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Linear rigidity of stationary stochastic processes

Alexander I. Bufetov * Yoann Dabrowski † Yanqi Qiu ‡

Abstract

We consider stationary stochastic processes $X_n$, $n \in \mathbb{Z}$ such that $X_0$ lies in the closed linear span of $X_n$, $n \neq 0$; following Ghosh and Peres, we call such processes linearly rigid. Using a criterion of Kolmogorov, we show that it suffices, for a stationary stochastic process to be rigid, that the spectral density vanish at zero and belong to the Zygmund class $\Lambda^*_1$. We next give sufficient condition for stationary determinantal point processes on $\mathbb{Z}$ and on $\mathbb{R}$ to be rigid. Finally, we show that the determinantal point process on $\mathbb{R}^2$ induced by a tensor square of Dyson sine-kernels is not linearly rigid.

1 Introduction

This paper is devoted to rigidity of stationary determinantal point processes.

Recall that stationary determinantal point processes are strongly chaotic: they have the Kolmogorov property (Lyons [9]) and the Bernoulli property (Lyons and Steif [10]); and they satisfy the Central Limit Theorem (Costin and Lebowitz [2], Soshnikov[13]). On the other hand, Ghosh [5] and Ghosh-Peres [6] proved, for the determinantal point processes such as Dyson sine process and Ginibre point process, that number of particles in a finite window is measurable with respect to the
completion of the sigma-algebra describing the configurations outside that finite window. Their argument is spectral: they construct, for any small $\varepsilon$, a compactly supported smooth function $\varphi_\varepsilon$, such that $\varphi_\varepsilon$ equals 1 in a fixed finite window and the linear statistic corresponding to $\varphi_\varepsilon$ has variance smaller than $\varepsilon$.

In the same spirit, we consider general stationary stochastic processes (in broad sense) $X_n$, $n \in \mathbb{Z}$ such that $X_0$ lies in the closed linear span of $X_n$, $n \neq 0$; following Ghosh and Peres, we call such processes linearly rigid. Using a criterion of Kolmogorov, we show that it suffices, for a stationary stochastic process to be rigid, that the spectral density vanish at zero and belong to the Zygmund class $\Lambda^*(1)$. We next give sufficient condition for stationary determinantal point processes on $\mathbb{Z}$ and on $\mathbb{R}$ to be rigid. Finally, we show that the determinantal point process on $\mathbb{R}^2$ induced by a tensor square of Dyson sine-kernels is not linearly rigid.

We now turn to more precise statements. Let $X = \{X_n : n \in \mathbb{Z}^d\}$ be a multi-dimensional time stationary stochastic process of real-valued random variables defined on a probability space $(\Omega, \mathbb{P})$. Let $H(X) \subset L^2(\Omega, \mathbb{P})$ denote the closed subspace linearly spanned by $\{X_n : n \in \mathbb{Z}^d\}$ and let $\check{H}_0(X)$ denote the one linearly spanned by $\{X_n : n \in \mathbb{Z}^d \setminus \{0\}\}$.

**Definition 1.1.** The stochastic process $X$ is said to be linearly rigid if

$$X_0 \in \check{H}_0(X).$$

Let $\text{Conf}(\mathbb{R}^d)$ be the set of configurations on $\mathbb{R}^d$. For a bounded Borel subset $B \subset \mathbb{R}^d$, we denote $N_B : \text{Conf}(\mathbb{R}^d) \to \mathbb{N} \cup \{0\}$ the function defined by

$$N_B(X) := \text{the cardinality of } B \cap X.$$

The space $\text{Conf}(\mathbb{R}^d)$ is equipped with the Borel $\sigma$-algebra which is the smallest $\sigma$-algebra making all $N_B$’s measurable. Recall that a point process with phase space $\mathbb{R}^d$ is, by definition, a Borel probability measure on the space $\text{Conf}(\mathbb{R}^d)$. For the background on point process, the reader is referred to Daley and Vere-Jones’ book [3].

