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Non-Arbitrage under a Class of Honest Times

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Abstract This paper quantifies the interplay between the non-arbitrage notion of No-Unbounded-Profit-with-Bounded-Risk (NUPBR hereafter) and additional information generated by a random time. This study complements the one of Aksamit/Choulli/Deng/Jeanblanc [1] in which the authors studied similar topics for the case of stopping with the random time instead, while herein we are concerned with the part after the occurrence of the random time. Given that all the literature —up to our knowledge— proves that the NUPBR notion is always violated after honest times that avoid stopping times in a continuous filtration, herein we propose a *new class of honest times* for which the NUPBR notion can be preserved for some models. For this family of honest times, we elaborate two principal results. The first main result characterizes the pairs of initial market and honest time for which the resulting model preserves the NUPBR property, while the second main result characterizes the honest times that preserve the NUPBR property for any quasi-left continuous model. Furthermore, we construct explicitly “the-after- τ ” local martingale deflators for a large class of initial models (i.e., models in the small filtration) that are already risk-neutralized.

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1 Introduction

Since the earliest studies in modern finance and mathematical finance, there were clear evidence about the foundational rôle of non arbitrage in the financial modelling, as well as the rôle of information in any market (financial market and/or insurance market). This paper complements the study we started in [1] about the quantification of the exact interplay between an extra information/uncertainty and arbitrage for quasi-left-continuous models¹. Similarly as in [1], our focus resides in the non-arbitrage concept of No-Unbounded-Profit-with-Bounded-Risk (NUPBR hereafter), in the case where the extra information is the time of the occurrence of a random time, when it occurs. In order to keep this section as short as possible, we refer the reader to [1] for detailed financial and mathematical motivations of these choices. Throughout the whole the paper, an arbitrage is an No-Unbounded-Profit-with-Bounded-Risk opportunity, and a model is said to be arbitrage-free if it fulfills the NUPBR condition.

1.1 What are the Main Goals and their Related Literature?

Throughout the paper, we consider given a stochastic basis $(\Omega, \mathcal{G}, \mathbb{F}, P)$, where $\mathcal{F}_\infty := \bigcup_{t \geq 0} \mathcal{F}_t \subseteq \mathcal{G}$, and the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual hypotheses (i.e., right continuity and completeness) and models the flow of “public” information that all agents receive through time. Throughout the paper, the initial financial market is defined on this basis and is represented by a **d -dimensional semimartingale** S and a riskless asset, with null interest rate. In addition to this initial model, we consider a fixed random time (a non-negative random variable) denoted by τ . This random time can represent the death time of an insurer, the default time of a firm, or any occurrence time of an influential event that can impact the market somehow. In this setting, our ultimate aim lies in answering the following.

If $(\Omega, \mathbb{F}, S)$ is arbitrage-free, then what can be said about $(\Omega, \mathbb{F}, S, \tau)$?

After mathematically modeling the *new informational system* (i.e. modeling the incorporation of the additional uncertainty/information into the system), this question translates into whether the model $(\Omega, \mathbb{G}, S)$ is arbitrage free or not. Here $\mathbb{G}$, that will be specified mathematically in the next section, is the new flow of information that incorporates the flow $\mathbb{F}$ and τ , as soon as it occurs, and makes τ a $\mathbb{G}$ -stopping time. Thanks to [20] (see also [7] for the continuous case and [18] for the one dimensional case), one can easily prove that $(\Omega, \mathbb{G}, S)$ satisfies the NUPBR condition if and only if both models $(\Omega, \mathbb{G}, S^\tau)$ and $(\Omega, \mathbb{G}, S - S^\tau)$ fulfill the NUPBR condition. In Aksamit et al. [1], the authors focused on $(\Omega, \mathbb{G}, S^\tau)$, while the second part $(\Omega, \mathbb{G}, S - S^\tau)$ constitutes the main objective of this paper. As it will be mathematically specified

¹ A quasi-left-continuous model/process is a process that does not jump on predictable stopping times

in the next section, the NUPBR notion consists, roughly speaking, of “controlling” in some sense the gain processes –resulting from financial strategies (predictable processes)– that have their negative parts bounded uniformly in time and randomness by the constant one. Mathematically speaking, this gain processes are stochastic integrals with respect to the asset’s price process. Thus, due to the Dellacherie-Mokobodski criterion, the first challenge in investigating the NUPBR condition for $(\Omega, \mathbb{G}, S - S^\tau)$, lies in assuring that this model constitutes an adequate integrator for “admissible” but complex (not only buy-and-hold) financial strategies. Equivalently, we need to make sure that this model is a semimartingale (see Theorem 80 in [9] page 401). This is our main leitmotif for assuming the “honest” assumption on the random time τ , as we are not interested in the semimartingale issue under enlargement of filtration on the one hand. On the other hand, it is known that (see [17, Théorème 4.14]), in contrast to $(\mathbb{G}, S^\tau)$, the semimartingale structures might fail for $(\mathbb{G}, S - S^\tau)$ when τ is arbitrary general. Therefore, for the rest of the paper, τ is assumed to be honest, a fact that will be mathematically defined in the next section.

The quasi-majority of the literature about informational markets (i.e. markets with two groups of agents, where one group receives more information than the other) addresses the investment problem, and assumes that the random time has hazard rate, or has intensity, or satisfies Jacod’s assumption of $P(\tau \in dx | \mathcal{F}_t) \ll P(\tau \in dx)$ for any $t \geq 0$. All these random time models can not be honest times, and the only studies –up to our knowledge– that address arbitrage and honest times are [14] and [12]. More importantly, these papers –where the honest times are assumed to avoid stopping times and the filtration is Brownian– prove that the NUPBR property fails for $(S - S^\tau, \mathbb{G})$. Up to our knowledge, there is no single result –of any form in the literature– that proposes a class (or an example) of honest time for which the NUPBR condition holds after τ . Thus, our first goal is to answer the following

Is there any τ for which NUPBR is preserved for some models? (1.1)

In the case where the answer to this question is positive, our next goals can be summarized as follows.

For which pairs (S, τ) the process $S - S^\tau$ fulfills NUPBR ? (1.2)

and

for which τ , is $(S - S^\tau, \mathbb{G})$ arbitrage-free for any arbitrage-free S ? (1.3)

Throughout the paper, by *arbitrages* we mean those financial strategies that produce unbounded profit with bounded risk, and by *signal process* we refer to the process $P(\tau < t | \mathcal{F}_t) =: 1 - \tilde{Z}_t$. This process is the only information at time t , about whether τ is below time t or not, that the agents endowed with the public information receive.

