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To cite this version:
Cédric Milliet. ON THE RADICALS OF A GROUP THAT DOES NOT HAVE THE INDEPENDENCE PROPERTY. 2016. <hal-01256813>

HAL Id: hal-01256813
https://hal.archives-ouvertes.fr/hal-01256813
Submitted on 15 Jan 2016

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ON THE RADICALS OF A GROUP THAT DOES NOT HAVE THE
INDEPENDENCE PROPERTY

CÉDRIC MILLET

Abstract. We give an example of a pure group that does not have the independence
property, whose Fitting subgroup is neither nilpotent nor definable and whose soluble radical
is neither soluble nor definable. This answers a question asked by E. Jaligot in May 2013.

The Fitting subgroup of a stable group is nilpotent and definable (F. Wagner [Wag95]).
More generally, the Fitting subgroup of a group that satisfies the descending chain condition
on centralisers is nilpotent (J. Derakhshan, F. Wagner [DW97]) and definable (F. Wagner
[Wag99, Corollary 2.5], see also [OH13] and [AB14]). The soluble radical of a superstable
group is soluble and definable (A. Baudish [Bau90]). Whether this also holds for a stable
group is still an open question.

Inspired by [MT12], we provide an example of a pure group that does not have the indepen-
dence property, whose Fitting subgroup is neither nilpotent nor definable and whose soluble
radical is neither soluble nor definable. The proofs require some algebra because we have
decided to provide a precise computation of the Fitting subgroup and soluble radical of the
group considered.

Definition 1 (independence property). Let $M$ be a structure. A formula $\varphi(x, y)$ has the
independence property in $M$ if for all $n \in \omega$, there are tuples $a_1, \ldots, a_n$ and $(b_J)_{J \subseteq \{1, \ldots, n\}}$ of
$M$ such that $(M \models \varphi(a_i, b_J)) \iff i \in J$. $M$ does not have the independence property (or
is NIP for short) if no formula has the independence property in $M$.

Let $L$ be a first order language, $M$ an $L$-structure. A set $X$ is interpretable in $M$ if there
is a definable subset $Y \subseteq M^n$ in $M$ and a definable equivalence relation $E$ on $X$ such that
$X = Y/E$. A family $\{Y_i/E_i : i \in I\}$ of interpretable sets in $M$ is uniformly interpretable in
$M$ if the corresponding families $\{Y_i : i \in I\}$ and $\{E_i : i \in I\}$ are uniformly definable in $M$.

Let $L$ be yet another first order language. An $L$-structure $N$ is interpretable in $M$ if its
domain, functions, relations and constants are interpretable sets in $M$. A family of $L$-
structures $\{N_i : i \in I\}$ is uniformly interpretable in $M$ if the family of domains is uniformly
interpretable in $M$, as well as, for each symbol $s$ of the language $L$, the family $\{s_i : i \in I\}$ of
interpretations of $s$ in $N_i$.

Lemma 2 (D. Macpherson, K. Tent [MT12]). Let $M$ be an $L$-structure that does not have
the independence property and let $\{N_i : i \in I\}$ be a family of $L$-structures that is uniformly
interpretable in $M$. For every ultrafilter $U$ on $I$, the $L$-structure $\prod_{i \in I} N_i/U$ does not have the
independence property.
Corollary 3. Let \( m \) and \( n \) be natural numbers and \( p \) a prime number. Let us consider the general linear group \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) over the finite ring \( \mathbb{Z}/p^n\mathbb{Z} \). Let \( \mathcal{U} \) be an ultrafilter on \( \mathbb{N} \) and let \( G \) be the ultraproduct

\[
G = \prod_{n \in \mathbb{N}} \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) / \mathcal{U}.
\]

The pure group \( G \) does not have the independence property.

