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This paper completes the studies undertaken in [3, 4] and [8], where the authors quantify the impact of a random time on the No-Unbounded-Risk-with-Bounded-Profit concept (called NUPBR hereafter) for quasi-left-continuous models and discrete-time market models respectively. Herein, we focus on the NUPBR for semimartingales models that live on thin predictable sets only and when the extra information about the random time is added progressively over time. For this setting, we explain how far the NUPBR property is affected when one stops the model by an arbitrary random time or when one incorporates fully an honest time into the model. Furthermore, we show how to construct explicitly local martingale deflator under the bigger filtration from those of the smaller filtration. As consequence, by combining the current results on the thin case and those of [3, 4], we elaborate universal results for general semimartingale models.

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1. Introduction. We consider a stochastic basis $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, where $\mathbb{F}$ is a filtration satisfying the usual hypotheses (i.e., right continuity and completeness), and $\mathcal{F}_\infty \subseteq \mathcal{G}$. Financially speaking, the filtration $\mathbb{F}$ represents the flow of public information through time. On this basis, we consider an arbitrary but fixed d-dimensional càdlàg semimartingale, S, which represents the price processes of d-stocks, while the riskless asset's price is assumed to be constant. Beside the initial model $(\Omega, \mathcal{G}, \mathbb{F}, P, S)$, we consider a random time τ, i.e. a non-negative $\mathcal{G}$-measurable random variable. At the practical level, this random time can model the death time of an insurer, the default time of a firm, or any occurrence time of an event that might affect the market in some way.

The main goal of this paper lies in discussing whether the new model $(S, \mathbb{F}, \tau)$ is arbitrage free or not. To address this question rigourously, we need to specify the non-arbitrage concept adopted herein on the one hand, as arbitrage in continuous time has competing definitions. On the other hand, one need to model the flow of information that catch both the flow $\mathbb{F}$ and the information represented by τ. To this random time, we associate the process D and the filtration $\mathbb{G}$ given by

$$(1) \qquad D := I_{[\![\tau,+\infty[\![}, \quad \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}\,, \quad \mathcal{G}_t = \bigcap_{s>t}\Big(\mathcal{F}_s \vee \sigma(D_u, u \leq s)\Big).$$

The filtration $\mathbb{G}$ is the smallest right-continuous filtration which contains $\mathbb{F}$ and makes τ a stopping time. In the probabilistic literature, $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ with τ. In this setting of enlarged filtration, most of the literature in mathematical finance addressed the utility maximization problem/optimal portfolio (see [6], [18], [29], [31], and the references therein for details). Very recently, there has been an upsurge interest in the fundamental topic of arbitrage theory under the variation of information (see [1], [2], [3], [4], [5], [8], and [17]). To define mathematically the non-arbitrage condition that we shall study in this paper, we need to give some notations that will be useful throughout the paper. The remaining part of this section is divided into three subsections. The first subsection provides some mathematical definitions and notations, while the second subsection defines the non-arbitrage concept adopted in this paper and recall some of its very useful properties and characterizations. The last subsection describes briefly our innovative contributions and the organization of the paper.

1.1. *Some Notations and Definitions.* Throughout the paper, $\mathbb{H}$ denotes a filtration satisfying the usual hypotheses and Q a probability measure on

the filtered probability space $(\Omega, \mathbb{H})$. The set of martingales for the filtration $\mathbb{H}$ under Q is denoted by $\mathcal{M}(\mathbb{H}, Q)$. When $Q = P$, we simply denote $\mathcal{M}(\mathbb{H})$. As usual, $\mathcal{A}^+(\mathbb{H})$ denotes the set of increasing, right-continuous, $\mathbb{H}$-adapted and integrable processes.
If $\mathcal{C}(\mathbb{H})$ is a class of $\mathbb{H}$-adapted processes, we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}(\mathbb{H})$ the set of processes X such that there exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$ and the stopped processes X^{T_n} belong to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0,loc}(\mathbb{H}) = \mathcal{C}_0(\mathbb{H}) \cap \mathcal{C}_{loc}(\mathbb{H})$.
For a process K with $\mathbb{H}$-locally integrable variation, we denote by $K^{o,\mathbb{H}}$ its dual optional projection. The dual predictable projection of K (also called the $\mathbb{H}$-dual predictable projection) is denoted $K^{p,\mathbb{H}}$. For a process X, we denote ${}^{o,\mathbb{H}}X$ (resp. ${}^{p,\mathbb{H}}X$) its optional (resp. predictable) projection with respect to $\mathbb{H}$. **Throughout the paper, stochastic processes have arbitrary finite dimension**, when it is not specified.
For an $\mathbb{H}$- semi-martingale Y, $L(Y, \mathbb{H})$ is the set of $\mathbb{H}$-predictable processes having the same dimension as Y that are integrable w.r.t. Y. For $H \in L(Y, \mathbb{H})$, the resulting integrable of H w.r.t Y is a real-valued process, and is denoted by $H \centerdot Y_t := \int_0^t H_s dY_s$.
As usual, for a process X and a random time ϑ, we denote by X^{ϑ} the stopped process. To distinguish the effect of filtration, we will denote $\langle ., . \rangle^{\mathbb{F}}$, or $\langle ., . \rangle^{\mathbb{G}}$ the sharp bracket (predictable covariation process) calculated in the filtration $\mathbb{F}$ or $\mathbb{G}$, if confusion may rise. We recall that, for general semimartingales X and Y, the sharp bracket is (if it exists) the dual predictable projection of the covariation process $[X, Y]$. For other unexplained notations and concepts in stochastic calculus and/or (semi)martingale theory, the reader might refer to [13], [19], [20], or [21].

We recall the definition of thin processes/sets for the reader's convenience

Definitions 1.1. *A set $A \subset \Omega \times [0, \infty[$ is thin if, for all $\omega \in \Omega$, the set $A(\omega)$ is countable. A process X is called thin if there exists a sequence of random variables ξ_n and a sequence of random times T_n such that $X = \sum_{n=1}^{\infty} \xi_n I_{[\![T_n,\infty[\![}$. Its paths vary on a thin set only, and hence*

$$X = I_{\cup_{n=1}^{\infty}[\![T_n]\!]} \centerdot X = \sum_{n=1}^{\infty} I_{[\![T_n]\!]} \centerdot X = \sum_{n=1}^{\infty} \Delta X_{T_n} I_{[\![T_n,+\infty[\![}.$$

1.2. *The non-arbitrage concept .* We introduce the non-arbitrage notion that will be addressed in the paper.

DEFINITIONS 1.2. *An $\mathbb{H}$-semimartingale X satisfies the No-Unbounded-Profit-with-Bounded-Risk condition under $(\mathbb{H}, Q)$ (called NUPBR$(\mathbb{H}, Q)$ hereafter) if for any $T \in (0, +\infty)$ the set*

$$\mathcal{K}_T(X, \mathbb{H}) := \Big\{ (H \centerdot S)_T \;|\quad H \in L(X, \mathbb{H}) \text{ and } H \centerdot X \geq -1 \Big\}$$

is bounded in probability under Q. When $Q \sim P$, we simply write, with an abuse of language, X satisfies NUPBR$(\mathbb{H})$.

This non-arbitrage concept of NUPBR appeared naturally with other non-arbitrage concepts (classical non-arbitrage and no-free-lunch-and-vanishing-risk) in the seminal papers [15] and [24]. In [32], the author considered an other version of the NUPBR that he linked to positive supermartingale deflators instead of positive local martingale deflators. The current definition of the NUPBR was given in [3], together with the following .

PROPOSITION 1.3. *Let X be an $\mathbb{H}$-semimartingale. Then the following assertions are equivalent.*
(a) *X satisfies NUPBR$(\mathbb{H})$.*
(b) *There exist a real-valued and positive $\mathbb{H}$-local martingale, Y and a real-valued and $\mathbb{H}$-predictable process θ such that $0 < \theta \leq 1$ and $Y(\theta \centerdot X)$ is a local martingale.*

For any $\mathbb{H}$-semimartingale X, the real-valued local martingales Y fulfilling the assertion (b) of Proposition 1.3 are called σ-martingale densities for X (or positive local martingale deflator). The set of these σ-martingale densities will be denoted throughout the paper by

$$\mathcal{L}(\mathbb{H}, X) := \{ Y \in \mathcal{M}_{loc}(\mathbb{H}) |\; Y > 0,\; \exists \theta \in \mathcal{P}(\mathbb{H}),\, 0 < \theta \leq 1,\; Y(\theta \centerdot X) \in \mathcal{M}_{loc}(\mathbb{H}) \} \tag{2}$$

where, as usual, $\mathcal{P}(\mathbb{H})$ stands for the predictable σ-field on $\Omega \times [0, \infty)$ and by abuse of notation $\theta \in \mathcal{P}(\mathbb{H})$ means that θ is $\mathcal{P}(\mathbb{H})$-measurable. The main idea of Proposition 1.3 appeared, for the first time, in [10] for continuous processes. Then, it was extended to general one dimensional semimartingales and multi-dimensional semimartingales by [27] and [35] respectively. The vital role of the NUPBR for the existence of the numéraire portfolio and/or for market's viability is detailed, at different levels of generality, in [25], [30], [33], and [12]. In the following, we state, without proof, an obvious lemma.

LEMMA 1.4. *For any $\mathbb{H}$-semimartingale X and any $Y \in \mathcal{L}(\mathbb{H}, X)$, one has ${}^{p,\mathbb{H}}(Y|\Delta X|) < \infty$ and ${}^{p,\mathbb{H}}(Y \Delta X) = 0$.*

Below, we state a result that was proved in [3], and will be frequently used throughout the paper.

PROPOSITION 1.5. *Let X be an $\mathbb{H}$-adapted process. Then, the following assertions are equivalent.*
(a) *There exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$, such that for each $n \geq 1$, there exists a probability Q_n on $(\Omega, \mathbb{H}_{T_n})$ such that $Q_n \sim P$ and X^{T_n} satisfies NUPBR$(\mathbb{H})$ under Q_n.*
(b) *X satisfies NUPBR$(\mathbb{H})$.*
(c) *There exists a real-valued and $\mathbb{H}$-predictable process ϕ, such that $0 < \phi \leq 1$ and $(\phi \centerdot X)$ satisfies NUPBR$(\mathbb{H})$.*

We end this section with two lemmas and one theorem that are simple but useful. The first lemma deals with predictable process with finite variation, and states the following.

LEMMA 1.6. *Let X be an $\mathbb{H}$-predictable process with finite variation. Then X satisfies NUPBR$(\mathbb{H})$ if and only if $X \equiv X_0$ (i.e. the process X is constant).*

The following lemma play important role in simplifying the proofs for the NUPBR property by splitting the underlying process into two subprocesses with distinct features. To this end, we recall that for an $\mathbb{H}$-semimartingale, X, we associate a sequence of $\mathbb{H}$-predictable stopping times $(T_n^X)_{n\geq 1}$ that exhausts the accessible jump times of X, and put $\Gamma_X := \bigcup_{n=1}^{\infty} [\![T_n^X]\!]$. Then, we can decompose X as follows.

$$X = X^{(qc)} + X^{(a)},\ X^{(a)} := I_{\Gamma_X} \centerdot X,\ X^{(qc)} := X - X^{(a)}. \tag{3}$$

The process $X^{(a)}$ (the accessible part of X) is a thin process with accessible jumps only, while $X^{(qc)}$ is a $\mathbb{H}$-quasi-left-continuous process (the quasi-left-continuous part of X).

LEMMA 1.7. *Let X be an $\mathbb{H}$-semimartingale. Then X satisfies NUPBR$(\mathbb{H})$ if and only if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR$(\mathbb{H})$.*

PROOF. Thanks to Proposition 1.3, X satisfies NUPBR$(\mathbb{H})$ if and only if there exist an $\mathbb{H}$-predictable real-valued process ϕ and a positive real-valued $\mathbb{H}$-local martingale Y such that $0 < \phi \leq 1$ and $Y(\phi \centerdot X)$ is an $\mathbb{H}$- local martingale. Then, by putting $\widetilde{\Omega} := \Omega \times [0, +\infty)$, it is easy to see that $Y(\phi I_{\Gamma_X} \centerdot X)$ and $Y(\phi I_{\widetilde{\Omega}\setminus\Gamma_X} \centerdot X)$ are both $\mathbb{H}$-local martingales. This proves that both $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR$(\mathbb{H})$.

Conversely, if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR($\mathbb{H}$), then there exist two $\mathbb{H}$-predictable real-valued processes $\phi_1, \phi_2 > 0$ and two real-valued and positive $\mathbb{H}$-local martingales $D_1 = \mathcal{E}(N_1), D_2 = \mathcal{E}(N_2)$ such that $D_1(\phi_1 \centerdot X^{(a)}) = D_1(\phi_1 I_{\Gamma_X} \centerdot X)$ and $D_2(\phi_2 \centerdot X^{(qc)}) = D_2(\phi_2 I_{\widetilde{\Omega}\setminus\Gamma_X} \centerdot X)$ are both $\mathbb{H}$-local martingales. Remark that there is no loss of generality in assuming $N_1 = I_{\Gamma_X} \centerdot N_1$ and $N_2 = I_{\widetilde{\Omega}\setminus\Gamma_X} \centerdot N_2$, and put

$$N := I_{\Gamma_X} \centerdot N_1 + I_{\widetilde{\Omega}\setminus\Gamma_X} \centerdot N_2 \quad \text{ and } \quad \psi := \phi_1 I_{\Gamma_X} + \phi_2 I_{\widetilde{\Omega}\setminus\Gamma_X}.$$

Obviously, $\mathcal{E}(N) > 0$, $\mathcal{E}(N)$ and $\mathcal{E}(N)(\psi \centerdot X)$ are $\mathbb{H}$-local martingales, ψ is real-valued, $\mathbb{H}$-predictable, and $0 < \psi \leq 1$. This ends the proof of the lemma. □

The last result of this subsection gives characterizations of the NUPBR for single jump processes that will be useful in the forthcoming sections.

THEOREM 1.8. *Let T be an $\mathbb{H}$-predictable stopping time ($\mathbb{H} := (\mathcal{H}_t)_{t\geq 0}$), ξ is an $\mathcal{H}_T$-measurable random variable, and $X := \xi I_{[\![T,+\infty[\![}$. Then the following assertions are equivalent.*
(1) X satisfies NUPBR($\mathbb{H}$).
(2) X satisfies the "classical" non-arbitrage (i.e. For any $\mathbb{H}$-predictable process H such that $H \centerdot X \geq -1$, the property $(H \centerdot X)_\infty \geq 0 \quad P-a.s.$ implies that $(H \centerdot X)_\infty \geq 0 \quad P-a.s.$).
(3) There exists a real-valued and positive $\mathcal{H}_T$-measurable random variable, $Y^{\mathbb{H}}$, such that on $\{T < +\infty\}$, P-almost surely, we have

$$E(Y^{\mathbb{H}}|\mathcal{H}_{T-}) = 1, \quad E(Y^{\mathbb{H}}|\xi||\mathcal{H}_{T-}) < +\infty, \text{ and } E(Y^{\mathbb{H}}\xi|\mathcal{H}_{T-}) = 0.$$

(4) There exists a real-valued, bounded, and positive $\mathcal{H}_T$-measurable random variable, $Y^{\mathbb{H}}$, such that on $\{T < +\infty\}$, P-almost surely, we have

$$E(Y^{\mathbb{H}}|\mathcal{H}_{T-}) = 1, \quad E(Y^{\mathbb{H}}|\xi||\mathcal{H}_{T-}) < +\infty, \text{ and } E(Y^{\mathbb{H}}\xi|\mathcal{H}_{T-}) = 0.$$

PROOF. Due to the nature of the process X (for any $\mathbb{H}$-predictable process H, we have $H \centerdot X = H_T X$), the proof of (1)$\Longleftrightarrow$(2) is obvious and will be omitted. The proof of (1)$\Longleftrightarrow$ (3) follows immediately from Proposition 1.3 and using again the simple fact that $\phi \centerdot X = \phi_T X$, while the implication (4)$\Longrightarrow$ (3) is obvious. Thus, the only statement that deserves a proof is (3)$\Longrightarrow$ (4). To this end, we set

$$K_0 = 0, \quad K_1 := \xi I_{\{T<+\infty\}} + I_{\{T=+\infty\}}, \quad \mathcal{K}_0 := \mathcal{H}_{T-}, \quad \mathcal{K}_1 := \mathcal{H}_T.$$

Then, when assertion (3) holds, the one-step discrete-time model

$$(K = (K_i)_{i=0,1}, (\mathcal{K}_i)_{i=0,1}, P)$$

satisfies the "classical non-arbitrage", NA. Therefore, a direct application of Theorem 3.3 of [14] (see also [8]) to this model, we deduce that the random variable $Y^{\mathbb{H}}$ in assertion (3) can be chosen to be bounded. This proves assertion (4), and the proof of the theorem is completed. □

1.3. *Our Achievements.* Given the new flow of information, our main goal lies in understanding when $(S, \mathbb{G})$ satisfies the NUPBR for the $\mathbb{F}$-semimartingale S. In [3] and [4], the authors considered the quasi-left-continuous models, and addressed the NUPBR for $(S^\tau, \mathbb{G})$ and $(S - S^\tau, \mathbb{G})$ respectively. Herein, in virtue of Lemma 1.7, we complete these studies by focusing on the case where S is a thin $\mathbb{F}$-semimartingale with accessible jumps only. It is important to mention that the framework of the current paper can not be incorporated into the setting of [3, 4]. For the accessible thin setting, we study the impact of additional uncertainty generated by τ on the NUPBR. To this end, our achievements are numerous. For the pair (S, τ), where S is a thin semimartingale with accessible jumps only, on one hand, we explicitly construct two $\mathbb{F}$-adapted functionals, $(t, \omega, x) \in [0, +\infty) \times \Omega \times \mathbb{R}^d \longrightarrow \mathcal{T}_a(t, \omega, x)$ and $(t, \omega, x) \in [0, +\infty) \times \Omega \times \mathbb{R}^d \longrightarrow \mathcal{T}_b(t, \omega, x)$, such that the following characterizations hold:
1) The model $(S^\tau, \mathbb{G})$fulfills the NUPBR if and only if $(\mathcal{T}_b(S), \mathbb{F})$ does.
2) When τ is honest and satisfies some mild assumptions, $(S - S^\tau, \mathbb{G})$fulfills the NUPBR if and only if $(\mathcal{T}_a(S), \mathbb{F})$ does.
On the other hand, we characterize all the model of τ that preserve the NUPBR after stopping (respectively after total incorporation of the honest time). Furthermore, for single jump processes with predictable jump times, we give other equivalent characterizations for the fulfillment of the NUPBR of $(S^\tau, \mathbb{G})$ (respectively $(S - S^\tau, \mathbb{G})$). In these characterizations, we work with S itself and we opt for an absolute change of probability instead.
For both cases, the single jump case and the thin case, we construct explicitly deflators for $(S^\tau, \mathbb{G})$ (respective;y $(S - S^\tau, \mathbb{G})$) when S belongs to a subclass of thin $\mathbb{F}$-local martingales.
All these achievements represent the innovative core of the paper. Thus, by combining these contributions and those of [3, 4], we elaborate unified statements for the general semimartingale models.