1.2 Our Financial and Mathematical Achievements

Our first original contribution lies in answering (1.1) positively, and thus proposing a **new class of honest times** for which there is a real hope for the resulting informational market to possess the NUPBR condition after τ . Our family of honest times includes all the $\mathbb{F}$ -stopping times as well as many examples of non $\mathbb{F}$ -stopping times. By considering this subclass of honest times throughout the paper, our remaining novelties reside in answering (1.2) and (1.3) in terms of processes adapted to the flow of “public” information only. Among these contributions, we prove that, under our assumptions on the random time, any market model whose underlying assets’ price process has continuous paths fulfills the NUPBR condition after τ , and the extra information (generated by the random time) might induce arbitrages only if the initial market jumps. Furthermore, via practical examples, we conclude that existence of arbitrages for $(S - S^\tau, \mathbb{G})$ is not related at all to whether S jumps or not at τ itself. Furthermore, we show that the jumps of S , that occur at the same time when $\tilde{Z}$ ($\tilde{Z}_t := P(\tau \geq t | \mathcal{F}_t)$) jumps to one, play central rôle in generating arbitrages after τ . At the quantitative finance level, our paper quantifies –with extreme precision– the jumpy part of the signal process, $\tilde{Z}$, that is responsible for arbitrages when they occur at the time as those of S . We also show how to construct explicitly a deflator for $(S - S^\tau, \mathbb{G})$, when $(S, \mathbb{F})$ is already risk-neutralized and does not jump when $\tilde{Z}$ jumps to one.

This paper is organized as follows. In the following section (Section 2), we present our main results, their immediate consequences, and/or their economic and financial interpretations. In this section, we also develop many practical examples and show how the main ideas came into play. Section 3 deals with the derivation of explicit local martingale deflators for a class of processes. The last section (Section 4) focuses on proving the main theorems announced in Section 2 without proof. The paper contains also an appendix where some of the existing and/or new technical results are summarized.

2 The Main Results and their Financial Interpretations

This section contains three subsections. The first subsection defines notations and the NUPBR concept, while the second subsection develops simple examples of informational markets and explains how some ingredients of the main results play natural and important rôles. The last subsection announces the principal results, their applications, and gives their financial meanings as well.

2.1 Notations and Preliminaries

In what follows, $\mathbb{H}$ denotes a filtration satisfying the usual hypotheses. The set of $\mathbb{H}$ -martingales is denoted by $\mathcal{M}(\mathbb{H})$. As usual, $\mathcal{A}^+(\mathbb{H})$ denotes the set of

increasing, right-continuous, $\mathbb{H}$ -adapted and integrable processes.

If $\mathcal{C}(\mathbb{H})$ is a class of $\mathbb{H}$ -adapted processes, we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}(\mathbb{H})$ the set of processes X such that there exists a sequence $(T_n)_{n \geq 1}$ of $\mathbb{H}$ -stopping times that increases to $+\infty$ and the stopped processes X^{T_n} belong to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0,loc} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$.

For a process K with $\mathbb{H}$ -locally integrable variation, we denote by $K^{o,\mathbb{H}}$ its dual optional projection. The dual predictable projection of K is denoted $K^{p,\mathbb{H}}$. For a process X , we denote ${}^{o,\mathbb{H}}X$ (resp. ${}^{p,\mathbb{H}}X$) its optional (resp. predictable) projection with respect to $\mathbb{H}$.

For a finite-dimensional $\mathbb{H}$ -semi-martingale Y , the set $\mathcal{L}(Y, \mathbb{H})$ is the set of $\mathbb{H}$ -predictable processes having the same dimension as Y and being integrable w.r.t. Y and for $H \in \mathcal{L}(Y, \mathbb{H})$, the resulting integral is the one-dimensional process denoted by $H \cdot Y_t := \int_0^t H_s dY_s$. **Throughout the paper, stochastic processes have arbitrary finite dimension** (in case it is not specified). We recall the notion of non-arbitrage that is addressed in this paper.

Definition 2.1 An $\mathbb{H}$ -semimartingale X satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition under $(\mathbb{H}, Q)$ (hereafter called $\text{NUPBR}(\mathbb{H}, Q)$) if for any finite deterministic horizon T' , the set

$$\mathcal{K}_{T'}(X) := \left\{ (H \cdot X)_{T'} \mid H \in \mathcal{L}(X, \mathbb{H}), \text{ and } H \cdot X \geq -1 \right\}$$

is bounded in probability under Q . When $Q \sim P$, we simply write $\text{NUPBR}(\mathbb{H})$ and say that X satisfies $\text{NUPBR}(\mathbb{H})$ for short.

For more details about this non-arbitrage condition and its relationship to the literature, we refer the reader to Aksamit et al. [1]. The NUPBR property is intimately related to the existence of a σ -martingale density. Below, we recall the definition of σ -martingale and σ -martingale density for a process.

Definition 2.2 An $\mathbb{H}$ -adapted process X is called an $(\mathbb{H}, \sigma)$ -martingale if there exists a real-valued $\mathbb{H}$ -predictable process ϕ such that

$$0 < \phi \leq 1, \quad \text{and } \phi \cdot X \text{ is an } \mathbb{H}\text{-martingale.}$$

If X is $\mathbb{H}$ -adapted, we call $(\mathbb{H}, \sigma)$ -martingale density for X , any real-valued positive $\mathbb{H}$ -local martingale L such that XL is an $(\mathbb{H}, \sigma)$ -martingale. The set of all $(\mathbb{H}, \sigma)$ -martingale densities for X is denoted by

$$\mathcal{L}_\sigma(X, \mathbb{H}) := \left\{ L \in \mathcal{M}_{loc}(\mathbb{H}) \mid L > 0, \quad LX \text{ is an } (\mathbb{H}, \sigma)\text{-martingale} \right\}.$$

The equivalence between $\text{NUPBR}(\mathbb{H})$ for a process X and $\mathcal{L}_\sigma(X, \mathbb{H}) \neq \emptyset$ is established in [1] (see Proposition 2.3) when the horizon may be infinite, and in [20] for the case of finite horizon.

Beside the initial model $(\Omega, \mathbb{F}, P, S)$ in which S is assumed to be **quasi-left-continuous semimartingale**, we consider a finite random time τ , to which we associate the process D and the filtration $\mathbb{G}$ given by

$$D := I_{[\tau, +\infty[}, \quad \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, \quad \mathcal{G}_t := \bigcap_{s > t} \left(\mathcal{F}_s \vee \sigma(D_u, u \leq s) \right).$$

The filtration $\mathbb{G}$ is the smallest right-continuous filtration which contains $\mathbb{F}$ and makes τ a stopping time. In the probabilistic literature, $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ with τ . In addition to $\mathbb{G}$ and D , we associate to τ two important $\mathbb{F}$ -supermartingales: the $\mathbb{F}$ -optional projection of $I_{\llbracket 0, \tau \rrbracket}$ denoted Z , and the $\mathbb{F}$ -optional projection of $I_{\llbracket 0, \tau \rrbracket}$, denoted $\tilde{Z}$, which satisfy

$$Z_t := P(\tau > t \mid \mathcal{F}_t) \text{ and } \tilde{Z}_t := P(\tau \geq t \mid \mathcal{F}_t). \quad (2.1)$$

The supermartingale Z is right-continuous with left limits, while $\tilde{Z}$ admits right limits and left limits. An important $\mathbb{F}$ -martingale, denoted by m , is given by

$$m := Z + D^{o, \mathbb{F}}, \quad (2.2)$$

where $D^{o, \mathbb{F}}$ is the $\mathbb{F}$ -dual optional projection of $D = I_{\llbracket \tau, \infty \rrbracket}$ (Note that Z is bounded and $D^{o, \mathbb{F}}$ is nondecreasing and integrable).