Proof. Consider the group \( \text{GL}_m(\mathbb{Z}_p) \) over the ring \( \mathbb{Z}_p \) of \( p \)-adic integers, and the normal subgroups \( 1 + p^n \text{M}_m(\mathbb{Z}_p) \) for every \( n \geq 1 \). One has the group isomorphism

\[
\text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \simeq \text{GL}_m(\mathbb{Z}_p)/1 + p^n \text{M}_m(\mathbb{Z}_p).
\]

Therefore, the family of groups \( \{ \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) : n \in \mathbb{N} \} \) is uniformly interpretable in the ring \( \text{M}_m(\mathbb{Q}_p) \), which is interpretable in the field \( \mathbb{Q}_p \) of \( p \)-adic numbers, hence NIP by [Mat93]. By Lemma 2, the group \( G \) does not have the independence property. \( \square \)

1. Preliminaries on the normal structure of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \)

Given a field \( k \), the normal subgroups of the general linear group \( \text{GL}_m(k) \) are precisely the subgroups of the centre and the subgroups containing the special linear group \( \text{SL}_m(k) \) (J. Dieudonné [Die55]). In particular, the maximal normal soluble subgroup of \( \text{GL}_m(k) \) is the centre, except for the two soluble groups \( \text{GL}_2(\mathbb{F}_2) \) and \( \text{GL}_2(\mathbb{F}_3) \). The situation is different for the general linear group \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) over the ring \( \mathbb{Z}/p^n\mathbb{Z} \), whose normal subgroups are classified by J. Brenner [Bre38]. We follow also W. Klingenberg [Kli60] who deals with the normal subgroups of the general linear group over a local ring \( R \), which applies in particular to \( \mathbb{Z}/p^n\mathbb{Z} \).

The centre of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) is the subgroup of homotheties \( (\mathbb{Z}/p^n\mathbb{Z})^\times \cdot 1 \). The general congruence subgroup of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) of order \( \ell \) is

\[
\text{GC}_m(\ell) = (\mathbb{Z}/p^n\mathbb{Z})^\times \cdot 1 + p^\ell \text{M}_m(\mathbb{Z}/p^n\mathbb{Z}).
\]

It is a normal subgroup of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \). For every element \( g \) of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \), there is a maximal \( \ell \leq n \) such that \( g \) belongs to \( \text{GC}_m(\ell) \). We call \( \ell \) the \emph{congruence order} of \( g \).

The special linear subgroup of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) of matrices having determinant 1 is written \( \text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \). An \emph{elementary transvection} is an element of \( \text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) of the form \( 1 + re_{ij} \) for \( r \in \mathbb{Z}/p^n\mathbb{Z} \) and \( i \neq j \). A \emph{transvection} is a conjugate of an elementary transvection.

Proposition 4 (J. Brenner [Bre38, Theorem 1.5]). Let \( \tau \) a transvection of congruence order \( \ell \). The normal subgroup of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) generated by \( \tau \) is

\[
\left< \tau^{\text{GL}_m(\mathbb{Z}/p^n\mathbb{Z})} \right> = \text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \left( 1 + p^\ell \text{M}_m(\mathbb{Z}/p^n\mathbb{Z}) \right).
\]

Theorem 5 (J. Brenner [Bre38]). Let \( mp \geq 6 \) and \( g \) an element of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) of congruence order \( \ell \). The normal subgroup \( \left< g^{\text{GL}_m(\mathbb{Z}/p^n\mathbb{Z})} \right> \) of \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \) generated by \( g \) satisfies

\[
\text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \left( 1 + p^\ell \text{M}_m(\mathbb{Z}/p^n\mathbb{Z}) \right) \subset \left< g^{\text{GL}_m(\mathbb{Z}/p^n\mathbb{Z})} \right> \subset (\mathbb{Z}/p^n\mathbb{Z})^\times \cdot 1 + p^\ell \text{M}_m(\mathbb{Z}/p^n\mathbb{Z}).
\]

For any real number \( x \), we write \( [x] \) for the \emph{floor of} \( x \), that is \( [x] \) is the greatest integer \( k \) such that \( k \leq x \).
Lemma 6. For any \(m \geq 2, \ell \geq 1\) and \(n \geq 1\), the group \(1 + p^f M_m(\mathbb{Z}/p^n\mathbb{Z})\) is a normal nilpotent subgroup of \(GL_m(\mathbb{Z}/p^n\mathbb{Z})\) of nilpotency class \(\left\lceil \frac{n - 1}{\ell} \right\rceil\).