This paper is organized as follows. The next section (Section 2) addresses the case of stopping at τ (i.e. deals with the model $(S^\tau, \mathbb{G})$), while Section

3 focuses on the model $(S - S^\tau, \mathbb{G})$. Sections 4 and 5 prove the main results elaborated in Sections 2 and 3. Section 5 is the most technical part of the paper. We conclude this paper with an appendix, where we recall some useful technical results.

2. The Case of Stopping at $\boldsymbol{\tau}$. This section discusses the NUPBR for the model $(S^\tau, \mathbb{G})$ in four subsections. The first three subsections state our principal results –as well as their immediate consequences and/or their applications– for single jump processes, thin semimartingales and general semimartingales respectively. The fourth subsection proposes a method to construct explicitly $\mathbb{G}$-local martingale deflators from $\mathbb{F}$-local martingale deflators. To this end, in addition to $\mathbb{G}$ and D defined in (1), we associate to τ two important $\mathbb{F}$-supermartingales given by

$$Z_t := P\Big(\tau > t \mid \mathcal{F}_t\Big) \text{ and } \quad \widetilde{Z}_t := P\Big(\tau \geq t \mid \mathcal{F}_t\Big). \tag{4}$$

The supermartingale Z is right-continuous with left limits and coincides with the $\mathbb{F}$-optional projection of $I_{[\![0,\tau[\![}$, while $\widetilde{Z}$ admits right limits and left limits only and is the $\mathbb{F}$-optional projection of $I_{[\![0,\tau]\!]}$. The decomposition of Z leads to an important $\mathbb{F}$-martingale m, given by

$$m := Z + D^{o,\mathbb{F}}, \tag{5}$$

where $D^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of D (see [22] for more details).

2.1. *The main results for single jump processes.* In this subsection, we outline the main results on the NUPBR condition for the stopped single jump $\mathbb{F}$-semimartingales (with predictable jump time only) with τ. The following gives many characterisation for the NUPBR($\mathbb{G}$) of S^τ, when S is a single jump process.

THEOREM 2.1. *Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable variable ξ satisfying $E(|\xi||\mathcal{F}_{T-}) < +\infty$ P-a.s. on $\{T < +\infty\}$. If $S := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$, then the following assertions are equivalent:*
(a) *S^τ satisfies NUPBR($\mathbb{G}$).*
(b) *The process $\widetilde{S} := \xi I_{\{\widetilde{Z}_T>0\}} I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![} = I_{\{\widetilde{Z}>0\}} \centerdot S$ satisfies NUPBR($\mathbb{F}$).*
(c) *S satisfies NUPBR($\mathbb{F}, \widetilde{Q}_T$), where $\widetilde{Q}_T$ is*

$$\frac{d\widetilde{Q}_T}{dP} := \frac{\widetilde{Z}_T}{Z_{T-}} I_{\{Z_{T-}>0\}} + I_{\{Z_{T-}=0\}}, \tag{6}$$

(d) *S satisfies NUPBR$(\mathbb{F}, Q_T)$, where Q_T is defined by*

$$\frac{dQ_T}{dP} := \frac{I_{\{\widetilde{Z}_T>0\}\cap\Gamma_0}}{P(\widetilde{Z}_T > 0|\ \mathcal{F}_{T-})} + I_{\Omega\setminus\Gamma_0},\ \Gamma_0 := \{P(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) > 0\ \&\ T < +\infty\}. \tag{7}$$

The proof of this theorem is long and requires an intermediate result from the next subsection. Thus, this proof is delegated to Section 4.

Remark 2.2. *(i) The importance of Theorem 2.1 goes beyond its vital role, as a building block for the more general result, and provides two main characterizations for the NUPBR$(\mathbb{G})$ of S^τ. The characterizations (c) and (d) are expressed in term of the NUPBR$(\mathbb{F})$ of S under absolute continuous change of measure, while the characterization (a) proposes a transformation of S without any change of measure. Furthermore, Theorem 2.1 can be easily extended to the case of countably many ordered predictable jump times $T_0 = 0 \le T_1 \le T_2 \le ...$ with $\sup_n T_n = +\infty$ $P-a.s.$.*
(ii) In Theorem 2.1, the choice of S having the form $S := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$ is not restrictive at all. This can be understood from the fact that any single jump process S can be decomposed as follows

$$S := \xi I_{[\![T,+\infty[\![} = \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![} + \xi I_{\{Z_{T-}=0\}} I_{[\![T,+\infty[\![} =: \overline{S} + \widehat{S}.$$

Thanks to $\{T \le \tau\} \subset \{Z_{T-} > 0\}$, we have $\widehat{S}^\tau = \xi I_{\{Z_{T-}=0\}} I_{\{T\le\tau\}} I_{[\![T,+\infty[\![} \equiv 0$ is (obviously) a $\mathbb{G}$-martingale. Thus, the only part of S that requires careful attention is $\overline{S} := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$.

The following proposition describes the models of τ for which any single jump process X (that jumps at fixed $\mathbb{F}$-predictable stopping time T) satisfying the NUPBR$(\mathbb{F})$, X^τ satisfies the NUPBR$(\mathbb{G})$.

Proposition 2.3. *Let T be an $\mathbb{F}$-predictable stopping time. Then, the following assertions are equivalent:*
(a) *On $\{T < +\infty\}$, we have*

$$\left\{\widetilde{Z}_T = 0\right\} \subset \left\{Z_{T-} = 0\right\}. \tag{8}$$

(b) *For any $M := \xi I_{[\![T,+\infty[\![}$ where $\xi \in L^\infty(\mathcal{F}_T)$ such that $E(\xi|\mathcal{F}_{T-}) = 0$, M^τ satisfies NUPBR$(\mathbb{G})$.*
(c) *For any $X = \xi I_{[\![T,+\infty[\![}$ such that $\xi \in L^\infty(\mathcal{F}_T)$ and X satisfies NUPBR$(\mathbb{F})$, X^τ satisfies NUPBR$(\mathbb{G})$.*

PROOF. The proof is outlined in two steps, where we prove (a)$\Longleftrightarrow$ (b) and (c)$\Longleftrightarrow$ (b) respectively.

1) Here, we prove (a)$\Longleftrightarrow$ (b) and we start by proving (a) $\Longrightarrow$ (b). Suppose that (8) holds. Then, due to Remark 2.2–(b), we can restrict our attention to the case where $M := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$ with $\xi \in L^\infty(\mathcal{F}_T)$ and $E(\xi|\mathcal{F}_{T-}) = 0$. Since assertion (a) is equivalent to $[\![T]\!] \cap \{\widetilde{Z} = 0 \ \& \ Z_- > 0\} = \emptyset$, we get

$$\widetilde{M} := \xi I_{\{\widetilde{Z}_T>0\}} I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![} = M \qquad \text{is an } \mathbb{F}\text{-martingale.}$$

Therefore, a direct application of Theorem 2.1 (to M) allows us to conclude that M^τ satisfies the NUPBR($\mathbb{G}$). This ends the proof of (a)$\Longrightarrow$ (b). To prove the reverse implication, we suppose that assertion (b) holds and consider

$$M := \xi I_{[\![T,+\infty[\![}, \quad \text{where } \ \xi := \left(I_{\{\widetilde{Z}_T=0\}} - P(\widetilde{Z}_T = 0|\mathcal{F}_{T-})\right) I_{\{T<+\infty\}}.$$

Since $\{T \leq \tau\} \subset \{\widetilde{Z}_T > 0\} \subset \{Z_{T-} > 0\}$, then we get

$$M^\tau = -P(\widetilde{Z}_T = 0|\mathcal{F}_{T-}) I_{\{T\leq\tau\}} I_{[\![T,+\infty[\![},$$

and this process is $\mathbb{G}$-predictable. Therefore, M^τ satisfies NUPBR($\mathbb{G}$) if and only if it is a constant process equal to $M_0 = 0$ (see Lemma 1.6). This is equivalent to

$$0 = E\Big[P(\widetilde{Z}_T = 0|\mathcal{F}_{T-}) I_{\{T\leq\tau\}} I_{\{T<+\infty\}}\Big] = E\left(Z_{T-} I_{\{\widetilde{Z}_T=0 \ \& \ T<+\infty\}}\right).$$

It is obvious that this equality is equivalent to (8), and assertion (a) follows.

2) It is obvious that (c)$\Longrightarrow$ (b), and hence the rest of the proof focuses on the reverse sense. Suppose that assertion (b) holds, and consider $X = \xi I_{[\![T,+\infty[\![}$ satisfying NUPBR($\mathbb{F}$) and $\xi \in L^\infty(\mathcal{F}_T)$. Then, thanks to Theorem 1.8, there exists a positive real-valued $\mathcal{F}_T$-measurable random variable $Y^{\mathbb{F}}$ such that

$$E(Y^{\mathbb{F}}|\mathcal{F}_{T-}) = 1, \quad \text{and } E(Y^{\mathbb{F}}\xi|\mathcal{F}_{T-}) = 0 \quad P-a.s. \text{ on } \{T < +\infty\}.$$

Put

$$\overline{X} := \frac{\xi Y^{\mathbb{F}}}{1 + E(Y^{\mathbb{F}}|\xi||\mathcal{F}_{T-})} I_{[\![T,+\infty[\![}.$$

Then, it is easy to check that $\overline{X}$ is an $\mathbb{F}$-martingale, and a direct application of assertion (b) to $\overline{X}$ allows us to deduce that $\overline{X}^\tau$ satisfies NUPBR($\mathbb{G}$). Thus, again Theorem 1.8 implies the existence of a real valued, bounded, and positive $\mathcal{G}_T$-measurable random variable, $Y^{\mathbb{G}}$, such that

$$E(Y^{\mathbb{G}}Y^{\mathbb{F}}|\xi||\mathcal{G}_{T-}) < +\infty \ \text{ and } E(Y^{\mathbb{F}}Y^{\mathbb{G}}\xi|\mathcal{G}_{T-}) = 0 \ \ P-a.s. \text{ on } \{T \leq \tau < +\infty\}. \tag{9}$$

Thus, $Z^{\mathbb{G}} := \frac{Y^{\mathbb{G}}Y^{\mathbb{F}}}{E(Y^{\mathbb{G}}Y^{\mathbb{F}}|\mathcal{G}_{T-})} I_{\{T<+\infty\}} + I_{\{T=+\infty\}}$ is a well defined, real-valued and positive $\mathcal{G}_T$-measurable random variable such that

$$E(Z^{\mathbb{G}}|\mathcal{G}_{T-}) = 1, \quad E(Z^{\mathbb{G}}|\xi||\mathcal{G}_{T-}) < +\infty \ \text{ and } E(Z^{\mathbb{F}}\xi|\mathcal{G}_{T-}) = 0 \ \ P{-}a.s. \text{ on } \{T \leq \tau < +\infty\}.$$

By combining this fact with Theorem 1.8, we conclude that X^τ satisfies NUPBR($\mathbb{G}$), and the proof of the theorem is completed. □

2.2. *The main results for thin processes.* This subsection deals with the case where S is a thin $\mathbb{F}$-semimartingale with $\mathbb{F}$-accessible jumps only. We start by extending Theorem 2.1 to this setting as follows.

THEOREM 2.4. *Let S be a thin process with accessible jumps only. Then, the following assertions are equivalent.*
(a) *The process S^τ satisfies the NUPBR($\mathbb{G}$).*
(b) *For any $\delta > 0$, there exists a positive $\mathbb{F}$-local martingale, Y, such that P-a.s on $\{Z_- \geq \delta\}$ we have*

$$(10) \qquad {}^{p,\mathbb{F}}\left(Y|\Delta S|I_{\{\widetilde{Z}>0\}}\right) < +\infty \ \text{ and } \ {}^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}>0\}}\right) = 0.$$

(c) *For any δ, the process*

$$(11) \qquad S^{(0)} := \sum \Delta S I_{\{\widetilde{Z}>0 \ \& \ Z_- \geq \delta\}},$$

satisfies the NUPBR($\mathbb{F}$).

The proof of this theorem is technically involved, especially the proof of (a)$\Longrightarrow$(c), and thus it is postponed to Subsection 4.1.

REMARK 2.5. *i) It is important to notice that, in Theorem 2.4, we did not assume any condition on S. Thus, if S is a thin process –with accessible jumps only– and furthermore satisfies NUPBR($\mathbb{F}$) and*

$$\{\widetilde{Z} = 0 < Z_-\} \cap \{\Delta S \neq 0\} = \emptyset,$$

then S^τ satisfies NUPBR($\mathbb{G}$). This follows immediately from Theorem 2.4 by using $Y \in \mathcal{L}(S,\mathbb{F})$ and Lemma 1.4.
ii) It is worthy to mention that $S^{(0)}$ coincides with $\left(I_{\{Z_-\geq\delta\}} \centerdot S\right)^{\widetilde{R}_1-}$, where

$$(12) \qquad \widetilde{R}_1 = R I_{A_1} + (+\infty) I_{\Omega\setminus A_1}, \quad R := \inf\{t \geq 0 \mid Z_t = 0\},$$

and $A_1 := \{\widetilde{Z}_R = 0 < Z_{R-}\} \cap \{R < +\infty\}$.
It is important to mention that this remark resembles to Remark 2.16–(1)

in [3] for the quasi-left-continuous case. However, there is one important difference which lies in the event A, described in [3], which is in general a subset of A_1 only.
iii) The assertion (b) of Theorem 2.4, that characterizes the NUPBR($\mathbb{G}$) for S^τ, is very specific to the thin case with accessible jumps, while it is irrelevant for the quasi-left-continuous models. Indeed, for these models, (10) holds on the whole Ω and for any positive $\mathbb{F}$-martingale Y.

The following extends Proposition 2.3 to the case of countably many jumps that might not be ordered in any way.

THEOREM 2.6. *The following assertions are equivalent.*
(a) *The set $\{\widetilde{Z} = 0 > Z_-\}$ is totally inaccessible.*
(b) *X^τ satisfies the NUPBR($\mathbb{G}$) for any thin process X with accessible jumps satisfying NUPBR($\mathbb{F}$).*

PROOF. The proof of the theorem will be achieved in two parts, namely part 1) and part 2) where we prove (b)$\Longrightarrow$(a) and (a)$\Longrightarrow$(b) respectively.
1) Suppose that assertion (b) holds. Then, thanks to Proposition 2.3, we deduce that for any $\mathbb{F}$-predictable stopping time T,

$$[\![T]\!] \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset \tag{13}$$

on the one hand. On the other hand, since $\{\widetilde{Z} = 0 < Z_-\}$ is thin, there exists a sequence of $\mathbb{F}$-stopping times $(\sigma_k)_{k\geq 1}$ with disjoint graphs such that

$$\{\widetilde{Z} = 0 < Z_-\} = \bigcup_{k=1}^{+\infty} [\![\sigma_k]\!]. \tag{14}$$

Recall that, for each σ_k, there exist two $\mathbb{F}$-stopping times (σ_k^i and σ_k^a that are totally inaccessible and accessible respectively) and a sequence of $\mathbb{F}$-predictable stopping times $(T_l^{(k)})_{l\geq 1}$ such that

$$[\![\sigma_k]\!] = [\![\sigma_k^i]\!] \cup [\![\sigma_k^a]\!], \qquad [\![\sigma_k^a]\!] \subset \bigcup_{l=1}^{+\infty} [\![T_l^{(k)}]\!].$$

Thus, by combining these with $\left(\bigcup_{k=1}^{+\infty} [\![\sigma_k^i]\!]\right) \cap \left(\bigcup_{k=1,l=1}^{+\infty} [\![T_l^{(k)}]\!]\right) = \emptyset$, (14) and (13), we derive

$$\bigcup_{k=1}^{+\infty} [\![\sigma_k^a]\!] = \left(\bigcup_{k=1,l=1}^{+\infty} [\![T_l^{(k)}]\!]\right) \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset.$$

This proves that $\{\widetilde{Z}=0<Z_-\}$ is a totally inaccessible set and the proof of (b)$\Longrightarrow$(a) is completed.

2) To prove the reverse sense, we assume that assertion (a) holds, and consider $X=\sum \xi_n I_{[\![T_n,+\infty[\![}$ satisfying NUPBR($\mathbb{F}$), where T_n are $\mathbb{F}$-predictable stopping times and ξ_n are bounded $\mathcal{F}_{T_n}$-measurable random variable. Since $\{\Delta X\neq 0\}\subset \bigcup_{n=1}^{+\infty}[\![T_n]\!]\in\mathcal{P}(\mathbb{F})$, we get $\{\widetilde{Z}=0<Z_-\}\cap\{\Delta X\neq 0\}=\emptyset$, and hence, from Remark 2.5–(i), X^τ satisfies the NUPBR($\mathbb{G}$). This proves assertion (b), and the proof of the theorem is completed. $\square$

2.3. *The main results for general semimartingales.* This subsection summarizes the main results, for general semimartingales, by combining the results of Subsection 2.2 and those of [3]. Recall that, for any càdlàg process X and any stopping time σ, we denote $X^{\sigma-}:=XI_{]\!]0,\sigma[\![}+X_{\sigma-}I_{[\![\sigma,+\infty[\![}$.