To distinguish the effect of filtration, we will denote $\langle \cdot, \cdot \rangle^{\mathbb{F}}$, or $\langle \cdot, \cdot \rangle^{\mathbb{G}}$ to specify the sharp bracket (predictable covariation process) calculated in the filtration $\mathbb{F}$ or $\mathbb{G}$, if confusion may rise. We recall that, for general semi-martingales X and Y , the sharp bracket is (if it exists) the dual predictable projection of the covariation process $[X, Y]$. For the reader's convenience, we recall the definition of honest time.

Definition 2.3 A random time σ is honest, if for any t , there exists an $\mathcal{F}_t$ measurable r.v. σ_t such that $\sigma I_{\{\sigma < t\}} = \sigma_t I_{\{\sigma < t\}}$.

We refer to Jeulin [17, Chapter 5] and Barlow [6] for more information about honest times. In this paper, we restrict our study to the following subclass $\mathcal{H}$ of random times:

$$\mathcal{H} := \{\tau \text{ is an honest time satisfying } Z_\tau I_{\{\tau < +\infty\}} < 1, \quad P - a.s.\} \quad (2.3)$$

Remark 2.4 1) It is clear that any $\mathbb{F}$ -stopping time belongs to $\mathcal{H}$ (we even have $Z_\tau I_{\{\tau < +\infty\}} = 0$), and hence our subclass of honest times is not empty.

2) In the case where $\mathbb{F}$ is the completed Brownian filtration, we consider the following $\mathbb{F}$ -stopping times

$$U_0^\epsilon = V_0^\epsilon = 0, \quad U_n^\epsilon := \inf\{t \geq V_{n-1}^\epsilon : B_t = \epsilon\}, \quad V_n^\epsilon := \inf\{t \geq U_n^\epsilon : B_t = 0\},$$

where $\epsilon \in (0, 1)$ and B is a one dimensional standard Brownian motion. Then,

$$\tau := \sup\{V_n^\epsilon : V_n^\epsilon \leq T_1\},$$

where $T_1 := \inf\{t \geq 0 : B_t = 1\}$, is a honest time which is not a stopping time, and belongs to $\mathcal{H}$ (see [3] for detailed proof). Other examples of elements of $\mathcal{H}$ that are not stopping times are given in the next subsection.

We conclude this subsection with the following lemma, obtained in [1].

Lemma 2.5 *Let X be an $\mathbb{H}$ -predictable process with finite variation. Then X satisfies $NUPBR(\mathbb{H})$ if and only if $X \equiv X_0$ (i.e. the process X is constant).*

2.2 Particular Cases and Examples

In this subsection, by analysing particular cases and examples, we obtain some results vital for understanding the exact interplay between the features of the initial markets and the honest time under consideration. The following simple lemma plays a key role in this analysis.

Lemma 2.6 *The following assertions hold.*

(a) *Let M be an $\mathbb{F}$ -local martingale, and τ be an honest time. Then the process $\widehat{M}$, defined as*

$$\widehat{M} := M - M^\tau + (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \llbracket} \cdot \langle M, m \rangle^\mathbb{F}, \quad (2.4)$$

is a $\mathbb{G}$ -local martingale.

(b) *If $\tau \in \mathcal{H}$, then the $\mathbb{G}$ -predictable process $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \llbracket}$ is $\mathbb{G}$ -locally bounded.*

Proof 1) Assertion (a) is a standard result on progressive enlargement of filtration with honest times (see [6, 10, 17]).

2) Herein we prove assertion (b). It is known [10, Chapter XX] that $Z = \widetilde{Z}$ on $\llbracket \tau, +\infty \llbracket$, and

$$\llbracket \tau, +\infty \llbracket \subset \{Z_- < 1\} \cap \{\widetilde{Z} < 1\} \subset \{Z_- < 1\} \cap \{Z < 1\}.$$

Then, since $\tau \in \mathcal{H}$, we deduce that $\llbracket \tau, +\infty \llbracket \subset \{Z < 1\}$, and hence the process

$$X := (1 - Z)^{-1} I_{\llbracket \tau, +\infty \llbracket},$$

is càdlàg $\mathbb{G}$ -adapted with values in $[0, +\infty)$ (finite values). Combining these with $\llbracket \tau, +\infty \llbracket \subset \{Z_- < 1\}$, we can prove easily that

$$T_n := \inf\{t \geq 0 : X_t \geq n\} \uparrow +\infty \text{ and } \max(X^{T_n-}, X_-^{T_n}) \leq n, \quad P - a.s..$$

Thus, $X_- = (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \llbracket}$ is locally bounded, and the proof of the lemma is completed. $\square$

Theorem 2.7 *Suppose that $\tau \in \mathcal{H}$. If S is continuous and satisfies NUPBR($\mathbb{F}$), then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*

Remark 2.8 This theorem follows from one of our principal result stated in the next subsection. However, due to the simplicity of its proof that does not require any further technicalities, we opted for detailing this proof below.

Proof of Theorem 2.7: Let $S = (S^1, \dots, S^d)$ be a d -dimensional continuous process satisfying NUPBR($\mathbb{F}$). Then, there exists a positive $\mathbb{F}$ -local martingale L such that LS is an $(\mathbb{F}, \sigma)$ -martingale. Since S is continuous and L is a local martingale, we deduce that $\sup_{u \leq \cdot} |S_u| \sup_{u \leq \cdot} |\Delta L_u|$ is locally integrable. Thus, thanks to Proposition 3.3 in [4] and $\sum_{i=1}^d \Delta(LS^i) = \sum_{i=1}^d S^i \Delta L \geq -d \sup_{u \leq \cdot} |S_u| \sup_{u \leq \cdot} |\Delta L_u|$, we conclude that LS is an $\mathbb{F}$ -local martingale instead. Consider a sequence of $\mathbb{F}$ -stopping times $(T_n)_{n \geq 1}$ that increases to

infinity such that both L^{T_n} and $L^{T_n}S^{T_n}$ are martingales, and put $Q_n := (L_{T_n}/L_0) \cdot P \sim P$. Then, $S^{(n)} := S^{T_n}$ is an $(\mathbb{F}, Q_n)$ -martingale on the one hand. On the other hand, in virtue of Proposition A.1, $S - S^\tau$ satisfies NUPBR($\mathbb{G}$) if and only if $S^{(n)} - (S^{(n)})^\tau$ satisfies NUPBR($\mathbb{G}$) under Q_n , for all $n \geq 1$. This shows that, without loss of generality, one need to prove the theorem only when S is an $\mathbb{F}$ -martingale. Thus, for the rest of the proof, we assume that S is an $\mathbb{F}$ -martingale. Thanks to Lemma 2.6, the process $Y^\mathbb{G} := \mathcal{E}((1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot \widehat{m}^c)$ is a well defined continuous real-valued and positive $\mathbb{G}$ -local martingale, where m^c is the continuous $\mathbb{F}$ -local martingale part of m , and $\widehat{m}^c$ is defined as in (2.4). Thanks to the continuity of S and (2.4), we get

$$\begin{aligned} S - S^\tau + \left[S - S^\tau, \frac{I_{\llbracket \tau, +\infty \rrbracket}}{1 - Z_-} \cdot \widehat{m}^c \right] &= S - S^\tau + (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot \langle S, m \rangle^\mathbb{F} \\ &= \widehat{S} \in \mathcal{M}_{loc}(\mathbb{G}). \end{aligned}$$

