f_0(x, v)$, periodic in $x$, is assumed to be a measurable function with values in $[0, \frac{1}{\alpha}]$, and such that

$$(1 + |v|^2)f_0(x, v) \in L^1([0, 1] \times \mathbb{R}^2), \quad \sup_{x \in [0, 1]} f_0(x, v) dv = c_0 < \infty, \quad \inf_{x \in [0, 1]} f_0(x, v) > 0, \text{ a.a.} v \in \mathbb{R}^2. $$ (2.4)

With $v_1$ denoting the component of $v$ in the $x$-direction, consider for functions periodic in $x$, the initial value problem

$$\partial_t f(t, x, v) + v_1 \partial_x f(t, x, v) = Q(f)(t, x, v), \quad f(0, x, v) = f_0(x, v), \quad (t, x, v) \in \mathbb{R}_+ \times [0, 1] \times \mathbb{R}^2. $$ (2.5)

The main results of the present paper are given in the following theorem.

**Theorem 2.1**

Assume (2.1)-(2.2)-(2.3). There exists a strong solution $f \in C([0, \infty]; L^1([0, 1] \times \mathbb{R}^2))$ of (2.5) with $0 < f(t, \cdot, \cdot) < \frac{1}{\alpha}$ for $t > 0$. There is $t_m > 0$ such that for any $T > t_m$, there is $\eta_T > 0$ so that

$$f(t, \cdot, \cdot) \leq \frac{1}{\alpha} - \eta_T, \quad t \in [t_m, T].$$

The solution is unique and depends continuously in $C([0, T]; L^1([0, 1] \times \mathbb{R}^2))$ on the initial $L^1$-datum. It conserves the mass, momentum and energy.

**Remarks.**

The above results seem to be new also in the fermion case where $\alpha = 1$. Indeed, whereas global existence of weak solutions in the 3D fermionic case is proved in [11] and [14], we here prove the global existence and the uniqueness of strong solutions in the 2D case.

This paper is restricted to the slab case, since the proof below uses an estimate for the Bony functional only valid in one space dimension.
Due to the filling factor $F(f)$, the proof in an essential way depends on the two-dimensional velocity frame, which corresponds to the anyon context. It does not extend to Haldane statistics in three or higher velocity dimensions.

The approach in the paper can also be used to obtain regularity results. The control of $\int f(t, x, v)dv$ in the present space-dependent setting is non-trivial. An entropy for (2.5) is

$$
\int \left( f \log f + \left( \frac{1}{\alpha} - f \right) \log(1 - \alpha f) - \left( \frac{1}{1 - \alpha} + f \right) \log(1 + (1 - \alpha)f)^{1 - \alpha} \right) dx dv.
$$

The asymptotic behaviour of the solution when $t \to \infty$ is an interesting still open problem, as is the behaviour of (2.5) beyond the anyon frame, i.e. for higher $v$-dimensions under Haldane statistics. It seems likely that a close to equilibrium approach as in the classical case, could work with fairly general kernels $B$ for close to equilibrium initial values $f_0$ with some regularity and strong decay conditions for large velocities. Any progress on the large data case in several space-dimensions under Haldane statistics would be quite interesting.

The paper is organized as follows. The lack of Lipschitz continuity of $F(f)$ when $f$ is in a neighborhood of $\frac{1}{\alpha}$ requires some care. Since the gain term vanishes when $f = \frac{1}{\alpha}$ and the derivative becomes negative there, $f$ should start decreasing before reaching this value. The proof that this takes place uniformly over phase-space and approximations, is based on a good control of $\int f(t, x, v)dv$ in the integration of the gain and loss parts of $Q$. That is a main topic in Section 3 together with the study of a family of approximating equations with large velocity cut-off.

Section 4 starts with an initial value analysis, that shows that $f(t, \cdot, \cdot) < \frac{1}{\alpha} - b_1 t$ for some constant $b_1 > 0$ on an initial layer and that $f$ remains far from $\frac{1}{\alpha}$ afterwards. This is crucial for handling the Hölder continuity of $F(f)$ for values of $f$ close to $\frac{1}{\alpha}$, $F(f)$ being Lipschitz continuous away from $\frac{1}{\alpha}$. Based on this control of the values of $f$, the well-posedness of the problem and the conservation properties of the solution are proven.

### 3 Approximations and control of mass density.

The conditions (2.1)-(2.2)-(2.3) for the kernel $B$ and (2.4) are assumed throughout this section. For any $j \in \mathbb{N}$, denote by $\psi_j$, the cut-off function with

$$
\psi_j(r) = 0 \quad \text{if } r > j \quad \text{and} \quad \psi_j(r) = 1 \quad \text{if } r \leq j,
$$

and set

$$
\chi_j(v, v_s, v', v'_s) = \psi_j(|v|)\psi_j(|v_s|)\psi_j(|v'|)\psi_j(|v'_s|).
$$

Let the uniformly bounded function $F_j$ be defined on $[0, \frac{1}{\alpha}]$ by

$$
F_j(y) = \frac{1 - \alpha y}{(\frac{j}{1 - \alpha})^{1 - \alpha}}(1 + (1 - \alpha)y)^{1 - \alpha}.
$$
Denote by $Q_j$ (resp. $Q^+_j$, and $Q^-_j$ to be used later), the operator

$$Q_j(f)(v) := \frac{1}{\pi} \int B(|v - v_*|, \theta) \chi_j(v, v_*') (f' f_* F_j(f) F_j(f_*)) dv_* d\theta,$$

(resp. its gain part $Q^+_j(f)(v) := \frac{1}{\pi} \int B(|v - v_*|, \theta) \chi_j(v, v_*', v'_*) f' f_* F_j(f) F_j(f_*') dv_* d\theta$, and its loss part $Q^-_j(f)(v) := \frac{1}{\pi} \int B(|v - v_*|, \theta) \chi_j(v, v_*', v'_*) f f_* F_j(f') F_j(f'_*) dv_* d\theta$).

For $j \in \mathbb{N}$, let a mollifier $\varphi_j$ be defined by $\varphi_j(x, v) = j^3 \varphi(jx, jv)$, where

$$\varphi \in C_0^\infty(\mathbb{R}^3), \quad \text{support}(\varphi) \subset [0, 1] \times \{v \in \mathbb{R}^2; |v| \leq 1\}, \quad \varphi \geq 0, \quad \int \varphi(x, v) dx dv = 1.$$

Let $f_{0,j}$ be the restriction to $[0, 1] \times \{v; |v| \leq j\}$ of $(\min\{f_0, \frac{1}{\alpha} - \frac{1}{J}\}) * \varphi_j.$

The following lemma concerns a corresponding approximation of (2.5).

**Lemma 3.1** For $T > 0$, there is a unique solution $f_j \in C([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\}))$ to

$$\partial_t f_j + v_1 \partial_x f_j = Q_j(f_j), \quad f_j(0, \cdot, \cdot) = f_{0,j}.$$

(3.1)

There is $\eta_j > 0$ such that $f_j$ takes its values in $[0, \frac{1}{\alpha} - \eta_j].$

The solution conserves mass, first $v$-moment and energy.