Given a stationary point process on $\mathbb{R}^d$ and $\lambda > 0$, we introduce the stationary stochastic process $N^{(\lambda)} = (N^{(\lambda)}_n)_{n \in \mathbb{Z}^d}$ by the formula

$$N^{(\lambda)}_n(X) := \text{the cardinality of } X \cap (n\lambda + [-\lambda/2, \lambda/2]^d).$$

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Definition 1.2. A stationary point process \( \mathbb{P} \) on \( \mathbb{R}^d \) is called **linearly rigid**, if for any \( \lambda > 0 \), the stationary stochastic process \( N^{(\lambda)} = (N^{(\lambda)}_n)_{n \in \mathbb{Z}^d} \) is linearly rigid, i.e.,

\[ N^{(\lambda)}_0 \in \tilde{H}_0(N^{(\lambda)}). \]

The above definition is motivated by the definition due to Ghosh and Peres of rigidity of point processes on \( \mathbb{R}^d \), see [5] and [6]. Given a Borel subset \( C \subset \mathbb{R}^d \), we will denote

\[ \mathcal{F}_C = \sigma(\{N_B : B \subset C, B \text{ bounded Borel}\}) \]

the \( \sigma \)-algebra generated by all random variables of the form \( N_B \) where \( B \subset C \) ranges over all bounded Borel subsets of \( C \). Let \( \mathbb{P} \) be a point process on \( \mathbb{R} \), i.e., \( \mathbb{P} \) is a Borel probability on \( \text{Conf}(\mathbb{R}^d) \), and denote \( \mathcal{F}^\mathbb{P}_C \) for the \( \mathbb{P} \)-completion of \( \mathcal{F}_C \).

**Definition 1.3** (Ghosh [5], Ghosh-Peres [6]). A point process \( \mathbb{P} \) on \( \mathbb{R}^d \) is called **rigid**, if for any bounded Borel set \( B \subset \mathbb{R}^d \) with Lebesgue-negligible boundary \( \partial B \), the random variable \( N_B \) is \( \mathcal{F}^\mathbb{P}_{\mathbb{R}^d \setminus B} \)-measurable.

**Remark 1.1.** Of course, in the above definition, it suffices to take Borel sets \( B \) of the form \( [-\gamma, \gamma]^d \) for \( \gamma > 0 \), cf. [6].

A linear rigid stationary point process on \( \mathbb{R}^d \) is of course rigid in the sense of Ghosh and Peres. Observe that proofs for rigidity in [5], [6] and [1] in fact establish linear rigidity. We would like also to mention a notion of insertion-deletion tolerance studied by Holroyd and Soo in [7], which is in contrast to the notion of rigidity property.

### 2 The Kolmogorov criterion for linear rigidity

In this note, the Fourier transform of a function \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) is defined as

\[ \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i2\pi x \cdot \xi} dx. \]

Denote by \( \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \) the \( d \)-dimensional torus. In what follows, we identify \( \mathbb{T}^d \) with \([ -1/2, 1/2]^d \). The Fourier coefficients of a measure \( \mu \) on \( \mathbb{T}^d \) are given, for any \( k \in \mathbb{Z}^d \), by the formula

\[ \hat{\mu}(k) = \int_{\mathbb{T}^d} e^{-i2\pi k \cdot \theta} d\mu_X(\theta), \text{ where } k \cdot \theta := k_1 \theta_1 + \cdots + k_d \theta_d. \]
Denote by $\mu_X$ the spectral measure of $X$, i.e.,
\[ \forall k \in \mathbb{Z}^d, \quad \mathbb{E}(X_0X_k) = \mathbb{E}(X_nX_{n+k}) = \int_{T^d} e^{-i2\pi k \cdot \theta} d\mu_X(\theta) = \hat{\mu}_X(k). \quad (3) \]

Recall that we have the following natural isometric isomorphism
\[ H(X) \simeq L^2(T^d, \mu_X), \quad (4) \]
by assigning to $X_n \in H(X)$ the function $\theta \mapsto e^{i2\pi n \cdot \theta} \in L^2(T^d, \mu_X)$.
Let $\mu_X = \mu_a + \mu_s$ be the Lebesgue decomposition of $\mu_X$ with respect to the normalized Lebesgue measure $m(d\theta) = d\theta_1 \cdots d\theta_d$ on $T^d$, i.e., $\mu_a$ is absolutely continuous with respect to $m$ and $\mu_s$ is singular to $m$. Set
\[ \omega_X(\theta) := \frac{d\mu_a}{dm}(\theta). \]

**Lemma 2.1 (The Kolmogorov Criterion).** We have
\[ \text{dist}(X_0, \tilde{H}_0(X)) = \left( \int_{T^d} \omega_X^{-1} dm \right)^{-1/2}. \]

The right side is to be interpreted as zero if $\int_{T^d} \omega_X^{-1} dm = \infty$.

