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Je rêve d'un jour où l'égoïsme ne régnera plus dans les sciences, où on s'associera pour étudier, au lieu d'envoyer aux académiciens des plis cachetés, on s'empressera de publier ses moindres observations pour peu qu'elles soient nouvelles, et on ajoutera «je ne sais pas le reste».
Évariste Galois (1811-1832)

Abstract

In this short report, we address the problem of constructing a right circular cylinder from a given set of five 3D points. The idea is to be able to construct a cylinder in a similar way as one can construct a plane from three points, or a sphere from four points. This would be particularly useful for cylinder robust fitting and cylinder extraction. However, this leads to a much more complex situation than for the plane or the sphere, since the equations involved are nonlinear with respect to the parameters. Our approach is to simplify the initial system of equations in order to get a more tractable computational problem. The system arrived at in this paper consists of three polynomial equations in three unknowns, of degree (2, 2, 3), which is simpler than the system found in related works. This system has been tested numerically using an interval analysis software.

1 Introduction

Geometric primitive extraction and robust fitting methods have found new applications with 3D data generated from laser range scanners. A popular method in geometric primitive extraction and robust fitting is to use minimal subsets and Monte-Carlo methods [4, 5] [1]. Suppose we want to extract a plane in a set of 3D points. We know that a nonaligned set of 3 points defines a unique plane. In other words, the minimal subset for the plane consists of 3 noncollinear points. Since each nondegenerate subset of 3 points define a plane, one can generate many local surface hypotheses merely by taking triplets at random in the point cloud. The idea is that if we take many samples and manage to retain the *best* plane candidate, then we can extract it from a cloud containing outliers and other surfaces. Typically, the general framework described in [4] is to evaluate the surface instance as the *best* according to a cost (or merit) function. For the case of the plane, this type of framework lead to efficient extraction methods in practise [4] (see also the plane extraction applied on the Gaussian image in [2]). In our applications, when dealing with digitized industrial environments, many pipes are present, and hence not only planes but also cylinders need to be extracted.

A plane is defined using three unconstrained parameters: two angles for the unit normal vector, and the signed distance of the origin to the plane. Consequently, at least three points are required to define a plane. Moreover, three points are enough: one may define a plane using a nondegenerate set of three points lying on it. In this case, a nondegenerate set simply corresponds to a set of three noncollinear points. More precisely, the equation of the plane through points X_1, X_2, X_3 writes:

$$\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x & y & z \end{vmatrix} = 0$$

With a nondegenerate set, developping this determinant using the last row gives the equation coefficients and hence the parameters of the plane.

A right circular cylinder may be defined by five free parameters. Usually, a cylinder is rather defined using intuitive parameters such as: a point P on the axis, a unit direction vector u and a radius r (see figure 1). However, without loss of generality, the point P may be

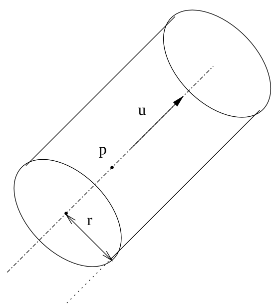

Figure 1: Intuitive parameters for defining a right circular cylinder

constrained to be the orthogonal projection of the origin onto the axis. P may then be defined using only two coordinates on the plane defined perpendicular to u going through the origin.

Moreover, the direction vector, being unitary, may be defined by two angles only. Thus, this amounts to five free parameters.

This means that at least five equations are needed. Thus, a set of at least five points is required. Further, we will see that a nondegenerate set of five points is enough to define a cylinder. The exact meaning of 'nondegenerate' is not yet clear, but will be made more precise later on.

Let us first note that there is no algebraic expression — i.e. that may be written as a polynomial function being equal to zero —, that specifically describes the class of the cylinders: the smallest class of algebraic surfaces containing the cylinders is the class of the quadrics.

One could obviously look for the quadric defined by nine points, then check whether this one is a cylinder or not. Indeed, any implicit surface whose equation writes

$$a^T q(\mathbf{x}) = 0$$

with $q(\mathbf{x}) = (q_1(x,y,z), ..., q_r(x,y,z))$ and $a \in \mathbb{R}^r$ can be defined using a nondegenerate set of $r-1$ points. If these $r-1$ points belong to the surface, then the equation of the surface can be written as follows:

$$\begin{vmatrix} q_1(\mathbf{x_1}) & \dots & q_r(\mathbf{x_1}) \\ \vdots & & \vdots \\ q_1(\mathbf{x_{r-1}}) & \dots & q_r(\mathbf{x_{r-1}}) \\ q_1(\mathbf{x}) & \dots & q_r(\mathbf{x}) \end{vmatrix} = 0$$

The parameter vector a can then be derived by developing this determinant using the last row. The plane and the sphere are particular cases of this framework. For the case of the quadric, nine points are needed, and $q = (1, x, y, z, x^2, xy, xz, y^2, yz, z^2)$.

