

Effective irrationality measures for quotients of logarithms of rational numbers

Yann Bugeaud

Abstract. We establish uniform irrationality measures for the quotients of the logarithms of two rational numbers which are very close to 1. Our proof is based on a refinement in the theory of linear forms in logarithms which goes back to a paper of Shorey.

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1. Introduction and result

Let ξ be an irrational real number. The real number μ is an irrationality measure of ξ if, for every positive ε , there are a positive number $C(\xi, \varepsilon)$ and at most finitely many rational numbers p/q with $q \geq 1$ and

$$\left| \xi - \frac{p}{q} \right| < \frac{C(\xi, \varepsilon)}{q^{\mu+\varepsilon}}.$$

If, moreover, the constant $C(\xi, \varepsilon)$ is effectively computable for every positive ε , then μ is an effective irrationality measure of ξ . We denote by $\mu(\xi)$ (resp. $\mu_{\text{eff}}(\xi)$) the infimum of the irrationality measures (resp. effective irrationality measures) of ξ . It follows from the theory of continued fractions that $\mu(\xi) \geq 2$ for every irrational real number ξ and an easy covering argument shows that there is equality for almost all ξ , with respect to the Lebesgue measure.

The following statement is a straightforward consequence of Baker's theory of linear forms in logarithms (see e.g. [Wa00] and the references therein). By definition, two positive rational numbers are multiplicatively independent if the quotient of their logarithms is irrational.

Theorem 1.1 *Let a_1, a_2, b_1, b_2 be positive integers with $a_1 > a_2$ and $b_1 > b_2$. Assume that a_1/a_2 and b_1/b_2 are multiplicatively independent. There exists an absolute, effectively computable, constant C such that*

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq C(\log a_1)(\log b_1).$$

The purpose of this note is to show how a known refinement in the theory of linear forms in logarithms in the special case where the rational numbers are very close to 1, which goes back to Shorey's paper [Sho74], allows one to considerably improve Theorem 1.1 in this special case. Several spectacular applications to Diophantine problems and to Diophantine equations of this idea of Shorey have already been found; see for example [Wa78, Wa00] and the survey [Bu08]. Quite surprisingly, it seems that it has not yet been noticed that it can be used to give uniform upper bounds for irrationality measures of roots of rational numbers (see [Bu16]) and of quotients of logarithms of rational numbers, under some suitable assumptions.

Our main result is the following.

Theorem 1.2 *Let a_1, a_2, b_1, b_2 be positive integers such that*

$$\max\{16, a_2\} < a_1 < 6a_2/5 \quad \text{and} \quad \max\{16, b_2\} < b_1 < 6b_2/5. \quad (1.1)$$

Define η by $a_1 - a_2 = a_1^{1-\eta}$ and ν by $b_1 - b_2 = b_1^{1-\nu}$. If a_1/b_1 and a_2/b_2 are multiplicatively independent, then we have

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq 1 + 27638 \frac{(\log a_1)(\log b_1)}{\min\{\eta \log a_1, \nu \log b_1\}^2}. \quad (1.2)$$

We display an immediate corollary of Theorem 1.2 which deals with the case $\eta > 1/2$ and $\nu > 1/2$. It illustrates the strength of the theory of linear forms in logarithms.

Corollary 1.3 *Let a_1, a_2, b_1, b_2 be positive integers such that*

$$36 \leq a_2 < a_1 < a_2 + \sqrt{a_1}, \quad 36 \leq b_2 < b_1 < b_2 + \sqrt{b_1}, \quad \text{and} \quad \sqrt{b_1} < a_1 < b_1^2.$$

If a_1/b_1 and a_2/b_2 are multiplicatively independent, then we have

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq 221105. \quad (1.3)$$

It is apparent from the proof of Theorem 1.2 that the numerical constants in (1.2) and (1.3) can be reduced (roughly, divided by 3) if a_1 and b_1 are sufficiently large. No particular significance has to be attached to the numerical constant $6/5$ in (1.1).

Let a, b and d be positive integers with $a \neq b$ and $\max\{a, b\} < d$. Under certain conditions, Rhin [Rh83] (see also [RhTo86]) obtained explicit upper bounds for

$$\mu_{\text{eff}}\left(\frac{\log(1 + a/d)}{\log(1 + b/d)}\right).$$

His approach, which gives better numerical results than ours, heavily uses the fact that the two rational numbers a/d and b/d have the same denominator. It seems to us that Theorem 1.2, which applies without any specific restriction on the denominators a_2 and b_2 of the rational numbers, is new and cannot be straightforwardly derived from the methods of [Rh83, RhTo86].

2. Proof of Theorem 1.2

We reproduce with some simplification Corollary 2.4 of Guillon [Gou06] in the special case where the algebraic numbers involved are rational numbers. We replace his assumption $E \geq 2$ by $E \geq 15$, to avoid trouble with the quantity $\log \log \log E$ occurring in the definition of E^* in Corollary 2.4 of [Gou06], which is not defined if E is too small.