When the measure $\mu$ is assumed to be absolutely continuous with respect to $m$, Lemma 2.1 is a result of Kolmogorov, see Remark 5.17 in Lyons-Steif [10].

**Corollary 2.2.** The stationary stochastic process $X = (X_n)_{n \in \mathbb{Z}^d}$ is linearly rigid if and only if
\[ \int_{T^d} \omega_X^{-1} dm = \infty. \]

**Proof of Lemma 2.1.** We follow the argument of Lyons-Steif [10]. By the Lebesgue decomposition of $\mu$, we may take a subset $A \subset T^d$ of full Lebesgue measure $m(A) = 1$, such that $\mu_a(A) = 1$ and $\mu_s(A) = 0$.

Denote
\[ L_0 = \text{span} L^2(T^d, \mu_X)[e^{i2\pi n \cdot \theta} : n \neq 0]. \]
By the isometric isomorphism (4), it suffices to show that
\[ \text{dist}(1, L_0) = \left( \int_{T^d} \omega_X^{-1} dm \right)^{-1/2}, \quad (5) \]
where 1 is the constant function taking value 1. Write
\[ 1 = p + h, \text{ such that } p \perp L_0, h \in L_0. \]

Modifying, if necessary, the values of \( p \) and \( h \) on a \( \mu \)-negligible subset, we may assume that
\[ 1 = p(\theta) + h(\theta) \text{ for all } \theta \in \mathbb{T}^d. \]

Since \( p \perp L_0 \), we have
\[ 0 = \langle p, e^{i2\pi n \cdot \theta} \rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} p(\theta)e^{-i2\pi n \cdot \theta}d\mu(\theta), \text{ for any } n \in \mathbb{Z}^d \setminus 0. \tag{6} \]

Therefore, the complex measure \( p \cdot d\mu \) is a multiple of Lebesgue measure, i.e., there exists \( \xi \in \mathbb{C} \), such that
\[ p \cdot d\mu = \xi dm. \tag{7} \]

It follows that \( p \) must vanish almost everywhere with respect to the singular component \( \mu_s \) of \( \mu \), and \( p(\theta)\omega_X(\theta) = \xi \) for \( m \)-almost every \( \theta \in \mathbb{T}^d \). Thus we have
\[ \| p \|_{L^2(d\mu)} = \| p \|_{L^2(d\mu_\omega)}. \tag{7} \]

and
\[ h(\theta) = 1 - \xi \omega_X(\theta)^{-1} \text{ for } m\text{-almost every } \theta \in \mathbb{T}^d. \tag{8} \]

**Case 1:** \( \int_{\mathbb{T}^d} \omega_X^{-1}dm < \infty \).

Define a function \( f : \mathbb{T}^d \to \mathbb{C} \) by \( f = \omega_X^{-1}\chi_A \). Then \( f \in L^2(d\mu) \ominus L_0 \). Indeed,
\[ \| f \|^2_{L^2(d\mu)} = \int_{\mathbb{T}^d} \omega_X^{-2}\chi_A d\mu = \int_{\mathbb{T}^d} \omega_X^{-2}d\mu_a = \int_{\mathbb{T}^d} \omega_X^{-3}dm < \infty. \]

And, for all \( n \in \mathbb{Z}^d \setminus 0 \),
\[ \langle f, e^{i2\pi n \cdot \theta} \rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} \omega_X(\theta)^{-1}\chi_A(\theta)e^{-i2\pi n \cdot \theta}d\mu(\theta) = \int_{\mathbb{T}^d} e^{-i2\pi n \cdot \theta}dm(\theta) = 0. \]

It follows that \( f \perp h \), i.e.,
\[ 0 = \langle h, f \rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} h\omega_X^{-1}\chi_A d\mu = \int_{\mathbb{T}^d} hdm. \]

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By (8), we get
\[ \int_{T^d} (1 - \xi \omega_X^{-1}) dm = 0, \]
and hence
\[ \xi = (\int_{T^d} \omega_X^{-1} dm)^{-1}. \]
It follows that
\[ \text{dist}(1, L_0)^2 = \|p\|^2_{L^2(\mu)} = \|p\|^2_{L^2(\mu_a)} = \xi^2 \int_{T^d} \omega_X \omega_X dm = \xi. \]
This shows the desired equality (5).