**Proof of Lemma 3.1.**

Let $T > 0$ be given. We shall first prove by contraction that for $T_1 > 0$ and small enough, there is a unique solution

$$f_j \in C([0, T_1] \times [0, 1]; L^1(\{v; |v| \leq j\})) \cap \{f; f \in [0, \frac{1}{\alpha}]\}$$

to (3.1). Let the map $C$ be defined on periodic in $x$ functions in

$$C([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\})) \cap \{f; f \in [0, \frac{1}{\alpha}]\}$$

by $C(f) = g,$ where $g$ is the unique solution of the following linear differential equation

$$\partial_t g + v_1 \partial_x g = \frac{1}{\pi} (1 - \alpha g) \left(\frac{1 + (1 - \alpha) f}{\frac{1}{f} + 1 - \alpha f}\right)^{1 - \alpha} \int B \chi_j f' f_* F_j(f_*') dv_* d\theta - \frac{g}{\pi} \int B \chi_j f_* F_j(f') F_j(f_*') dv_* d\theta,$$

$g(0, \cdot, \cdot) = f_{0,j}.$

It follows from the linearity of the previous partial differential equation that it has a unique periodic solution $g$ in $C([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\})).$ For $f$ with values in $[0, \frac{1}{\alpha}], g$ takes its values in $[0, \frac{1}{\alpha}].$

Indeed, denoting by

$$g^\sharp(t, x, v) = g(t, x + tv_1, v),$$

it holds that

$$g^\sharp(t, x, v) = f_{0,j}(x, v) e^{-\int_0^t \sigma^\sharp_j(r, x, v) dr}$$

$$+ \frac{1}{\pi} \int_0^t ds \left( (1 - \alpha g) \left(\frac{1 + (1 - \alpha) f}{\frac{1}{f} + 1 - \alpha f}\right)^{1 - \alpha} \int B \chi_j f' f_* F_j(f_*') dv_* d\theta \right)^2 (s, x, v) e^{-\int_s^t \sigma^\sharp_j(r, x, v) dr}$$

$$\geq f_{0,j}(x, v) e^{-\int_0^t \sigma^\sharp_j(r, x, v) dr} > 0,$$
and
\[
(1 - \alpha g)^2(t, x, v) = (1 - \alpha f_{0,j})(x, v)e^{-\int_0^t \tilde{\sigma}_j^2(r, x, v)dr} + \frac{\alpha}{\pi} \int_0^t \left( g \int B_{\chi_j} f_s F_j(f_s')(f_s')dv_s d\theta \right)(s, x, v)e^{-\int_s^t \tilde{\sigma}_j^2(r, x, v)dr} ds
\]
\[
\geq (1 - \alpha f_{0,j})(x, v)e^{-\int_0^t \tilde{\sigma}_j^2(r, x, v)dr} > 0.
\]

Here,
\[
\tilde{\sigma}_j := \frac{1}{\pi} \int B_{\chi_j} f_s F_j(f_s')(f_s')dv_s d\theta,
\]
\[
\tilde{\sigma}_j := \frac{\alpha}{\pi} \left( \frac{1 + (1 - \alpha)f}{\frac{1}{j} + 1 - \alpha f} \right)^{1-\alpha} \int B_{\chi_j} f' f_s F_j(f_s)dv_s d\theta.
\]

C is a contraction on \( C([0, T_1] \times [0, 1]; L^1(\{v; |v| \leq j\})) \cap \{f; f \in [0, \frac{1}{\alpha}]\} \), for \( T_1 > 0 \) small enough only depending on \( j \), since the derivative of the map \( F_j \) is bounded on \([0, \frac{1}{\alpha}]\). Let \( f_j \) be its fixed point, i.e. the solution of (3.1) on \([0, T_1]\). The argument can be repeated and the solution can be continued up to \( t = T \). By Duhamel’s form for \( f_j \) (resp. \( 1 - \alpha f_j \)),
\[
f_j(t, x, v) \geq f_{0,j}(x, v)e^{-\int_0^t \tilde{\sigma}_j^2(r, x, v)dr} > 0, \quad t \in [0, T], \; x \in [0, 1], \; |v| \leq j,
\]
(resp.
\[
(1 - \alpha f_j)^2(t, x, v) \geq (1 - \alpha f_{0,j})(x, v)e^{-\int_0^t \tilde{\sigma}_j^2(r, x, v)dr}
\]
\[
\geq \frac{1}{j e^{c\beta T}}, \quad t \in [0, T], \; x \in [0, 1], \; |v| \leq j.
\]

Consequently, for some \( \eta_j > 0 \), there is a periodic in \( x \) solution \( f_j \in C([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\})) \) to (3.1) with values in \([0, \frac{1}{\alpha} - \eta_j]\).

If there were another nonnegative local solution \( \tilde{f}_j \) to (3.1), defined on \([0, T']\) for some \( T' \in [0, T] \), then by the exponential form it would stay below \( \frac{1}{\alpha} \). The difference \( f_j - \tilde{f}_j \) would for some constant \( c_{T'} \) satisfy
\[
\int |(f_j - \tilde{f}_j)^2(t, x, v)|dxdv \leq c_{T'} \int_0^t |(f_j - \tilde{f}_j)^2(s, x, v)|dsdxdv, \; t \in [0, T'], \quad (f_j - \tilde{f}_j)^2(0, x, v) = 0,
\]
implying that the difference would be identically zero on \([0, T']\). Thus \( f_j \) is the unique solution on \([0, T] \) to (3.1), and has its range contained in \([0, \frac{1}{\alpha} - \eta_j]\).

To go from anyon to boson BN we start from a fixed iv \( f_0 \) bdd by \( 2^{-\alpha} \) with \( L \in \mathbb{N} \) (i.e. we shall go through the rest of the paper under this extra hypothesis, already knowing the final existence results from the anyon case for fixed \( \alpha \), in particular that the existence of \( f_0^\alpha \) for all times is known).

We shall prove that there it a time \( T > 0 \) independent of \( j \) and \( 0 < \alpha \ll 2^{-\alpha} \), so that the solutions are bounded by \( 2^{\alpha+1} \) on \([0, T]\). We then prove that the limit of the solutions \( f_0^\alpha \) when \( \alpha \rightarrow 0 \) solves the corresponding bosonic BN problem. The proof is a rehash of the anyon proof from here on. In the present sketch I just insert in red the changes in the old proofs.

Iterating the result from \( T \) on, it follow that the solution \( f_0^\alpha \) exists up to the first time \( T_\infty \) when
\[
\lim_{t \to T_\infty} \sup_{(x,v)} f_\alpha(t) = \infty. \quad \text{The steps below with red comments added, should be perceived in the setting just described. The old anyon paper lemmas are thereby sharpened to prove that the constants only depend on mass, energy and L. We first study the case of a fixed \( j \) and a (small) \( \alpha \). Up to Lemma 3.5 included, the time interval when the solution does not exceed }\ 2L+1, \text{ may be } \alpha\text{-dependent. Then we prove a new lemma, implying that this time interval can be chosen independent of } j \text{ and } \alpha. \\

\text{The remaining part of this section is devoted to obtaining a uniform control with respect to } j \in \mathbb{N} \text{ of} \quad \int \sup_{t \in [0,T], x \in [0,1]} f_\alpha^j(t,x,v) dv.
\]

It relies on the following four lemmas, where the first is an estimate of the Bony functionals,
\[
\bar{B}_j(t) := \int_0^1 \int |v - v_*|^2 B \chi_j f_j \chi_j f_j^* F_j F_j^* f_j' F_j^* F_j^* dvdv d\theta dx, \quad t \geq 0.
\]

**Lemma 3.2**

*For \( T > 0 \) such that \( f(t) \leq 2^{L+1} \) for \( 0 \leq t \leq T \), it holds that
\[
\int_0^T \bar{B}_j(t) dt \leq c'_0(1 + T), \quad j \in \mathbb{N},
\]

with \( c'_0 \) independent of \( j \) and (small) \( \alpha \), and only depending on } \int f_0(x,v) dx dv, \int |v|^2 f_0(x,v) dx dv \text{ and } L. 

**Proof of Lemma 3.2.**

Denote \( f_j \) by \( f \) for simplicity. The proof is an extension of the classical one (cf [4], [5]), together with the control of the filling factor when \( v \in \mathbb{R}^2 \), as follows.