The issue is that a cylinder is a singular case of a quadric. With a set of 9 theoretically perfect points lying on a cylinder, and ignoring the numerical computation issues, then the quadric obtained would be a cylinder. However, in practise, a quadric is very unlikely to be a cylinder, all the more in the context of real world noisy points.

Therefore, we aim at dealing with the case of the cylinder specifically. Although straightforward to state, the problem of constructing the cylinder(s) defined by five points in $\mathbb{R}^3$ is complex and seems to have received interest only very recently [6, 3]. As we will see, this problem does not in general lead to a closed-form solution and thus requires a numerical resolution.

Some attempts have been made to handle the system obtained in the case of the cylinder using elimination theory [4]. However, this solution encountered numerical difficulties.

Let us also note that the case of the cylinder may be dealt with in a simpler way, introducing the normal estimates in [2]. Some noise is added by introducing new estimates even though the influence of the noise is lowered by treating the points and the normals separatly.

In this report, we propose to address the problem by working with the mere points, without introducing normal estimates. The initial system of equations is processed symbolically. This leads to a simpler system with three polynomial equations in three unknowns. The system obtained here has a smaller degree that the one obtained in [3], and hence will be simpler to deal with numerically. This final system has been tested by numerical solving on sample cases, using interval analysis methods. Experiments for applying this method to a cylinder extraction algorithm have also been carried out.

2 Expressing the system

Let us consider a cylinder defined by a directional unit vector u, a point P on the axis and a radius r. A point X belongs to this cylinder if its distance to the axis $P + \mathbb{R}u$ equals the radius r, that is if:

$$d(X, \text{Axis})^2 = r^2$$

Expliciting the distance of point X to the axis $P + \mathbb{R}u$, one gets:

$$(X - P)^2 - ((X - P)^T u)^2 = r^2$$

where $a^T b$ denotes the scalar product of a and b in $\mathbb{R}^3$ and $a^2 = a^T a$. Let us consider five points X_i on the cylinder. The following conditions hold:

$$\begin{cases} d(X_i, \text{Axis}) = r (1 \leq i \leq 5) \\ P : \text{projection of the origin 0 on the axis} \end{cases}$$

which may be written:

$$\begin{cases} (X_i - P)^2 - ((X_i - P)^T u)^2 = r^2 (1 \leq i \leq 5) \\ P^T u = 0 \\ u^T u = 1 \end{cases}$$

$$\begin{aligned} (X_1 - P)^2 - ((X_1 - P)^T u)^2 &= r^2 \\ (X_2 - P)^2 - ((X_2 - P)^T u)^2 &= r^2 \\ (X_3 - P)^2 - ((X_3 - P)^T u)^2 &= r^2 \\ (X_4 - P)^2 - ((X_4 - P)^T u)^2 &= r^2 \\ (X_5 - P)^2 - ((X_5 - P)^T u)^2 &= r^2 \\ P^T u &= 0 \\ u^T u &= 1 \end{aligned}$$

This represents a system of seven polynomial equations of total degree 4, 2 in P, 2 in u, 2 in r, with seven unknowns.

3 Simplifying this system

Elimination of r and P First, the radius r may be eliminated:

$$r = \sqrt{(X_1 - P)^2 - ((X_1 - P)^T u)^2}$$

and

$$\begin{cases} (X_2 - P)^2 - (X_1 - P)^2 - ((X_2 - P)^T u)^2 + ((X_1 - P)^T u)^2 = 0 \\ (X_3 - P)^2 - (X_1 - P)^2 - ((X_3 - P)^T u)^2 + ((X_1 - P)^T u)^2 = 0 \\ (X_4 - P)^2 - (X_1 - P)^2 - ((X_4 - P)^T u)^2 + ((X_1 - P)^T u)^2 = 0 \\ (X_5 - P)^2 - (X_1 - P)^2 - ((X_5 - P)^T u)^2 + ((X_1 - P)^T u)^2 = 0 \\ P^T u = 0 \\ u^T u - 1 = 0 \end{cases} \tag{1}$$

The equation $(X_i - P)^2 - (X_1 - P)^2 - ((X_i - P)^T u)^2 + ((X_1 - P)^T u)^2 = 0$ may be written as

$$(X_i - P)^T (X_i - P) - (X_1 - P)^T (X_1 - P) - ((X_i - P)^T u)^2 + ((X_1 - P)^T u)^2 = 0$$

that is (since $a^T a - b^T b = (a - b)^T (a + b)$):

$$(X_i - X_1)^T (X_i + X_1 - 2P) - ((X_i - X_1)^T u)((X_i + X_1 - 2P)^T u) = 0$$

so

$$(X_i - X_1)^T (X_i + X_1 - 2P) - ((X_i - X_1)^T u)(u^T (X_i + X_1 - 2P)) = 0$$

i.e.