Proof. For every \(x\) in \(M_m(\mathbb{Z}/p^n\mathbb{Z})\), one has

\[
(1 + px)^p = 1 + \sum_{k=1}^{p} (px)^k C_p^k = 1 + p^2 y.
\]

It follows that \(1 + pM_m(\mathbb{Z}/p^n\mathbb{Z})\) is a nilpotent \(p\)-group. Its iterated centres are

\[
Z(H_n) = (1 + pZ/p^nZ) \cdot 1 + p^{n-1} M_m(Z/p^nZ)
\]

\[
Z_2(H_n) = (1 + pZ/p^nZ) \cdot 1 + p^{n-2} M_m(Z/p^nZ)
\]

\[
\vdots
\]

\[
Z_{n-2}(H_n) = (1 + pZ/p^nZ) \cdot 1 + p^2 M_m(Z/p^nZ)
\]

\[
Z_{n-1}(H_n) = 1 + pM_m(Z/p^nZ),
\]

so the nilpotency class of \(1 + pM_m(\mathbb{Z}/p^n\mathbb{Z})\) is \(n - 1\) when \(n \geq 1\). For every natural number \(q\) satisfying \(n - q\ell \geq \ell\), one has

\[
Z_q \left( 1 + p^f M_m(\mathbb{Z}/p^n\mathbb{Z}) \right) = \left( 1 + p^f Z/p^nZ \right) \cdot 1 + p^{n-q\ell} M_m(Z/p^nZ),
\]

so the greatest \(q\) such that the above \(q\)th centre is a proper subgroup is the greatest \(q\) satisfying \(n - q\ell > \ell\). As one has

\[
n - q\ell > \ell \iff n - 1 - q\ell \geq \ell \iff q \leq \left\lceil \frac{n - 1}{\ell} \right\rceil - 1,
\]

this greatest \(q\) is precisely \(\left\lceil \frac{n - 1}{\ell} \right\rceil - 1\). \(\square\)

For any real number \(x\), we write \([x]\) for the ceiling of \(x\), that is \([x]\) is the least integer \(k\) such that \(k \geq x\).

Lemma 7. For any \(1 \leq \ell \leq n\) and \(m \geq 3\), the group \(1 + p^f M_m(\mathbb{Z}/p^n\mathbb{Z})\) is soluble of derived length \(\left\lceil \log_2 \frac{n}{\ell} \right\rceil\).

Proof. Let us write \(PC_m(\ell) = 1 + p^f M_m(\mathbb{Z}/p^n\mathbb{Z})\) and show that

\[
\left( SL_m(\mathbb{Z}/p^n\mathbb{Z}) \cap PC_m(\ell) \right)' = PC_m(\ell)' = SL_m(\mathbb{Z}/p^n\mathbb{Z}) \cap PC_m(2\ell).
\]

Let \(\alpha = 1 - p^f \gamma\) and \(\beta = 1 - p^f \delta\) be two elements of \(1 + p^f M_m(\mathbb{Z}/p^n\mathbb{Z})\). Then

\[
\alpha \beta \alpha^{-1} \beta^{-1} = \left( 1 - p^f \gamma \right) \left( 1 - p^f \delta \right) \left( 1 + p^f \gamma + \cdots + p^{n\ell} \gamma^n \right) \left( 1 + p^f \delta + \cdots + p^{n\ell} \delta^n \right)
\]

\[
= 1 + p^{2\ell} (\gamma \delta - \gamma \delta) + p^{3\ell} (\cdots) + \cdots,
\]