The integral over time of the momentum \( \int v_1 f(t,0,v) dv \) (resp. the momentum flux \( \int v_2 f(t,0,v) dv \)) is first controlled. Let \( \beta \in C^1([0,1]) \) be such that \( \beta(0) = -1 \) and \( \beta(1) = 1 \). Multiply (3.1) by \( \beta(x) \) (resp. \( v_1 \beta(x) \)) and integrate over \([0,t] \times [0,1] \times \mathbb{R}^2 \). It gives
\[
\int_0^t \int v_1 f(\tau,0,v) dv d\tau = \frac{1}{2} \left( \int \beta(x) f_0(x,v) dx dv - \int \beta(x) f(t,x,v) dx dv \right.
\]
\[
\left. + \int_0^t \int \beta'(x)v_1 f(\tau,x,v) dx dv d\tau \right),
\]

(resp.
\[
\int_0^t \int v_2^2 f(\tau,0,v) dv d\tau = \frac{1}{2} \left( \int \beta(x) v_1 f_0(x,v) dx dv - \int \beta(x) v_1 f(t,x,v) dx dv \right.
\]
\[
\left. + \int_0^t \int \beta'(x)v_2 f(\tau,x,v) dx dv d\tau \right). 
\]

Consequently, using the conservation of mass and energy of \( f \),
\[
\int_0^t \int v_1 f(\tau,0,v) dv d\tau + \int_0^t \int v_2^2 f(\tau,0,v) dv d\tau \leq c(1 + t).
\]

\( (3.2) \)
Here $c$ is of magnitude of mass plus energy uniformly in $\alpha$. Let

$$I(t) = \int_{x<y} (v_1 - v_{s1}) f(t, x, v) f(t, y, v_s) dv dy dv v_s.$$  

It results from

$$I'(t) = - \int (v_1 - v_{s1})^2 f(t, x, v) f(t, x, v_s) dx dv v_s + 2 \int v_{s1} (v_1 - v_1) f(t, 0, v_s) f(t, x, v) dx dv v_s,$$

and the conservations of the mass, momentum and energy of $f$ that

$$\int_0^t \int_0^1 (v_1 - v_{s1})^2 f(s, x, v) f(s, x, v_s) dx dv v_s, ds$$

$$\leq 2 \int f_0(x, v) dx dv \int |v_1| f_0(x, v) dv + 2 \int f(t, x, v) dx dv \int |v_1| f(t, x, v) dx dv$$

$$+ 2 \int_0^t \int v_{s1} (v_1 - v_1) f(\tau, 0, v_s) f(\tau, x, v) dx dv v_s, d\tau$$

$$\leq 2 \int f_0(x, v) dx dv \int (1 + |v|^2) f_0(x, v) dv + 2 \int f(t, x, v) dx dv \int (1 + |v|^2) f(t, x, v) dx dv$$

$$+ 2 \int_0^t (\int v_{s1}^2 f(\tau, 0, v_s) dv_s) d\tau \int f_0(x, v) dx dv - 2 \int_0^t (\int v_{s1} f(\tau, 0, v_s) dv_s) d\tau \int v_1 f_0(x, v) dx dv$$

$$\leq c \left( 1 + \int_0^t \int v_{s1}^2 f(\tau, 0, v) dv d\tau + \int_0^t \int v_1 f(\tau, 0, v) dv d\tau \right).$$

And so, by (3.2),

$$\int_0^t \int_0^1 (v_1 - v_{s1})^2 f(\tau, x, v) f(\tau, x, v_s) dx dv v_s, d\tau \leq c(1 + t).$$  \hspace{1cm} (3.3)

Denote by $u_1 = \frac{f v_1 dv}{f dv}$. Recalling (2.1) it holds

$$\int_0^t \int_0^1 (v_1 - u_1)^2 B \chi \int f f_j f_j'(f_j)' F_j(f_j)(s, x, v, v_s, \theta) dv dv v_s, d\theta, dx ds$$

$$\leq c \int_0^t \int_0^1 (v_1 - u_1)^2 f f_j s, x, v, v_s) dv dv v_s, dx ds$$

$$= \frac{c}{2} \int_0^t \int_0^1 (v_1 - v_{s1})^2 f f_j s, x, v, v_s) dv dv v_s, dx ds$$

$$\leq c(1 + t).$$  \hspace{1cm} (3.4)

The new $c$ also contains sup $F_j(f_j')(f_j)' F_j(f_j)$ which is of magnitude bounded by $2^L$ (so $c$ of magnitude $2^L$(mass+energy) and uniformly in $j$ and $\alpha$. Multiply equation (3.1) for $f$ by $v_1^2$, integrate and use that $\int v_1^2 Q_1(f) dv = \int (v_1 - u_1)^2 Q_1(f) dv$ and (3.4). It results

$$\frac{1}{\pi} \int_0^t \int_0^1 (v_1 - u_1)^2 B \chi_j f f_j'(f_j)' F_j(f_j)(s, x, v, v_s, \theta) dv dv v_s, d\theta, dx ds$$

$$= \int v_1^2 f(t, x, v) dx dv - \int v_1^2 f_0(x, v) dx dv + \frac{1}{\pi} \int_0^t \int_0^1 (v_1 - u_1)^2 B \chi_j f f_j'(f_j)' F_j(f_j)' dv dv v_s, d\theta, dx ds$$

$$\leq c_0(1 + t),$$
where $c_0$ is a constant of magnitude $2^L$(mass+energy).

After a change of variables the left hand side can be written

$$
\frac{1}{\pi} \int_{0}^{t} (v_1' - u_1')^2 B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds
$$

$$
= \frac{1}{\pi} \int_{0}^{t} (c_1 - n_1 [(v - v_s) \cdot n])^2 B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds,
$$

where $c_1 = v_1 - u_1$. And so,

$$
\int_{0}^{t} \int n_1^2 [(v - v_s) \cdot n]^2 B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds
$$

$$
\leq \pi c_0 (1 + t) + 2 \int_{0}^{t} \int c_1 n_1 [(v - v_s) \cdot n] B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds.
$$

The term containing $n_1^2 [(v - v_s) \cdot n]^2$ is estimated from below. When $n$ is replaced by an orthogonal (direct) unit vector $n_\perp$, $v'$ and $v'_s$ are shifted and the product $f f_s F_j(f')F_j(f'_s)$ is unchanged. In $\mathbb{R}^2$ the ratio between the sum of the integrand factors $n_1^2 [(v - v_s) \cdot n]^2 + n_\perp^2 [(v - v_s) \cdot n_\perp]^2$ and $|v - v_s|^2$, is, outside of the angular cut-off (2.2), uniformly bounded from below by $\gamma'^2$. Indeed, if $\theta_1$ denotes the angle between $\frac{v - v_s}{|v - v_s|}$ and $n$,

$$
n_1^2 \left[ \frac{v - v_s}{|v - v_s|} \cdot n \right]^2 + n_\perp^2 \left[ \frac{v - v_s}{|v - v_s|} \cdot n_\perp \right]^2 = \cos^2 \theta \cos^2 \theta_1 + \sin^2 \theta \sin^2 \theta_1
$$

$$
\geq \gamma'^2 \cos^2 \theta_1 + \gamma' (2 - \gamma') \sin^2 \theta_1
$$

$$
\geq \gamma'^2, \quad \gamma' < |\cos \theta| < 1 - \gamma', \quad \theta_1 \in [0, 2\pi].
$$

This is where the condition $v \in \mathbb{R}^2$ is used.

That leads to the lower bound

$$
\int_{0}^{t} \int n_1^2 [(v - v_s) \cdot n]^2 B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds
$$

$$
\geq \frac{\gamma'^2}{2} \int_{0}^{t} \int |v - v_s|^2 B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds.
$$

And so,

$$
\gamma'^2 \int_{0}^{t} \int |v - v_s|^2 B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds
$$

$$
\leq 2\pi c_0 (1 + t) + 4 \int_{0}^{t} \int (v_1 - u_1) n_1 [(v - v_s) \cdot n] B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds
$$

$$
\leq 2\pi c_0 (1 + t) + 4 \int_{0}^{t} \int (v_1 (v_2 - v_s) n_1 n_2) B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta ds,
$$

since

$$
\int u_1 (v_1 - v_s) n_1^2 B_{X_2} f f_s F_j(f')F_j(f'_s) dv dv_s d\theta dx
$$

$$
= \int u_1 (v_2 - v_s) n_1 n_2 x B f f_s F (i f') F_j(f'_s) dv dv_s d\theta dx = 0,
$$