**Case 2:** \( \int_{T^d} \omega_X^{-1} dm = \infty. \)

We claim that \( \xi = 0. \) If the claim were verified, then we would get the desired identity in this case
\[ \text{dist}(1, L_0) = 0. \]
So let us turn to the proof of the claim. We argue by contradiction. If \( \xi \neq 0, \) then \( p \neq 0 \) and
\[ \|p\|^2_{L^2(\mu)} = \|p\|^2_{L^2(\mu_a)} = \xi^2 \|\omega_X^{-1}\|^2_{L^2(\mu_a)} = \xi^2 \int_{T^d} \omega_X^{-1} dm = \infty. \]
This contradicts the fact that \( p \in L^2(\mu). \)

**Remark 2.1.** The same argument shows that, in the case of one-dimensional time, the following assertions are equivalent:

- \[ \sum_{k=-n}^{n} X_k \in \text{span}\{X_j : |j| \geq n + 1\}; \]
- for any \( \alpha_1, \ldots, \alpha_n \in (-1/2, 1/2) \setminus \{0\}, \) we have
\[ \int_{\mathbb{R}} \prod_{j=1}^{n} \frac{|e^{i2\pi \theta} - e^{i2\pi \alpha_j}|^2 |e^{i2\pi \theta} - e^{-i2\pi \alpha_j}|^2}{\omega_X(\theta)} dm(\theta) = \infty. \]

It would be interesting to find a similar characterization for multi-dimensional time as well.
Denote by \( \text{Cov}(U, V) \) the covariance between two random variables \( U \) and \( V \):
\[
\text{Cov}(U, V) = \mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V).
\]

If \( X = (X_n)_{n \in \mathbb{Z}^d} \) is a stochastic process such that
\[
\sum_{n \in \mathbb{Z}^d} |\text{Cov}(X_0, X_n)| < \infty,
\]
then we may define a continuous function on \( \mathbb{T}^d \) by the formula
\[
\omega_X(\theta) := \sum_{n \in \mathbb{Z}^d} \text{Cov}(X_0, X_n)e^{i2\pi n \cdot \theta}.
\]

**Lemma 2.3.** Let \( X = (X_n)_{n \in \mathbb{Z}^d} \) be a stationary stochastic process satisfying condition (9). Then we have the following explicit Lebesgue decomposition of \( \mu_X \):
\[
\mu_X = (\mathbb{E}X_0)^2 \cdot \delta_0 + \omega_X \cdot m,
\]
where \( \delta_0 \) is the Dirac measure on the point \( 0 \in \mathbb{T}^d \) and \( \omega_X \) is the function on \( \mathbb{T}^d \) defined by (10).

**Proof.** Note that, under the assumption (9), the function \( \omega_X(\theta) \) is well-defined and continuous on \( \mathbb{T}^d \). For proving the decomposition (11), it suffices to show that the Fourier coefficients of \( \mu_X \) coincide with those of \( \nu_X := (\mathbb{E}X_0)^2 \cdot \delta_0 + \omega_X \cdot m \). But if \( n \in \mathbb{Z}^d \), then
\[
\hat{\nu}_X(n) = (\mathbb{E}X_0)^2 + \text{Cov}(X_0, X_n) = \mathbb{E}(X_0X_n) = \hat{\mu}_X(n).
\]
The lemma is completely proved. \( \square \)

### 3 A sufficient condition for linear rigidity

**Theorem 3.1.** Let \( X = (X_n)_{n \in \mathbb{Z}} \) be a stationary stochastic process. If
\[
\sup_{N \geq 1} \left( N \sum_{|n| \geq N} |\text{Cov}(X_0, X_n)| \right) < \infty,
\]
and
\[
\sum_{n \in \mathbb{Z}} \text{Cov}(X_0, X_n) = 0.
\]
Then \( X \) is linearly rigid.
Remark 3.1. The condition (12) is a sufficient condition such that the spectral density $\omega_X$ is a function in the Zygmund class $\Lambda_*(1)$, see below for definition. The condition (13) implies in particular that $\omega_X$ vanishes at the point $0 \in \mathbb{T}$.

We shall apply a result of F. Móricz [12, Thm. 3] on absolutely convergent Fourier series and Zygmund class functions. Recall that a continuous 1-periodic function $\varphi$ defined on $\mathbb{R}$ is said to be in the Zygmund class $\Lambda_*(1)$, if there exists a constant $C$ such that

$$|\varphi(x + h) - 2\varphi(x) + \varphi(x - h)| \leq Ch$$

for all $x \in \mathbb{R}$ and for all $h > 0$.