Therefore, a combination of this and Itô's formula applied to $(S - S^\tau)Y^\mathbb{G}$, we conclude that this latter process is a $\mathbb{G}$ -local martingale. This proves NUPBR($\mathbb{G}$) for $S - S^\tau$, and the proof of the theorem is achieved. $\square$

Remark 2.9 The above theorem asserts clearly that, if $\tau \in \mathcal{H}$, the jumps of S have significant impact on $\mathbb{G}$ -arbitrages for $S - S^\tau$. Thus, the following natural question arises:

$$\text{Does the condition } \{\Delta S \neq 0\} \cap \llbracket \tau \rrbracket = \emptyset \text{ impact } \mathbb{G}\text{-arbitrages?} \quad (2.5)$$

Example 2.10 Suppose that $\mathbb{F}$ is generated by a Poisson process N with intensity one. Consider two real numbers $a > 0$ and $\mu > 1$, and set

$$\tau := \sup\{t \geq 0 : Y_t := \mu t - N_t \leq a\}, \quad M_t := N_t - t. \quad (2.6)$$

It can be proved easily, see [3], that $\tau \in \mathcal{H}$ is finite almost surely, and the associated processes Z and $\widetilde{Z}$ are given by

$$Z = \Psi(Y - a)I_{\{Y \geq a\}} + I_{\{Y < a\}} \quad \text{and} \quad \widetilde{Z} = \Psi(Y - a)I_{\{Y > a\}} + I_{\{Y \leq a\}}.$$

Here $\Psi(u) := P(\sup_{t \geq 0} Y_t > u)$ is the ruin probability associated to the process Y (see [5]). As a result we have

$$1 - Z_- = [1 - \Psi(Y_- - a)] I_{\{Y_- > a\}}, \quad (2.7)$$

and we can prove that

$$m = m_0 + \phi \cdot M, \quad \text{where} \quad (2.8)$$

$$\phi := [\Psi(Y_- - a - 1) - \Psi(Y_- - a)] I_{\{Y_- > 1+a\}} + [1 - \Psi(Y_- - a)] I_{\{a < Y_- \leq 1+a\}}.$$

Suppose that $S = I_{\{a \leq Y_- < a+1\}} \cdot M$. Then, in virtue of Lemma 2.5, the process $S - S^\tau$ (which is not null) violates NUPBR($\mathbb{G}$) if it is $\mathbb{G}$ -predictable with finite variation. This latter fact is equivalent to $\widehat{S}$ ($\mathbb{G}$ -local martingale part of $S - S^\tau$)

being null, or equivalently $\langle \widehat{S}, \widehat{S} \rangle^{\mathbb{G}} \equiv 0$. By using Lemma 2.6 and Itô's lemma and putting $V_t = t$, we derive

$$\begin{aligned} [\widehat{S}, \widehat{S}] &= I_{\llbracket \tau, +\infty \llbracket} \cdot [S] = I_{\llbracket \tau, +\infty \llbracket} \cdot S + I_{\{a < Y_- \leq a+1\}} I_{\llbracket \tau, +\infty \llbracket} \cdot V \\ &= I_{\llbracket \tau, +\infty \llbracket} \cdot \widehat{S} + I_{\{a < Y_- \leq a+1\}} I_{\llbracket \tau, +\infty \llbracket} \left(1 - \frac{\phi}{1 - Z_-} \right) \cdot V, \quad (2.9) \\ &= I_{\llbracket \tau, +\infty \llbracket} \cdot \widehat{S} \text{ is a } \mathbb{G}\text{-local martingale.} \end{aligned}$$

The last equality is due to $\phi \equiv 1 - Z_-$ on $\{a \leq Y_- < a+1\} \cap \llbracket \tau, +\infty \llbracket$. This proves that $\widehat{S} \equiv 0$, and hence $S - S^\tau$ violates $\text{NUPBR}(\mathbb{G})$.

Example 2.11 Consider the same setting and notations as Example 2.10, except for the initial market model that we suppose having the form of $S = I_{\{Y_- > a+1\}} \cdot M$ instead. Then, by combining Lemma 2.6, Itô's lemma and similar calculation as in (2.9), we deduce that both $Y^{\mathbb{G}} := \mathcal{E}(\xi \cdot \widehat{S})$ and $Y^{\mathbb{G}}(S - S^\tau)$ are $\mathbb{G}$ -local martingales and $Y^{\mathbb{G}} > 0$. Here ξ is given by

$$\xi := \frac{\Psi(Y_- - a - 1) - 1}{2 - \Psi(Y_- - a) - \Psi(Y_- - a - 1)} I_{\{Y_- > a+1\}} I_{\llbracket \tau, +\infty \llbracket}.$$

This proves that $S - S^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$.

Remark 2.12 1) The economics/financial meaning of Examples 2.10 and 2.11 reside in the following: The random time defined in (2.6) represents the last time the cash reserve of a firm does not exceed the level a . Then, in Example 2.10 (respectively in Example 2.11) one can consider a security whose price process lives on $\{a \leq Y_- < 1+a\}$ (respectively on $\{Y_- > 1+a\}$).

2) It is important to notice that, in both Examples 2.10 and 2.11, the graph of the random time τ is included in a thin and $\mathbb{F}$ -predictable set (i.e., the union of countable graphs of predictable stopping times). Hence, due to the quasi-left-continuity of S , we immediately conclude that the set $\{\Delta S \neq 0\} \cap \llbracket \tau \rrbracket$ is empty for both examples. This clearly answers (2.5), and suggests that one should look at completely different direction in order to understand the key fact behind eliminating $\mathbb{G}$ -arbitrages. Thus, the question of how can we assess the occurrence or not of $\mathbb{G}$ -arbitrages for $S - S^\tau$, arises naturally.

2.3 Main Results and Their Applications

The following, which is our first main result, answers (1.2).

Theorem 2.13 *Suppose that S is an $\mathbb{F}$ -quasi-left-continuous semimartingale, and $\tau \in \mathcal{H}$ is finite. Then, there exists an $\mathbb{F}$ -local martingale, denoted by $m^{(1)}$, which is pure jumps (i.e. its continuous local martingale part is null) satisfying*

$$\Delta m^{(1)} \in \{1 - Z_-, 0\}, \quad \{\Delta m^{(1)} \neq 0\} \subset \{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\}, \quad (2.10)$$

and $S - S^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$ if and only if $\mathcal{T}_a(S)$ satisfies $\text{NUPBR}(\mathbb{F})$, where

$$\mathcal{T}_a(S) := (1 - Z_-) \cdot S - [S, m^{(1)}]. \quad (2.11)$$

The proof of this theorem is technical, and hence it is postponed to Section 4. Below, we describe the rôles as well as the meaning of each of the ingredients (i.e. $m^{(1)}$, and $\mathcal{T}_a(S)$) of the theorem above.