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by an exchange of the variables $v$ and $v_*$. Moreover, exchanging first the variables $v$ and $v_*$,

$$
2 \int_0^t \int v_1(v_2 - v_*^2)n_1n_2B\chi_j f_f_*^s F_j(f')F_j(f'_*)dv_*d\theta ds
$$

$$
= \int_0^t \int (v_1 - v_*)_n_1n_2B\chi_j f f_*^s F_j(f')F_j(f'_*)dv_*d\theta ds
$$

$$
\leq \frac{8}{\gamma^2} \int_0^t \int (v_1 - v_*)_n_1^2\gamma^2 2n_1n_2B\chi_j f f_*^s F_j(f')F_j(f'_*)dv_*d\theta ds
$$

$$
+ \frac{\gamma^2}{8} \int_0^t \int (v_2 - v_*^2)^2 n_2^2B\chi_j f f_*^s F_j(f')F_j(f'_*)dv_*d\theta ds
$$

$$
\leq \frac{8c_0}{\gamma^2}(1 + t) + \frac{\gamma^2}{8} \int_0^t \int (v_2 - v_*)^2 n_2^2B\chi_j f f_*^s F_j(f')F_j(f'_*)dv_*d\theta ds.
$$

It follows that

$$
\int_0^t \int |v - v_*|^2B\chi_j f f_*^s F_j(f')F_j(f'_*)dv_*d\theta ds \leq c'_0(1 + t),
$$

with $c'_0$ uniformly in $j$ and $\alpha$ of the same magnitude as $c_0$, only depending on $\int f_0(x,v)dx dv$, $\int |v|^2 f_0(x,v) dx dv$ and $L$. This completes the proof of the lemma.

---

**Lemma 3.3**

*Given $T > 0$ such that $f(t) \leq 2^{L+1}$ for $0 \leq t \leq T$, the solution $f_j$ of (3.1) satisfies*

$$
\int \sup_{t \in [0,T]} f_j^x(t, x, v)dx dv < c_1 + c_2 T, \quad j \in \mathbb{N},
$$

*where $c_1$ and $c_2$ are independent of $T$, $j$ and (small) $\alpha$, and only depend on $\int f_0(x,v) dx dv$, $\int |v|^2 f_0(x,v) dx dv$ and $L$.*

**Proof of Lemma 3.3.**

Denote $f_j$ by $f$ for simplicity. Since

$$
f_j^x(t, x, v) = f_0(x, v) + \int_0^t Q_j(f)(s, x + sv_1, v)ds,
$$

it holds that

$$
\sup_{t \in [0,T]} f_j^x(t, x, v) \leq f_0(x, v) + \int_0^T Q_j^+(f)(t, x + tv_1, v)dt.
$$

(3.5)

Integrating (3.5) with respect to $(x, v)$ and using Lemma 3.2, gives

$$
\int \sup_{0 \leq t \leq T} f_j^x(t, x, v)dx dv \leq \int f_0(x, v) dx dv + \frac{1}{\pi} \int_0^T B\chi_j f(t, x + tv_1, v')f(t + t_1, v')F_j(f)(t, x + tv_1, v)F_j(f)(t, x + tv_1, v_*)dv_*d\theta dt
$$

$$
\leq \int f_0(x, v) dx dv + \frac{1}{\gamma^2} \int_0^T B\chi_j |v - v_*|^2
$$

$$
f(t, x, v')f(t, x, v'_*)F_j(f)(t, x, v)F_j(f)(t, x, v_*|dv_*d\theta dt
$$

$$
\leq \int f_0(x, v) dx dv + \frac{c'_0(1 + T)}{\gamma^2} := \frac{C_1 + C_2 T}{\gamma^2}.
$$

---
Lemma 3.4
Given $T > 0$ such that $f(t) \leq 2^{L+1}$ for $0 \leq t \leq T$, and $\delta_1 > 0$, there exist $\delta_2 > 0$ and $t_0 > 0$ independent of $j$ and $\alpha$ and only depending on $\int f_0(x,v)dx$ and $\int |v|^2 f_0(x,v)dx$ and $L$, such that for $t \leq T$

$$\sup_{x_0 \in [0,1]} \int_{|x-x_0|<\delta_2} \sup_{t \leq s \leq t+t_0} f^\sharp_j(s,x,v)dx dv < \delta_1, \quad j \in \mathbb{N}.$$ 

Proof of Lemma 3.4.
Denote $f$ by $f$ for simplicity. For $s \in [t, t+t_0]$ it holds,

$$f^\sharp(s,x,v) = f^\sharp(t + t_0, x, v) - \int_s^{t+t_0} Q_j^+(f)(\tau, x + \tau v_1, v) d\tau$$

$$\leq f^\sharp(t + t_0, x, v) + \int_s^{t+t_0} Q_j^-(f)(\tau, x + \tau v_1, v) d\tau.$$ 

And so

$$\sup_{t \leq s \leq t+t_0} f^\sharp(s,x,v) \leq f^\sharp(t + t_0, x, v) + \int_t^{t+t_0} Q_j^+(f)(s, x + sv_1, v) ds.$$ 

Integrating with respect to $(x,v)$, using Lemma 3.2 and the bound $\frac{1}{\alpha}$ from above of $f$, gives

$$\int_{|x-x_0|<\delta_2} \sup_{t \leq s \leq t+t_0} f^\sharp(s,x,v) dx dv$$

$$\leq \int_{|x-x_0|<\delta_2} f^\sharp(t + t_0, x, v) dx dv$$

$$+ \frac{1}{\pi} \int_t^{t+t_0} \int B \chi \int f^\sharp(s,x,v)f(s, x + sv_1, v)v \mathcal{F}_j(f)(s, x + sv_1, v)dv dv d\theta dx ds$$

$$\leq \frac{1}{\delta_2} \int_{|v-v_1|<\delta_2} f^\sharp(t + t_0, x, v) dx dv + \frac{1}{\lambda^2} \int_t^{t+t_0} \int_{|v-v_1|<\delta_2} B \chi \int f^\sharp(s,x,v)f(s, x + sv_1, v)dv dv d\theta dx ds$$

$$+ c2^{2L} \int_t^{t+t_0} \int_{|v-v_1|<\delta_2} B \chi \int f^\sharp(s,x,v)f(s, x + sv_1, v)dv dv d\theta dx ds$$

$$\leq \frac{1}{\delta_2} \int_{|v-v_1|<\delta_2} f^\sharp(t + t_0, x, v) dx dv + \frac{c_0(1 + t_0)}{\lambda^2} + c2^{2L} t_0 \lambda^2 \int f_0(x,v) dv$$

Depending on $\delta_1$, suitably choosing $\Lambda$ and then $\delta_2$, $\lambda$ and then $t_0$, the lemma follows. 

The previous lemmas imply for fixed $j$ and $\alpha$ a bound for the $v$-integral of $f^\sharp_j$ only depending on $\int f_0(x,v)dx$, on $\int |v|^2 f_0(x,v)dx$ and on $L$ as will now be proved.
Lemma 3.5
With $T'$ defined as the maximum time for which $f_j^\alpha \leq 2^{L+1}$, take $T = \min\{1, T'\}$. The solution $f_j^\alpha$ of (3.1) satisfies
\[
\int \sup_{(t,x) \in [0,T] \times [0,1]} f_j^\alpha z_j(t,x,v)dv \leq c_1,
\]
where $c_1$ only depends on $\int f_0(x,v)dx dv$, $\int |v|^2 f_0(x,v)dx dv$ and $L$.