so \(PC_m(\ell)\) is included in \(SL_m(\mathbb{Z}/p^n\mathbb{Z}) \cap PC_m(2\ell)\). Conversely, consider the two elementary transvections \(\sigma = 1 + p^f e_{12}\) and \(\tau = 1 + p^f e_{21}\). One has

\[
\sigma \tau \sigma^{-1} \tau^{-1} = (1 + p^f e_{12})(1 + p^f e_{21})(1 - p^f e_{12})(1 - p^f e_{21})
\]

\[
= 1 + p^{2\ell} e_{11} - p^{2\ell} e_{22} - p^{3\ell} e_{12} + p^{3\ell} e_{21}.
\]
It follows that \( (\text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \text{PC}_m(\ell))^\prime \) contains an element that lies in \( \text{PC}_m(2\ell) \setminus \text{PC}_m(2\ell + 1) \).

As \( (\text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \text{PC}_m(\ell))^\prime \) is a characteristic subgroup of \( \text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \text{PC}_m(\ell) \), it is normal in \( \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z}) \). By Theorem 5, \( (\text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \text{PC}_m(\ell))^\prime \) contains \( \text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \text{PC}_m(2\ell) \).

We have thus shown that for every natural number \( k \), the \( k \)th derived subgroup of \( \text{PC}_m(\ell) \) is

\[
\text{PC}_m(\ell)^{(k)} = \text{SL}_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \text{PC}_m(2^k\ell).
\]

The derived length of \( \text{PC}_m(\ell) \) is the least \( k \) such that \( 2^k\ell \geq n \).

**Lemma 8.** For natural numbers \( k \) and \( n \geq k^2 + k \), let \( \ell_n(k) = 1 + \left\lfloor \frac{n}{k+1} \right\rfloor \). Then \( \ell_n(k) \) is the smallest natural number satisfying the equality \( \left\lfloor \frac{n}{\ell_n(k)} \right\rfloor = k \).

**Proof.** Let \( n = q(k+1)+r \) be the Euclidean division of \( n \) by \( k+1 \), with \( q \geq k \) and \( 0 \leq r < k+1 \). Then one has

\[
0 < \frac{k+1-r}{1+q} \leq 1 \quad \text{hence} \quad \left\lfloor \frac{n}{\ell_n(k)} \right\rfloor = \left\lfloor \frac{(k+1)q+r}{1+q} \right\rfloor = \left\lfloor k+1 - \frac{k+1-r}{1+q} \right\rfloor = k,
\]

so \( \ell_n(k) \) satisfies the equality. It is the smallest such, as one has

\[
\left\lfloor \frac{n}{\ell_n(k)} \right\rfloor = \left\lfloor \frac{n}{k+1} \right\rfloor = \left\lfloor \frac{n}{q} \right\rfloor = \left\lfloor k+1 + \frac{r}{q} \right\rfloor \geq k+1. \quad \square
\]

**Lemma 9.** For natural numbers \( k \geq 1 \) and \( n \geq 2^k \), let \( d_n(k) = \left\lfloor \frac{n}{2^k} \right\rfloor \). Then \( d_n(k) \) is the smallest natural number satisfying the equality \( \left\lfloor \log_2 \frac{n}{d_n(k)} \right\rfloor = k \).

**Proof.** One has

\[
\frac{n}{2^k} \leq \left\lfloor \frac{n}{2^k} \right\rfloor < \frac{n}{2^k} + 1
\]

hence

\[
k - 1 \leq k - \log_2 \left(1 + \frac{2^k}{n}\right) < \log_2 \left(\frac{n}{2^k}\right) \leq k,
\]

so that \( d_n(k) \) satisfies the equality. It is the smallest such, as

\[
\left\lfloor \log_2 \left(\frac{n}{2^k} - 1\right) \right\rfloor = \left\lfloor \log_2 \left(\frac{2^k}{n} \frac{n}{2^k} - \frac{2^k}{n}\right) \right\rfloor = \left\lfloor k - \log_2 \left(\frac{2^k}{n} \left\lfloor \frac{n}{2^k} \right\rfloor - \frac{2^k}{n}\right) \right\rfloor \geq k+1. \quad \square
\]