Remark 2.14 (a) It is important to mention that $m^{(1)}$ is constructed explicitly for any pair (S, τ) . However, this construction requires technical notations, and it is delegated together with the proof of the theorem to Section 4 for the sake of simplicity. The martingale $m^{(1)}$ quantifies exactly the jumpy part of $1 - \tilde{Z}$ that plays key rôle in generating $\mathbb{G}$ -arbitrages for the process $S - S^\tau$.

(b) **The process** $\mathcal{T}_a(S)$ is the part of S that does not jump at the same as $m^{(1)}$. In fact, due to the first property in (2.10), it is easy to verify that $[\mathcal{T}_a(S), m^{(1)}] = (1 - Z_-) \cdot [S, m^{(1)}] - \Delta m^{(1)} \cdot [S, m^{(1)}] \equiv 0$.

(d) Theorem 2.13 claims that $S - S^\tau$ is arbitrage-free under $\mathbb{G}$ if and only if the part $\mathcal{T}_a(S)$ (of S) is arbitrage-free under $\mathbb{F}$. In virtue of this theorem, one can calculate $m^{(1)}$ as instructed in Section 4, then check the NUPBR($\mathbb{F}$) for $(1 - Z_-) \cdot S - [S, m^{(1)}]$ afterwards and conclude whether $S - S^\mathbb{G}$ is arbitrage free or not. Thus, this theorem also furnishes practical cases, as outlined in the forthcoming Corollary 2.15 and Theorem 2.17.

Corollary 2.15 *Suppose that S is $\mathbb{F}$ -quasi-left-continuous, and $\tau \in \mathcal{H}$ is finite. Then the following assertions hold:*

- (a) *If $(S, [S, m^{(1)}])$ satisfies NUPBR($\mathbb{F}$), then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*
- (b) *If S satisfies NUPBR($\mathbb{F}$) and $m^{(1)} \equiv 0$ (or equivalently $[S, m^{(1)}] \equiv 0$), then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*
- (c) *If S satisfies NUPBR($\mathbb{F}$) and $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset$, then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*

Remark 2.16 1) Assertion (b) asserts that if S does not jump at the same time as $m^{(1)}$ (i.e. $[S, m^{(1)}] \equiv 0$), then no arbitrage under $\mathbb{G}$ will occur in the part “after- τ ”. Assertion (c) claims that the same conclusion remains valid whenever S does not jump when $1 - \tilde{Z}$ jumps to zero. Thus, assertion (a) assumes much weaker assumption than assertion (b), since assertion (a) assumes that $Z[S, m^{(1)}] \in \mathcal{M}_{loc}(\mathbb{F})$ for some risk-neutral density Z of S , while assertion (b) assumes that $[S, m^{(1)}]$ is null.

Proof of Corollary 2.15: It is obvious that assertion (a) follows directly from combining $(1 - Z_-) \cdot S - S^{(1)} = (1 - Z_-, -1) \cdot (S, [S, m^{(1)}])$ and Theorem 2.13. Due to $m^{(1)} \equiv 0 = [S, m^{(1)}]$, assertion (b) follows from assertion (a).

Thanks to (2.10), we deduce that $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 1 > Z_-\} = \emptyset$ implies $m^{(1)} \equiv 0$, and assertion (c) follows from assertion (b), and the proof of the corollary is achieved. $\square$

In the spirit of further applicability of Theorem 2.13, we state the following

Theorem 2.17 *Suppose that $\tau \in \mathcal{H}$. Let μ be the optional random measure associated to the jumps of S , and $\nu^\mathbb{F}$ and $\nu^\mathbb{G}$ be the $\mathbb{F}$ -compensator and the $\mathbb{G}$ -compensator of μ and $I_{\tau, +\infty} \cdot \mu$ respectively. If S satisfies NUPBR($\mathbb{F}$) and*

$$I_{\tau, +\infty} \cdot \nu^\mathbb{F} \text{ is equivalent to } \nu^\mathbb{G} \quad P - a.s., \quad (2.12)$$

then $S - S^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$.

The proof of this theorem will follow from Corollary 2.15–(b) as long as we can argue that (2.12) implies $m^{(1)} \equiv 0$. Thus, this proof is delegated to Section 4.

Remark 2.18 Remark that we always have the absolute continuity $\nu^{\mathbb{G}} \ll I_{\llbracket \tau, +\infty \rrbracket} \cdot \nu^{\mathbb{F}} P - a.s.$ This follows from the fact that $\nu^{\mathbb{G}}$ is absolutely continuous with respect to $\nu^{\mathbb{F}}$ and it lives on $\llbracket \tau, +\infty \rrbracket$ only.

(a) **The Lévy Case:** Suppose that S is a Lévy process and $F(dx)$ is its Lévy measure, then $\nu^{\mathbb{F}}(dt, dx) = F(dx)dt$ and $\nu^{\mathbb{G}}(dt, dx) = I_{\llbracket \tau, +\infty \rrbracket} F_t^{\mathbb{G}}(dx)dt$, where $F_t^{\mathbb{G}}(dx)$ is a sort of generalized Lévy measure. Thus, Theorem 2.17 asserts that if $P \otimes \lambda$ almost every (ω, t) ($\lambda(dt) = dt$), $F_t^{\mathbb{G}}(\omega, dx) = f_t(x, \omega)F(dx)$ for some real-valued functional $f_t(x, \omega) > 0$ $P \otimes \lambda - a.e.$, then $S - S^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$. For more practical Lévy cases, we refer the reader to [11].

(b) **Examples 2.10–2.11 versus Theorem 2.17:** In the context of Example 2.10, we easily calculate $\nu^{\mathbb{F}}(dt, dx) = I_{\{a < Y_{t-} \leq a+1\}} \delta_1(dx)dt$ and $\nu^{\mathbb{G}}(dt, dx) = I_{\llbracket \tau, +\infty \rrbracket}(t) I_{\{a < Y_{t-} \leq a+1\}} (1 - \phi_t/(1 - Z_{t-})) \delta_1(dx)dt \equiv 0$ which is not equivalent to $I_{\llbracket \tau, +\infty \rrbracket} \cdot \nu^{\mathbb{F}}$. This example shows that (2.12) can be violated. Therefore, in those circumstances, we can not conclude whether $S - S^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$ or not directly from Theorem 2.17.