Proof of Lemma 3.5.
Denote by $E(x)$ the integer part of $x \in \mathbb{R}$, $E(x) \leq x < E(x) + 1$.
As in the proof of Lemma 3.3,
\[
\sup_{s \leq t} f^\alpha(s,x,v) \leq f_0(x,v) + \int_0^t Q_j^+(f)(s,x + sv_1,v)ds = f_0(x,v)
\]
\[
+ \int_0^t \int B_{X_j} f(s,x + sv_1, v') f(s,x + sv_1, v'_s) F_j(f)(s,x + sv_1,v) F_j(f)(s,x + sv_1,v_s) dv_s dv \theta ds
\]
\[
\leq f_0(x,v) + c 2^{2L} A,
\]  \hspace{1cm} (3.6)
where
\[
A = \int_0^t \int B_{X_j} \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_s), v'_s) dv_s dv \theta ds.
\]
For $\theta$ outside of the angular cutoff (2.2), let $n$ be the unit vector in the direction $v - v'$, and $n_\perp$ the orthogonal unit vector in the direction $v - v'_s$. With $e_1$ a unit vector in the $x$-direction,
\[
\max(|n \cdot e_1|, |n_\perp \cdot e_1|) \geq \frac{1}{\sqrt{2}}.
\]
For $\delta > 0$ that will be fixed later, split $A$ into $A_1 + A_2 + A_3 + A_4$, where
\[
A_1 = \int_0^t \int_{|n \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| \geq \delta} B_{X_j} \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_s), v'_s) dv_s dv \theta ds,
\]
\[
A_2 = \int_0^t \int_{|n \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| \leq \delta} B_{X_j} \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_s), v'_s) dv_s dv \theta ds,
\]
\[
A_3 = \int_0^t \int_{|n_\perp \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| \geq \delta} B_{X_j} \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_s), v'_s) dv_s dv \theta ds,
\]
\[
A_4 = \int_0^t \int_{|n_\perp \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| \leq \delta} B_{X_j} \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_s), v'_s) dv_s dv \theta ds.
\]
In $A_1$ and $A_2$, bound the factor $\sup_{\tau \in [0,t]} f^\#(\tau, x + s(v_1 - v'_s), v'_s)$ by its supremum over $x \in [0,1]$, and make the change of variables
\[
s \to y = x + s(v_1 - v'_1).
\]
with Jacobian
\[
\frac{Ds}{Dy} = \frac{1}{|v_1 - v'_1|} = \frac{1}{|v - v_s|} \frac{1}{(|n_\perp \cdot e_1|) |n_1|} \leq \frac{\sqrt{2}}{\gamma'}.\]
It holds that
\[ A_1 \leq \int_{|v_1 - v'| > \delta_2} \frac{Bx_j}{|v_1 - v'|} \left( \int_{y \in (x,x+t(v_1 - v'))} \sup_{\tau \in [0,t]} f^\#(\tau, y, v') dy \right) \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v'_*) dv_* d\theta, \]
and
\[ A_2 \leq \frac{\sqrt{\gamma'}}{\gamma} \int_{|y - x| < \delta_2} Bx_j \left( \int_{y \in (x,x+t(v_1 - v'))} \sup_{\tau \in [0,t]} f^\#(\tau, y, v') dy \right) \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v'_*) dv_* d\theta. \]

Then, performing the change of variables \((v, v_*, n) \rightarrow (v', v'_*, -n)\),
\[
\int \sup_{x \in [0,1]} A_1 dv \\
\leq \int \sup_{x \in [0,1]} A_1 dv \\
= \int \sup_{x \in [0,1]} \frac{Bx_j}{|v_1 - v'|} \left( \int_{y \in (x,x+t(v_1 - v'))} \sup_{\tau \in [0,t]} f^\#(\tau, y, v) dy \right) \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_* v' dv_* d\theta \\
\leq B_0 \pi t(1 + \frac{1}{\delta_2}) \int \sup_{x \in [0,1]} \int_{\tau \in [0,t]} f^\#(\tau, y, v) dv \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v'_*) dv_* \\
\text{Apply Lemma 3.3, so that}
\int \sup_{x \in [0,1]} A_1 dv \leq B_0 \pi (1 + \frac{1}{\delta_2})(c'_1 + c'_2) \int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*). \tag{3.7}
\]

Moreover, performing the change of variables \((v, v_*, n) \rightarrow (v'_*, v', -n)\),
\[
\int \sup_{x \in [0,1]} A_2 dv \leq \frac{B_0 \pi \sqrt{2}}{\gamma \gamma'} \sup_{x \in [0,1]} \left( \int_{|y - x| < \delta_2} \sup_{\tau \in [0,t]} f^\#(\tau, y, v) dv_* \right) \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v) dv.
\]

Given \(\delta_1 = \frac{\gamma \gamma'}{4 B_0 \pi \sqrt{2}}\), apply Lemma 3.4 with the corresponding \(\delta_2\) and \(t_0\), so that for \(t \leq \min\{T, t_0\}\),
\[
\int \sup_{x \in [0,1]} A_2 dv \leq \frac{1}{4} \int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v) dv. \tag{3.8}
\]

The terms \(A_3\) and \(A_4\) are treated similarly, with the change of variables \(s \rightarrow y = x + s(v_1 - v'_s)\). Using (3.7)-(3.8) and the corresponding bounds obtained for \(A_3\) and \(A_4\) leads to
\[
\int \sup_{(s,x) \in [0,t] \times [0,1]} f^\#(s, x, v) dv \leq 2 \int \sup_{x \in [0,1]} f_0(x, v) dv \\
+ 4 B_0 \pi (1 + \frac{1}{\delta_2})(c'_1 + c'_2) \int \sup_{(s,x) \in [0,t] \times [0,1]} f^\#(s, x, v) dv, \quad t \leq \min\{T, t_0\}. 
\]
Hence
\[
\int \sup_{(s,x)\in [0,T]\times [0,1]} f^\#(s,x,v)dv \leq 4 \int \sup_{x\in [0,1]} f_0(x,v)dv, \quad t \leq \min\{t_0, \frac{\delta_2}{8B_0\pi(\delta_2 + 1)(c_1' + c_2')}\}.
\]

Since \( t_0, c_1' \) and \( c_2' \) only depend on \( \int f_0(x,v)dx dv, \int |v|^2 f_0(x,v)dx dv \) and \( L \), it follows that the argument can be repeated up to \( t = T \) with the number of steps uniformly bounded for all \( j \) and (small) \( \alpha \). This completes the proof of the lemma. \( \blacksquare \)

We shall next prove a new lemma making the previous \( T \) independent of \( j \) and \( \alpha \).

**Remark.**

Lemmas 3.2-3.5 also hold with essentially the same proofs, for strong solutions of (2.5) with locally bounded energy.

The following two preliminary lemmas are needed for the control of large velocities.

**Lemma 3.6**

*Given* \( T > 0 \), the solution \( f_j \) of (3.1) satisfies
\[
\int_0^1 \int_\{|v|>\lambda\} |v| \sup_{t \in [0,T]} f^\#_j(t,x,v)dvdx \leq \frac{c_T}{\lambda}, \quad j \in \mathbb{N},
\]

*where* \( c_T \) *only depends on* \( T \), \( \int f_0(x,v)dx dv \) *and* \( \int |v|^2 f_0(x,v)dx dv \).

*Proof of Lemma 3.6.*

For convenience \( j \) is dropped from the notation \( f_j \). As in (3.5),
\[
\sup_{t \in [0,T]} f^\#(t,x,v) \leq f_0(x,v) + \int_0^T Q_j^+(f)(s,x+sv_1,v)ds.
\]

Integration with respect to \((x,v)\) for \(|v|>\lambda\), gives
\[
\int_0^1 \int_{|v|>\lambda} |v| \sup_{t \in [0,T]} f^\#(t,x,v)dvdx \leq \int_{|v|>\lambda} |v|f_0(x,v)dvdx + \frac{1}{\pi} \int_0^T \int_{|v|>\lambda} B_{\lambda j} |v|f(s,x+sv_1,v')f(s,x+sv_1,v'^*F(f)(s,x+sv_1,v)F(f)(s,x+sv_1,v'^*dvdv_d\theta dx ds.
\]

Here in the last integral, either \(|v'|\) *or* \(|v'^*|\) *is the largest and larger than* \( \lambda \). The two cases are symmetric, and we discuss the case \(|v'| \geq |v'^*|\). After a translation in *x*, the integrand is estimated from above by
\[
c|v'|f^\#(s,x,v') \sup_{(t,x)\in [0,T]\times [0,1]} f^\#(t,x,v'^*).
\]

The change of variables \((v,v_*,n) \rightarrow (v'^*, -n)\), the integration over
\[(s,x,v,v_*,\omega) \in [0,T] \times [0,1] \times \{v \in \mathbb{R}^2; |v| > \lambda \} \times \mathbb{R}^2 \times [-\pi/2, \pi/2],\]

and Lemma 3.5 give the bound
\[
\frac{c}{\lambda} \left( \int_0^T \int |v|^2 f^\#(s,x,v)dx dv ds \right) \left( \int \sup_{(t,x)\in [0,T]\times [0,1]} f^\#(t,x,v_*)dv_* \right) \leq \frac{cTc_1(T)}{\lambda} \int |v|^2 f_0(x,v)dx dv.
\]

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The lemma follows.