THEOREM 2.7. *Let R and $\widetilde{R}_1$ be the stopping times defined in (12), and $A_1:=\{\widetilde{Z}_R=0<Z_{R-}\}\cap\{R<+\infty\}$. Then, there exists an event $A\in\mathcal{F}_R$ such that $A\subset A_1$ and the following assertions are equivalent.*
(a) S^τ satisfies the NUPBR($\mathbb{G}$)
(b) Both $\left(S^{(qc)}\right)^{\widetilde{R}-}$ and $\left(S^{(a)}\right)^{\widetilde{R}_1-}$ satisfy the NUPBR($\mathbb{F}$).
Here $\widetilde{R}:=RI_A+(+\infty)I_{\Omega\setminus A}$, $S^{(qc)}$ and $S^{(a)}$ are the quasi-left-continuous and thin accessible parts of S respectively defined in (3).

PROOF. Thanks to Lemma 3, $(S^\tau,\mathbb{G})$ satisfies the NUPBR if and only if both models, $\left((S^{(qc)})^\tau,\mathbb{G}\right)$ and $\left((S^{(a)})^\tau,\mathbb{G}\right)$, fulfill the NUPBR. Here, $S^{(qc)}$ and $S^{(a)}$ are the quasi-left-continuous part and the thin accessible part of S respectively. Thus, a direct application of [3] (respect. Theorem 2.4) to $S^{(qc)}$ (respect. to $S^{(a)}$) , we conclude that $(S^{(qc)})^\tau$ (respect. $(S^{(a)})^\tau$) satisfies the NUPBR($\mathbb{G}$) if and only if $(S^{(a)})^{\widetilde{R}_1-}$ (respect. $(S^{(qc)})^{\widetilde{R}-}$) satisfies the NUPBR($\mathbb{F}$).Therefore, the proof of theorem follows immediately from combining these remarks. $\square$

The complete general result, in the spirit of describing models for τ that preserve the NUPBR after stopping with τ, is the following.

THEOREM 2.8. *The following assertions are equivalent.*
(a) *The set $\{\widetilde{Z}=0<Z_-\}$ is evanescent.*
(b) *X^τ satisfies the NUPBR($\mathbb{G}$) for any X satisfying the NUPBR($\mathbb{F}$).*

PROOF. The proof follows immediately from the combination of Lemma 3, Theorem 2.6 and Proposition 2.22 in [3] (where the authors prove that the thin set $\{\widetilde{Z}=0<Z_-\}$ is accessible if and only if assertion (b) holds for any $\mathbb{F}$-quasi-left-continuous process X) . □

2.4. *Explicit local martingale deflators.* This section discusses how to construct explicitly $\mathbb{G}$-local martingale deflators from $\mathbb{F}$-deflators for a class of processes. This is achieved, for single jump processes and general thin processes afterwards, by considering $\mathbb{F}$-local martingales.

PROPOSITION 2.9. *Let $M := \xi I_{[\![T,+\infty[\![}$ be an $\mathbb{F}$-martingale, where T is an $\mathbb{F}$-predictable stopping time, and ξ is an $\mathcal{F}_T$-measurable random variable. Then the following assertions are equivalent.*
(a) *M is an $\mathbb{F}$-martingale under Q_T given by (7).*
(b) *On the set $\{T<+\infty\}$, we have*

$$E\left(M_T I_{\{\widetilde{Z}_T=0\}}|\ \mathcal{F}_{T-}\right)=0,\quad P-a.s. \tag{15}$$

(c) *M^τ is a $\mathbb{G}$-martingale under $Q_T^{\mathbb{G}}$ given by*

$$\frac{dQ_T^{\mathbb{G}}}{dP}:=\frac{U^{\mathbb{G}}(T)}{E(U^{\mathbb{G}}(T)|\ \mathcal{G}_{T-})}\ \text{ where } U^{\mathbb{G}}(T):=I_{\{T>\tau\}}+I_{\{T\leq\tau\}}\frac{Z_{T-}}{\widetilde{Z}_T}. \tag{16}$$

PROOF. The proof will be achieved in two steps where we prove (a)$\Longleftrightarrow$(b) and (a)$\Longleftrightarrow$(c) respectively.
Step 1. Here, we prove (a)$\Longleftrightarrow$(b). For simplicity we denote by $Q:=Q_T$, where Q_T is defined in (7), and remark that on $\{Z_{T-}=0\}$, Q coincides with P and (15) holds, due to $\{Z_{T-}=0\}\subset\{\widetilde{Z}_T=0\}$. Thus, it is enough to prove (a)$\Longleftrightarrow$(b) on the set $\{T<+\infty\ \&\ Z_{T-}>0\}$. On this set, due to $E(\xi|\mathcal{F}_{T-})=0$ (since M is an $\mathbb{F}$-martingale), we derive

$$\begin{aligned} E^Q(\xi|\mathcal{F}_{T-}) &= E(\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-})\Big(P(\widetilde{Z}_T>0|\mathcal{F}_{T-})\Big)^{-1} \\ &= -E(\xi I_{\{\widetilde{Z}_T=0\}}|\mathcal{F}_{T-})\left(P(\widetilde{Z}_T>0|\mathcal{F}_{T-})\right)^{-1}. \end{aligned}$$

Therefore, assertion (a) (or equivalently $E^Q(\xi|\mathcal{F}_{T-})=0$) is equivalent to (15). This ends the proof of (a) $\Longleftrightarrow$ (b).
Step 2. To prove (a)$\Longleftrightarrow$(c), we notice that due to $\{T\leq\tau\}\subset\{\widetilde{Z}_T>0\}\subset\{Z_{T-}>0\}$, on $\{T\leq\tau\}$ we have

$$\begin{aligned} P\left(\widetilde{Z}_T>0|\mathcal{F}_{T-}\right)E^{Q_T^{\mathbb{G}}}(\xi|\mathcal{G}_{T-}) &= E\left(\frac{Z_{T-}}{\widetilde{Z}_T}\xi I_{\{T\leq\tau\}}|\mathcal{G}_{T-}\right)=E\left(\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right) \\ &= E^Q(\xi|\mathcal{F}_{T-})\,P\left(\widetilde{Z}_T>0|\mathcal{F}_{T-}\right). \end{aligned}$$

This equality proves that $M^\tau \in \mathcal{M}(Q^{\mathbb{G}}, \mathbb{G})$ if and only if $M \in \mathcal{M}(Q, \mathbb{F})$, and the proof of (a)$\Longleftrightarrow$(c) is completed. This ends the proof of the theorem. $\square$

To generalize this proposition to the case of infinitely many jumps that might not be ordered at all, we need to introduce some notations and recall some facts from [3]. First of all, we refer to [13] (Chapter VIII.2 sections 32-35 pages 356-361) and [20] (Chapter III.4.b, Definition 3(3.8), pages 106-109) for the optional stochastic integration (see also Definition 3.4 in [3]).

Definitions 2.10. *Let N be an $\mathbb{H}$-local martingale with continuous part N^c and K be an $\mathbb{H}$-optional process. K is said to be integrable with respect to N if ${}^{p,\mathbb{H}}(K)$ is N^c-integrable, ${}^{p,\mathbb{H}}(|K\Delta N|) < +\infty$ and*

$$\left(\sum \left(K\Delta N - {}^{p,\mathbb{H}}(K\Delta N)\right)^2\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{H}).$$

Put

$$K^{\mathbb{G}} := \frac{Z_-^2 \widetilde{Z}^{-1}}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}} I_{]\!]0,\tau]\!]}, \quad V^{\mathbb{G}} := \sum {}^{p,\mathbb{F}}(I_{\{\widetilde{Z}=0\}}) I_{]\!]0,\tau]\!]}, \tag{17}$$

and to any $\mathbb{F}$-local martingale M, we associate the $\mathbb{G}$-local martingale part of M^τ given by

$$\widehat{M} := M^\tau - Z_-^{-1} I_{[\![0,\tau]\!]} \cdot \langle M, m\rangle^{\mathbb{F}}. \tag{18}$$

Below, we recall some useful results of [3].

Proposition 2.11. *The following assertions hold.*
(a) The $\mathbb{G}$-optional process $K^{\mathbb{G}}$ is $\widehat{m}$-integrable in the sense of the above definition. Here $\widehat{m} := m^\tau - Z_-^{-1} I_{]\!]0,\tau]\!]} \cdot \langle m\rangle^{\mathbb{F}}$. Furthermore, the resulting integral

$$\widetilde{L}^{(b)} := \mathcal{E}\left(-\frac{K^{\mathbb{G}}}{1-\Delta V^{\mathbb{G}}} \odot \widehat{m}\right), \tag{19}$$

is a positive (i.e. $\widetilde{L}^{(b)} > 0$) $\mathbb{G}$-local martingale satisfying $[\widetilde{L}^{(b)}, M] \in \mathcal{A}_{loc}(\mathbb{G})$ for any $\mathbb{F}$-local martingale M.
(b) $V^{\mathbb{G}} \in \mathcal{A}^+_{loc}(\mathbb{G})$ and $(1-\Delta V^{\mathbb{G}})^{-1}$ is $\mathbb{G}$-locally bounded.

The proof of this proposition can be found in [3] (see Lemma 3.3 and Proposition 3.6). The extension of Proposition 2.9 goes through connecting the random variable $U^{\mathbb{G}}(T)$ defined in (16) to the process $\widetilde{L}^{(b)}$ as follows.

Remark 2.12. *In virtue of the calculation performed in [3] (see equation (B.1) where the authors calculate the jumps of $K^{\mathbb{G}} \odot \widehat{m}$), on $]\!]0, \tau]\!]$ we have*

$$(1-\Delta V^{\mathbb{G}})\left(1-\frac{\widetilde{L}^{(b)}}{\widetilde{L}^{(b)}_-}\right) = -\frac{(1-\Delta V^{\mathbb{G}})\Delta \widetilde{L}^{(b)}}{\widetilde{L}^{(b)}_-} = \Delta\left(K^{\mathbb{G}} \odot \widehat{m}\right) = 1-\frac{Z_-}{\widetilde{Z}}-\Delta V^{\mathbb{G}}.$$

Thus, for an $\mathbb{F}$-predictable stopping time T, on $\{T \leq \tau\}$ we get

$$U^{\mathbb{G}}_T = \frac{Z_{T-}}{\widetilde{Z}_T} = (1-\Delta V^{\mathbb{G}}_T)\frac{\widetilde{L}^{(b)}_T}{\widetilde{L}^{(b)}_{T-}}.$$

This proves that assertions (a) and (b) of Proposition 2.9 are equivalent to

(20) $\widetilde{L}^{(b)} M^\tau$ *is a $\mathbb{G}$-martingale for any single jump $\mathbb{F}$-martingale M.*

Theorem 2.13. *Consider $\widetilde{L}^{(b)}$ defined in (19) and let M be a thin $\mathbb{F}$-local martingale, with accessible jumps only, satisfying*

$$ {}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}=0<Z_-\}}\right) \equiv 0. \tag{21}$$

Then, $\widetilde{L}^{(b)} M^\tau$ is a $\mathbb{G}$-local martingale.

Proof. We start by remarking that it is enough to prove the existence of a real-valued $\mathbb{G}$-predictable process φ such that $0 < \varphi \leq 1$ and $\widetilde{L}^{(b)}(\varphi \cdot M^\tau)$ is a $\mathbb{G}$-martingale (local martingale). This means that $\widetilde{L}^{(b)} \in \mathcal{L}(M^\tau, \mathbb{G})$ (i.e it is a σ-martingale density for M^τ under $\mathbb{G}$). This remark, which simplifies the proof, is based on the fact that $[\widetilde{L}^{(b)}, M^\tau]$ is locally integrable (see Proposition 2.11–(a)), and Proposition 3.3 and Corollary 3.5 of [7] (These assert that a σ-martingale, whose negative part is locally integrable, is in fact a local martingale). Again, thanks to $[\widetilde{L}^{(b)}, M^\tau] \in \mathcal{A}_{loc}(\mathbb{G})$, we deduce that ${}^{p,\mathbb{G}}\left(\widetilde{L}^{(b)}|\Delta M^\tau|\right) < +\infty$, and consider the following $\mathbb{G}$-predictable process

$$\phi := \left[1 + {}^{p,\mathbb{G}}\left(|\Delta M^\tau|\right) + {}^{p,\mathbb{G}}\left(\widetilde{L}^{(b)}|\Delta M^\tau|\right)\right]^{-1}\left[I_{\widetilde{\Omega}\setminus(\cup_n [\![T_n]\!])} + \sum_{n=1}^{+\infty} 2^{-n} I_{[\![T_n]\!]}\right],$$

where $\widetilde{\Omega} := \Omega \times [0, +\infty)$ and $(T_n)_{n\geq 1}$ is the sequence of $\mathbb{F}$-predictable stopping times that exhausts the jumps of M. Thus, it is easy to check that $0 < \phi \leq 1$, and both processes $\phi \cdot M^\tau$ and $(\widetilde{L}^{(b)}_-\phi) \cdot M^\tau + [\widetilde{L}^{(b)}, \phi \cdot M^\tau] = \sum \widetilde{L}^{(b)}\phi\Delta M^\tau$ have integrable variations on the one hand. On the other hand, since $\sum \widetilde{L}^{(b)}\phi\Delta M^{\tau}$ jumps on the predictable stopping times $(T_n)_{n\geq 1}$ only, its $\mathbb{G}$-compensator is

$$\sum {}^{p,\mathbb{G}}\left(\widetilde{L}^{(b)}\phi\Delta M^{\tau}\right) = \sum_{n=1}^{+\infty} \phi_{T_n}\ {}^{p,\mathbb{G}}\left(\widetilde{L}^{(b)}\Delta M^{\tau}\right)_{T_n} I_{[\![T_n,+\infty[\![} \equiv 0.$$

The last equality follows from ${}^{p,\mathbb{G}}\left(\widetilde{L}^{(b)}\Delta M^{\tau}\right)_{T_n} = E\left(\widetilde{L}^{(b)}_{T_n}\Delta M_{T_n}\middle|\mathcal{G}_{T_n-}\right) I_{\{T_n\leq\tau\}} = 0$. Thus, this proves that $(\widetilde{L}^{(b)}_{-}\phi)\cdot M^{\tau} + [\widetilde{L}^{(b)}, \phi\cdot M^{\tau}]$ is a $\mathbb{G}$-local martingale or equivalently $\widetilde{L}^{(b)}(\phi\cdot M^{\tau})$ is a $\mathbb{G}$-local martingale. Hence, the proof of the theorem is completed. □

COROLLARY 2.14. *For any thin $\mathbb{F}$-martingale with accessible jumps only, M, such that $\{\Delta M\neq 0\}\cap\{\widetilde{Z}=0<Z_{-}\}$ is evanescent, $\widetilde{L}^{(b)}M^{\tau}$ is a $\mathbb{G}$-local martingale.*

PROOF. The proof of the corollary follows immediately from Theorem 2.13, as the condition $\{\Delta M\neq 0\}\cap\{\widetilde{Z}=0<Z_{-}\}=\emptyset$ implies (21). □

3. The part after τ. Herein, we focus on the process $S-S^{\tau}$. Since in general, the model $(S-S^{\tau},\mathbb{G})$ might fail to preserve the semimartingale structure, we restrict our study for this part to the case where τ is a honest time (i.e. for any $t\geq 0$, there exists an $\mathcal{F}_t$-measurable random variable τ_t such that $\tau=\tau_t$ on $\{\tau<t\}$) (for more details about honest time, we refer the reader to [9] and [22]). In the same spirit of Section 2, we summarize the results of this part in four subsections. The first three subsections outline the principal results for single jump processes, thin processes and general semimartingales respectively. The last subsection explains how to obtain $\mathbb{G}$-local martingale deflators for $S-S^{\tau}$ from the $\mathbb{F}$-deflators of S when S varies in a class of processes. However, in many parts of this section, we consider the following assumption on τ:

$$\tau \text{ is an honest time and} \qquad Z_{\tau}<1 \quad P-a.s. \tag{22}$$

3.1. *The main results for single jump processes.* This subsection presents our main results on the NUPBR for $(S-S^{\tau},\mathbb{G})$, for single jump models.

THEOREM 3.1. *Suppose that τ is an honest time. Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable r.v. ξ such that $E(|\xi||\ \mathcal{F}_{T-})<+\infty$ P-a.s. on $\{T<+\infty\}$.*
If $S:=\xi I_{\{Z_{T-}<1\}}I_{[\![T,+\infty[\![}$, then the following are equivalent:

(a) $S-S^\tau$ *satisfies the NUPBR*$(\mathbb{G})$.
(b) S *satisfies the NUPBR*$(\mathbb{F},\widetilde{Q}'_T)$, *where*

$$\widetilde{Q}'_T := \Big(\frac{1-\widetilde{Z}_T}{1-Z_{T-}} I_{\{Z_{T-}<1\}} + I_{\{Z_{T-}=1\}}\Big)\cdot P. \tag{23}$$

(c) S *satisfies the NUPBR*$(\mathbb{F},Q'_T)$, *where for* $\Gamma_1(T) := \{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})>0\ \&\ T<+\infty\}$ *we set*

$$Q'_T := \Big(\frac{I_{\{\widetilde{Z}_T<1\}\cap\Gamma_1(T)}}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})} + I_{\Omega\setminus\Gamma_1(T)}\Big)\cdot P. \tag{24}$$

(d) $\widetilde{S} := \xi I_{\{\widetilde{Z}_T<1\}} I_{[\![T,+\infty[\![}$ *satisfies the NUPBR*$(\mathbb{F})$.