**Theorem 3.2 (Móricz, [12]).** If $\{c_n\}_{n \in \mathbb{Z}} \in \mathbb{C}$ is such that

$$\sup_{N \geq 1} \left( N \sum_{|n| \geq N} |c_n| \right) < \infty,$$

then the function $\varphi(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{i2\pi n \theta}$ is in the Zygmund class $\Lambda_*(1)$.

**Proof of Theorem 3.1.** First, in view of (10), our assumption (13) implies

$$\omega_X(0) = 0.$$

Next, by Theorem 3.2, under the assumption (12), we have

$$\omega_X \in \Lambda_*(1).$$

Since all Fourier coefficients of $\omega_X$ are real, we have

$$\omega_X(\theta) = \omega_X(-\theta).$$

Consequently, there exists $C > 0$, such that

$$\omega_X(\theta) = \frac{\omega_X(\theta) + \omega_X(-\theta)}{2} = \frac{\omega_X(\theta) + \omega_X(-\theta) - 2\omega_X(0)}{2} \leq C|\theta|,$$

whence

$$\int_{\mathbb{T}} \omega_X^{-1} dm = \infty,$$

and the stochastic process $X = (X_n)_{n \in \mathbb{Z}}$ is linearly rigid by the Kolmogorov criterion. $\square$
4 Applications to stationary determinantal point processes

In this section, we first give a sufficient condition for linear rigidity of stationary determinantal point processes on $\mathbb{R}$ and then give an example of a very simple stationary, but not linearly rigid, determinantal point process on $\mathbb{R}^2$. We briefly recall the main definitions. Let $B \subset \mathbb{R}^d$ be a bounded Borel subset. Let $K_B : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the operator of convolution with the Fourier transform $\hat{\chi}_B$ of the indicator function $\chi_B$. In other words, the kernel of $K_B$ is

$$K_B(x, y) = \hat{\chi}_B(x - y). \quad (16)$$

In particular, if $d = 1$ and $B = (-1/2, 1/2)$, then we find the well-known Dyson sine kernel

$$K_{\text{sine}}(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}.$$

Note that we always have $K_B(x, x) = K_B(0, 0)$. Denote by $\mathbb{P}_{K_B}$ the determinantal point process induced by $K_B$. For the background on the determinantal point processes, the reader is referred to [8], [9], [11], [13].

**Proposition 4.1.** Let $\mathbb{P}_{K_B}$ be the stationary determinantal point process on $\mathbb{R}^d$ induced by the kernel $K_B$ in (16). For any $\lambda > 0$, denote by $N^{(\lambda)} = (N^{(\lambda)}_n)_{n \in \mathbb{Z}^d}$ the stationary stochastic process associated to $\mathbb{P}_{K_B}$ as in (2). Then

$$\sum_{n \in \mathbb{Z}^d} |\text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})| < \infty \quad (17)$$

and

$$\sum_{n \in \mathbb{Z}^d} \text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) = 0. \quad (18)$$

**Proof.** Fix a number $\lambda > 0$, for simplifying the notation, let us denote $N_n^{(\lambda)}$ by $N_n$. Denote for any $n \in \mathbb{Z}^d$,

$$Q_n = n\lambda + [-\lambda/2, \lambda/2)^d.$$

By definition of a determinantal point process, we have

$$\mathbb{E}(N_n) = \mathbb{E}(N_0) = \int_{Q_0} K_B(x, x)dx = \lambda^d K_B(0, 0).$$
If \( n \neq 0 \), we have
\[
\mathbb{E}(N_0 N_n) = \int \int \chi_{Q_0}(x)\chi_{Q_n}(y) \left| \begin{array}{cc} K_B(x, x) & K_B(x, y) \\ K_B(y, x) & K_B(y, y) \end{array} \right| dxdy
\]
\[
= \lambda^2 K_B(0, 0)^2 - \int \int_{Q_0 \times Q_n} |K_B(x, y)|^2 dxdy,
\]
whence
\[
\text{Cov}(N_0, N_n) = -\int \int_{Q_0 \times Q_n} |K_B(x, y)|^2 dxdy. \tag{19}
\]