For the case of Example 2.11, we have $\nu^{\mathbb{F}}(dt, dx) = I_{\{Y_{t-} > a+1\}} \delta_1(dx)dt$ and $\nu^{\mathbb{G}}(dt, dx) = I_{\llbracket \tau, +\infty \rrbracket}(t) I_{\{Y_{t-} > a+1\}} (1 - \phi_t/(1 - Z_{t-})) \delta_1(dx)dt$ which is equivalent to $I_{\llbracket \tau, +\infty \rrbracket} \cdot \nu^{\mathbb{F}}$ since $\{Y_{t-} > a+1\} \subset \{\phi < 1 - Z_{t-}\}$ $P \otimes dt - a.e.$ Thus, here Theorem 2.17 allows us to conclude that $S - S^\tau$ fulfills the $\text{NUPBR}(\mathbb{G})$.

In the remaining part of this subsection, we focus on answering (1.3).

Theorem 2.19 Assume that $\tau \in \mathcal{H}$. Then, the following are equivalent.

- (a) The thin set $\{\tilde{Z} = 1 > Z_{-}\}$ is accessible (i.e. it is contained in a countable union of graphs of $\mathbb{F}$ -predictable stopping times).
- (b) For every (bounded) $\mathbb{F}$ -quasi-left-continuous martingale X , the process $X - X^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$.
- (b') For any probability $Q \sim P$ and every (bounded) $\mathbb{F}$ -quasi-left-continuous $X \in \mathcal{M}(Q, \mathbb{F})$, the process $X - X^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$.
- (c) For every (bounded) $\mathbb{F}$ -quasi-left-continuous process X satisfying $\text{NUPBR}(\mathbb{F})$, the process $X - X^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$.

Proof The proof of the proposition is organized in three parts, where we prove (a) $\iff$ (b), (b) $\iff$ (b') and (b') $\iff$ (c) respectively.

1) We start by proving that (a) $\implies$ (b). Suppose that the thin set $\{\tilde{Z} = 1 > Z_{-}\}$ is accessible. Then, for any $\mathbb{F}$ -quasi-left-continuous martingale X , we have $\{\Delta X \neq 0\} \cap \{\tilde{Z} = 1 > Z_{-}\} = \emptyset$. Hence, thanks to Corollary 2.15–(d), we deduce that $X - X^\tau$ satisfies $\text{NUPBR}(\mathbb{G})$. This completes the proof of (a) $\implies$ (b). To prove the reverse, assuming that assertion (b) holds, we consider a sequence of stopping times $(T_n)_{n \geq 1}$ that exhausts the thin set $\{\tilde{Z} = 1 > Z_{-}\}$ (i.e., $\{\tilde{Z} = 1 > Z_{-}\} = \bigcup_{n=1}^{+\infty} \llbracket T_n \rrbracket$). Then, each T_n – that we denote by T for the sake

of simplicity— can be decomposed into a totally inaccessible part T^i and an accessible part T^a as $T = T^i \wedge T^a$. Consider the following quasi-left-continuous $\mathbb{F}$ -martingale

$$M := V - V^{p,\mathbb{F}} =: V - \tilde{V},$$

where $V := I_{\llbracket T^i, +\infty \rrbracket}$. Then, since $\{T^i < +\infty\} \subset \{\tilde{Z}_{T^i} = 1\}$, we deduce that $\{T^i < +\infty\} \subset \{\tau \geq T^i\}$ and hence

$$I_{\llbracket \tau, +\infty \rrbracket} \cdot M = -I_{\llbracket \tau, +\infty \rrbracket} \cdot \tilde{V} \quad \text{is } \mathbb{G}\text{-predictable.}$$

Then, the finite variation and $\mathbb{G}$ -predictable process, $I_{\llbracket \tau, +\infty \rrbracket} \cdot M$, satisfies NUPBR($\mathbb{G}$) if and only if it is null, or equivalently

$$0 = E \left(I_{\llbracket \tau, +\infty \rrbracket} \cdot \tilde{V}_\infty \right) = E \left(\int_0^\infty (1 - Z_{s-}) d\tilde{V}_s \right) = E \left((1 - Z_{T^i-}) I_{\{T^i < +\infty\}} \right).$$

Therefore, we conclude that $T^i = +\infty$, P -a.s., and the stopping time T is an accessible stopping time. This ends the proof of (a) $\iff$ (b).

2) It is easy to see that the implication (b') $\implies$ (b) follows from taking $Q = P$. To prove the reverse sense, we suppose given $Q \sim P$ and an $\mathbb{F}$ -quasi-left-continuous $X \in \mathcal{M}(\mathbb{F}, Q)$. Then, put

$$Z_t^\mathbb{F} := E \left(\frac{dQ}{dP} \middle| \mathcal{F}_t \right) =: \mathcal{E}_t(N), \quad Y := \left(\frac{\mathcal{E}(N^{(qc)})X}{\mathcal{E}(N^{(qc)})} \right) \quad \text{and} \quad N^{(qc)} := N - I_{\bigcup_n \llbracket \sigma_n \rrbracket} \cdot N,$$

where $(\sigma_n)_n$ is the sequence of $\mathbb{F}$ -predictable stopping times that exhausts all the predictable jumps of N . In other words, $N^{(qc)}$ is the $\mathbb{F}$ -quasi-left-continuous local martingale part of N . Then, due to the quasi-left-continuity of X , simple calculations show that Y is an $\mathbb{F}$ -quasi-left-continuous martingale. Therefore, by a directly applying assertion (b) to Y , we conclude that $Y - Y^\tau = \left(\frac{\mathcal{E}(N^{(qc)})(X - X^\tau) + X^\tau(\mathcal{E}(N^{(qc)}) - \mathcal{E}(N^{(qc)})^\tau)}{\mathcal{E}(N^{(qc)}) - \mathcal{E}(N^{(qc)})^\tau} \right)$ satisfies NUPBR($\mathbb{G}$). This implies the existence of a real-valued positive $\mathbb{G}$ -local martingale $Z^\mathbb{G}$ such that both processes $Z^\mathbb{G} \mathcal{E}(N^{(qc)})(X - X^\tau)$ and $Z^\mathbb{G} \mathcal{E}(N^{(qc)})$ are σ -martingales under $(\mathbb{G}, P)$. Since $Z^\mathbb{G} \mathcal{E}(N^{(qc)})$ is positive and thanks to Proposition 3.3 and Corollary 3.5 of [4] (which states that a non-negative σ -martingale is a local martingale), we deduce that $Z^\mathbb{G} \mathcal{E}(N^{(qc)})$ is a real-valued positive element of $\mathcal{M}_{loc}(\mathbb{G}, P)$ such that $Z^\mathbb{G} \mathcal{E}(N^{(qc)})(X - X^\tau)$ is a σ -martingale. This proves that $X - X^\tau$ satisfies NUPBR($\mathbb{G}$), and the proof of (b) $\iff$ (b') is completed.