The proof of this theorem is long and requires intermediate results. Thus, we postpone this proof to Subsection 4.1.

REMARK 3.2. *(i) The theorem above is valid for any single jump process* S *with jump time* T, *and the condition* $I_{\{Z_{T-}=1\}}\equiv 0$ *is not restrictive at all. In fact for any single jump process* $S=\xi I_{[\![T,+\infty[\![}$, *we have*

$$S = \xi I_{\{Z_{T-}<1\}} I_{[\![T,+\infty[\![} + \xi I_{\{Z_{T-}=1\}} I_{[\![T,+\infty[\![} =: S^{(1)} + S^{(2)},$$

and $S^{(2)} - \big(S^{(2)}\big)^\tau \equiv 0$ *is a* $\mathbb{G}$*-local martingale. Thus the only part that requires careful study is* $S^{(1)} := \xi I_{\{Z_{T-}<1\}} I_{[\![T,+\infty[\![}$.
(ii) Theorem 3.1 provides two equivalent (and conceptually different) characterisations for the condition that $S-S^\tau$ *satisfies NUPBR*$(\mathbb{G})$. *One of these characterisations uses the NUPBR*$(\mathbb{F})$ *property under* P *for a transformation of* S, *while the other characterisation is essentially based on the NUPBR*$(\mathbb{F})$ *for* S *under an absolutely continuous probability measure.*

The next theorem describes the models for τ that preserve the NUPBR$(\mathbb{G})$ after τ for all single jump $\mathbb{F}$-martingales that jump at a predictable time T.

THEOREM 3.3. *Suppose that* τ *is an honest and consider an* $\mathbb{F}$*-predictable stopping time* T. *Then, the following assertions are equivalent:*
(a) *On* $\{T<+\infty\}$, *we have*

$$\Big\{\widetilde{Z}_T = 1\Big\} \subset \{Z_{T-}=1\}\,. \tag{25}$$

(b) *For any* $\xi\in L^\infty(\mathcal{F}_T)$ *such that* $E(\xi|\ \mathcal{F}_{T-})=0$ *P-a.s on* $\{T<+\infty\}$, *the process* $M-M^\tau$ *satisfies NUPBR*$(\mathbb{G})$, *where* $M:=\xi I_{[\![T,+\infty[\![}$.
(c) *For any* $\xi\in L^\infty(\mathcal{F}_T)$ *such that* $X:=\xi I_{[\![T,+\infty[\![}$ *satisfies NUPBR*$(\mathbb{F})$, *the process* $X-X^\tau$ *satisfies NUPBR*$(\mathbb{G})$.

Proof. The proof of (b)$\Longleftrightarrow$ (c) mimics the part-2) in the proof of Proposition 2.3. Thus this proof is omitted, and the rest of the proof focuses on proving (a)$\Longleftrightarrow$ (b). Suppose that assertion (a) holds, and consider $\xi \in L^\infty(\mathcal{F}_T)$ such that $E(\xi \mid \mathcal{F}_{T-}) = 0, \quad P-a.s.$ on $\{T < +\infty\}$. By decomposing M into

$$M = I_{\{Z_{T-}<1\}}\xi I_{[\![T,+\infty[\![} + I_{\{Z_{T-}=1\}}\xi I_{[\![T,+\infty[\![} := M^{(1)} + M^{(2)},$$

and noting that $M^{(2)} - (M^{(2)})^\tau = 0$, we can restrict our attention to the case where $M = M^{(1)}$ on the one hand. On the other hand, since $\{Z_{T-} = 1\} \subset \{\widetilde{Z}_T = 1\}$ P-a.s. on $\{T < +\infty\}$, it is obvious that (25) implies $\{\widetilde{Z}_T < 1\} = \{Z_{T-} < 1\}$ on $\{T < +\infty\}$, and hence

$$\widetilde{M} := I_{\{\widetilde{Z}_T<1\}}M = M \quad \text{is an } \mathbb{F}\text{-martingale.}$$

Thus, assertion (b) follows from a direct application of Theorem 3.1 to M. This ends the proof of (a)$\Longrightarrow$ (b). To prove the converse, we assume that assertion (b) holds, and we consider the $\mathcal{F}_T$-measurable and bounded r.v. $\xi := (I_{\{\widetilde{Z}_T=1\}} - P(\widetilde{Z}_T = 1|\mathcal{F}_{T-}))I_{\{T<+\infty\}}$ and the bounded $\mathbb{F}$-martingale $M := \xi I_{[\![T,+\infty[\![}$. Then, on the one hand, $M - M^\tau$ satisfies NUPBR($\mathbb{G}$). On the other hand, due to $\{T > \tau\} \subset \{\widetilde{Z}_T < 1\}$, the finite variation process

$$M - M^\tau = -P(\widetilde{Z}_T = 1|\mathcal{F}_{T-})I_{\{T>\tau\}}I_{[\![T,+\infty[\![} \quad \text{is } \mathbb{G}-\text{predictable.}$$

Thus, it is null, or equivalently $\{Z_{T-} < 1\} \subset \{\widetilde{Z}_T < 1\}$ $P-a.s.$ on $\{T < +\infty\}$. This proves assertion (a), and the proof of the theorem is completed. $\square$

3.2. *The main results for thin semimartingales.* Herein, we extend the results of the previous subsection to the case of thin semimartingale S. To this end, we start by extending Theorem 3.1 to this framework.

Theorem 3.4. *Suppose that τ satisfies (22), and S is a thin process with accessible jumps only. Then, the following assertions are equivalent.*
(a) *The process $S - S^\tau$ satisfies the NUPBR($\mathbb{G}$).*
(b) *For any $\delta > 0$, there exists a positive $\mathbb{F}$-local martingale Y, such that*

$$(26) \quad {}^{p,\mathbb{F}}\left(Y|\Delta S|I_{\{\widetilde{Z}<1\}}\right) < +\infty \;\&\; {}^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1\}}\right) = 0 \text{ on } \{1 - Z_- \geq \delta\}.$$

(c) *For any δ, the process*

$$(27) \qquad S^{(1)} := \sum \Delta S I_{\{\widetilde{Z}<1 \;\&\; 1-Z_-\geq\delta\}},$$

satisfies the NUPBR($\mathbb{F}$).

The proof of this theorem is long and is based on a result of the next subsection. Thus, this proof is postponed to Subsection 5.2.

REMARK 3.5. *1) The process $S^{(1)}$ defined in (27) is a thin semimartingale. In fact, we have $S^{(1)} = I_{\{1-Z_-\geq\delta\}} \cdot S - \sum \Delta S I_{\{\widetilde{Z}=1\ \&\ 1-Z_-\geq\delta\}}$, and*

$$\sum I_{\{\widetilde{Z}=1\ \&\ 1-Z_-\geq\delta\}} \leq \delta^{-2}\sum(\Delta m)^2 \leq \delta^{-2}[m,m] \in \mathcal{A}^+_{loc}(\mathbb{F}).$$

Furthermore, since $S = \sum \Delta S$ as S is a thin process, it is easy to see that the process $S^{(1)}$ takes the form of $S^{(1)} = (1-Z_-)^{-1} I_{\{Z_-\leq 1-\delta\}} \bullet \mathcal{T}_a(S,m)$. Here, for any pair of semimartingales (X,Y), we denote

$$\mathcal{T}_a(X,Y) := (1-Z_-) \bullet X - I_{\{\widetilde{Z}=1\}} \bullet [X,Y]. \tag{28}$$

In virtue of the above transformation, Theorem 3.4 states that $S-S^\tau$ satisfies the NUPBR($\mathbb{G}$) if and only if for any $\delta>0$, the process $(1-Z_-)^{-1}I_{\{Z_-\leq 1-\delta\}}\bullet \mathcal{T}_a(S,m)$ fulfills the NUPBR($\mathbb{F}$). This equivalence is similar to Theorem 2.12 in [4] for the quasi-left-continuous case. The only difference lies in the choice of the pair (X,Y) to which the operator $\mathcal{T}_a(.,.)$ is applied to. In fact in [4], we considered $(S,m^{(1)})$, where $m^{(1)}$ is a precise jumpy part $\mathbb{F}$-martingale of m, while herein we consider (S,m) itself.
2) It is important to mention that the "accessible thin" assumption on S is crucial for the theorem to be valid, as assertion (b) holds for any special quasi-left-continuous semimartingale. Thus, in this quasi-left-continuous case, this assertion does not characterise the NUPBR($\mathbb{G}$) for $S-S^\tau$ in any way!
3) The proof of (a)$\Longrightarrow$(b) is the very technical part in the proof of the theorem, while the rest is easy and is postponed to keep this section short.

THEOREM 3.6. *The following assertions are equivalent.*
(a) *The set $\{\widetilde{Z}=1>Z_-\}$ is totally inaccessible.*
(b) *$X-X^\tau$ satisfies the NUPBR($\mathbb{G}$) for any thin process X with predictable jumps satisfying NUPBR($\mathbb{F}$).*

PROOF. Suppose that assertion (a) holds, and consider a thin process with accessible jump times, X, satisfying NUPBR($\mathbb{F}$). Thus, $\{\Delta X\neq 0\}$ is a thin accessible set, and hence $\{\widetilde{Z}=1>Z_-\}\cap\{\Delta X\neq 0\}=\emptyset$. Therefore, we conclude that

$$X^{(1)} := \sum \Delta X I_{\{\widetilde{Z}<1\ \&\ 1-Z_-\geq\delta\}} = I_{\{1-Z_-\geq\delta\}}\cdot X \text{ satisfies NUPBR}(\mathbb{F}).$$

Then, a direct application of Theorem 3.4 leads to the NUPBR($\mathbb{G}$) of $X-X^\tau$. This proves (a)$\Longrightarrow$(b). To prove the reverse, we remark that the set $\{\widetilde{Z}=$

$1 > Z_-\}$ is thin, and we mimic exactly the part 1) of the proof of Theorem 2.6. This ends the proof of theorem. □

3.3. *The main results for general semimartingales.* This subsection combines the results of the previous subsection with those of [4].

THEOREM 3.7. *Suppose that τ satisfies (22), and S is an arbitrary $\mathbb{F}$-semimartingale. Then, there exists a pure jump and quasi-left-continuous $\mathbb{F}$-local martingale, $m^{(1)}$ satisfying*

$$\Delta m^{(1)} \in \{1 - Z_-, 0\},\ \{\Delta m^{(1)} \neq 0\} \subset \{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\},$$

and the following assertions are equivalent.
(a) *The process $S - S^\tau$ satisfies the NUPBR($\mathbb{G}$).*
(b) *For $\delta > 0$, both $I_{\{Z_- \leq 1-\delta\}} \centerdot \mathcal{T}_a(S^{(qc)}, m^{(1)})$ and $I_{\{Z_- \leq 1-\delta\}} \centerdot \mathcal{T}_a(S^{(a)}, m)$ satisfy the NUPBR($\mathbb{F}$).*
Here $\mathcal{T}_a(.,.)$ is given by (28) and $S^{(qc)}$ and $S^{(a)}$ are defined via (3).

PROOF. The proof follows immediately from a combination of Theorem 3.4 (or Remark 3.5-(1) instead), Theorem 2.12 in [4], and Lemma 1.7. □

THEOREM 3.8. *The following assertions are equivalent.*
(a) *The set $\{\widetilde{Z} = 1 > Z_-\}$ is evanescent.*
(b) *$X - X^\tau$ satisfies the NUPBR($\mathbb{G}$) for any X satisfying NUPBR($\mathbb{F}$).*

PROOF. The proof follows immediately from the combination of Lemma 3, Theorem 3.6 and Proposition 2.18 in [4](where the authors proved that the set $\{\widetilde{Z} = 1 > Z_-\}$ is accessible if and only if assertion (b) of the theorem holds for any quasi-left-continuous process X (i.e. X does not jump at predictable stopping times). □

3.4. *Explicit construction of local martingale deflators.* To construct $\mathbb{G}$-deflators for thin $\mathbb{F}$-local martingale, we start by illustrating this construction for single jump $\mathbb{F}$-martingales.

THEOREM 3.9. *Let τ be an honest time. Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable r.v. ξ such that $E[|\xi||\mathcal{F}_{T-}] < +\infty, P$-a.s. on $\{T < +\infty\}$. Define $M := \xi I_{\{Z_{T-}<1\}} I_{[\![T,+\infty[\![}$,*

$$\frac{dQ_T^{\mathbb{F}}}{dP} := D^{\mathbb{F}} := \frac{I_{\{\widetilde{Z}_T < 1\ \&\ P(\widetilde{Z}_T<1|\mathcal{F}_{T-})>0\}}}{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} + I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}, \quad and$$

$$\frac{dQ_T^{\mathbb{G}}}{dP} := D^{\mathbb{G}} := \frac{1 - Z_{T-}}{(1-\widetilde{Z}_T)P(\widetilde{Z}_T < 1|\mathcal{F}_{T-})} I_{\{T>\tau\}} + I_{\{T\leq\tau\}}. \tag{29}$$

Then the following assertions are equivalent.
(a) M *is a* $(Q^{\mathbb{F}}_T,\mathbb{F})$*-martingale.*
(b) *On* $\{Z_{T-}<1\}$*, we have*

$$E\left(\xi I_{\{\widetilde{Z}_T<1\}}\mid \mathcal{F}_{T-}\right)=0,\qquad P-a.s. \tag{30}$$

(c) $(M-M^\tau)$ *is a* $(Q^{\mathbb{G}}_T,\mathbb{G})$*-martingale.*

Proof. For the sake of simplicity, throughout the proof, we put $Q_1:=Q^{\mathbb{F}}_T$ and $Q_2:=Q^{\mathbb{G}}_T$. The proof of the theorem will be given in two steps.
1) Here, we prove (a)$\Longleftrightarrow$(b). Thanks to $\{\widetilde{Z}_T<1\}\subset\{Z_{T-}<1\}$ and $E[D^{\mathbb{F}}|\mathcal{F}_{T-}]=1$ on $\{T<+\infty\}$, we derive

$$E^{Q_1}[\xi I_{\{Z_{T-}<1\}}|\mathcal{F}_{T-}]\;=\;E\left[D^{\mathbb{F}}\xi I_{\{Z_{T-}<1\}}|\mathcal{F}_{T-}\right]=\frac{E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right]}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})}I_{\{Z_{T-}<1\}}.$$

Therefore, (a)$\Longleftrightarrow$(b) follows from combining this equality and the fact that M is a $(Q_1,\mathbb{F})$-martingale if and only if $E^{Q_1}(M_T\mid\mathcal{F}_{T-})I_{\{T<+\infty\}}=0$.
2) Here, we prove (b)$\Longleftrightarrow$ (c). To this end, we first notice that $M-M^\tau=\xi I_{\{Z_{T-}<1\ \&\ T>\tau\}}I_{[\![\tau,+\infty[\![}$ is a $(Q_2,\mathbb{G})$-martingale if and only if $E^{Q_2}[\xi I_{\{Z_{T-}<1\ \&\ T>\tau\}}|\mathcal{G}_{T-}]I_{\{T<+\infty\}}=0$. Then, using the fact that $E[D^{\mathbb{G}}|\mathcal{G}_{T-}]=1$ on $\{T<+\infty\}$, we get

$$\begin{aligned}
&E^{Q_2}[\xi I_{\{Z_{T-}<1\ \&\ T>\tau\}}|\mathcal{G}_{T-}]=E\left[D^{\mathbb{G}}\xi I_{\{Z_{T-}<1\}}I_{\{T>\tau\}}|\mathcal{G}_{T-}\right]\\
&=E\left[\frac{\xi I_{\{T>\tau\}}}{1-\widetilde{Z}_T}\Big|\mathcal{G}_{T-}\right]\frac{1-Z_{T-}}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})}I_{\{Z_{T-}<1\ \&\ T>\tau\}}\\
&=\frac{E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right]}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})}I_{\{Z_{T-}<1\}}I_{\{T>\tau\}}, \qquad (31)
\end{aligned}$$

where the equality (31) follows from the fact that, τ being honest and

$$E\left(H\mid\mathcal{G}_{T-}\right)I_{\{T>\tau\}}=E\left(H(1-\widetilde{Z}_T)\mid\mathcal{F}_{T-}\right)(1-Z_{T-})^{-1}I_{\{T>\tau\}},$$

for any $\mathcal{F}_T$-measurable random variable H such that the above conditional expectations exist (see Proposition 5.3 of [22]). Therefore, if assertion (b) holds, then assertion (c) follows immediately from (31). Conversely, if assertion (c) holds, then $E^{Q_2}[\xi I_{\{Z_{T-}<1\}}I_{\{T>\tau\}}|\mathcal{G}_{T-}]=0$. Thus, a combination of this with (31) leads to $E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right](1-Z_{T-})=0$. This proves assertion (b), and the proof of the theorem is completed. ∎

REMARK 3.10. *Theorem 3.9 can be viewed as continuous-time version of Theorem 4.5 in [8], and it can be generalized easily to the case of a finite number of ordered $\mathbb{F}$-predictable stopping times on the one hand. On the other hand, when extending this theorem to the case of general thin semimartingales, the main difficulty lies in finding a positive $\mathbb{F}$-local martingale, L such that the density of $Q_T^{\mathbb{F}}$ defined in (29) coincides with L_T for any $\mathbb{F}$-predictable stopping time T. This difficulty remains an open problem and we are unable to see how to approach it. In contrast to $Q_T^{\mathbb{F}}$, the probability $Q_T^{\mathbb{G}}$ –given also in (29)– satisfies $dQ_T^{\mathbb{G}}/dP = \widetilde{L}_T^{(a)}/\widetilde{L}_{T-}^{(a)}$, where $\widetilde{L}^{(a)}$ is a positive $\mathbb{G}$-local martingale that will be described below. To this end we need to introduce some notations and recall some results from [4].*

Throughout the rest of this subsection, we consider the following three processes.

$$W^{\mathbb{G}} := \sum {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right) I_{]\!]\tau,+\infty[\![}, \tag{32}$$

$$K^{(a)} := \frac{(1-Z_-)^2(1-\widetilde{Z})^{-1}}{(1-Z_-)^2+\Delta\langle m\rangle^{\mathbb{F}}} I_{]\!]\tau,+\infty[\![}, \tag{33}$$

and for any $M \in \mathcal{M}_{loc}(\mathbb{F})$

$$\widehat{M}^{(a)} := M - M^\tau + (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![} \cdot \langle m, M\rangle^{\mathbb{F}} \in \mathcal{M}_{loc}(\mathbb{G}), \tag{34}$$

In the following, we recall a useful result from [4].