$$(X_i - X_1)^T (I - uu^T)(X_i + X_1 - 2P) = 0$$

which writes

$$(X_i - X_1)^T (I - uu^T)(X_i + X_1) - 2(X_i - X_1)^T (I - uu^T) P = 0 \tag{2}$$

Now, given that $Pu^T = 0$, there is $(I - uu^T)P = P - u(u^T P) = P$. Under this constraint, the former equation 2 thus gives :

$$(X_i - X_1)^T (I - uu^T)(X_i + X_1) - 2(X_i - X_1)^T P = 0 \, (\text{for } i = 2, 3, 4, 5) \tag{3}$$

that is

$$(X_i - X_1)^T P = \frac{1}{2}(X_i - X_1)^T (I - uu^T)(X_i + X_1) \, (\text{for } i = 2, 3, 4, 5)$$

This holds in particular for $i = 2, 3, 4$. Let M denote:

$$M = [X_2, X_3, X_4]$$

then

$$M - X_1 \mathbf{1}^T = [X_2 - X_1, X_3 - X_1, X_4 - X_1]$$

(with $\mathbf{1} = [1, 1, 1]^T$), and

$$V = \begin{pmatrix} (X_2 - X_1)^T (I - uu^T)(X_2 + X_1) \\ (X_3 - X_1)^T (I - uu^T)(X_3 + X_1) \\ (X_4 - X_1)^T (I - uu^T)(X_4 + X_1) \end{pmatrix}$$

there is

$$((M - X_1 \mathbf{1}^T)^T) P = \frac{1}{2} V$$

hence

$$P = \frac{1}{2}((M - X_1 \mathbf{1}^T)^T)^{-1} V \tag{4}$$

when $(M - X_1 \mathbf{1}^T)$ is invertible, which is the case when the points X_1, X_2, X_3, X_4 are not coplanar. If these four points are coplanar, then P cannot be determined this way. In the remainder, we suppose that the points X_1, X_2, X_3, X_4 are not coplanar. If this is not the case, it may be possible to choose another set of four points among the five points that fulfills this condition. If the five points are such that every subset of four points are coplanar, which means that the five points are coplanar, then we cannot process the system of equations this way. We call such a case a *degenerate* set.

Remarque:
$((M - X_1\mathbf{1}^T)^T)^{-1}$ does not contain any term in u. The components of V are quadratic functions in the components of u.

Remarque:
Let α and β be two vectors of $\mathbb{R}^n$. In general, the following properties hold:

$$\alpha^T\beta = \beta^T\alpha \text{ (scalars)}$$
$$\alpha\beta^T \neq \beta\alpha^T \text{ (n×n matrices)}$$

Consequently, in general:

$$u^T\alpha\beta^T v \neq u^T\beta\alpha^T v$$

However, it can be proven that, $\forall u \in \mathbb{R}^n$:

$$u^T\alpha\beta^T u = u^T\beta\alpha^T u$$

This derives from the fact that:

$$\alpha^T uu^T\beta = \beta^T uu^T\alpha$$

and that

$$u^T\alpha\beta^T u = \alpha^T uu^T\beta$$

($\alpha^T uu^T\beta = (\alpha^T u)(u^T\beta) = u^T\beta\alpha^T u$)

Writing the system in u When the points X_1, X_2, X_3, X_4 are not coplanar, the system may be written (using equations 3, 4 and 1):

$$\begin{cases} (X_5 - X_1)^T(I - uu^T)(X_5 + X_1) - (X_5 - X_1)^T((M - X_1\mathbf{1}^T)^T)^{-1}V = 0 \\ u^T((M - X_1\mathbf{1}^T)^T)^{-1}V = 0 \\ u^Tu - 1 = 0 \end{cases} \tag{5}$$

This may be written as a system of the following form:

$$\begin{cases} u^TAu + \alpha = 0 \ \textbf{(Equation E1)} \\ u_x(u^TB_1u) + u_y(u^TB_2u) + u_z(u^TB_3u) + u^Tb = 0 \ \textbf{(Equation E2)} \\ u^Tu - 1 = 0 \ \textbf{(Equation E3)} \end{cases}$$

Let us try and write this, i.e. express the terms $A, \alpha, B_1, B_2, B_3, b$ introduced above. Recall that

$$M - X_1\mathbf{1}^T = [X_2 - X_1, X_3 - X_1, X_4 - X_1] = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ z_2 - z_1 & z_3 - z_1 & z_4 - z_1 \end{pmatrix}$$