**Lemma 3.7**

*Given* $T > 0$, the solution $f_j$ of (3.1) satisfies

$$
\int_{|v| > \lambda} \sup_{(t,x) \in [0,T] \times [0,1]} f_j^2(t,x,v) dv \leq \frac{c'_T}{\sqrt{\lambda}}, \quad j \in \mathbb{N},
$$

where $c'_T$ only depends on $T$, $\int f_0(x,v) dv$ and $\int |v|^2 f_0(x,v) dv$.

**Proof of Lemma 3.7.**

Take $\lambda > 2$. As above,

$$\int_{|v| > \lambda} \sup_{(t,x) \in [0,T] \times [0,1]} f_j^2(t,x,v) dv \leq \int_{|v| > \lambda} \sup_{x \in [0,1]} f_0(x,v) dv + cC, \quad (3.9)$$

where

$$C = \int_{|v| > \lambda} \sup_{x \in [0,1]} \int_{0}^{T} B \chi_j f^\#(s, x + s(v_1 - v_1'), v') f^\#(s, x + s(v_1 - v_1'), v_1') dv d\theta ds.$$  

For $v', v_1'$ outside of the angular cutoff (2.2), let $n$ be the unit vector in the direction $v - v'$, and $n_\perp$ the orthogonal unit vector in the direction $v - v_1'$. Let $e_1$ be a unit vector in the $x$-direction.

Split $C$ as $C = \sum_{4 \leq i \leq 6} C_i$, where $C_1$ (resp. $C_2$, $C_3$) refers to integration with respect to $(v_1, \theta)$ on

$$\{(v_1, \theta); \quad n \cdot e_1 \geq \frac{1}{\sqrt{2}}, \quad |v'| \geq |v_1'|\},$$

(resp. $\{(v_1, \theta); n \cdot e_1 \geq \sqrt{1 - \frac{1}{\lambda}}, |v'| \leq |v_1'|\}$, $\{(v_1, \theta); n \cdot e_1 \in \left[\frac{1}{\sqrt{2}}, \sqrt{1 - \frac{1}{\lambda}}\right], |v'| \leq |v_1'|\}$),

and analogously for $C_i$, $4 \leq i \leq 6$, with $n$ replaced by $n_\perp$. By symmetry, $C_i$, $4 \leq i \leq 6$ can be treated as $C_i$, $1 \leq i \leq 3$, so we only discuss the control of $C_i$, $1 \leq i \leq 3$.

By the change of variables $(v, v_1, n) \to (v', v_1', -n)$, and noticing that $|v'| \geq \frac{\lambda}{\sqrt{2}}$ in the domain of integration of $C_1$, it holds that

$$C_1 \leq \int_{|v| > \frac{1}{\sqrt{2}}} \sup_{x \in [0,1]} \int_{0}^{T} B \chi_j f^\#(s, x + s(v_1 - v_1'), v) f^\#(s, x + s(v_1 - v_1'), v_1') dv d\theta ds dv$$

$$\leq \int_{|v| > \frac{1}{\sqrt{2}}} \sup_{x \in [0,1]} \int_{0}^{T} B \chi_j \sup_{\tau \in [0,T]} f^\#(\tau, x + s(v_1 - v_1'), v) \sup_{(\tau,X) \in [0,T] \times [0,1]} f^\#(\tau, X, v_1') dv d\theta ds dv.$$  

With the change of variables $s \to y = x + s(v_1' - v_1)$,

$$C_1 \leq \int_{|v| > \frac{1}{\sqrt{2}}} \sup_{x \in [0,1]} \int_{n \cdot e_1 \geq \frac{1}{\sqrt{2}}} \int_{y \in (x, x + T(v_1' - v_1))} B \chi_j \frac{f^\#(\tau, y, v)}{|v_1' - v_1|} \sup_{\tau \in [0,T]} f^\#(\tau, X, v_1') \sup_{(\tau,X) \in [0,T] \times [0,1]} f^\#(\tau, X, v_1') dy dv d\theta ds dv$$

$$\leq \int_{|v| > \frac{1}{\sqrt{2}}} \int_{n \cdot e_1 \geq \frac{1}{\sqrt{2}}} \frac{|E(T(v_1' - v_1)) + 1|}{|v_1' - v_1|} \int_{0}^{T} B \chi_j \sup_{\tau \in [0,T]} f^\#(\tau, y, v) \sup_{(\tau,X) \in [0,T] \times [0,1]} f^\#(\tau, X, v_1') dy dv d\theta ds dv.$$  

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Moreover,

\[ |E(T(v'_1 - v_1)) + 1)| \leq T|v'_1 - v_1| + 1 \leq (T + \frac{\sqrt{2}}{\gamma^2})|v'_1 - v_1|, \]

where \( \gamma \) and \( \gamma' \) were defined in (2.2). Consequently,

\[
C_1 \leq c(T + 1) \int_0^1 \int_{|v| > \frac{\sqrt{2}}{\lambda}} \sup_{\tau \in [0,T]} f^\#(\tau, y, v) dy dv \int_{(\tau, X) \in [0,T] \times [0,1]} \sup_{(\tau, v_*) \in [0,T]} f^\#(\tau, X, v_*) dv_* \\
\leq \frac{c(T + 1)}{\lambda} \int_0^1 \int_{|v| > \frac{\sqrt{2}}{\lambda}} |v| \sup_{\tau \in [0,T]} f^\#(\tau, y, v) dy dv \int_{(\tau, X) \in [0,T] \times [0,1]} \sup_{(\tau, v_*) \in [0,T]} f^\#(\tau, X, v_*) dv_* \\
\leq \frac{c}{\sqrt{\lambda}} (T + 1)^2 c_1(T),
\]

By Lemma 3.5 and Lemma 3.6,

\[ C_3 \leq \frac{c}{\sqrt{\lambda}} (T + 1)^2 c_1(T). \]

The lemma follows.
4 Proof of the main theorem.

This section is devoted to the proof of Theorem 2.1. It consists in four steps. In the first step, we prove the existence of an initial layer \([0, t_m]\), with \(t_m\) independent on \(j\), where \(f_j^2(t, \cdot, \cdot) < \frac{1}{\alpha} - b_1 t\).

In a second step, we prove the existence of a solution \(f\) to (2.5). In the third step, we prove its uniqueness and the stability result stated in Theorem 2.1. Finally, the fourth step proves the conservations of mass, momentum and energy of the solution.

First step: analysis of an initial layer.

Denote by
\[
\tilde{v}_j(f) := \frac{1}{\pi} \int B \chi_j f' f_j (f_*) dv_* d\theta, \quad v_j(f) := \frac{1}{\pi} \int B \chi_j f_* F_j(f') dv_* d\theta,
\]
so that
\[
Q_j(f) = F_j(f) \tilde{v}_j(f) - f v_j(f).
\]

Consider
\[
v_j(f_j)^2(t, x, v) = \frac{1}{\pi} \int B \chi_j f_j(t, x + tv_1, v_*) F_j(f_j(t, x + tv_1, v')) F_j(f_j(t, x + tv_1, v_*')) dv_* d\theta.
\]

With the angular cut-off (2.2), \(v_* \to v'\) and \(v_* \to v'_*\) are changes of variables. Indeed, if the polar coordinates of \(v_* - v\) are \((r_*, \varphi)\) and \(\theta\) is the angle between \(v_* - v\) and \(n\), then the polar coordinates of \(v' - v\) (resp. \(v_*' - v\)) are \((|r_* \cos \theta|, \varphi + \theta)\) (resp. \((|r_* \sin \theta|, \varphi + \theta + \frac{\pi}{2})\)). It follows from the angular cut-off (2.2), that the Jacobians \(\frac{D v_*}{D v'} = \frac{1}{|\cos \theta|}\) (resp. \(\frac{D v'_*}{D v'} = \frac{1}{|\sin \theta|}\)) are bounded. Using these changes of variables and Lemma 3.5, for \(\omega\) outside the integration cut-off, the measure of the set
\[
Z_{(j, t, x, v, \omega)} := \{v_*; f(t, x + tv_1, v_*) > \frac{1}{2} \text{ or } f(t, x + tv_1, v'_*) > \frac{1}{2}\} \tag{4.1}
\]
is uniformly bounded with respect to \((x, v, \omega), t \leq T,\) and \(j \in \mathbb{N}\). Take \(j_T\) so large that \(\pi j_T^2\) is at least eight times this uniform bound. Notice that here \(j_T\) only depends on \(T\) and \(\int (1 + |v|^2) f_0(x, v) dxdv\).