PROPOSITION 3.11. *The following assertions hold.*
(a) The positive process $(1-\Delta W^{\mathbb{G}})^{-1}$ is $\mathbb{G}$-locally bounded.
(b) The $\mathbb{G}$-optional process, $K^{(a)}$, is $\widehat{m}^{(a)}$-integrable (with respect to Definition 2.10). The resulting integral

$$\widetilde{L}^{(a)} := \mathcal{E}\left(K^{(a)}(1-\Delta W^{\mathbb{G}})^{-1} \odot \widehat{m}^{(a)}\right), \tag{35}$$

is a positive $\mathbb{G}$-local martingale such that $[\widetilde{L}^{(a)}, \widehat{M}^{(a)}] \in \mathcal{A}_{loc}(\mathbb{G})$, for any $M \in \mathcal{M}_{loc}(\mathbb{F})$.

In order to extend Theorem 3.9 to general thin semimartingales, we start by connecting the probability $Q_T^{\mathbb{G}}$ and $\widetilde{L}_T^{(a)}$, for any $\mathbb{F}$-predictable stopping time T, as follows.

REMARK 3.12. *Put $L^{\mathbb{G}} := K^{(a)} \odot \widehat{m}^{(a)}$. Then, we derive*

$$D^{\mathbb{G}}(T) : \;=\; \frac{1-Z_{T-}}{1-\widetilde{Z}_T}\frac{I_{\{T>\tau\}}}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})} + I_{\{T\le\tau\}} = \left(1+\frac{\Delta m_T}{1-\widetilde{Z}_T}\right) I_{\{T>\tau\}} + I_{\{T\le\tau\}}$$

$$= \quad \frac{1+\Delta L^{\mathbb{G}}-\Delta V^{\mathbb{G}}}{1-\Delta V^{\mathbb{G}}} = 1+\Delta \widetilde{L}^{(a)} = \frac{\widetilde{L}^{(a)}_T}{\widetilde{L}^{(a)}_{T-}}.$$

As a result, assertions (a) and (b) of Theorem 3.9 are equivalent to

$$\widetilde{L}^{(a)}(M-M^{\tau}) \text{ is a } \mathbb{G}\text{-martingale.} \tag{36}$$

The following extends Theorem 3.9 to the general thin processes with accessible jump times.

THEOREM 3.13. *Suppose that M be a thin $\mathbb{F}$-local martingale with accessible jumps only such that*

$$^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}=1>Z_-\}}\right) \equiv 0. \tag{37}$$

Then, $\widetilde{L}^{(a)}\,(M-M^{\tau})$ is a $\mathbb{G}$-local martingale.

PROOF. Thanks to Itô's formula, it is immediate that $\widetilde{L}^{(a)}\,(M-M^{\tau})$ is a $\mathbb{G}$-local martingale if and only if

$$X^{\mathbb{G}} := \widetilde{L}^{(a)}_{-}\cdot(M-M^{\tau}) + [\widetilde{L}^{(a)}, M-M^{\tau}] \tag{38}$$

is a $\mathbb{G}$-local martingale. Thanks to Proposition 3.3 and Corollary 3.5 of [7] (which states that a special σ-martingale is a local martingale) combined with the fact that $X^{\mathbb{G}}$ is a $\mathbb{G}$-special semimartingale (it is easy to check it out as a direct application of Proposition 3.11–(b)), it is enough to prove that $X^{\mathbb{G}}$ is a σ-martingale under $\mathbb{G}$. Or equivalently prove that $\Phi\cdot X^{\mathbb{G}}$ is $\mathbb{G}$-local martingale for some $\mathbb{G}$-predictable process Φ such that $0<\Phi\le 1$. This is the focus of the remaining part of this proof. Since M is a thin process with accessible jumps only, there exists a sequence of $\mathbb{F}$-predictable stopping times, $(T_n)_{n\ge 1}$, that exhaust the jumps of M, and

$$X^{\mathbb{G}} = \sum \widetilde{L}^{(a)}\Delta M I_{]\!]\tau,+\infty[\![}.$$

As a result, $X^{\mathbb{G}}$ is thin and jumps on the sequence of stopping times $(T_n)_{n\ge 1}$ only on the one hand. On the other hand, due to Proposition 3.11 (assertion (b)), we have $^{p,\mathbb{G}}(\widetilde{L}^{(a)}|\Delta M|)I_{]\!]\tau,+\infty[\![} < +\infty$, and hence

$$\Phi := \left[\sum I_{[\![T_n]\!]}2^{-n} + I_{\widetilde{\Omega}\setminus(\cup_n [\![T_n]\!])}\right]\left(1+\ ^{p,\mathbb{G}}(\widetilde{L}^{(a)}|\Delta M|)I_{]\!]\tau,+\infty[\![}\right)^{-1},$$

where $\widetilde{\Omega} := \Omega \times [0, +\infty)$, is a $\mathbb{G}$-predictable process satisfying $0 < \Phi \leq 1$, $\Phi \cdot X^{\mathbb{G}} \in \mathcal{A}(\mathbb{G})$, and its $\mathbb{G}$-compensator is given by

$$(\Phi \cdot X^{\mathbb{G}})^{p,\mathbb{G}} = \sum_n \Phi^{\ p,\mathbb{G}}(\widetilde{L}^{(a)} \Delta M^{(n)}) I_{\rrbracket \tau, +\infty \llbracket} = 0.$$

Here $M^{(n)} := \Delta M_{T_n} I_{\llbracket T_n, +\infty \llbracket}$, while the last equality follows from (36) of Remark 3.12. This proves that $\Phi \cdot X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale, and the proof of the theorem is completed. $\square$

COROLLARY 3.14. *a) If M be a thin $\mathbb{F}$-local martingale, with accessible jumps only, such that $\{\Delta M \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$, then $\widetilde{L}^{(a)}(M - M^\tau)$ is a $\mathbb{G}$-local martingale.*
b) Suppose that S is thin with accessible jumps only, $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$, and S satisfies the NUPBR($\mathbb{F}$). Then $S - S^\tau$ satisfies the NUPBR($\mathbb{G}$).

PROOF. Since S satisfies the NUPBR($\mathbb{F}$), then there exist an $\mathbb{F}$-predictable process ϕ, a sequence of $\mathbb{F}$-stopping times $(T_n)_{n \geq 1}$ that increases to infinity, and a probability measure $Q_n \sim P$ on $(\Omega, \mathcal{F}_{T_n})$ such that

$$0 < \phi \leq 1, \quad \phi \centerdot S^{T_n} \in \mathcal{M}_{0,loc}(Q_n, \mathbb{F}).$$

Recall that for any $Q \sim P$, $\{\widetilde{Z} = 1\} = \{\widetilde{Z}^Q = 1\}$ where $\widetilde{Z}^Q_t := Q(\tau \geq t | \mathcal{F}_t)$. Thus, a combination of this fact with $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$ leads to

$$\{\Delta(\phi \centerdot S^{T_n}) \neq 0\} \cap \{\widetilde{Z}^{Q_n} = 1 > Z^{Q_n}_-\} = \emptyset.$$

Therefore, by applying directly Theorem 3.13 to $\phi \centerdot S^{T_n}$ under Q_n, we conclude that $\phi \centerdot S^{T_n} - (\phi \centerdot S^{T_n})^\tau$ (or equivalently $S^{T_n} - S^{T_n \wedge \tau}$) satisfies the NUPBR($\mathbb{G}, Q_n$). Hence, the corollary follows immediately from Proposition 1.5. This ends the proof of the corollary. $\square$

4. Proofs of Theorems 2.1 and 3.1. In this section, we prove Theorems 2.1 and 3.1. These proofs are long, but are not technical. For the undefined notations, the reader may refer to Theorems 2.1 and 3.1.

4.1. *Proof of Theorem 2.1.* The proof is achieved in four steps, where we prove (c) $\Longleftrightarrow$(d), (d)$\Longleftrightarrow$ (b), (a)$\Longrightarrow$ (c), and (b) $\Longrightarrow$ (a) respectively. To this end, we start with two simple but very useful remarks a) and b).
a) Since S is a single jump process with $\mathbb{F}$-predictable jump time T, then it is easy to see that for any $\mathcal{F}_{T-}$-measurable event A,

(39) S satisfies NUPBR($\mathbb{F}$) iff both $I_A S$ & $I_{\Omega \setminus A} S$ satisfy NUPBR($\mathbb{F}$).

b) Using the notations of Theorem 2.1, one can prove that

$$\begin{aligned}\{Z_{T-}=0 \ \&\ T<+\infty\} &= \left\{P(\widetilde{Z}_T>0|\mathcal{F}_{T-})=0 \ \&\ T<+\infty\right\}\\ &= (\Omega\setminus\Gamma_0)\cap\{T<+\infty\}\subset\{\widetilde{Z}_T=0 \ \&\ T<+\infty\}. \qquad (40)\end{aligned}$$

In fact, since $\{Z_{T-}=0 \ \&\ T<+\infty\}\subset\{\widetilde{Z}_T=0 \ \&\ T<+\infty\}$ and $E(\widetilde{Z}_T|\mathcal{F}_{T-})=Z_{T-}$ on $\{T<+\infty\}$, we derive

$$E\left(Z_{T-}I_{\Omega\setminus\Gamma_0}I_{\{T<+\infty\}}\right)=E\left(\widetilde{Z}_TI_{\Omega\setminus\Gamma_0}I_{\{T<+\infty\}}\right)=0,$$

and

$$\begin{aligned}0&=P\left(\{Z_{T-}=0\}\cap\{\widetilde{Z}_T>0\}\cap\{T<+\infty\}\right)\\ &=E\left(I_{\{Z_{T-}=0\}\cap\{T<+\infty\}}P\left(\widetilde{Z}_T>0|\mathcal{F}_{T-}\right)\right).\end{aligned}$$

This implies (40).

Step 1: In this step, we prove (c) $\Longleftrightarrow$ (d). Thanks to (39), it is enough to prove the equivalence between assertions (d) and (c) separately on the events $A:=\{Z_{T-}>0 \ \&\ T<+\infty\}$ and $\Omega\setminus A$. Thus, in virtue of (40) which proves that A and Γ_0 coincides, on $\Omega\setminus A=\Omega\setminus\Gamma_0$, the three probabilities P, Q_T and $\widetilde{Q}_T$ coincide, and the equivalence between assertions (c) and (d) becomes obvious. On $A=\Gamma_0:=\{T<+\infty \ \&\ P[\widetilde{Z}_T>0|\mathcal{F}_{T-}]>0\}$, one has $\widetilde{Q}_T\sim Q_T$, and the equivalence between (c) and (d) is also obvious. This achieves this first step.

Step 2: This step proves (d)$\Longleftrightarrow$ (b). Thanks to $\{Z_{T-}=0 \ \&\ T<+\infty\}\subset\{\widetilde{Z}_T=0 \ \&\ T<+\infty\}$, we deduce that on $\{Z_{T-}=0 \ \&\ T<+\infty\}\cup\{T=+\infty\}$, $\widetilde{S}\equiv S\equiv 0$ and Q_T coincides with P as well. Hence, the equivalence between assertions (d) and (b) is obvious on $\Omega\setminus\Gamma_0$. Thus, it is enough to prove the equivalence between these assertions on $\Gamma_0:=\{T<+\infty \ \&\ P(\widetilde{Z}_T>0|\mathcal{F}_{T-})>0\}$. Assume that (d) holds. Then, thanks to Theorem 1.8, there exists a real-valued $\mathcal{F}_T$-measurable random variable, Y, such that $Y>0$ $Q_T-a.s.$ and on $\{T<+\infty\}$, we have

$$E^{Q_T}(Y|\mathcal{F}_{T-})=1,\quad E^{Q_T}(Y|\xi||\mathcal{F}_{T-})<+\infty,\quad \&\quad E^{Q_T}(Y\xi|\mathcal{F}_{T-})=0.$$

Since $Y>0$ on $\{\widetilde{Z}_T>0\}$, by putting

$$Y_1:=YI_{\{\widetilde{Z}_T>0 \ \&\ T<+\infty\}}+I_{\{\widetilde{Z}_T=0 \ \&\ T<+\infty\}\cup\{T=+\infty\}} \text{ and } \widetilde{Y}_1:=\frac{Y_1}{E[Y_1|\mathcal{F}_{T-}]},$$

it is easy to check that $Y_1>0$, $\widetilde{Y}_1>0$ $P-a.s.$, and on $\{T<+\infty\}$

$$E\left[\widetilde{Y}_1|\mathcal{F}_{T-}\right]=1 \text{ and } E\left[\widetilde{Y}_1\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]=\frac{E\left[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{E[Y_1|\mathcal{F}_{T-}]}=0.$$

Thus, again Theorem 1.8 allows us to conclude that $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$). This ends the proof of (a)$\Longrightarrow$(b). To prove the reverse sense, we suppose that assertion (b) holds. Then, thanks again to Theorem 1.8, there exists a real-valued and positive $\mathcal{F}_T$-measurable random variable, Y, such that

$$E[Y|\xi|I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]<+\infty,\ E[Y|\mathcal{F}_{T-}]=1,\ \text{and}\ E[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]=0$$

on $\{Z_{T-}>0\ \&\ T<+\infty\}$. Then, by setting

$$Y_2:=\frac{YP(\widetilde{Z}_T>0|\mathcal{F}_{T-})}{E[YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]}I_{\{\widetilde{Z}_T>0\ \&\ T<+\infty\}}+I_{\{\widetilde{Z}_T=0\ \&\ T<+\infty\}\cup\{T=+\infty\}},$$

it is easy to check that $Q_T-a.s.\ Y_2>0$,

$$E^{Q_T}\left(Y_2|\mathcal{F}_{T-}\right)=1,\quad \text{and}\quad E^{Q_T}\left(Y_2\xi I_{\{Z_{T-}>0\}}|\mathcal{F}_{T-}\right)=\frac{E\left[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{E[YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]}=0.$$

Hence, the proof of assertion (d) follows immediately from these combined with Theorem 1.8. This completes the proof of (d)$\Longleftrightarrow$(b).
Step 3: This step proves (a) $\Longrightarrow$ (c). Suppose that S^τ satisfies NUPBR($\mathbb{G}$). Then, a direct application of Theorem 1.8 to $(S^\tau,\mathbb{G})$ implies the existence of a real-valued and positive $\mathcal{G}_T$-measurable random variable $Y^{\mathbb{G}}$ such that on $\{T<+\infty\}$

$$E[Y^{\mathbb{G}}I_{\{T\leq\tau\}}|\mathcal{G}_{T-}]=1,\ E[|\xi|Y^{\mathbb{G}}I_{\{T\leq\tau\}}|\mathcal{G}_{T-}]<+\infty,\ \text{and}\ E[\xi Y^{\mathbb{G}}I_{\{T\leq\tau\}}|\mathcal{G}_{T-}]=0.\tag{41}$$

By applying Lemma B.1–(c) to $Y^{\mathbb{G}}$, we deduce the existence of two real-valued and positive $\mathcal{F}_T$-measurable variables $Y_1^{\mathbb{F}}$ and $Y_2^{\mathbb{F}}$ such that $Y^{\mathbb{G}}I_{\{T\leq\tau\}}=Y_1^{\mathbb{F}}I_{\{T<\tau\}}+Y_2^{\mathbb{F}}I_{\{T=\tau\}}$. Then, by inserting these in (41), and putting

$$\widetilde{Y}:=Y_1^{\mathbb{F}}\frac{Z_T}{\widetilde{Z}_T}I_{\{\widetilde{Z}_T>0\}}+(1-\frac{Z_T}{\widetilde{Z}_T})I_{\{\widetilde{Z}_T>0\}}Y_2^{\mathbb{F}}+I_{\{\widetilde{Z}_T=0\}}>0,$$

we deduce that, on $\{T<+\infty\}$, this real-valued and positive $\mathcal{F}_T$-measurable random variable $\widetilde{Y}$ satisfies

$$E[|\xi|\widetilde{Y}|\mathcal{F}_{T-}]I_{\{Z_{T-}>)\}}=E[|S_T|\widetilde{Y}|\mathcal{F}_{T-}]<+\infty\ \text{(thanks to Lemma B.3–(a))},$$

and

$$0\ =\ E[\xi Y^{\mathbb{G}}I_{\{T\leq\tau\}}|\mathcal{G}_{T-}]=E\Big(\xi\left[Y_1^{\mathbb{F}}Z_T+(\widetilde{Z}_T-Z_T)Y_2^{\mathbb{F}}\right]\Big|\mathcal{F}_{T-}\Big)\frac{I_{\{T\leq\tau\}}}{Z_{T-}}.$$

Therefore, by taking conditional expectation in the above equality, we get

$$0 = E[\xi\widetilde{Y}\frac{\widetilde{Z}_T}{Z_{T-}}I_{\{Z_{T-}>0\}}|\mathcal{F}_{T-}] = E^{\widetilde{Q}_T}[\xi\widetilde{Y}|\mathcal{F}_{T-}]I_{\{Z_{T-}>0\}} = E^{\widetilde{Q}_T}[S_T\widetilde{Y}|\mathcal{F}_{T-}].$$

These latter remarks combined with Theorem 1.8 lead to the proof of assertion (c), and the proof of (a)$\Longrightarrow$(c) is completed.