3) Remark that (c) $\implies$ (b') is obvious, and hence we focus on proving the reverse only. Suppose that assertion (b') holds, and consider an $\mathbb{F}$ -quasi-left-continuous process X satisfying NUPBR($\mathbb{F}$). Then, there exists a real-valued and positive $\mathbb{F}$ -local martingale Y , and a real-valued and $\mathbb{F}$ -predictable process ϕ such that

$$0 < \phi \leq 1 \quad Y(\phi \cdot X) \quad \text{is an } \mathbb{F}\text{-martingale.}$$

Let (T_n) be a sequence of $\mathbb{F}$ -stopping times that increases to infinity (almost surely) such that Y^{T_n} is a martingale, and set

$$\bar{X} := \phi \cdot X, \quad Q_n := Y_{T_n}/Y_0 \cdot P \sim P.$$

By applying assertion (b') to $\bar{X}^{T_n}$ and $Q_n \sim P$ (since $\bar{X}^{T_n}$ is an $\mathbb{F}$ -quasi-left-continuous element of $\mathcal{M}(\mathbb{F}, Q_n)$), we conclude that $(\phi \cdot (X - X^\tau))^{T_n} = \bar{X}^{T_n} - (\bar{X}^{T_n})^\tau$ satisfies NUPBR($\mathbb{G}$). Hence, thanks -again- to Proposition A.1, NUPBR($\mathbb{G}$) for $X - X^\tau$ follows immediately. This ends the proof of (b) $\iff$ (c), and that of the proposition as well. $\square$

Theorem 2.20 *Suppose that $\tau \in \mathcal{H}$ and $\mathbb{F}$ is quasi-left-continuous. Then the following assertions are equivalent.*

- (a) *The thin set $\{\tilde{Z} = 1 > Z_-\}$ is evanescent.*
- (b) *For every (bounded) $\mathbb{F}$ -martingale X , the process $X - X^\tau$ satisfies NUPBR($\mathbb{G}$).*
- (b') *For any probability $Q \sim P$ and every (bounded) $\mathbb{F}$ -quasi-left-continuous $X \in \mathcal{M}(Q, \mathbb{F})$, the process $X - X^\tau$ satisfies NUPBR($\mathbb{G}$).*
- (c) *For every (bounded) X satisfying NUPBR($\mathbb{F}$), $X - X^\tau$ satisfies the NUPBR($\mathbb{G}$).*

Proof The proofs of both equivalences (b') $\iff$ (c) and (b) $\iff$ (b') follow the same arguments as the corresponding proofs in Theorem 2.19 (see parts 2) and 3)). Hence, we omit these proofs and the proof of (a) $\implies$ (b) as well, as this latter one follows immediately from Theorem 2.19-(a) or Corollary 2.15-(d). Thus, the remaining part of the proof focuses on proving (a) $\implies$ (b). To this end, we assume that assertion (b) holds, and recall that -when $\mathbb{F}$ is a quasi-left-continuous filtration- any accessible $\mathbb{F}$ -stopping time is predictable (see [8] or [13, Th. 4.26]). Then, since $\mathbb{F}$ is a quasi-left-continuous filtration, any $\mathbb{F}$ -martingale is quasi-left-continuous, and from Theorem 2.19 we deduce that the thin set, $\{\tilde{Z} = 1 < Z_-\}$, is predictable. Now take any $\mathbb{F}$ -predictable stopping time T such that

$$[T] \subset \{\tilde{Z} = 1 > Z_-\}.$$

This implies that $\{T < +\infty\} \subset \{\tilde{Z}_T = 1\}$, and due to $E(\tilde{Z}_T | \mathcal{F}_{T-}) = Z_{T-}$ on $\{T < +\infty\}$, we get

$$E(I_{\{T < +\infty\}}(1 - Z_{T-})) = E(I_{\{T < +\infty\}}(1 - \tilde{Z}_T)) = 0.$$

This leads to $T = +\infty$ P -a.s (since $\{T < +\infty\} \subset \{Z_{T-} < 1\}$), and the proof of the theorem is completed. $\square$

Remark 2.21 The conclusion of Theorem 2.20 remains valid without the quasi-left-continuous assumption on the filtration $\mathbb{F}$. This general case, that can be found in the earlier version [1], requires more technical arguments.

The proof of Theorem 2.13 is technical, and is delegated to Section 4. Herein, we provide the principal ideas of the proof and the main difficulties that we encountered when designing this proof, as well as the connection of the main

results to Section 3. To this end, we suppose that S is locally bounded, and put $\mathcal{T}_{\mathbb{G}}(S) := S - S^\tau$ and $\mathcal{T}_{\mathbb{F}}(S) = S - [S, m^{(1)}]$. Thus, in virtue of Definition 2.2, the NUPBR($\mathbb{H}$) for $\mathcal{T}_{\mathbb{H}}(S)$ (when $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$) boils down to find a positive $\mathbb{H}$ -local martingale, $Z^{\mathbb{H}}$, such that $Z^{\mathbb{H}}\mathcal{T}_{\mathbb{H}}(S)$ is also a local martingale. In other words, this reduces, roughly speaking, to find a “local setting” (i.e. $Z^{\mathbb{H}}$) under which $\mathcal{T}_{\mathbb{H}}(S)$ is a fair-game process, or equivalently it has a null drift. Thus, the two major difficulties are: (a) How to get the $\mathbb{G}$ -local setting from that of $\mathbb{F}$ and vice-versa. (b) Once, the setting issue is resolved, how the drift of $\mathcal{T}_{\mathbb{H}}(S)$ vary when $\mathbb{H}$ varies in $\{\mathbb{F}, \mathbb{G}\}$. As interesting practical cases, we address the cases when $\mathcal{T}_{\mathbb{F}}(S) = S$ and/or $Z^{\mathbb{F}} = 1$. For these cases, one can see how the two issues (a) and (b) can be addressed. This is the aim of the next section. The general case, however, requires a deep method that is based on the Jacod’s statistical parametrisation for local martingales. This starts with decomposing S into three parts: The continuous local martingale part, the pure jump local martingale and the drift. Then, more importantly, any local martingale deflator is parameterized and identified by a pair of processes $(\beta^{\mathbb{H}}, f^{\mathbb{H}})$ satisfying

$$(\beta^{\mathbb{H}}, f^{\mathbb{H}}) \in \mathcal{I}_{loc}(\mathbb{H}) \quad \text{and} \quad G_{\mathbb{H}}(\beta^{\mathbb{H}}, f^{\mathbb{H}}) \equiv 0.$$

Here $\mathcal{I}_{loc}(\mathbb{H})$ is a set of $\mathbb{H}$ -predictable functionals satisfying some local integrability/positivity/measurability assumptions, and $G_{\mathbb{H}}(.,.)$ is an $\mathbb{H}$ -predictable functional that corresponds to the zero-drift equation for the process $\mathcal{T}_{\mathbb{H}}(S)$ under $Z^{\mathbb{H}}$. Herein, the issues (a) and (b) reduce to see how the two pairs $(\mathcal{I}_{loc}(\mathbb{G}), G_{\mathbb{G}}(.,.))$ and $(\mathcal{I}_{loc}(\mathbb{F}), G_{\mathbb{F}}(.,.))$ are obtained from each other.