Using Duhamel’s form for the solution, one gets using (2.3) and Lemma 3.5 that
\[
f_j^2(t, x, v_*) \geq c_{1T} f_0(x, v_*) > 0, \quad j \geq j_T, \quad t \leq T, \tag{4.2}
\]
with \(c_{1T}\) independent of \(j \geq j_T\). It follows from (4.2) and the third assumption in (2.4) that
\[
v_j(f_j)^4(t, x, v) > c_{2T} > 0, \quad (t, x, v) \in [0, T] \times [0, 1] \times \{v \in \mathbb{R}^2; |v| \leq j\}, \tag{4.3}
\]
uniformly with respect to \(j \geq j_T\), and with \(c_{2T}\) only depending on \(T\) and \(f_0\). Using again the \(v_* \to v'\) change of variables together with Lemma 3.5, one obtains that for some constant \(c_{3T} > 0\),
\[
\tilde{v}_j^2(f_j)(t, x, v) \leq c_{3T}, \quad j \geq j_T, \quad (t, x, v) \in [0, T] \times [0, 1] \times \{v \in \mathbb{R}^2; |v| \leq j\}. \tag{4.4}
\]
The functions defined on \([0, \frac{1}{\alpha}]\) by \(x \to \frac{F_j(x)}{x}\) are uniformly bounded from above with respect to \(j\) by
\[
x \to c_{\alpha}^{\alpha-1} \left(1 - \alpha x \right)^{\alpha},
\]
that is continuous and decreasing to zero at \(x = \frac{1}{\alpha}\). Hence there is \(\mu \in [0, \frac{1}{\alpha}]\) such that

\[
x \in \left[\frac{1}{\alpha} - \mu, \frac{1}{\alpha}\right] \implies \frac{F_j(x)}{x} \leq \frac{c_2 T}{4c_3 T}, \quad j \geq j_T.
\]

Consequently, for \(j \geq j_T\),

\[
f^j_b(t, x, v) \in \left[\frac{1}{\alpha} - \mu, \frac{1}{\alpha}\right] \implies D_t f^j_b(t, x, v) = (F_j(f^j_b) - \frac{1}{2} f^j_b v_j^2) (t, x, v) - \frac{1}{2} f^j_b v_j^2 (t, x, v)
\leq \frac{1}{2} f^j_b v_j^2 (t, x, v)
\leq -\frac{1}{2} \left(\frac{1}{\alpha} - \mu\right)c_2 T := -b_1.
\]

This gives a maximum time \(t_1 = \frac{\mu}{b_1}\) for \(f^j_b\) to reach \(\frac{1}{\alpha} - \mu\) from an initial value \(f_0(x, v) \in \left[\frac{1}{\alpha} - \mu, \frac{1}{\alpha}\right]\). On this time interval \(D_t f^j_b \leq -b_1\). If \(t_1 \geq T\), then at \(t = T\) the value of \(f^j_b\) is bounded from above by \(\frac{1}{\alpha} - b_1 T := \frac{1}{\alpha} - \mu'\), with \(0 < \mu' \leq \mu\). Take \(t_m = \min(t_1, T)\), and from now on \(\mu = t_m b_1\). For any \((x, v),\) if \(f_j(0, x, v) < \frac{1}{\alpha} - \mu\) were to reach \(\frac{1}{\alpha} - \mu\) at \((t, x, v)\) with \(t \leq t_m\), then \(D_t f^j_b(t, x, v) \leq -b_1\), which excludes such a possibility. It follows that \(f_j \leq \frac{1}{\alpha} - \mu\) everywhere for \(t \in [t_m, T]\), and that

\[
f^j_b(t, x, v) \leq \frac{1}{\alpha} - b_1 t, \quad \text{for } t \in [0, t_m].
\]

The previous estimates leading to the definition of \(t_m\) are independent of \(j \geq j_T\).

**Second step: existence of a solution \(f\) to (2.5).**

Using the initial layer and the results in Section 3, we shall prove for any \(T > 0\) the convergence in \(C([0, T]; L^1([0, T] \times \mathbb{R}^2))\) of the sequence \((f_j)\) to a solution \(f\) of (2.5).

Let us prove that \((f_j)\) is a Cauchy sequence in \(L^1([0, T] \times [0, 1] \times \mathbb{R}^2)\) when \(j \to \infty\).

We shall prove that given \(\beta > 0\), there exists \(b \geq \max\{1, j_T\}\), such that

\[
\sup_{t \in [0, T]} \int |g_j(t, x, v)| dx dv < \beta, \quad j > b,
\]

where \(g_j = f_j - f_b\). The function \(g_j\) satisfies the equation

\[
\partial_t g_j + v_1 \partial_x g_j = \frac{1}{\pi} \int (\chi_j - \chi_b) B \left( f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* \right) dv \, d\theta
\]
\[
+ \frac{1}{\pi} \int \chi_b B \left( f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* \right) dv \, d\theta
\]
\[
- \frac{1}{\pi} \int \chi_b B \left( f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* \right) dv \, d\theta
\]
\[
+ \frac{1}{\pi} \int \chi_b B \left( f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* \right) dv \, d\theta
\]
\[
- \frac{1}{\pi} \int \chi_b B \left( f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* \right) dv \, d\theta
\]
\[
- \frac{1}{\pi} \int \chi_b B \left( f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* f_j^b f_j^* \right) dv \, d\theta
\].
Moreover, using Lemma 3.5

\[ \int (\chi_j - \chi_b) B \left( f_j^x f_j^y f_j(f_j) F_j(f_j) + f_j f_j^x F_j(f_j^x) F_j(f_j^x) \right) dx dv d\theta \]

\[ \leq c \int \left| f_j(t, x, v) \right| dx dv \]

\[ \leq \frac{c}{b^\alpha}, \text{ by the conservation of energy of } f_j, \]

\[ \int \chi_B |f_j f_j^x - f_b f_b^x| F_j(f_j^x) dxdvdv, d\omega \]

\[ \leq c \left( \int_{(t,x)\in[0,T] \times [0,1]} f_j^x(t, x, v) dv + \int_{(t,x)\in[0,T] \times [0,1]} f_b^x(t, x, v) dv \right) \int \left| (f_j^x - f_b^x)(t, x, v) \right| dx dv \]

\[ \leq c \int \left| (f_j^x - f_b^x)(t, x, v) \right| dx dv. \]

Next,

\[ \int \chi_B \left( f_j f_j^x F_j(f_j^x) |F_j(f_b) - F_b(f_b)| \right)^2 dxdvdv, d\theta \]

\[ = \int \chi_B f_j f_j^x F_j(f_j^x)(1 - \alpha f_b)(1 + 1 - \alpha f_b)^{\alpha-1} \left( \frac{1}{b} + 1 - \alpha f_b \right)^{\alpha-1} dxdvdv, d\theta. \]

By Lemma 3.3 and Lemma 3.5, this integral restricted to the set where \( 1 - \alpha f_b(t, x, v) \leq \frac{2}{b} \), hence where

\[ (1 - \alpha f_b)\left( \frac{1}{j} + 1 - \alpha f_b \right)^{\alpha-1} - \left( \frac{1}{b} + 1 - \alpha f_b \right)^{\alpha-1} \leq \frac{2^{\alpha+1}}{b^\alpha}, \]

is bounded by \( \frac{c}{b^\alpha} \) for some constant \( c > 0 \).