Moreover,

$$(M - X_1\mathbf{1}^T)^{-1} = \frac{1}{\det(M - X_1\mathbf{1}^T)}\text{Com}(M - X_1\mathbf{1}^T)^T$$

so

$$((M - X_1\mathbf{1}^T)^T)^{-1} = \frac{1}{\det(M - X_1\mathbf{1}^T)}\text{Com}(M - X_1\mathbf{1}^T)$$

that is

$$((M - X_1\mathbf{1}^T)^T)^{-1} = \frac{1}{\Delta_{234}} \begin{pmatrix} +\Delta_{34}^{yz} & -\Delta_{24}^{yz} & +\Delta_{23}^{yz} \\ -\Delta_{34}^{xz} & +\Delta_{24}^{xz} & -\Delta_{23}^{xz} \\ +\Delta_{34}^{xy} & -\Delta_{24}^{xy} & +\Delta_{23}^{xy} \end{pmatrix} \tag{6}$$

with

$$\Delta_{234} = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ z_2 - z_1 & z_3 - z_1 & z_4 - z_1 \end{vmatrix}$$

and

$$\Delta_{ij}^{xy} = \begin{vmatrix} x_i - x_1 & x_j - x_1 \\ y_i - y_1 & y_j - y_1 \end{vmatrix}$$

Remarque:
We suppose that $\Delta_{234} \neq 0$, which expresses that the four points are not coplanar.

Notation: Given two vectors X and Y in $\mathbb{R}^3$, we use the following notations

$$\begin{aligned} k(X,Y) &= (X+Y)^T(X-Y) \\ K(X,Y) &= (X+Y)(X-Y)^T \end{aligned}$$

($k(X,Y) \in \mathbb{R}$, $K(X,Y) \in M_3(\mathbb{R})$). Note that:

$$\begin{aligned} k(X,-Y) &= k(X,Y) \\ K(X,-Y) &= K(X,Y)^T \neq K(Y,X) \text{ in general} \end{aligned}$$

but:

$$\begin{aligned} \forall u \in \mathbb{R}^3, u^T K(X,Y) u &= u^T(X+Y)(X-Y)^T u \\ &= (X+Y)^T u u^T (X-Y) \\ &= (X-Y)^T u u^T (X+Y) \\ &= u^T K(X,-Y) u \end{aligned}$$

This means that one might either write $u^T(X+Y)(X-Y)^T u$ or $u^T(X-Y)(X+Y)^T u$.
These notations finally enable us to rewrite the equations:

Equation E1 Multiplying by Δ_{234}, there holds:

$$\Delta_{234}[k(X_5,X_1) - u^T K(X_5,X_1)u] \\ -(X_5 - X_1)^T \begin{pmatrix} +\Delta_{34}^{yz} & -\Delta_{24}^{yz} & +\Delta_{23}^{yz} \\ -\Delta_{34}^{xz} & +\Delta_{24}^{xz} & -\Delta_{23}^{xz} \\ +\Delta_{34}^{xy} & -\Delta_{24}^{xy} & +\Delta_{23}^{xy} \end{pmatrix} \begin{pmatrix} k(X_2,X_1) - u^T K(X_2,X_1)u \\ k(X_3,X_1) - u^T K(X_3,X_1)u \\ k(X_4,X_1) - u^T K(X_4,X_1)u \end{pmatrix} = 0$$

so

$$\Delta_{234} k(X_5,X_1) - \Delta_{234} u^T K(X_5,X_1) u \\ -(X_5 - X_1)^T \begin{pmatrix} +\Delta_{34}^{yz} k(X_2,X_1) - \Delta_{24}^{yz} k(X_3,X_1) + \Delta_{23}^{yz} k(X_4,X_1) \\ -\Delta_{34}^{xz} k(X_2,X_1) + \Delta_{24}^{xz} k(X_3,X_1) - \Delta_{23}^{xz} k(X_4,X_1) \\ +\Delta_{34}^{xy} k(X_2,X_1) - \Delta_{24}^{xy} k(X_3,X_1) + \Delta_{23}^{xy} k(X_4,X_1) \end{pmatrix} \\ +(X_5 - X_1)^T \begin{pmatrix} +\Delta_{34}^{yz} & -\Delta_{24}^{yz} & +\Delta_{23}^{yz} \\ -\Delta_{34}^{xz} & +\Delta_{24}^{xz} & -\Delta_{23}^{xz} \\ +\Delta_{34}^{xy} & -\Delta_{24}^{xy} & +\Delta_{23}^{xy} \end{pmatrix} \begin{pmatrix} u^T K(X_2,X_1)u \\ u^T K(X_3,X_1)u \\ u^T K(X_4,X_1)u \end{pmatrix} = 0$$