We also have
\[
\mathbb{E}(N_0^2) = \mathbb{E} \left[ \sum_{x,y \in X} \chi_{Q_0}(x)\chi_{Q_0}(y) \right]
\]
\[
= \mathbb{E} \left[ \sum_{x \in X} \chi_{Q_0}(x) \right] + \mathbb{E} \left[ \sum_{x, y \in X, x \neq y} \chi_{Q_0}(x)\chi_{Q_0}(y) \right]
\]
\[
= \int_{Q_0} K_B(x, x)dx + \int \int \chi_{Q_0}(x)\chi_{Q_0}(y) \left| \begin{array}{cc} K_B(x, x) & K_B(x, y) \\ K_B(y, x) & K_B(y, y) \end{array} \right| dxdy
\]
\[
= \lambda^d K_B(0, 0) + \lambda^2 K_B(0, 0)^2 - \int \int_{Q_0 \times Q_0} |K_B(x, y)|^2 dxdy,
\]
whence
\[
\text{Cov}(N_0, N_0) = \text{Var}(N_0) = \lambda^d K_B(0, 0) - \int \int_{Q_0 \times Q_0} |K_B(x, y)|^2 dxdy. \tag{20}
\]

Now recall that \( K_B \) is an orthogonal projection. Thus we have
\[
K_B(0, 0) = K_B(x, x) = \int |K_B(x, y)|^2 dy = \sum_{n \in \mathbb{Z}^d} \int_{Q_n} |K_B(x, y)|^2 dy. \tag{21}
\]

The identities (19), (20) and (21) imply that
\[
\sum_{n \in \mathbb{Z}^d} \text{Cov}(N_0, N_n) = \lambda^d K_B(0, 0) - \int_{Q_0} dx \sum_{n \in \mathbb{Z}^d} \int_{Q_n} |K_B(x, y)|^2 dy
\]
\[
= \lambda^d K_B(0, 0) - \lambda^d K_B(0, 0) = 0.
\]

Moreover, the above series converge absolutely. Proposition 4.1 is completely proved.
Remark 4.1. By Lemma 2.3 and Proposition 4.1, we see that for any stationary determinantal point process induced by a projection, the spectral density of the associated stochastic process \( N^{(\lambda)} \) always vanishes at 0.

### 4.1 Stationary determinantal point processes on \( \mathbb{R} \)

**Theorem 4.2.** Assume that \( B \subset \mathbb{R} \) satisfies

\[
\sup_{R > 0} \left( R \int_{|\xi| \geq R} |\widehat{\chi}_B(\xi)|^2 d\xi \right) < \infty. \tag{22}
\]

Then the stationary determinantal point process \( \mathbb{P}_{K_B} \) is linearly rigid.

**Proof.** By definition of linear rigidity, we need to show that for any \( \lambda > 0 \), the stochastic process \( N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}} \) is linearly rigid. As in the proof of Proposition 4.1, we denote \( N_n^{(\lambda)} \) by \( N_n \). By Theorem 3.1, it suffices to show that

\[
\sup_{N \geq 1} \left( N \sum_{|n| \geq N} |\text{Cov}(N_0, N_n)| \right) < \infty, \tag{23}
\]

and

\[
\sum_{n \in \mathbb{Z}} \text{Cov}(N_0, N_n) = 0. \tag{24}
\]

By Proposition 4.1, the identity (24) holds in general case. It remains to prove (23). By (19), we have

\[
\sup_{N \geq 1} \left( N \sum_{|n| \geq N} |\text{Cov}(N_0, N_n)| \right) = \sup_{N \geq 1} N \int \bigcup_{|n| \geq N} Q_n |\widehat{\chi}_B(x - y)|^2 dx dy
\]

\[
\leq \sup_{N \geq 1} \lambda N \int_{|\xi| \geq (N - 1)\lambda} |\widehat{\chi}_B(\xi)|^2 d\xi < \infty
\]

where in the last inequality, we used our assumption (22). Theorem 4.2 is proved completely. \( \square \)

**Remark 4.2.** When \( B \) is a finite union of finite intervals on the real line, the rigidity of the stationary determinantal point process \( \mathbb{P}_{K_B} \) is due to Ghosh [5].
4.2 Tensor product of sine kernels

In higher dimension, the situation becomes quite different. Let
\[ S = I \times I = (-1/2, 1/2) \times (-1/2, 1/2) \subset \mathbb{R}^2. \]

Then the associate kernel \( K_S \) has a tensor form: \( K_S = K_{\text{sine}} \otimes K_{\text{sine}} \), that is, for \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( \mathbb{R}^2 \), we have
\[
K_S(x, y) = K_{\text{sine}}(x_1, y_1)K_{\text{sine}}(x_2, y_2) = \frac{\sin(\pi(x_1 - y_1)) \sin(\pi(x_2 - y_2))}{\pi(x_1 - y_1) \pi(x_2 - y_2)}.
\]