2. **Radicals of G**

We now consider

\[
G = \prod_{n \in \mathbb{N}} \text{GL}_m(\mathbb{Z}/p^n\mathbb{Z})/\mathcal{U}.
\]

We call *Fitting subgroup of G* and write \( F(G) \) the subgroup generated by all its normal nilpotent subgroups. By Zorn’s Lemma, any nilpotent subgroup of \( G \) of nilpotency class \( k \) is contained in a maximal such, which might not be unique.
Lemma 10. There is a first order formula $\varphi_k$ in the language of groups such that, for any group $N$, $N$ is nilpotent of class $k$ if and only if $N \models \varphi_k$.

Proof. Consider the formula
$$\forall x_1 \cdots \forall x_k [x_1, [x_2, \ldots, [x_{k-1}, x_k] \cdots]] = 1 \land \exists y_1 \cdots \exists y_{k-1} [y_1, [y_2, \ldots, [y_{k-2}, y_{k-1}] \cdots]] \neq 1.$$  

\[ \Box \]

Theorem 11 (Łos). Let $(M_i)_{i \in \mathbb{N}}$ be a collection of $L$-structure, $\mathcal{U}$ an ultrafilter on $\mathbb{N}$ and $M$ the ultraproduct $\prod_i M_i/\mathcal{U}$. One has $M \models \varphi$ if and only if $\{ i \in \mathbb{N} : M_i \models \varphi \}$ is in $\mathcal{U}$.

Theorem 12 (Fitting subgroup of $G$). If the ultrafilter $\mathcal{U}$ is non-principal, for every natural number $k$, $G$ has a unique maximal normal nilpotent subgroup $N_k$ of nilpotency class $k$

$$N_k = \prod_{n \in \mathbb{N}} \left( \left( \mathbb{Z}/p^n\mathbb{Z} \right)^{\times} \cdot 1 + p^{1+\left\lfloor \frac{n-1}{k+1} \right\rfloor} M_m(\mathbb{Z}/p^n\mathbb{Z}) \right) / \mathcal{U},$$

hence the Fitting subgroup of $G$ is

$$F(G) = \bigcup_{k=1}^{\infty} N_k.$$

$F(G)$ is neither nilpotent, nor definable.

Proof. Let $k$ be a fixed natural number. By Lemma 6 and Lemma 8, the normal subgroup

$$\left( \mathbb{Z}/p^n\mathbb{Z} \right)^{\times} \cdot 1 + p^{1+\left\lfloor \frac{n-1}{k+1} \right\rfloor} M_m(\mathbb{Z}/p^n\mathbb{Z})$$

of $GL_m(\mathbb{Z}/p^n\mathbb{Z})$ has nilpotency class $k$ for all but finitely many $n$. As $\mathcal{U}$ contains the Fréchet filter and as being of nilpotency class $k$ is expressible by a first order formula in the pure language of groups according to Lemma 10, by Łos Theorem, the ultraproduct

$$\prod_{n \in \mathbb{N}} \left( \left( \mathbb{Z}/p^n\mathbb{Z} \right)^{\times} \cdot 1 + p^{1+\left\lfloor \frac{n-1}{k+1} \right\rfloor} M_m(\mathbb{Z}/p^n\mathbb{Z}) \right) / \mathcal{U}$$

is a normal nilpotent subgroup of class $k$ of $G$. Reciprocally, if $g$ belongs to a normal nilpotent subgroup of class $k$, then $g^G$ generates a normal nilpotent subgroup of class at most $k$. By Łos Theorem, there is a set $I \in \mathcal{U}$ such that for all $n \in I$, the conjugacy class $g_n^G$ generates a nilpotent normal subgroup $\langle g_n^G \rangle$ of $G_n$ of class at most $k$. Let $n \in I$ be fixed. By Theorem 5, there is a unique natural number $1 \leq \ell \leq n$ such that

$$SL_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \left( 1 + p^\ell M_m(\mathbb{Z}/p^n\mathbb{Z}) \right) \subset \langle g_n^G \rangle \subset (\mathbb{Z}/p^n\mathbb{Z})^{\times} \cdot 1 + p^\ell M_m(\mathbb{Z}/p^n\mathbb{Z}).$$