Theorem G *Let a_1, a_2, b_1, b_2 be positive integers such that a_1/a_2 and b_1/b_2 are multiplicatively independent and greater than 1. Let A and B be real numbers such that*

$$A \geq \max\{a_1, e\}, \quad B \geq \max\{b_1, e\}.$$

Let x and y be positive integers and set

$$X' = \frac{x}{\log A} + \frac{y}{\log B}.$$

Set

$$E = 1 + \min \left\{ \frac{\log A}{\log(a_1/a_2)}, \frac{\log B}{\log(b_1/b_2)} \right\}$$

and

$$\log X = \max\{\log X' + \log E, 265 \log E, 600 + 150 \log E\}.$$

Assume furthermore that $15 \leq E \leq \min\{A^{3/2}, B^{3/2}\}$. Then,

$$\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -8550(\log A)(\log B)(\log X)(4 + \log E)(\log E)^{-3}.$$

It is crucial for our proof that the dependence on X comes through the factor $(\log X)$ and not through $(\log X)^2$, as in [LMN95]. Instead of Theorem G we could use an earlier result of Waldschmidt [Wa93] (see also Theorem 9.1 of [Wa00]), but the numerical constants in (1.2) and (1.3) would then be slightly larger.

Proof of Theorem 1.2. Our aim is to estimate from below the quantity

$$\left| \frac{\log(a_1/a_2)}{\log(b_1/b_2)} - \frac{x}{y} \right|,$$

for large positive integers x, y . We will establish a lower bound of the form

$$\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -C_1(C_2 + \log \max\{x, y\}), \quad (2.1)$$

for some quantities C_1, C_2 which depend at most on a_1, a_2, b_1 and b_2 . This will show that $1 + C_1$ is an irrationality measure for $\log(a_1/a_2)/\log(b_1/b_2)$.

We apply Theorem G and set $A = a_1$ and $B = b_1$.

Observe that

$$\log(a_1/a_2) = \log(1 + ((a_1 - a_2)/a_2)) \leq (a_1 - a_2)/a_2 = a_1^{1-\eta} a_2^{-1},$$

thus

$$\frac{\log a_1}{\log(a_1/a_2)} \geq (\log a_1) \frac{a_2}{a_1} a_1^\eta \geq \max\{14, a_1^\eta\},$$

since $a_1 \geq 17$ and $a_1^\eta = a_1/(a_1 - a_2) \geq 6$. Likewise, we check that

$$\frac{\log b_1}{\log(b_1/b_2)} \geq \max\{14, b_1^\nu\}.$$

Furthermore, since $a_2 \geq 14$ and $a_1 \geq 17$, we get

$$1 + \frac{\log a_1}{\log(a_1/a_2)} \leq 1 + \frac{\log a_1}{\log(1 + 1/a_2)} \leq 1 + 1.1 a_2 \log a_1 \leq a_1^{3/2},$$

and a similar upper bound holds for $1 + (\log b_1)/(\log(b_1/b_2))$, thus we have proved that

$$E := 1 + \min \left\{ \frac{\log a_1}{\log(a_1/a_2)}, \frac{\log b_1}{\log(b_1/b_2)} \right\}$$

satisfies

$$15 \leq E \leq \min\{A^{3/2}, B^{3/2}\}.$$

Note also that the quantity

$$E' = \min\{a_1^\eta, b_1^\nu\}$$

satisfies $6 \leq E' \leq E$, thus $(4 + \log E')/(\log E') \leq (4 + \log 6)/(\log 6)$. It then follows from Theorem G that

$$\log |y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -27638(\log A)(\log B)(\log \max\{x, y\} + \log E)(\log E')^{-2},$$

when $\max\{x, y\}$ is sufficiently large. This lower bound is of the form (2.1), with explicit values for C_1 and C_2 , thus we have proved that

$$\mu_{\text{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq 1 + 27638 \frac{(\log a_1)(\log b_1)}{\min\{\eta \log a_1, \nu \log b_1\}^2}.$$

This completes the proof of Theorem 1.2. \square

References

- [Bu08] Y. Bugeaud, *Linear forms in the logarithms of algebraic numbers close to 1 and applications to Diophantine equations*, Proceedings of the Number Theory conference DION 2005, Mumbai, pp. 59–76, Narosa Publ. House, 2008.
- [Bu16] Y. Bugeaud, *Effective irrationality measures for roots of rational numbers close to 1*. In preparation.
- [Gou06] N. Gouillon, *Explicit lower bounds for linear forms in two logarithms*, J. Théor. Nombres Bordeaux **18** (2006), 125–146.
- [LMN95] M. Laurent, M. Mignotte et Y. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation*, J. Number Theory **55** (1995), 285–321.
- [Rh83] G. Rhin, *Sur l'approximation diophantienne simultanée de deux logarithmes de nombres rationnels*. In: Approximations diophantiennes et nombres transcendants, Progress in Mathematics, Colloque de Luminy 1982 (4th ed.), Vol. 31, Birkhäuser, Basel (1983), pp. 247–258.
- [RhTo86] G. Rhin and Ph. Toffin, *Approximants de Padé simultanés de logarithmes*, J. Number Theory **24** (1986), 284–297.
- [Sho74] T. N. Shorey, *Linear forms in the logarithms of algebraic numbers with small coefficients I*, J. Indian Math. Soc. (N. S.) **38** (1974), 271–284.
- [Wa78] M. Waldschmidt, *Transcendence measures for exponentials and logarithms*, J. Austral. Math. Soc. Ser. A **25** (1978), 445–465.
- [Wa93] M. Waldschmidt, *Minorations de combinaisons linéaires de logarithmes de nombres algébriques*, Canadian J. Math. **45** (1993), 176–224.
- [Wa00] M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups. Transcendence Properties of the Exponential Function in Several Variables*, Grundlehren Math. Wiss. 326, Springer, Berlin, 2000.

Yann Bugeaud
 Université de Strasbourg
 Mathématiques
 7, rue René Descartes
 67084 STRASBOURG (FRANCE)

e-mail: bugeaud@math.unistra.fr