Thus

$$\begin{array}{c}
\Delta_{234} k(X_5, X_1) \\
-(X_5 - X_1)^T \begin{pmatrix} +\Delta_{34}^{yz} k(X_2, X_1) - \Delta_{24}^{yz} k(X_3, X_1) + \Delta_{23}^{yz} k(X_4, X_1) \\ -\Delta_{34}^{xz} k(X_2, X_1) + \Delta_{24}^{xz} k(X_3, X_1) - \Delta_{23}^{xz} k(X_4, X_1) \\ +\Delta_{34}^{xy} k(X_2, X_1) - \Delta_{24}^{xy} k(X_3, X_1) + \Delta_{23}^{xy} k(X_4, X_1) \end{pmatrix} \\
-\Delta_{234} u^T K(X_5, X_1) u \\
+(X_5 - X_1)^T \begin{pmatrix} u^T[+\Delta_{34}^{yz} K(X_2, X_1) - \Delta_{24}^{yz} K(X_3, X_1) + \Delta_{23}^{yz} K(X_4, X_1)]u \\ u^T[-\Delta_{34}^{xz} K(X_2, X_1) + \Delta_{24}^{xz} K(X_3, X_1) - \Delta_{23}^{xz} K(X_4, X_1)]u \\ u^T[+\Delta_{34}^{xy} K(X_2, X_1) - \Delta_{24}^{xy} K(X_3, X_1) + \Delta_{23}^{xy} K(X_4, X_1)]u \end{pmatrix} \\
= 0
\end{array}$$

- Constant part of this equation:

$$\begin{aligned}
E_C &= \Delta_{234} k(X_5, X_1) \\
&\quad -(x_5 - x_1)[+\Delta_{34}^{yz} k(X_2, X_1) - \Delta_{24}^{yz} k(X_3, X_1) + \Delta_{23}^{yz} k(X_4, X_1)] \\
&\quad -(y_5 - y_1)[-\Delta_{34}^{xz} k(X_2, X_1) + \Delta_{24}^{xz} k(X_3, X_1) - \Delta_{23}^{xz} k(X_4, X_1)] \\
&\quad -(z_5 - z_1)[+\Delta_{34}^{xy} k(X_2, X_1) - \Delta_{24}^{xy} k(X_3, X_1) + \Delta_{23}^{xy} k(X_4, X_1)]
\end{aligned}$$

that is

$$\begin{aligned}
E_C &= \Delta_{234} k(X_5, X_1) \\
&\quad +k(X_2, X_1)[-(x_5 - x_1)\Delta_{34}^{yz} + (y_5 - y_1)\Delta_{34}^{xz} - (z_5 - z_1)\Delta_{34}^{xy}] \\
&\quad +k(X_3, X_1)[+(x_5 - x_1)\Delta_{24}^{yz} - (y_5 - y_1)\Delta_{24}^{xz} + (z_5 - z_1)\Delta_{24}^{xy}] \\
&\quad +k(X_4, X_1)[-(x_5 - x_1)\Delta_{23}^{yz} + (y_5 - y_1)\Delta_{23}^{xz} - (z_5 - z_1)\Delta_{23}^{xy}]
\end{aligned}$$

that is

$$\begin{aligned}
E_C &= -\Delta_{345} k(X_2, X_1) + \Delta_{245} k(X_3, X_1) \\
&\quad -\Delta_{235} k(X_4, X_1) + \Delta_{234} k(X_5, X_1)
\end{aligned}$$

using the previous notation to define $\Delta_{235}, \Delta_{245}, \Delta_{345}$.

Lest us thus write

$$\begin{aligned}
c_{2345} &= -\Delta_{345} k(X_2, X_1) + \Delta_{245} k(X_3, X_1) \\
&\quad -\Delta_{235} k(X_4, X_1) + \Delta_{234} k(X_5, X_1)
\end{aligned}$$

- Quadratic part of this equation:

$$\begin{aligned}
Q &= u^T[-\Delta_{234} K(X_5, X_1) \\
&\quad +(x_5 - x_1)[+\Delta_{34}^{yz} K(X_2, X_1) - \Delta_{24}^{yz} K(X_3, X_1) + \Delta_{23}^{yz} K(X_4, X_1)] \\
&\quad +(y_5 - y_1)[-\Delta_{34}^{xz} K(X_2, X_1) + \Delta_{24}^{xz} K(X_3, X_1) - \Delta_{23}^{xz} K(X_4, X_1)] \\
&\quad +(z_5 - z_1)[+\Delta_{34}^{xy} K(X_2, X_1) - \Delta_{24}^{xy} K(X_3, X_1) + \Delta_{23}^{xy} K(X_4, X_1)] \\
&\quad]u
\end{aligned}$$

so (same method as before):

$$\begin{aligned} Q &= -u^T[-\Delta_{345}K(X_2,X_1) + \Delta_{245}K(X_3,X_1) \\ &\quad -\Delta_{235}K(X_4,X_1) + \Delta_{234}K(X_5,X_1)]u \end{aligned}$$

if the following 3×3 matrix is introduced:

$$\begin{aligned} M_{2345} &= -\Delta_{345}K(X_2,X_1) + \Delta_{245}K(X_3,X_1) \\ &\quad -\Delta_{235}K(X_4,X_1) + \Delta_{234}K(X_5,X_1) \end{aligned}$$

then equation E1 may be written:

$$\boxed{c_{2345} - u^T M_{2345} u = 0}$$

(NB: as such the matrix M_{2345} is not symmetrical, but may be transformed to be so)