For the remaining domain of integration where \( 1 - \alpha f_b(t, x, v) \geq \frac{2}{b} \), it holds

\[ |F_j(f_b) - F_b(f_b)| \leq c(1 - \alpha f_b)^\alpha \left( \frac{1}{j} - (1 - \alpha f_b) \right)^{\alpha-1} - \left( \frac{1}{b} - (1 - \alpha f_b) \right)^{\alpha-1} \]

\[ = c \left( \frac{1}{j} - \frac{1}{b} \right)(1 - \alpha f_b)^{\alpha-1} \lambda^{\alpha-2} \text{ where } \lambda \in [1, \frac{3}{2}], \]

\[ \leq \frac{2^{\alpha-1}c}{b^\alpha}. \]

And so,

\[ \int \chi_B \left( f_j f_j^x F_j(f_j^x) |F_j(f_b) - F_b(f_b)| \right)^2 dxdvdv, d\theta \leq \frac{c}{b^\alpha}. \]

Finally

\[ \int \chi_B \left( f_j f_j^x F_j(f_j^x) |F_j(f_b)| \right)^2(t, x, v) dxdvdv, d\theta \leq c \int |F_j(f_j)|^2(t, x, v) dxdv. \]

Split the \((x, v)\)-domain of integration of the latest integral into

\[ D_1 := \{(x, v); (f_j^x(t, x, v), f_b^x(t, x, v)) \in [0, \frac{1}{\alpha} - \mu]^2\}, \]

\[ D_2 := \{(x, v); (f_j^x(t, x, v), f_b^x(t, x, v)) \in [\frac{1}{\alpha} - \mu, \frac{1}{\alpha}]^2\}, \]

\[ D_3 := \{(x, v); (f_j^x, f_j^y)(t, x, v) \in [\frac{1}{\alpha} - \mu, \frac{1}{\alpha}] \times [0, \frac{1}{\alpha} - \mu] \text{ or } (f_j^x, f_b^x)(t, x, v) \in [0, \frac{1}{\alpha} - \mu] \times [\frac{1}{\alpha} - \mu, \frac{1}{\alpha}]\}. \]
It holds that
\[
\int_{D_1} |F_j(f_j) - F_j(f_0)|^2(t, x, v)dx dv \leq c(\alpha \mu)^{\alpha - 1} \int_{D_1} |g_j^2(t, x, v)| dx dv,
\]
\[
\int_{D_2} |F_j(f_j) - F_j(f_0)|^2(t, x, v)dx dv \leq c t^{\alpha - 1} \int_{D_2} |g_j^2(t, x, v)| dx dv, \quad \text{by (4.5)},
\]
\[
\int_{D_3} |F_j(f_j) - F_j(f_0)|^2(t, x, v)dx dv \leq c((\alpha \mu)^{\alpha - 1} + t^{\alpha - 1}) \int_{D_3} |g_j^2(t, x, v)| dx dv.
\]

The remaining terms to the right in (4.8) are of the same types as the ones just estimated. Consequently,
\[
\frac{d}{dt} \int |g_j^2(t, x, v)| dx dv \leq \frac{c}{b^2} + c(1 + t^{\alpha - 1}) \left( \int |g_j^2(t, x, v)| dx dv \right), \quad (4.9)
\]

And so,
\[
\int |g_j^2(t, x, v)| dx dv \leq \left( \int_{|v| > b} f_0(x, v) dx dv + \frac{c T}{b^2} \right) e^{c(T + \frac{c}{\alpha})},
\]

which tends to zero when \( b \to +\infty \), uniformly w.r.t. \( j \geq b \). This proves that \((f_j)_{j \in \mathbb{N}}\) is a Cauchy sequence in \( L^1([0, T] \times [0, 1] \times \mathbb{R}^2) \) and ends the proof of the existence of a solution \( f \) to (2.5).

Third step: uniqueness of the solution to (2.5) and stability results.

The previous line of arguments can be followed to obtain that the solution is unique. Namely, assuming the existence of two solutions \( f_1 \) and \( f_2 \) to (2.5) with locally bounded energy, (4.5) holds for both solutions. The difference \( g = f_1 - f_2 \) satisfies
\[
\partial_t g + v_1 \partial_x g = \frac{1}{\pi} \int B(f_1' f_1^* - f_2' f_2^*) F(f_1) F(f_1^*) dv_* d\theta - \frac{1}{\pi} \int B(f_1 f_1^* - f_2 f_2^*) F(f_1') F(f_1^*) dv_* d\theta \\
+ \frac{1}{\pi} \int B f_1 f_2' \left( F(f_1')(F(f_1) - F(f_2)) + F(f_2)(F(f_1^*) - F(f_2^*)) \right) dv_* d\theta \\
- \frac{1}{\pi} \int B f_2 f_2' \left( F(f_1)(F(f_1') - F(f_2')) + F(f_2')(F(f_1^*) - F(f_2^*)) \right) dv_* d\theta.
\]

The first line in the r.h.s. of the former equation gives rise to \( c \int |f^2(t, x, v)| dx dv \) in the bound from above of \( \frac{d}{dt} |g^2(t, x, v)| dx dv \), whereas the two last lines in the r.h.s. of the former equation give rise to the bound \( c(1 + t^{\alpha - 1}) \int |g^2(t, x, v)| dx dv \). Consequently,
\[
\frac{d}{dt} \int |g^2(t, x, v)| dx dv \leq c(1 + t^{\alpha - 1}) \int |g^2(t, x, v)| dx dv.
\]

This implies that \( \int |g^2(t, x, v)| dx dv \) is identically zero, since it is zero initially.

The proof of stability is similar.

Fourth step: conservations of mass, momentum and energy.

The conservation of mass and first momentum of \( f \) follows from the boundedness of the total energy.
The energy is non-increasing by the construction of $f$. Energy conservation will follow if the energy is non-decreasing. Taking $\psi = \frac{|v|^2}{1+|v|^2}$ as approximation for $|v|^2$, it is enough to bound

$$\int Q(f, f)(t, x, v)\psi(v)dxdv = \frac{1}{\pi} \int B\psi\left(f'_s F(f) F(f_s) - f f_s F(f'_s) F(f_s)\right)dxdvdv_s d\theta$$

from below by zero in the limit $\epsilon \to 0$. Similarly to [13],

$$\int Q(f, f)\psi dxdv = \frac{1}{2\pi} \int B f f_s F(f'_s) F(f'_s)\left(\psi(v') + \psi(v'_s) - \psi(v) - \psi(v_s)\right)dxdvdv_s d\theta \geq - \frac{1}{\pi} \int B f f_s F(f'_s) F(f'_s) \frac{\epsilon|v|^2|v_s|^2}{(1+\epsilon|v|^2)(1+\epsilon|v_s|^2)}dxdvdv_s d\theta.$$

The previous line, with the integral taken over a bounded set in $(v, v_s)$, converges to zero when $\epsilon \to 0$. In integrating over $|v|^2 + |v_s|^2 \geq 2\lambda^2$, there is symmetry between the subset of the domain with $|v|^2 > \lambda^2$ and the one with $|v_s|^2 > \lambda^2$. We discuss the first sub-domain, for which the integral in the last line is bounded from below by

$$-c \int |v_s|^2 f(t, x, v_s)dxdv_s \int_{|v| \geq \lambda} B \sup_{(s, x) \in [0, t] \times [0, 1]} f^\#(s, x, v)dvd\theta \geq -c \int_{|v| \geq \lambda} \sup_{0 \leq (s, x) \in [0, t] \times [0, 1]} f^\#(s, x, v)dv.$$

It follows from Lemma 3.7 that the right hand side tends to zero when $\lambda \to \infty$. This implies that the energy is non-decreasing, and bounded from below by its initial value. That completes the proof of the theorem. 

References