Step 4: This last step proves (b)$\Longrightarrow$(a). Suppose that $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$), and use Theorem 1.8 to guarantee the existence of a real-valued and positive $\mathcal{F}_T$-measurable random variable Y such that, on $\{T < +\infty\}$, we have

$$E[Y|\xi|I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < +\infty, \text{ and } E[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0. \tag{42}$$

Then, consider the real-valued and positive $\mathcal{G}_T$-measurable random variable $Y^{\mathbb{G}}$ given by

$$Y^{\mathbb{G}} := Y\left(\widetilde{Z}\right)^{-1} I_{\{T\leq\tau\}} + I_{\{T>\tau\}}.$$

Thus, it is easy from (42) to see that $E\left(Y^{\mathbb{G}}|S^\tau_T||\mathcal{G}_{T-}\right) < +\infty$ and derive

$$\begin{aligned} E\left(Y^{\mathbb{G}}S^\tau_T|\mathcal{G}_{T-}\right) &= E\left(Y^{\mathbb{G}}\xi I_{\{T\leq\tau\}}|\mathcal{G}_{T-}\right) \\ &= E\left(Y^{\mathbb{G}}\xi I_{\{T\leq\tau\}}|\mathcal{F}_{T-}\right)\frac{1}{Z_{T-}}I_{\{T\leq\tau\}} \\ &= E\left(Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right)\frac{1}{Z_{T-}}I_{\{T\leq\tau\}} = 0. \end{aligned}$$

A combination of these latter facts with Theorem 1.8 leads to assertions (a). This ends the fourth step and the proof of the theorem is completed. □

4.2. *Proof of Theorem 3.1.* Due to

$$\{Z_{T-} = 1\} = \{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\} \subset \{\widetilde{Z}_T = 1\} \text{ on } \{T < +\infty\},$$

it is obvious that $\widetilde{Q}'_T \sim Q'_T \ll P$, and the proof of (b)$\Longleftrightarrow$(c) follows immediately. Thus, the remaining part of the proof consists of three steps, where we prove (c)$\Longrightarrow$(d), (d)$\Longrightarrow$(a) and (a)$\Longrightarrow$(b) respectively.

Step 1:(c)$\Longrightarrow$(d). Suppose (c) holds. Then, thanks to Theorem 1.8, there exists a real-valued $\mathcal{F}_T$-measurable random variable $Y > 0$, Q'_T-a.s. such that $E^{Q'_T}[|S_T|Y|\mathcal{F}_{T-}] < +\infty$ and

$$E^{Q'_T}[S_TY|\mathcal{F}_{T-}] = E[\xi Y I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}]I_{\{Z_{T-}<1\}} = 0.$$

Since $I_{\{Z_{T-}=1\}}\widetilde{S} \equiv 0$, it is enough to focus on the part corresponding to $\widetilde{S}I_{\{Z_{T-}<1\}}$. Set

$$\widetilde{Y} := YI_{\{\widetilde{Z}_T<1\}} + I_{\{\widetilde{Z}_T=1 \text{ or } T=+\infty\}},$$

and thanks to the previous equalities, we obtain

$$E[\widetilde{Y}|\xi|I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] < +\infty \text{ and } E[\widetilde{Y}|\widetilde{S}_T|\mathcal{F}_{T-}I_{\{Z_{T-}<1\}} = E[\widetilde{Y}\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] = 0.$$

Therefore, by combining these with Theorem 1.8, we end the proof of assertion (d).

Step 2: (d)$\Longrightarrow$ (a). Since $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$), again Theorem 1.8 asserts the existence of a real-valued and positive $\mathcal{F}_T$-measurable random variable $Y_3 > 0$ such that

$$E[Y_3|\xi|I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] < +\infty \text{ and } E[Y_3\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] = 0. \tag{43}$$

Then, put

$$Y^{\mathbb{G}} := Y_3\left(1-\widetilde{Z}\right)^{-1} I_{\{\tau<T<+\infty\}} + I_{\{T\leq\tau \text{ or } T=+\infty\}}.$$

Then it is easy to see that $Y^{\mathbb{G}}$ is a real-valued and positive $\mathcal{G}_T$-measurable random variable. Furthermore, due to (43), one can easily deduce that $E\left(Y^{\mathbb{G}}|(S-S^\tau)_T|\Big|\mathcal{G}_{T-}\right) < +\infty$ and

$$\begin{aligned}
E\left(Y^{\mathbb{G}}(S-S^\tau)_T\Big|\mathcal{G}_{T-}\right) &= E\left(Y^{\mathbb{G}}\xi I_{\{T>\tau\}}\Big|\mathcal{G}_{T-}\right)\\
&= E\left(Y^{\mathbb{G}}\xi I_{\{T>\tau\}}\Big|\mathcal{F}_{T-}\right)(1-Z_{T-})^{-1}I_{\{T>\tau\}}\\
&= E\left(Y_3\left(1-\widetilde{Z}\right)^{-1}\xi I_{\{T>\tau\}}\Big|\mathcal{F}_{T-}\right)(1-Z_{T-})^{-1}I_{\{T>\tau\}}\\
&= E\left(Y_3\xi I_{\{\widetilde{Z}_T<1\}}\Big|\mathcal{F}_{T-}\right)(1-Z_{T-})^{-1}I_{\{T>\tau\}} = 0.
\end{aligned}$$

Therefore, by combining these latter facts on $Y^{\mathbb{G}}$ and Theorem 1.8, we conclude that $S-S^\tau$ satisfies NUPBR($\mathbb{G}$).

Step 3: (a)$\Longrightarrow$ (b). Suppose $S-S^\tau$ satisfies NUPBR($\mathbb{G}$), and use Theorem 1.8 again to guarantee the existence of a real-valued and positive $\mathcal{G}_T$-measurable $Y^{\mathbb{G}} > 0$ such that

$$E[|\xi|Y^{\mathbb{G}}I_{\{T>\tau\}}|\mathcal{G}_{T-}] < +\infty \text{ and } E[\xi Y^{\mathbb{G}}I_{\{T>\tau\}}|\mathcal{G}_{T-}] = 0. \tag{44}$$

Then, Lemma B.2–(c), asserts the existence of a positive $\mathcal{F}_T$-measurable $Y^{\mathbb{F}}$ such that $Y^{\mathbb{G}}I_{\{T>\tau\}} = Y^{\mathbb{F}}I_{\{T>\tau\}}$. By inserting this in (44), and thanks to Lemma B.3–(b),

$$E^{\widetilde{Q}'(T)}[|S_T|Y^{\mathbb{F}}|\mathcal{F}_{T-}] = E^{\widetilde{Q}'(T)}[|\xi|Y^{\mathbb{F}}|\mathcal{F}_{T-}]I_{\{Z_{T-}<1\}} < +\infty,$$

and we calculate

$$\begin{aligned} 0 &= E[\xi Y^{\mathbb{G}} I_{\{T>\tau\}}|\mathcal{G}_{T-}] = E[\xi Y^{\mathbb{F}}(1-\widetilde{Z}_T)|\mathcal{F}_{T-}]\frac{I_{\{T>\tau\}}}{1-Z_{T-}} \\ &= E^{\widetilde{Q}'(T)}\left(\xi Y^{\mathbb{F}}|\ \mathcal{F}_{T-}\right) I_{\{T>\tau\}}. \end{aligned}$$

Therefore, by taking conditional expectation, we obtain

$(1-Z_{T-})E^{\widetilde{Q}'(T)}[\xi Y^{\mathbb{F}}|\mathcal{F}_{T-}] = 0$, or equivalently $E^{\widetilde{Q}'(T)}[S_T Y^{\mathbb{F}}|\mathcal{F}_{T-}] = 0$ $P-a.s.$

This proves assertion (b), and the proof of the theorem is achieved. □

5. Proof of Theorems 2.4 and 3.4. This section is devoted to the proofs of Theorems 2.4 and 3.4. These proofs are technical and require some notations on random measures and semimartingale characteristics. For any filtration $\mathbb{H}$, we denote

$$\widetilde{\mathcal{O}}(\mathbb{H}) := \mathcal{O}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d), \qquad \widetilde{\mathcal{P}}(\mathbb{H}) := \mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field on $\mathbb{R}^d$. To a càdlàg $\mathbb{H}$-adapted process X, we associate the following optional random measure μ_X defined by

$$\mu_X(dt, dx) := \sum_{u>0} I_{\{\Delta X_u \neq 0\}} \delta_{(u,\Delta X_u)}(dt, dx)\,. \tag{45}$$

For a product-measurable functional $W \geq 0$ on $\Omega \times [0, +\infty[\times \mathbb{R}^d$, we denote $W \star \mu_X$ (or sometimes, with abuse of notation $W(x) \star \mu_X$) the process

$$(W \star \mu_X)_t := \int_0^t \int_{\mathbb{R}^d-\{0\}} W(u, x)\mu_X(du, dx) = \sum_{0<u\leq t} W(u, \Delta X_u) I_{\{\Delta X_u \neq 0\}}. \tag{46}$$

Definitions 5.1. *Consider a càdlàg $\mathbb{H}$-adapted process X, and its optional random measure μ_X.*
(a) We denote by $\mathcal{G}^1_{loc}(\mu_X, \mathbb{H})$, the set of all $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functions, W, such that

$$\left[\sum_{t\leq \cdot}\left(W(t, \Delta S_t) I_{\{\Delta S_t \neq 0\}} - \int W_t(x)\nu_X(\{t\}, dx)\right)^2\right]^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{H}).$$

(b) The set $\mathcal{H}^1_{loc}(\mu_X, \mathbb{H}))$ is the set of all $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable functions, W, such that $(W^2 \star \mu_X)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{H})$.

Also on $\Omega \times [0,+\infty[\times \mathbb{R}^d$, we define the measure $M^P_{\mu_X} := P \otimes \mu_X$ by

$$M^P_{\mu_X}(W) := \int W dM^P_{\mu_X} := E\left[(W \star \mu_X)_\infty\right],$$

(when the expectation is well defined). The conditional "expectation" given $\widetilde{\mathcal{P}}(\mathbb{H})$ of a product-measurable functional W, denoted by $M^P_{\mu_X}(W|\widetilde{\mathcal{P}}(\mathbb{H}))$, is the unique $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional $\widetilde{W}$ satisfying

$$E\left[(WI_\Sigma \star \mu_X)_\infty\right] = E\left[(\widetilde{W} I_\Sigma \star \mu_X)_\infty\right], \quad \text{for all } \Sigma \in \widetilde{\mathcal{P}}(\mathbb{H}).$$

When $X = S$, for the sake of simplicity, we denote $\mu := \mu_S$. Then, the $\mathbb{F}$-canonical decomposition of S is

$$S = S_0 + h \star (\mu - \nu) + b \cdot A + (x - h) \star \mu, \tag{47}$$

where h, defined as $h(x) := xI_{\{|x|\leq 1\}}$, is the truncation function. We associate to μ defined in (46) when $X = S$, its predictable compensator random measure ν. A direct application of Theorem A.1 in [3] (see also Theorem 3.75 in [20] (page 103), or Lemma 4.24 in [21] (Chap III)), to the martingale m defined in (5), leads to the existence of a local martingale $m^\perp$ as well as $f_m \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$, $g_m \in \mathcal{H}^1_{loc}(\mu, \mathbb{F})$ and $\beta_m \in L(S^c, \mathbb{F})$ such that

$$m = \beta_m \centerdot S^c + f_m \star (\mu - \nu) + g_m \star \mu + m^\perp. \tag{48}$$

The corresponding canonical decomposition of S^τ under $\mathbb{G}$ is given by

$$S^\tau = S_0 + h \star (\mu^{\mathbb{G}}_b - \nu^{\mathbb{G}}_b) + h\frac{f_m}{Z_-} I_{]\!]0,\tau]\!]} \star \nu + b \centerdot A^\tau + (x - h) \star \mu^{\mathbb{G}}_b \tag{49}$$

where (β_m, f_m) is given by (48) and $\mu^{\mathbb{G}}_b$ and $\nu^{\mathbb{G}}_b$ are given by

$$\mu^{\mathbb{G}}_b(dt, dx) := I_{[\![0,\tau]\!]}(t)\mu(dt, dx), \quad \nu^{\mathbb{G}}_b(dt, dx) := (1 + Z^{-1}_- f_m)I_{[\![0,\tau]\!]}(t)\nu(dt, dx). \tag{50}$$

5.1. *Proof of Theorem 2.4.* This proof consists of four steps, where we prove (b)$\Longleftrightarrow$(c), (b)$\Longrightarrow$(a), and (a)$\Longrightarrow$(b) respectively. The last step is the only step that is technically involved.

Step 1: Here, we prove (b)$\Longleftrightarrow$(c). Remark that (c)$\Longrightarrow$(b) follows immediately from Lemma 1.4. Hence, the rest of this step focuses on proving the reverse sense. Suppose that assertion (b) holds, and consider the following real-valued $\mathbb{F}$-predictable process

$$\varphi := \left[1 + \ {}^{p,\mathbb{F}}\left(Y|\Delta S|I_{\{\widetilde{Z}>0\}}\right)\right]^{-1}\left[I_{\widetilde{\Omega}\setminus(\cup_n[\![T_n]\!])} + \sum 2^{-n} I_{[\![T_n]\!]}\right],$$

where $\widetilde{\Omega} := \Omega \times [0, +\infty)$ and $(T_n)_n$ a sequence of $\mathbb{F}$-predictable stopping times such that $\{\Delta S \neq 0\} \subset \bigcup_{n=1}^{+\infty} [\![T_n]\!]$. Then, it is easy to see that $) < \varphi \leq 1$ and

$$X := Y_- \varphi \cdot S^{(0)} + [\varphi \cdot S^{(0)}, Y] = \sum Y \varphi \Delta S I_{\{\widetilde{Z}<1 \ \& \ Z_- \geq \delta\}}$$

has an integrable variation. Furthermore, the $\mathbb{F}$-compensator of X is given by (due to the fact it is a pure jump process with finite variation and it jumps on predictable stopping times only)

$$X^{p,\mathbb{F}} = \sum {}^{p,\mathbb{F}}\left(Y \varphi \Delta S I_{\{\widetilde{Z}>0\}}\right) I_{\{Z_- \geq \delta\}} \equiv 0.$$

Thus, $Y(\varphi \cdot S^{(0)})$ is an $\mathbb{F}$-local martingale, and $S^{(0)}$ satisfies the NUPBR($\mathbb{F}$). This ends the proof of (b)$\Longleftrightarrow$(c).

Step 2: Here, we prove (b)$\Longrightarrow$ (a). Suppose that assertion (b) holds, and consider a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_n$ that increases to infinity such that Y^{σ_n} is an $\mathbb{F}$-martingale, and $Q_n := Y_{\sigma_n}/Y_0 \cdot P \sim P$. Then, (10) implies that $(S^{(0)})^{\sigma_n}$ is a Q_n-local martingale and satisfies (21) under Q_n due to

$$(51) \{\widetilde{Z}^Q_T = 0\} = \{\widetilde{Z}_T = 0\}, \text{ for any } Q \sim P \text{ and any } \mathbb{F}\text{-stopping time } T,$$

where $\widetilde{Z}^Q_t := Q[\tau \geq t | \mathcal{F}_t]$. This latter fact follows from

$$E\left[\widetilde{Z}_T I_{\{\widetilde{Z}^Q_T=0\}}\right] = E\left[I_{\{\tau \geq T\}} I_{\{\widetilde{Z}^Q_T=0\}}\right] = 0,$$

(which implies $\{\widetilde{Z}^Q_T = 0\} \subset \{\widetilde{Z}_T = 0\}$) and the symmetric role of Q and P. Thus, a direct application of Theorem 2.13 to $\left((S^{(0)})^{\sigma_n}, Q_n\right)$ leads to the NUPBR($\mathbb{G}, Q_n$) of $(S^{(0)})^{\sigma_n \wedge \tau} = \left(I_{\{Z_- \geq \delta\}} \cdot S\right)^{\sigma_n \wedge \tau}$. Thanks to Proposition 1.5, this implies the NUPBR($\mathbb{G}$) of $\left(I_{\{Z_- \geq \delta\}} \cdot S\right)^{\tau}$ for any $\delta > 0$. Since $Z_-^{-1} I_{[\![0,\tau]\!]}$ is $\mathbb{G}$-locally bounded, there exists a family of $\mathbb{G}$-stopping times τ_δ that increases to infinity when δ decreases to zero, and $[\![0, \tau \wedge \tau_\delta]\!] \subset \{Z_- \geq \delta\}$. Therefore, we conclude that $S^{\tau \wedge \tau_\delta} = \left(I_{\{Z_- \geq \delta\}} \cdot S\right)^{\tau \wedge \tau_\delta}$ satisfies the NUPBR($\mathbb{G}$). Hence, again Proposition 1.5 implies finally that S^τ satisfies the NUPBR($\mathbb{G}$). This ends the proof of (b)$\Longrightarrow$(a).