Equation E2

$$u^T \begin{pmatrix} \Delta_{34}^{yz} & -\Delta_{24}^{yz} & \Delta_{23}^{yz} \\ -\Delta_{34}^{xz} & \Delta_{24}^{xz} & -\Delta_{23}^{xz} \\ \Delta_{34}^{xy} & -\Delta_{24}^{xy} & \Delta_{23}^{xy} \end{pmatrix} \begin{pmatrix} k(X_2,X_1) - u^T K(X_2,X_1)u \\ k(X_3,X_1) - u^T K(X_3,X_1)u \\ k(X_4,X_1) - u^T K(X_4,X_1)u \end{pmatrix} = 0$$

$$u^T \begin{pmatrix} \Delta_{34}^{yz} & -\Delta_{24}^{yz} & \Delta_{23}^{yz} \\ -\Delta_{34}^{xz} & \Delta_{24}^{xz} & -\Delta_{23}^{xz} \\ \Delta_{34}^{xy} & -\Delta_{24}^{xy} & \Delta_{23}^{xy} \end{pmatrix} \left[\begin{pmatrix} k(X_2,X_1) \\ k(X_3,X_1) \\ k(X_4,X_1) \end{pmatrix} - \begin{pmatrix} u^T K(X_2,X_1)u \\ u^T K(X_3,X_1)u \\ u^T K(X_4,X_1)u \end{pmatrix} \right] = 0$$

that is

$$\boxed{-u_x[u^T M_{234}^{yz} u] - u_y[u^T M_{234}^{xz} u] - u_z[u^T M_{234}^{xy} u] + u^T V_{234} = 0}$$

with

$$V_{234} = \begin{pmatrix} \Delta_{34}^{yz} & -\Delta_{24}^{yz} & \Delta_{23}^{yz} \\ -\Delta_{34}^{xz} & \Delta_{24}^{xz} & -\Delta_{23}^{xz} \\ \Delta_{34}^{xy} & -\Delta_{24}^{xy} & \Delta_{23}^{xy} \end{pmatrix} \begin{pmatrix} k(X_2,X_1) \\ k(X_3,X_1) \\ k(X_4,X_1) \end{pmatrix}$$

et

$$\begin{aligned} M_{234}^{yz} &= \Delta_{34}^{yz}K(X_2,X_1) - \Delta_{24}^{yz}K(X_3,X_1) + \Delta_{23}^{yz}K(X_4,X_1) \\ M_{234}^{xz} &= -\Delta_{34}^{xz}K(X_2,X_1) + \Delta_{24}^{xz}K(X_3,X_1) - \Delta_{23}^{xz}K(X_4,X_1) \\ M_{234}^{xy} &= \Delta_{34}^{xy}K(X_2,X_1) - \Delta_{24}^{xy}K(X_3,X_1) + \Delta_{23}^{xy}K(X_4,X_1) \end{aligned}$$

4 Final system obtained

When the points X_1, X_2, X_3, X_4 are not coplanar, there is:

- Vector notation

$$\left\{ \begin{array}{l} u^T M_{2345} u - c_{2345} = 0 \\ u_x[u^T M_{234}^{yz} u] + u_y[u^T M_{234}^{xz} u] + u_z[u^T M_{234}^{xy} u] - u^T V_{234} = 0 \\ u^T u - 1 = 0 \end{array} \right.$$

(where $u_x, u_y, u_z \in [-1, 1]$) with

$$\begin{array}{rcl} M_{2345} & = & -\Delta_{345} K(X_2, X_1) + \Delta_{245} K(X_3, X_1) \\ & & -\Delta_{235} K(X_4, X_1) + \Delta_{234} K(X_5, X_1) \\ c_{2345} & = & -\Delta_{345} k(X_2, X_1) + \Delta_{245} k(X_3, X_1) \\ & & -\Delta_{235} k(X_4, X_1) + \Delta_{234} k(X_5, X_1) \\ M_{234}^{yz} & = & \Delta_{34}^{yz} K(X_2, X_1) - \Delta_{24}^{yz} K(X_3, X_1) + \Delta_{23}^{yz} K(X_4, X_1) \\ M_{234}^{xz} & = & -\Delta_{34}^{xz} K(X_2, X_1) + \Delta_{24}^{xz} K(X_3, X_1) - \Delta_{23}^{xz} K(X_4, X_1) \\ M_{234}^{xy} & = & \Delta_{34}^{xy} K(X_2, X_1) - \Delta_{24}^{xy} K(X_3, X_1) + \Delta_{23}^{xy} K(X_4, X_1) \\ V_{234} & = & \begin{pmatrix} \Delta_{34}^{yz} & -\Delta_{24}^{yz} & \Delta_{23}^{yz} \\ -\Delta_{34}^{xz} & \Delta_{24}^{xz} & -\Delta_{23}^{xz} \\ \Delta_{34}^{xy} & -\Delta_{24}^{xy} & \Delta_{23}^{xy} \end{pmatrix} \begin{pmatrix} k(X_2, X_1) \\ k(X_3, X_1) \\ k(X_4, X_1) \end{pmatrix} \\ \Delta_{ijk} & = & \begin{vmatrix} x_i - x_1 & x_j - x_1 & x_k - x_1 \\ y_i - y_1 & y_j - y_1 & y_k - y_1 \\ z_i - z_1 & z_j - z_1 & z_k - z_1 \end{vmatrix} \\ \Delta_{ij}^{xy} & = & \begin{vmatrix} x_i - x_1 & x_j - x_1 \\ y_i - y_1 & y_j - y_1 \end{vmatrix} \text{ (idem } \Delta_{ij}^{xz} \text{ and } \Delta_{ij}^{yz}\text{).} \\ k(X, Y) & = & (X + Y)^T (X - Y) \\ K(X, Y) & = & (X + Y)(X - Y)^T \end{array}$$