As $1 + p^\ell M_m(\mathbb{Z}/p^n\mathbb{Z})$ is nilpotent of class $\left\lfloor \frac{n-1}{\ell} \right\rfloor$, one must have $k \geq \frac{n-1}{\ell}$. From Lemma 8, it follows that $\ell \geq \ell_{n-1}(k)$ for all but finitely many $n$ in $I$, so that $g$ belongs to the desired ultraproduct.

To show that the Fitting subgroup of $G$ is not definable, let $g_{n,\ell}$ be the elementary transvection $1 + p^\ell e_{ij}$ of $H_n$ for every $1 \leq \ell < n$. By Proposition 4, one has

$$\langle g_{n,\ell}^G \rangle = SL_m(\mathbb{Z}/p^n\mathbb{Z}) \cap \left( 1 + p^\ell M_m(\mathbb{Z}/p^n\mathbb{Z}) \right),$$

hence

$$Z_q \left( \langle g_{n,\ell}^G \rangle \right) = SL_m(\mathbb{Z}/p^n\mathbb{Z}) \cap Z_q \left( 1 + p^\ell M_m(\mathbb{Z}/p^n\mathbb{Z}) \right),$$
Lemma 13. There is a first order formula \( \varphi_\ell \) in the language of groups such that, for any group \( S \), \( S \) is soluble of derived length \( \ell \) if and only if \( S \models \varphi_\ell \).

Proof. Consider the term \( t_\ell(x_1, \ldots, x_{2^\ell}) \) defined inductively by setting \( t_1(x_1, x_2) = [x_1, x_2] \) and \( t_{\ell+1}(x_1, \ldots, x_{2^{\ell+1}}) = [t_\ell(x_1, \ldots, x_{2^\ell}), t_\ell(x_{2^\ell+1}, \ldots, x_{2^{\ell+1}})] \). Then consider the formula
\[
\forall x_1 \cdots \forall x_{2^\ell} t_\ell(x_1, \ldots, x_{2^\ell}) = 1 \land \exists y_1 \cdots \exists y_{2^\ell-1} t_{\ell-1}(y_1, \ldots, y_{2^\ell-1}) \neq 1.
\]

We call soluble radical of \( G \) and write \( R(G) \) the subgroup generated by all its normal soluble subgroups.

Theorem 14 (soluble radical of \( G \)). If the ultrafilter \( U \) is non-principal, for every natural number \( \ell \), \( G \) has a unique maximal normal soluble subgroup \( S_\ell \) of derived length \( \ell \)
\[
S_\ell = \prod_{n \in \mathbb{N}} \left( \left( \mathbb{Z}/p^n\mathbb{Z} \right)^G \cdot 1 + p^{\left\lceil \frac{n}{2^\ell} \right\rceil} M_{m/n}(\mathbb{Z}/p^n\mathbb{Z}) \right)/U,
\]
hence the soluble radical of \( G \) is
\[
R(G) = \bigcup_{\ell=1}^{\infty} S_\ell = F(G).
\]

\( R(G) \) is neither soluble, nor definable.

Proof. By Lemma 7, Lemma 9, Lemma 13 and Łos Theorem, \( S_\ell \) is a normal soluble subgroup of \( G \) of derived length \( \ell \). By Theorem 5 and Lemma 9, \( S_\ell \) is maximal such. Note that
\[
1 + \left\lceil \frac{n}{2^\ell} \right\rceil \leq \frac{n}{2^\ell}
\]
holds for every \( n \) and \( \ell \), so that one has \( S_\ell \subset N_{2^\ell} \) hence \( R(G) \) and \( F(G) \) coincide.

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