**Proposition 4.3.** The determinantal point process \( \mathbb{P}_{K_S} \) is not linearly rigid. More precisely, let \( N^{(1)} = (N^{(1)}_n)_{n \in \mathbb{Z}^2} \) be the stationary stochastic process given as in Definition 1.2, then
\[
N^{(1)}_0 \notin \mathcal{H}_0(N^{(1)}).
\]

To prove the above result, we need to introduce some extra notation. First, we define the multiple Zygmund class \( \Lambda^* \) as follows. A continuous function \( \varphi(x, y) \) periodic in each variable with period 1 is said to be in the multiple Zygmund class \( \Lambda^*(1, 1) \) if for the double difference difference operator \( \Delta_{2,2} \) of second order in each variable, applied to \( \varphi \), there exists a constant \( C > 0 \), such that for all \( x = (x_1, x_2) \in (-1/2, 1/2) \times (-1/2, 1/2) \) and \( h_1, h_2 > 0 \), we have
\[
|\Delta_{2,2}\varphi(x_1, x_2; h_1, h_2)| \leq Ch_1h_2, \tag{25}
\]
where
\[
\begin{align*}
\Delta_{2,2}\varphi(x_1, x_2; h_1, h_2) &= \varphi(x_1 + h_1, x_2 + h_2) + \varphi(x_1 - h_1, x_2 + h_2) \\
&\quad + \varphi(x_1 + h_1, x_2 - h_2) + \varphi(x_1 - h_1, x_2 - h_2) - 2\varphi(x_1 + h_1, x_2) \\
&\quad - 2\varphi(x_1 - h_1, x_2) - 2\varphi(x_1, x_2 + h_2) - 2\varphi(x_1, x_2 - h_2) + 4\varphi(x_1, x_2).
\end{align*}
\]

The following result is due to Fülop and Móricz [4, Thm 2.1 and Rem. 2.3]

**Theorem 4.4 (Fülop-Móricz).** If \( \{c_{jk}\}_{j,k \in \mathbb{Z}} \in \mathbb{C} \) is such that
\[
\sup_{N \geq 1, M \geq 1} \left( MN \sum_{|j| \geq N, |k| \geq M} |c_{jk}| \right) < \infty, \tag{26}
\]
then the function
\[
\varphi(\theta_1, \theta_2) = \sum_{j,k \in \mathbb{Z}} c_{jk} e^{i2\pi(j\theta_1 + k\theta_2)}
\]
is in the Zygmund class \( \Lambda^*(1, 1) \).
Let us turn to the study of the density function $\omega_N(1)$.

**Lemma 4.5.** There exists $c > 0$, such that for any $\theta_1, \theta_2 \in (-1/2, 1/2)$, we have

$$\omega_N(1)(\theta_1, \theta_2) \geq c(|\theta_1| + |\theta_2|).$$

**Proof.** To make notation lighter, in this proof we simply write $\omega$ for $\omega_N(1)$.

Denote $S_n = S \times (n + S)$ where $n + S := (-1/2 + n_1, 1/2 + n_1) \times (-1/2 + n_2, 1/2 + n_2)$. By the same argument as in the proof of Theorem 4.2, we obtain that for any $n = (n_1, n_2) \in \mathbb{Z}^2 \setminus 0$,

$$\hat{\omega}(n) = -\int_{S_n} |K_S(x, y)|^2 dxdy,$$

and

$$\hat{\omega}(0) = K_S(0, 0) - \int_{S_0} |K_S(x, y)|^2 dxdy.$$

The following properties can be easily checked.

- $\sum_{n \in \mathbb{Z}^2} \hat{\omega}(n) = 0$.
- $\hat{\omega}(\varepsilon_1 n_1, \varepsilon_2 n_2) = \hat{\omega}(n_1, n_2)$, where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.
- There exist $c, C > 0$, such that

$$\frac{c}{(1 + n_1^2)(1 + n_2^2)} \leq |\hat{\omega}(n_1, n_2)| \leq \frac{C}{(1 + n_1^2)(1 + n_2^2)}.$$

For instance, $\sum_{n \in \mathbb{Z}^2} \hat{\omega}(n) = 0$ follows from Proposition 4.1. These properties combined with Theorem 4.4 yield that

- $\omega(0, 0) = 0$.
- $\omega(\varepsilon_1 \theta_1, \varepsilon_2 \theta_2) = \omega(\theta_1, \theta_2)$ for any $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $\theta_1, \theta_2 \in (-1/2, 1/2)$.
- The function $\omega(\theta_1, \theta_2)$ is in the multiple Zygmund class $\Lambda_* (1, 1)$.