- Polynomial notation
If we set $u = [x, y, z]^T$ in order to lighten the writing, then the following equations in x, y, z are obtained:

equation E1:

$$\begin{array}{l} x^2(M_{2345}(0,0)) \\ +xy(M_{2345}(0,1) + M_{2345}(1,0)) \\ +xz(M_{2345}(0,2) + M_{2345}(2,0)) \\ +y^2(M_{2345}(1,1)) \\ +yz(M_{2345}(1,2) + M_{2345}(2,1)) \\ +z^2(M_{2345}(2,2)) \\ +1(-c_{2345}) \\ = 0 \end{array}$$

equation E2:

$$
\begin{aligned}
&x^3(M^{yz}_{234}(0,0))\\
&+x^2y(M^{yz}_{234}(0,1)+M^{yz}_{234}(1,0)+M^{xz}_{234}(0,0))\\
&+x^2z(M^{yz}_{234}(0,2)+M^{yz}_{234}(2,0)+M^{xy}_{234}(0,0))\\
&+xy^2(M^{yz}_{234}(1,1)+M^{xz}_{234}(0,1)+M^{xz}_{234}(1,0))\\
&+xyz(M^{yz}_{234}(1,2)+M^{yz}_{234}(2,1)+M^{xz}_{234}(0,2)\\
&+M^{xz}_{234}(2,0)+M^{xy}_{234}(0,1)+M^{xy}_{234}(1,0))\\
&+xz^2(M^{yz}_{234}(2,2)+M^{xy}_{234}(0,2)+M^{xy}_{234}(2,0))\\
&+y^3(M^{xz}_{234}(1,1))\\
&+y^2z(M^{xz}_{234}(1,2)+M^{xz}_{234}(2,1)+M^{xy}_{234}(1,1))\\
&+yz^2(M^{xz}_{234}(2,2)+M^{xy}_{234}(1,2)+M^{xy}_{234}(2,1))\\
&+z^3(M^{xy}_{234}(2,2))\\
&+x(-V_{234}.X)\\
&+y(-V_{234}.Y)\\
&+z(-V_{234}.Z)\\
&=0
\end{aligned}
$$

equation E3:

$$x^2+y^2+z^2-1=0$$

About this system A similar system is described in [3], except that the one written here has a lower degree (degree of the equations: (2,3,2) against (3,3,2)), and one can thus legitimately expect it to be simpler to solve numerically.

Following the reasoning used in [3], if you consider the obtained system as a system of two equations on a projective vector, then the Bezout theorem on algebraic equations tell us that this system has at most $2 \times 3 = 6$ projective solutions. In other words, there are at most six solutions u such that $\|u\| = 1$ (once the couples $(u, -u)$ have been eliminated). Furthermore, the upper bound of 6 solutions is reached in certain cases (see [3]).

NB: The third equation merely expressing the projective condition, this system may equivalently be written as a system of two trigonometric equations.

Computing the other parameters Once the system has been solved in u, we get the other parameters as follows. Using equation 6 and equation 2 of system 5, one gets

$$P = \frac{1}{2\Delta_{234}}\left[-\begin{pmatrix}u^T M^{yz}_{234} u\\ u^T M^{xz}_{234} u\\ u^T M^{xy}_{234} u\end{pmatrix} + V_{234}\right]$$

and

$$r = \sqrt{(X_1 - P)^2 - ((X_1 - P)^T u)^2}$$

5 Degenerate cases

The reasoning above is valid for the cases where the point X_1 does not lie on the plane defined by X_2, X_3, X_4. If the four points are coplanar, we can permute the five points so as to find another set of four points that are not coplanar and use the system above. The real degenerate cases are hence the cases where the five points X_1, X_2, X_3, X_4, X_5 are coplanar.

Let us first note that with digitized data, this case is extremely unlikely to occur in practise. However, with a set of nearly coplanar points, the system above might yield bad results due to numerical instability. Should this happen, this case may be treated separately. Indeed, five points on a plane define a planar conic section.