Step 3: In this step, we prove (a)$\Longrightarrow$(b). Suppose that S^τ satisfies NUPBR($\mathbb{G}$). Then, there exists a σ-martingale density under $\mathbb{G}$, for $I_{\{Z_- \geq \delta\}} \centerdot S^\tau$, $(\delta > 0)$, that we denote by $D^{\mathbb{G}}$. Then, from a direct application of Theorem A.1, we deduce the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional, $f^{\mathbb{G}} \in \mathcal{G}^1_{loc}(\mu^{\mathbb{G}}_b, \mathbb{G})$, such that $D^{\mathbb{G}} := \mathcal{E}(N^{\mathbb{G}}) > 0$, with

$$N^{\mathbb{G}} := W^{\mathbb{G}} \star (\mu^{\mathbb{G}}_a - \nu^{\mathbb{G}}_a), \ W^{\mathbb{G}} := f^{\mathbb{G}} - 1 + \frac{\widehat{f^{\mathbb{G}}} - a^{\mathbb{G}}}{1 - a^{\mathbb{G}}} I_{\{a^{\mathbb{G}}<1\}},$$

where $\nu^{\mathbb{G}}$ is defined in (50), and, introducing f_m defined in (48)

$$xf^{\mathbb{G}} I_{\{Z_-\geq\delta\}}\star\nu_a^{\mathbb{G}} = xf^{\mathbb{G}}\left(1+\frac{f_m}{Z_-}\right) I_{]\!]0,\tau]\!]} I_{\{Z_-\geq\delta\}}\star\nu\equiv 0. \tag{52}$$

Thanks to Lemma B.1, we conclude the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, f, such that $f^{\mathbb{G}} I_{]\!]0,\tau]\!]} = f I_{]\!]0,\tau]\!]}$. Thus, (52) becomes

$$U^{(b)} := xf\left(1+\frac{f_m}{Z_-}\right) I_{]\!]0,\tau]\!]} I_{\{Z_-\geq\delta\}}\star\nu\equiv 0.$$

Introduce the following notations

$$\begin{cases} \mu_0 := I_{\{\widetilde{Z}>0\ \&\ Z_-\geq\delta\}}\cdot\mu, \quad \nu_0 := h_0 I_{\{Z_-\geq\delta\}}\cdot\nu, \; h_0 := M^P_\mu\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right), \\ g := \frac{f(1+\frac{f_m}{Z_-})}{h_0} I_{\{h_0>0\}} + I_{\{h_0=0\}}, \quad a_0(t) := \nu_0(\{t\},\mathbb{R}^d), \end{cases} \tag{53}$$

and assume that

$$\sqrt{(g-1)^2\star\mu_0}\in\mathcal{A}^+_{loc}(\mathbb{F}). \tag{54}$$

Then, thanks to Lemma A.2, we deduce that $W := (g-1)/(1-a^0+\widehat{g})\in \mathcal{G}^1_{loc}(\mu_0,\mathbb{F})$, and the local martingales

$$N^{(0)} := \frac{g-1}{1-a^0+\widehat{g}}\star(\mu_0-\nu_0), \quad Y^{(0)} := \mathcal{E}(N^{(0)}), \tag{55}$$

are well defined satisfying $1+\Delta N^{(0)}>0$, $[N^{(0)},S]\in\mathcal{A}(\mathbb{F})$, and on $\{Z_->0\}$ we have

$$\begin{aligned} \frac{{}^{p,\mathbb{F}}\left(Y^{(0)}\Delta S I_{\{\widetilde{Z}>0\}}\right)}{Y^{(0)}_-} &= {}^{p,\mathbb{F}}\left((1+\Delta N^{(0)})\Delta S I_{\{\widetilde{Z}>0\}}\right) = {}^{p,\mathbb{F}}\left(\frac{g}{1-a^0+\widehat{g}}\Delta S I_{\{\widetilde{Z}>0\}}\right) \\ &= \Delta\frac{gxh_0}{1-a^0+\widehat{g}}\star\nu = \Delta\frac{xf(1+f_m/Z_-)}{1-a^0+\widehat{g}} I_{\{Z_->0\}}\star\nu \\ &= \frac{{}^{p,\mathbb{F}}\left(\Delta U^{(b)}\right)}{1-a^0+\widehat{g}}\equiv 0. \end{aligned}$$

This proves that assertion (b) holds under the assumption (54). The remaining part of the proof shows that this assumption holds. To this end, we start by noticing that on the set $\{h_0>0\}$,

$$\begin{aligned} g-1 &= \frac{f(1+\frac{f_m}{Z_-})}{h_0} - 1 = \frac{(f-1)(1+\frac{f_m}{Z_-})}{h_0} + \frac{f_m}{Z_- h_0} + \frac{M^P_\mu\left(I_{\{\widetilde{Z}=0\}}|\widetilde{\mathcal{P}}\right)}{h_0} \\ &:= \frac{(f-1)(1+\frac{f_m}{Z_-})}{h_0} + \frac{M^P_\mu\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_- h_0} =: g_1 + \frac{M^P_\mu\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_- h_0}. \end{aligned}$$

Since $\left((f-1)^2 I_{]\!]0,\tau]\!]}\star\mu\right)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{G})$, then due to Proposition A.3–(e)

$$\sqrt{(f-1)^2 I_{\{Z_-\geq\delta\}}\star(\widetilde{Z}\cdot\mu)}\in\mathcal{A}^+_{loc}(\mathbb{F}),\quad \text{for any}\quad \delta>0.$$

Then, a direct application of Proposition A.3–(a), for any $\delta>0$, we have

$$(f-1)^2 I_{\{|f-1|\leq\alpha\ \&\ Z_-\geq\delta\}}\star(\widetilde{Z}\cdot\mu),\quad |f-1|I_{\{|f-1|>\alpha\ \&\ Z_-\geq\delta\}}\star(\widetilde{Z}\cdot\mu)\in\mathcal{A}^+_{loc}(\mathbb{F}).$$

By stopping, without loss of generality, we assume these two processes and $[m,m]$ belong to $\mathcal{A}^+(\mathbb{F})$. Remark that $Z_-+f_m=M^P_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right)\leq M^P_\mu\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)=h_0$ that follows from $\widetilde{Z}\leq I_{\{\widetilde{Z}>0\}}$. Therefore, we derive

$$\begin{aligned}
E\left[g_1^2 I_{\{|f-1|\leq\alpha\}}\star\mu_0(\infty)\right] &= E\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2}I_{\{|f-1|\leq\alpha\}}\star\mu_0(\infty)\right]\\
&= E\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2}I_{\{|f-1|\leq\alpha\}}\star\nu_0(\infty)\right]\\
&\leq \delta^{-2}E\left[(f-1)^2(Z_-+f_m)I_{\{|f-1|\leq\alpha\ \&\ Z_-\geq\delta\}}\star\nu(\infty)\right]\\
&= \delta^{-2}E\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}}\star(\widetilde{Z}I_{\{Z_-\geq\delta\}}\cdot\mu)(\infty)\right]<+\infty,
\end{aligned}$$

and

$$\begin{aligned}
E\left[|g_1|I_{\{|f-1|>\alpha\}}\star\mu_0(\infty)\right] &= E\left[\frac{|f-1|(1+\frac{f_m}{Z_-})}{h_0}I_{\{|f-1|>\alpha\}}\star\mu_0(\infty)\right]\\
&= E\left[|f-1|(1+\frac{f_m}{Z_-})I_{\{|f-1|>\alpha\}}I_{\{Z_-\geq\delta\}}\star\nu_0(\infty)\right]\\
&\leq \delta^{-1}E\left[|f-1|I_{\{|f-1|>\alpha\}}\star(\widetilde{Z}I_{\{Z_-\geq\delta\}}\cdot\mu)(\infty)\right]<+\infty.
\end{aligned}$$

Here μ_0 and ν_0 are defined in (53). Therefore, again by Proposition A.3–(a), we conclude that $\sqrt{g_1^2\star\mu_0}\in\mathcal{A}^+_{loc}(\mathbb{F})$.

Due to $M^P_\mu(HK|\widetilde{\mathcal{P}}(\mathbb{F}))^2\leq M^P_\mu(H^2|\widetilde{\mathcal{P}}(\mathbb{F}))M^P_\mu(K^2|\widetilde{\mathcal{P}}(\mathbb{F}))$, we derive

$$\begin{aligned}
E\left[\frac{M^P_\mu\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)^2}{Z_-^2h_0^2}\star\mu_0(\infty)\right] &\leq E\left[\frac{M^P_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)M^P_\mu\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_-^2h_0^2}\star\mu_0(\infty)\right]\\
&= E\left[\frac{M^P_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)}{Z_-^2}I_{\{Z_-\geq\delta\}}\star\mu(\infty)\right]\\
&\leq \delta^{-2}E\left[[m,m]_\infty\right]<+\infty.
\end{aligned}$$

Hence, we conclude that $\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$. This ends the proof of (54), and the proof of the theorem is completed. $\square$

5.2. *Proof of Theorem 3.4.* Before proving the equivalence between the two assertions of the theorem, we will start outlining a number of remarks that simplify tremendously the proof. It is easy to prove that, for any $\mathbb{F}$-stopping time T, on $\{T < +\infty\}$ we have

$$\{\widetilde{Z}^Q_T = 1\} = \{\widetilde{Z}_T = 1\} \quad \text{for any} \quad Q \sim P, \tag{56}$$

where $\widetilde{Z}^Q_t := E^Q(\tau \geq t|\mathcal{F}_t)$. Indeed, due to

$$E\left[(1-\widetilde{Z}_T)I_{\{\widetilde{Z}^Q_T=1\}}\right] = E\left[I_{\{\tau<T\}}I_{\{\widetilde{Z}^Q_T=1\}}\right] = 0,$$

the inclusion $\{\widetilde{Z}^Q_T = 1\} \subset \{\widetilde{Z}_T = 1\}$ follows, while the reverse inclusion follows by symmetry. This proves (56).
Since S is a thin process with accessible jump times only, then there exists a sequence of $\mathbb{F}$-predictable stopping times, $(T_n)_{n\geq 1}$, such that

$$\{\Delta S \neq 0\} \subset \bigcup_{n=1}^{+\infty} [\![T_n]\!].$$

The proof of the theorem consists of three steps in which we prove (b)$\Longleftrightarrow$(c), (b)$\Longrightarrow$(a) and (a)$\Longrightarrow$(b) respectively.
Step 1: Here, we prove (b)$\Longleftrightarrow$(c). Thanks to Lemma 1.4, it is easy to see that (c)$\Longrightarrow$(b) follows immediately. To prove the reverse (i.e. (b)$\Longrightarrow$(c)), we consider the following real-valued and $\mathbb{F}$-predictable process

$$\varphi := \left[1 + {}^{p,\mathbb{F}}\left(Y|\Delta S|I_{\{\widetilde{Z}<1\}}\right)\right]^{-1}\left[I_{\widetilde{\Omega}\setminus(\bigcup_{n=1}^{+\infty}[\![T_n]\!])} + \sum_{n=1}^{+\infty} 2^{-n} I_{[\![T_n]\!]}\right],$$

where $\widetilde{\Omega} := \Omega \times [0, +\infty)$. It is easy to check that $0 < \varphi \leq 1$ and $U := Y_- \varphi \cdot S^{(1)} + [Y, \varphi \cdot S^{(1)}]$ is a process with integrable variation, and its compensator (since it is a pure jump process with finite variation and jumps on predictable stopping times $(T_n)_n$ only) is given by

$$U^{p,\mathbb{F}} = \sum \varphi\ {}^{p,\mathbb{F}}\left(Y \Delta S I_{\{\widetilde{Z}=1>Z_-\}}\right) \equiv 0.$$

This proves that Y is σ-martingale density for $S^{(1)}$ (i.e. $Y \in \mathcal{L}(S^{(1)}, \mathbb{F})$), and hence assertion (c) follows immediately.
Step 2: Here we prove (b)$\Longrightarrow$ (a). Suppose that assertion (b) holds, and

consider a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_n$ such that Y^{σ_n} is a martingale, and put $Q_n := (Y_{\sigma_n}/Y_0)\cdot P \sim P$. Then, since $\widetilde{S} := \sum \Delta S I_{\{\widetilde{Z}<1\}}$ is a thin process with accessible jumps only, the condition (26) translates into the fact that $\widetilde{S}^{\sigma_n}$ is a Q_n-local martingale satisfying

$$\{\Delta\widetilde{S}^{\sigma_n}\neq 0\}\cap\{\widetilde{Z}^{Q_n}=1>Z_-^{Q_n}\}=\{\Delta\widetilde{S}^{\sigma_n}\neq 0\}\cap\{\widetilde{Z}=1>Z_-\}=\emptyset,$$

due to (56). Therefore, thanks to Proposition 1.5, it is enough to prove that assertion (a) holds true under Q_n for S^{σ_n}. Therefore, without loss of generality, we assume $Y\equiv 1$ and hence $\widetilde{S}$ is a $\mathbb{F}$-local martingale satisfying (37). Thus, a direct application of Theorem 3.13 implies that $S-S^\tau=\widetilde{S}-\widetilde{S}^\tau$ satisfies the NUPBR($\mathbb{G}$).

Step 3: Here, we prove (a)$\Longrightarrow$(b). Suppose that $S-S^\tau$ satisfies the NUPBR($\mathbb{G}$). A direct application Theorem A.1 implies the existence of $f^{\mathbb{G}}\in\mathcal{G}^1_{loc}(\mu^{\mathbb{G}}_a,\mathbb{G})$ such that $f^{\mathbb{G}}>0$,

$$N^{\mathbb{G}} \;:=\; W^{\mathbb{G}}\star(\mu^{\mathbb{G}}_a-\nu^{\mathbb{G}}_a),\quad W^{\mathbb{G}}:=f^{\mathbb{G}}-1+\frac{\widehat{f^{\mathbb{G}}}-a^{\mathbb{G}}}{1-a^{\mathbb{G}}}I_{\{a^{\mathbb{G}}<1\}},$$

and

$$xf^{\mathbb{G}}\star\nu^{\mathbb{G}}_a=xf^{\mathbb{G}}\left(1-\frac{f_m}{1-Z_-}\right)I_{]\!]\tau,+\infty[\![}\star\nu\equiv 0. \tag{57}$$

Here $f_m:=M^P_\mu(\Delta m|\widetilde{\mathcal{P}}(\mathbb{F}))$ (given also by (48), and $\mu^{\mathbb{G}}_a$ and $\nu^{\mathbb{G}}_a$ are given by

$$\mu^{\mathbb{G}}_a:=I_{]\!]\tau,+\infty[\![}\cdot\mu,\ \nu^{\mathbb{G}}_a:=\left(1-\frac{f_m}{1-Z_-}\right)I_{]\!]\tau,+\infty[\![}\cdot\nu.$$

Thanks to Lemma B.1, there exists a $\mathcal{P}(\mathbb{F})$-measurable functional $f>0$ such that $f^{\mathbb{G}}=f$ on the stochastic interval $]\!]\tau,+\infty[\![$, and (57) becomes

$$xf\left(1-\frac{f_m}{1-Z_-}\right)I_{]\!]\tau,+\infty[\![}\star\nu\equiv 0. \tag{58}$$

Due to Proposition A.4 and $\mathbb{G}$-locally boundedness of $(1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}$, we could find a sequence of $\mathbb{F}$-stopping time $(\sigma^{\mathbb{F}}_n)_{n\geq 1}$ that increases to infinity and $(1-Z_-)^{-1}I_{[\![0,\sigma^{\mathbb{F}}_n]\!]}I_{]\!]\tau,+\infty[\![}$ is bounded by $(n+1)$. Also, since $\left((f-1)^2I_{]\!]\tau,+\infty[\![}\star\mu\right)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{G})$, thanks to Proposition A.4 (both assertions (c) and (a)) we deduce the existence of a sequence of $\mathbb{F}$-stopping times $(\tau_n)_n$ that increases to infinity such that the three processes $[m,m]^{\tau_n}$

$(f-1)^2I_{\{|f-1|\leq\alpha\ \&\ 1-Z_-\geq\delta\}}\star\overline{\mu})^{\tau_n}$ and $|f-1|I_{\{|f-1|>\alpha\&\ 1-Z_-\geq\delta\}}\star\overline{\mu})^{\tau_n}$

are integrable, where $\overline{\mu} := (1-\widetilde{Z})\cdot\mu$. Consider the following notations

$$\begin{aligned}\mu_1 &:= I_{\{\widetilde{Z}<1\ \&\ 1-Z_-\geq\delta\}}\cdot\mu,\quad \nu_1 := h_1 I_{\{1-Z_-\geq\delta\}}\cdot\nu,\ h_1 := M^P_\mu\left(I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right),\\ g &:= \frac{f(1-\frac{f_m}{1-Z_-})}{h_1}I_{\{h_1>0\ \&\ Z_-<1\}} + I_{\{h_1=0\text{ or } Z_-=1\}},\end{aligned}$$

and suppose that

$$W^{(1)}(t,x) := \frac{g_t(x)-1}{1-a^{(1)}_t+\widehat{g}_t}\in\mathcal{G}^1_{loc}(\mu_1,\mathbb{F}),\tag{59}$$

where $a^{(1)}_t := \nu_1(\{t\},\mathbb{R}^d)$ and $\widehat{g}_t := \int g_t(x)\nu_1(\{t\},dx)$.
Then, we can easily prove that assertion (b) holds. In fact, we take

$$N^{(1)} := \frac{g-1}{1-a^{(1)}+\widehat{g}}\star(\mu_1-\nu_1)\quad\text{and}\quad Y := \mathcal{E}(N^{(1)}).$$

Then, it is clear that

$$1+\Delta N^{(1)} = \frac{1}{1-a^{(1)}+\widehat{g}}I_{\{\Delta S=0\text{ or }\widetilde{Z}=1\}} + \frac{g(\Delta S)}{1-a^{(1)}+\widehat{g}}I_{\{\Delta S\neq 0\ \&\ \widetilde{Z}<1\}} > 0,$$

and on $\{Z_-<1\}$ we get

$$\begin{aligned}{}^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1\}}\right)_t &= Y_{t-}\ {}^{p,\mathbb{F}}\left((1+\Delta N^{(1)})\Delta S I_{\{\widetilde{Z}<1\}}\right)_t = \frac{Y_{t-}\ {}^{p,\mathbb{F}}\left(g(\Delta S)\Delta S I_{\{\widetilde{Z}<1\}}\right)_t}{1-a^{(1)}_t+\widehat{g}_t}\\ &= \frac{Y_{t-}}{1-a^{(1)}_t+\widehat{g}_t}\int g_t(x)xh_1(t,x)\nu(\{t\},dx)\\ &= \frac{Y_{t-}}{1-a^{(1)}_t+\widehat{g}_t}\int xf_t(x)\left(1-\frac{f_m(t,x)}{1-Z_-}\right)\nu(\{t\},dx)\equiv 0.\end{aligned}$$