Hence there exists $C > 0$, such that

$$|\omega(\theta_1, \theta_2) - \omega(\theta_1, 0) - \omega(0, \theta_2)| \leq C|\theta_1 \theta_2|. \quad (27)$$
Lemma 4.6. There exists $c > 0$, such that
\[
\omega(\theta_1, 0) \geq c|\theta_1| \text{ and } \omega(0, \theta_2) \geq c|\theta_2|.
\]
(28)

Let us postpone the proof of Lemma 4.6 and proceed to the proof of Lemma 4.5. The inequalities (27) and (28) imply that
\[
\omega(\theta_1, \theta_2) \geq c(|\theta_1| + |\theta_2|) - C|\theta_1\theta_2|.
\]

To prove the lower bound of type as in the lemma, it suffices to prove it when $|\theta_1|$ and $|\theta_2|$ are small enough, for instance, $2C|\theta_1| \leq c$, then we have
\[
\omega(\theta_1, \theta_2) \geq \frac{c}{2}(|\theta_1| + |\theta_2|).
\]

Now let us turn to the

Proof of Lemma 4.6. By symmetry, it suffices to prove that there exists $c > 0$, such that $\omega(\theta_1, 0) \geq |\theta_1|$. To this end, let us denote $\omega_1(\theta_1) := \omega(\theta_1, 0)$. Then $\omega_1(0) = 0$ and there exists $c > 0$ such that if $k \neq 0$, then
\[
\hat{\omega}_1(k) < 0 \text{ and } |\hat{\omega}_1(k)| \geq \frac{c}{(1 + k^2)},
\]
Indeed, we have
\[
\omega_1(\theta_1) = \sum_{k \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \hat{\omega}(k, n_2)e^{i2\pi k\theta_1},
\]
if $k \neq 0$, then $\hat{\omega}(k, n_2) < 0$ and hence
\[
|\hat{\omega}_1(k)| = \sum_{n_2 \in \mathbb{Z}} |\hat{\omega}(k, n_2)| \geq \sum_{n_2 \in \mathbb{Z}} \frac{c}{(1 + n_2^2)(1 + k^2)} \geq \frac{c'}{1 + k^2}.
\]
Note also that $\omega_1(0) = \omega(0, 0) = 0$, hence
\[
\sum_{k \in \mathbb{Z}} \hat{\omega}_1(k) = 0.
\]
It follows that
\[ \omega_1(\theta_1) = \sum_{k \in \mathbb{Z}} \hat{\omega}_1(k) e^{i2\pi k \theta_1} = \sum_{k \in \mathbb{Z}} \hat{\omega}_1(k) \left( \frac{e^{i2\pi k \theta_1} + e^{-i2\pi k \theta_1}}{2} - 1 \right) \]
\[ = \sum_{k \in \mathbb{Z}, k \neq 0} -\hat{\omega}_1(k) (1 - \cos(2\pi k \theta_1)) = \sum_{k \in \mathbb{Z}, k \neq 0} |\hat{\omega}_1(k)| (1 - \cos(2\pi k \theta_1)) \]
\[ \geq c'' \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} (1 - \cos(2\pi(2j-1) \theta_1)) \]

Combining with the classical formulae
\[ \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{\pi^2}{8}, \]
\[ |\alpha| = \frac{1}{4} - \frac{2}{\pi^2} \sum_{j=1}^{\infty} \frac{\cos(2(2j-1) \pi \alpha)}{(2j-1)^2}, \text{ for } \alpha \in (-1/2, 1/2); \]
we obtain that
\[ \omega_1(\theta_1) \geq c'' \frac{\pi^2}{2} |\theta_1|. \]

\[ \square \]

**Proof of Proposition 4.3.** By Lemma 2.1, it suffices to show that
\[ \int_{\mathbb{T}^2} \omega_{N(1)}^{-1} d\mu < \infty. \] (29)

By Lemma 4.5, the inequality (29) follows from the following elementary inequality
\[ \int_{|\theta_1| < 1/2, |\theta_2| < 1/2} \frac{1}{|\theta_1| + |\theta_2|} d\theta_1 d\theta_2 < \infty. \]
\[ \square \]
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