When this conic is a nondegenerate ellipse, there are two possible cylinders (Figure 2). Indeed, from the five points' coordinates, the polynomial equation of the ellipse may be derived

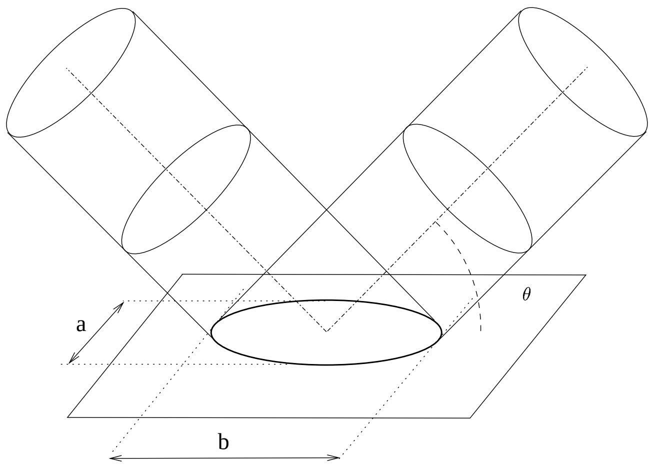

Figure 2: Two cylinders generating the same ellipse in the plane

in a straightforward way, as in the case of the circle defined by three points. One may then compute the angle θ, defined as

$$\cos\theta = \frac{a}{b}$$

where a is the minor length, and b is the major length. Each of the two possible cylinders has its axis intersecting the plane at the center of the ellipse. From the angle and the center, one gets a full definition of the axis. From the ellipse's dimension along its major axis, one gets the radius.

Even more degenerate cases occur when the five points are not only are coplanar, but do not define an ellipse in this plane. For example, the five points might be collinear. In this case, there might be an infinite number of cylinders satisfying the condition from the five points.

6 Conclusion

In this short report, we adressed the problem of constructing a right circular cylinder from a set of five 3D points. This generates a system of polynomial equations that is hard to directly compute numerically. Our calculus yielded a simpler system, consisting of three polynomial equations in three unknown, of degree (2,2,3). This shows that a nondegenerate set of five points is enough to build cylinders, and that the number of cylinders through five points ranges from zero to six.

Our formulation has two advantages. First we give an explicit expression of the coefficients of the system of equations. Secondly, the degree of the polynomial system obtained is smaller than others obtained in related works : (2,2,3) instead of (2,3,3) in [3].

6.1 Solving the system in u

It does not seem possible to further simplify this system in u symbolically. Consequently, in order to actually solve the problem of finding the cylinders' parameters, one has to compute a numerical solution of this final system. Usual numerical solving methods require an initial solution. It is not straightforward to provide such an initial solution here.

However, an interesting property here is that the coordinates of the vector u follow the constraint of equation 3, which enables us to limit the search to the volume $[-1, 1]^3$.

Given that the unknowns lie in a simple bounded domain, and that all the solutions are relevant *a priori*, this situation seemed particularly suited for interval analysis methods. This lead us to test the system numerically using the «*Interval_Solver*» software of the interval analysis library ALIAS developed at INRIA (see `http://www-sop.inria.fr/coprin/logiciels/ALIAS/`) and the interval analysis software developed at LSS lab, Supelec.

We tested the solving of this system on a few synthetic sets of five points (without noise) and a few sets of five points from real range scans. The results confirmed the validity of the equations, and the fact that interval analysis may be used for this purpose. Let us note that the execution times observed (on a PC Pentium II with Linux) were generally longer than 5s for reaching a precision of 10^{-3} on the coordinates. This turned out to be an issue in the context of our application.

6.2 Using this method for cylinder robust fitting

A Monte-Carlo method similar to the one in [4] has been implemented for cylinder robust fitting. In this context, at most six cylinders are found for each set of five points. Each of these cylinders was evaluated and the best one —with respect to the cost function— was kept. The cost function used was simply the number of inliers within a distance threshold. In that case, the scenes used were sets of points lying on a cylinder. While this usually lead to the proper solutions on synthetic cases, on real cases, no proper solution was reached in a reasonable time. More precisely, it seems that the algorithm was stuck due to certain cases, where the system solving took a very long time. We suspect this occurs for the sets of five points that are close to degeneracy. The Monte-Carlo method would allow such samples to be rejected without any computation, but it is still not totally clear to us how we can characterize such a near-degeneracy in a generic way, i.e. without introducing an artificial threshold .

A more promising approach may be to use some methods specifically dedicated to polynomial equations, such as the SYNAPS software developped at INRIA (see `http://www-sop.inria.fr/galaad/logiciels/synaps/`). In sample cases, this symbolic-numeric method seems to be able to find all the roots very efficiently (even with a more complex system) [3].

The main question in order to evaluate this solution for our application is to see how its performance behaves in the presence of noisy points and in cases of near-degeneracy. Future research on the case of the cylinder will consist in evaluating this method.

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