The last equality in the above string of equalities follows direct from (58). Therefore, assertion (b) follows immediately as long as (59) holds. Thus, the reaming part of this proof focuses on proving this condition. To this end, on $\{h_1>0\ \&\ Z_-<1\}$ we calculate

$$\begin{aligned}g-1 &= \frac{f(1-Z_--f_m)}{h_1(1-Z_-)}-1 = \frac{(f-1)(1-Z_--f_m)}{h_1(1-Z_-)} - \frac{M^P_\mu\left(\Delta m I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right)}{(1-Z_-)h_1}\\ &=: g_1+g_2,\end{aligned}$$

and remark that $\{1-Z_- - f_m > 0\} \subset \{h_0 > 0\}$ which is due to

$$1 - Z_- - f_m = 1 - M^P_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) \leq M^P_\mu\left(I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right) = h_1,$$

that is implied by $I_{\{\widetilde{Z}=1\}} \leq \widetilde{Z}$. Therefore, we derive that

$$\begin{array}{rcl}
E\left[g_1^2 I_{\{|f-1|\leq\alpha\}} \star \mu_0(\sigma_n\wedge\tau_n)\right] & = & E\left[\dfrac{(f-1)^2(1-Z_- - f_m)^2}{h_1^2(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} I_{\{\widetilde{Z}<1\}} \star \mu(\sigma_n\wedge\tau_n)\right] \\
& = & E\left[\dfrac{(f-1)^2(1-Z_- - f_m)^2}{h_1(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} \star \nu(\sigma_n\wedge\tau_n)\right] \\
& \leq & E\left[(f-1)^2 \dfrac{1-Z_- - f_m}{(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} \star \nu(\sigma_n\wedge\tau_n)\right] \\
& \leq & E\left[\dfrac{(f-1)^2}{(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} \star ((1-\widetilde{Z})\cdot\mu(\sigma_n\wedge\tau_n))\right] \\
& = & E\left[\dfrac{(f-1)^2}{(1-Z_-)^2} I_{\{|f-1|\leq\alpha\}} I_{]\!]\tau,+\infty[\![} \star \mu(\sigma_n\wedge\tau_n)\right] \\
& \leq & (n+1)^2 E\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} \star \mu(\tau_n)\right] < +\infty.
\end{array}$$

and

$$\begin{array}{rcl}
E\left[|g_1| I_{\{|f-1|>\alpha\}} \star \mu_1(\sigma_n\wedge\tau_n)\right] & = & E\left[\dfrac{|f-1|(1-Z_- - f_m)}{h_1(1-Z_-)} I_{\{|f-1|>\alpha\}} I_{\{\widetilde{Z}<1\}} \star \mu(\sigma_n\wedge\tau_n)\right] \\
& = & E\left[|f-1| \dfrac{1-Z_- - f_m}{1-Z_-} I_{\{|f-1|>\alpha\}} \star \nu(\sigma_n\wedge\tau_n)\right] \\
& \leq & E\left[\dfrac{|f-1|}{1-Z_-} I_{\{|f-1|>\alpha\}} \star ((1-\widetilde{Z})\cdot\mu)(\sigma_n\wedge\tau_n)\right] \\
& = & E\left[\dfrac{|f-1|}{1-Z_-} I_{\{|f-1|>\alpha\}} I_{]\!]\tau,+\infty[\![} \star \mu(\sigma_n\wedge\tau_n)\right] \\
& \leq & (n+1) E\left[|f-1| I_{\{|f-1|>\alpha\}} \star ((1-\widetilde{Z})\cdot\mu)(\tau_n)\right] < +\infty.
\end{array}$$

This proves that $\sqrt{g_1^2 \star \mu_1} \leq \sqrt{2}\sqrt{g_1^2 I_{\{|f-1|>\alpha\}} \star \mu_1} + \sqrt{2}\left(g_1 I_{\{|f-1|>\alpha\}} \star \mu_1\right)$ belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$. To prove $\sqrt{g_2^2 \star \mu_1} \in \mathcal{A}^+_{loc}(\mathbb{F})$, we derive

$$\begin{array}{rcl}
E\left[(g_2)^2 \star \mu_0(\sigma_n\wedge\tau_n)\right] & = & E\left[\dfrac{M^P_\mu\left(\Delta m I_{\{\widetilde{Z}<1\}}|\widetilde{\mathcal{P}}\right)^2}{(1-Z_-)^2 h_0^2} I_{\{\widetilde{Z}<1\ \&\ Z_-<1\}} \star \mu(\sigma_n\wedge\tau_n)\right] \\
& \leq & E\left[\dfrac{M^P_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right) M^P_\mu\left(I_{\{\widetilde{Z}<1\ \&\ Z_-<1\}}|\widetilde{\mathcal{P}}\right)^2}{(1-Z_-)^2 h_1^2} \star \nu(\sigma_n\wedge\tau_n)\right]
\end{array}$$

$$
\begin{aligned}
&= E\left[\frac{M^P_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)}{(1-Z_-)^2} I_{\{Z_-<1\}} \star \nu(\sigma_n\wedge\tau_n)\right]\\
&\le E\left[\frac{1}{(1-Z_-)^3} I_{]\!]\tau,+\infty[\![} \cdot \left(M^P_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)\star\nu\right)_{\sigma_n\wedge\tau_n}\right]\\
&\le (n+1)^3 E([m,m]_{\tau_n}) < +\infty.
\end{aligned}
$$

Hence, $\sqrt{(g-1)^2\star\mu_1}\in\mathcal{A}^+_{loc}(\mathbb{F})$ follows. Thanks to Lemma A.2, (59) follows immediately. This ends the proof of the theorem. □

APPENDIX A: INTEGRALITY RESULTS

THEOREM A.1. *Let S be a semi-martingale with predictable characteristic triplet $(b,c,\nu=A\otimes F)$, N is a local martingale such that $\mathcal{E}(N)>0$, and (β,f,g,N') are its Jacod's parameters. Then the following assertions hold.*
1) $\mathcal{E}(N)$ is a σ-martingale density of S if and only if the following two properties hold:

$$\int |x-h(x)+xf(x)|F(dx)<+\infty,\quad P\otimes A-a.e. \tag{60}$$

and

$$b+c\beta+\int\Big(x-h(x)+xf(x)\Big)F(dx)=0,\qquad P\otimes A-a.e. \tag{61}$$

2) In particular, we have

$$\int x(1+f_t(x)\nu(\{t\},dx)=\int x(1+f_t(x)F_t(dx)\Delta A_t=0,\qquad P-a.e. \tag{62}$$

PROOF. The proof can be found in Choulli et al. [11, Lemma 2.4] 2007. □

LEMMA A.2. *Let f be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional such that $f>0$ and*

$$\Big[(f-1)^2\star\mu\Big]^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{H}). \tag{63}$$

Then, the $\mathbb{H}$-predictable process $\left(1-a^{\mathbb{H}}+\widehat{f}^{\mathbb{H}}\right)^{-1}$ is locally bounded, and hence

$$W_t(x):=\frac{f_t(x)-1}{1-a^{\mathbb{H}}_t+\widehat{f}^{\mathbb{H}}_t}\in\mathcal{G}^1_{loc}(\mu,\mathbb{H}). \tag{64}$$

Here, $a^{\mathbb{H}}_t:=\nu^{\mathbb{H}}(\{t\},\mathbb{R}^d)$, $\widehat{f}^{\mathbb{H}}_t:=\int f_t(x)\nu^{\mathbb{H}}(\{t\},dx)$ and $\nu^{\mathbb{H}}$ is the $\mathbb{H}$-predictable random measure compensator of μ under $\mathbb{H}$.

PROOF. Put

$$U_t(x) = 1 - f_t(x), \quad \text{and} \quad \widehat{U}_t := a_t^{\mathbb{H}} - \widehat{f}_t^{\mathbb{H}}.$$

We start by remarking that (64) follows from the combination of (63) and the local boundedness of $1/(1-\widehat{U})$. Therefore, in what follows, we will focus on proving this latter fact. Consider $\delta \in (0,1)$, $\eta \in (0,1)$, and the stopping times and processes defined by

$$T_0 = 0, \quad T_{n+1} := \inf\left\{ t > T_n \mid \sum_{T_n < v \le t} \left(U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}}\right)^2 > \delta^2 \right\},$$

$$V_n(t) := \Big[\sum_{T_n < v \le t} \left(U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}}\right)^2\Big]^{1/2}$$

Remark that —since for each $n \ge 0$, the process $(V_n(t))^2$ is càdlàg (Right continuous with left limits) and nondecreasing real-valued process—- we have

$$(V_n(T_{n+1}))^2 := \sum_{T_n < v \le T_{n+1}} \left(U_v(\Delta S_v) I_{\{\Delta S_v \neq 0\}}\right)^2 \ge \delta^2 \quad \text{on} \quad \{T_{n+1} < +\infty\}.$$

This implies that T_n increases to $+\infty$ almost surely, and

$$V_n(t-) \le \delta, \quad P-a.s. \quad \text{for all } t \le T_{n+1}.$$

Due to $0 \le (1-\widehat{U})^{-1} I_{\{\widehat{U} < 1-\eta\}} \le \eta^{-1}$ and

$$(1-\widehat{U})^{-1} = (1-\widehat{U})^{-1} I_{\{\widehat{U} \ge 1-\eta\}} + (1-\widehat{U})^{-1} I_{\{\widehat{U} < 1-\eta\}},$$

we deduce that the proof of the lemma will achieved once we prove that

$$Y := \frac{1}{1-\widehat{U}} I_{\{\widehat{U} \ge 1-\eta\}}$$

is locally bounded. Thanks to [13], this fact is equivalent to

$$\sup_{0 \le u \le t} Y_u < +\infty \quad P-a.s. \quad \text{for any } t \in (0,+\infty).$$

Since T_n increases to ∞ almost surely, then this fact is implied by

$$\sup_{T_n \le u \le t \wedge T_{n+1}} Y_u < +\infty \quad P-a.s. \quad \text{on} \quad \{t > T_n\}.$$

Simple calculation leads to

$$\widehat{U}_s \leq V_n(s-) + {}^{p,\mathbb{H}}(\Delta V_n)_s, \quad \text{for all } \ T_n < s \leq T_{n+1}.$$

Thus, it is easy to see that for $\delta + \eta < 1$,

$$\{s \in]T_n, T_{n+1}] \mid \widehat{U}_s \geq 1 - \eta\} \subset \{s \in]T_n, T_{n+1}] \mid {}^{p,\mathbb{H}}(\Delta V_n)_s \geq 1 - \eta - V_n(s-)\}$$

$$\subset \{s \in]T_n, T_{n+1}] \mid \Delta\left((V_n)^{p,\mathbb{H}}\right) = {}^{p,\mathbb{H}}(\Delta V_n)_s \geq 1 - \eta - \delta\} =: \Gamma_n.$$

It is obvious that $\#(\Gamma_n \cap [0,t]) < +\infty \ P-a.s.$ since $(V_n)^{p,\mathbb{H}}$ is a càdlàg process. Thus, we deduce that

$$\sup_{T_n \leq u \leq t \wedge T_{n+1}} Y_u = \max_{T_n \leq u \leq t \wedge T_{n+1}} Y_u < +\infty.$$

This ends the proof of the lemma. □

Proposition A.3. *For any $\alpha > 0$, the following assertions hold:*
(a) *Let h be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional. Then, $\sqrt{(h-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{H})$ iff*

$$(h-1)^2 I_{\{|h-1| \leq \alpha\}} \star \mu \ \textit{and} \ \ |h-1| I_{\{|h-1| > \alpha\}} \star \mu \ \ \textit{belong to} \ \ \mathcal{A}^+_{loc}(\mathbb{H}).$$

(b) *Let V be an $\mathbb{F}$-predictable and non-decreasing process. Then, $V^\tau \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if $I_{\{Z_- \geq \delta\}} \centerdot V \in \mathcal{A}^+_{loc}(\mathbb{F})$ for any $\delta > 0$.*
(c) *Let h be a nonnegative and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional. Then, $hI_{]\!]0,\tau]\!]} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if for all $\delta > 0$, $hI_{\{Z_- \geq \delta\}} \star \mu^1 \in \mathcal{A}^+_{loc}(\mathbb{F})$, where $\mu^1 := \widetilde{Z} \centerdot \mu$.*
(d) *Let f be positive and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable, and $\mu^1 := \widetilde{Z}\centerdot\mu$. Then $\sqrt{(f-1)^2 I_{]\!]0,\tau]\!]} \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{G})$ iff $\sqrt{(f-1)^2 I_{\{Z_- \geq \delta\}} \star \mu^1} \in \mathcal{A}^+_{loc}(\mathbb{F})$, for all $\delta > 0$.*

Proposition A.4. *Suppose that τ is a finite honest time satisfying (22). Then, the following properties hold.*
(a) *Let $\Phi^{\mathbb{G}}$ a $\mathbb{G}$-predictable process and k a nonnegative and $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional such that $0 < \Phi^{\mathbb{G}} \leq 1$ and $\Phi^{\mathbb{G}} k \star \mu_{\mathbb{G}} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Then, $P \otimes A$-a.e.*

$$\int k(x)\,(1 - Z_- - f_m(x))\,F(dx) < +\infty \qquad \textit{on} \quad \{Z_- < 1\}. \tag{65}$$

(b) *Let f be a $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable and positive functional, and $\overline{\mu} := (1-\widetilde{Z}) \cdot \mu$. Then $\sqrt{(f-1)^2 I_{]\!]\tau,+\infty[\![} \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{G})$ if and only if $\sqrt{(f-1)^2 I_{\{1-Z_- \geq \delta\}} \star \overline{\mu}} \in \mathcal{A}^+_{loc}(\mathbb{F})$ for any $\delta > 0$.*

APPENDIX B: REPRESENTATION RESULTS

This section addresses the representation of $\mathbb{G}$-optional (respectively $\mathbb{G}$-predictable) processes in terms of $\mathbb{F}$-optional (respectively $\mathbb{F}$-predictable) processes. This is will be achieved in three lemmas. The first two lemmas can be found in [2], while the last lemma sounds to be new.

LEMMA B.1. *The following assertions hold.*
(a) *If $H^{\mathbb{G}}$ is a $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional, then there exist an $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional $H^{\mathbb{F}}$such that*

$$H^{\mathbb{G}}(\omega,t,x)I_{]\!]0,\tau]\!]} = H^{\mathbb{F}}(\omega,t,x)I_{]\!]0,\tau]\!]}. \tag{66}$$

(b) *If furthermore $H^{\mathbb{G}} > 0$ (respectively $H^{\mathbb{G}} \leq 1$), then we can choose $H^{\mathbb{F}} > 0$ (respectively $H^{\mathbb{F}} \leq 1$) such that*

$$H^{\mathbb{G}}(\omega,t,x)I_{]\!]0,\tau]\!]} = H^{\mathbb{F}}(\omega,t,x)I_{]\!]0,\tau]\!]}.$$

((c) *For any $\mathbb{F}$-stopping time, T, and any positive $\mathcal{G}_T$-measurable random variable $Y^{\mathbb{G}}$, there exist two positive $\mathcal{F}_T$-measurable random variables, $Y^{(1)}$ and $Y^{(2)}$, satisfying*

$$Y^{\mathbb{G}}I_{\{T\leq\tau\}} = Y^{(1)}I_{\{T<\tau\}} + Y^{(2)}I_{\{\tau=T\}}. \tag{67}$$

For the proof of the lamma, we refer the reader to [2].

LEMMA B.2. *Suppose that τ is honest. Let $H^{\mathbb{G}}$ be an $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional. Then the following assertions hold.*
(a) *There exist two $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional $H^{\mathbb{F}}$ and $K^{\mathbb{F}}$ such that*

$$H^{\mathbb{G}}(\omega,t,x) = H^{\mathbb{F}}(\omega,t,x)I_{]\!]0,\tau]\!]} + K^{\mathbb{F}}(\omega,t,x)I_{]\!]\tau,+\infty[\![}. \tag{68}$$

(b) *If furthermore $H^{\mathbb{G}} > 0$ (respectively $H^{\mathbb{G}} \leq 1$), then we can choose $K^{\mathbb{F}} > 0$ (respectively $K^{\mathbb{F}} \leq 1$) such that*

$$H^{\mathbb{G}}(\omega,t,x)I_{]\!]\tau,+\infty]\!]} = K^{\mathbb{F}}(\omega,t,x)I_{]\!]\tau,+\infty[\![}.$$

We end this paper by the following lemma that useful for the case of single jump processes.

LEMMA B.3. *Let T be an $\mathbb{F}$-predictable stopping time, and φ be a nonnegative $\mathbb{F}_{T-}$-measurable random variable. Then the following assertions hold.*
(a) *If $\varphi I_{\{T\leq\tau\}} < +\infty$ $P-a.s$, then $\varphi I_{\{Z_{T-}>0\}} < +\infty$ $P-a.s.$*
(b) *Suppose furthermore that τ is an honest time. If $\varphi I_{\{T>\tau\}} < +\infty$ $P-a.s$, then $\varphi I_{\{Z_{T-}<1\}} < +\infty$ $P-a.s.$*

PROOF. 1) Suppose that $\varphi I_{\{T\leq\tau\}} < +\infty\ P-a.s.$ Then, we calculate

$$0 = P\left((\varphi = +\infty)\cap(T\leq\tau)\right) = E\left(Z_{T-}I_{\{\varphi=+\infty\}}\right).$$

Thus, we obtain $Z_{T-}I_{\{\varphi=+\infty\}} = 0\ P-a.s$ or equivalently $\{Z_{T-} > 0\} \subset \{\varphi < +\infty\}$. This ernds the proof of assertion (a).
2) Suppose that τ is honest and $\varphi I_{\{T>\tau\}} < +\infty\ P-a.s.$ Then,

$$0 = P\left((T>\tau)\cap(\varphi = +\infty)\right) = E\left((1-Z_{T-})I_{\{\varphi=+\infty\}}\right).$$

Thus, assertion (b) follows immediately, and the proof of the lemma is completed. □

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