3 Explicit Deflators for a Class of $\mathbb{F}$ -Local Martingales

This section proposes explicit construction of $\mathbb{G}$ -local martingale deflators for $M - M^\tau$, when M belongs to a class of $\mathbb{F}$ -local martingales that we specify later. This goal is based essentially on understanding the exact relationship between the $\mathbb{G}$ -compensator and the $\mathbb{F}$ -compensator of a process when both exists. This is the aim of the first subsection, while the second and last subsection states the main results about deflators.

3.1 Dual Predictable Projections under $\mathbb{G}$ and $\mathbb{F}$

In the following, we start our study by writing the $\mathbb{G}$ -compensators/projections in terms of $\mathbb{F}$ -compensators/projections respectively.

Lemma 3.1 *Suppose that $\tau \in \mathcal{H}$. Then, the following assertions hold.*

(a) *For any $\mathbb{F}$ -adapted process V , with locally integrable variation we have*

$$I_{\tau, +\infty}[\cdot V^{p, \mathbb{G}}] = I_{\tau, +\infty}[(1 - Z_-)^{-1} \cdot ((1 - \tilde{Z}) \cdot V)^{p, \mathbb{F}}], \quad (3.1)$$

and on $]\tau, +\infty[$

$${}^{p,\mathbb{G}}(\Delta V) = (1 - Z_-)^{-1} {}^{p,\mathbb{F}}\left((1 - \tilde{Z})\Delta V\right). \quad (3.2)$$

(b) For any $\mathbb{F}$ -local martingale M , one has, on $]\tau, +\infty[$

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{1 - \tilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(\Delta M I_{\{\tilde{Z} < 1\}}\right)}{1 - Z_-}, \quad \text{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{1 - \tilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(I_{\{\tilde{Z} < 1\}}\right)}{1 - Z_-}. \quad (3.3)$$

(c) For any quasi-left-continuous $\mathbb{F}$ -local martingale M , one has

$${}^{p,\mathbb{G}}\left((\Delta M)(1 - \tilde{Z})^{-1} I_{]\tau, +\infty[}\right) = 0. \quad (3.4)$$

Proof The proof of the lemma will be achieved in three steps.

1) This step proves assertion (a). From Lemma 2.6

$$I_{]\tau, +\infty[} \cdot V - I_{]\tau, +\infty[} \cdot V^{p,\mathbb{F}} + I_{]\tau, +\infty[}(1 - Z_-)^{-1} \cdot \langle V, m \rangle^{\mathbb{F}}$$

is a $\mathbb{G}$ -local martingale, hence

$$\begin{aligned} (I_{]\tau, +\infty[} \cdot V)^{p,\mathbb{G}} &= I_{]\tau, +\infty[} \cdot V^{p,\mathbb{F}} - I_{]\tau, +\infty[}(1 - Z_-)^{-1} \cdot \langle V, m \rangle^{\mathbb{F}} \\ &= I_{]\tau, +\infty[} \cdot V^{p,\mathbb{F}} - I_{]\tau, +\infty[}(1 - Z_-)^{-1} \cdot (\Delta m \cdot V)^{p,\mathbb{F}} \\ &= I_{]\tau, +\infty[}(1 - Z_-)^{-1} \cdot \left((1 - Z_- - \Delta m) \cdot V\right)^{p,\mathbb{F}}, \end{aligned}$$

where the second equality follows from Yoeurp's lemma. This ends the proof of (3.1). The equality (3.2) follows immediately from (3.1) by taking the jumps in both sides, and using $\Delta K^{p,\mathbb{H}} = {}^{p,\mathbb{H}}(\Delta K)$ when both terms exist.

2) Now, we prove assertion (b). By applying (3.2) for $V_{\epsilon, \delta} \in \mathcal{A}_{loc}(\mathbb{F})$ given by

$$V_{\epsilon, \delta} := \sum (\Delta M)(1 - \tilde{Z})^{-1} I_{\{|\Delta M| \geq \epsilon, 1 - \tilde{Z} \geq \delta\}},$$

we get, on $]\tau, +\infty[$,

$${}^{p,\mathbb{G}}\left((\Delta M)(1 - \tilde{Z})^{-1} I_{\{|\Delta M| \geq \epsilon, 1 - \tilde{Z} \geq \delta\}}\right) = (1 - Z_-)^{-1} {}^{p,\mathbb{F}}\left(\Delta M I_{\{|\Delta M| \geq \epsilon, 1 - \tilde{Z} \geq \delta\}}\right).$$

Then, the first equality in (3.3) follows from letting ϵ and δ go to zero, and we get on $]\tau, +\infty[$

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{1 - \tilde{Z}}\right) = (1 - Z_-)^{-1} {}^{p,\mathbb{F}}\left(\Delta M I_{\{1 - \tilde{Z} > 0\}}\right) = (1 - Z_-)^{-1} {}^{p,\mathbb{F}}\left(\Delta M I_{\{\tilde{Z} < 1\}}\right).$$

To prove the second equality in (3.3), we write that, on $\llbracket \tau, +\infty \rrbracket$,

$$\begin{aligned} {}^{p,\mathbb{G}}\left(\frac{1}{1-\tilde{Z}}\right) &= (1-Z_-)^{-1} + (1-Z_-)^{-1} {}^{p,\mathbb{G}}\left(\frac{\Delta m}{1-\tilde{Z}}\right) \\ &= (1-Z_-)^{-1} + (1-Z_-)^{-2} {}^{p,\mathbb{F}}\left(\Delta m I_{\{1-\tilde{Z}>0\}}\right) \\ &= (1-Z_-)^{-1} - (1-Z_-)^{-1} {}^{p,\mathbb{F}}\left(I_{\{\tilde{Z}=1\}}\right) = (1-Z_-)^{-1} {}^{p,\mathbb{F}}\left(I_{\{\tilde{Z}<1\}}\right). \end{aligned}$$

The second equality is due to (3.2), and the third equality follows from combining ${}^{p,\mathbb{F}}(\Delta m) = 0$, and $\Delta m = \tilde{Z} - Z_-$. This ends the proof of assertion (b).

3) The proof of (3.4) follows immediately from assertion (b) and the fact that the thin process ${}^{p,\mathbb{F}}\left(\Delta M I_{\{\tilde{Z}<1\}}\right)$ may take nonzero values on countably many predictable stopping times only, on which ΔM already vanishes. This completes the proof of the lemma. $\square$

The next lemma focuses on the integrability of the process $(1-\tilde{Z})^{-1}I_{\llbracket \tau, +\infty \rrbracket}$ with respect to any process with $\mathbb{F}$ -locally integrable variation. As a result, we complete our comparison of $\mathbb{G}$ and $\mathbb{F}$ compensators. Recall that, due to [10, Chapter XX], $\tilde{Z} = Z$ on $\llbracket \tau, +\infty \rrbracket$.

Lemma 3.2 *Let τ be a honest time and V be a càdlàg and $\mathbb{F}$ -adapted process with finite variation. Then, the following assertions hold.*

(a) *The process*

$$U := (1-Z)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot V, \quad (3.5)$$

is a well defined process, that is $\mathbb{G}$ -adapted, càdlàg and has finite variation.

(b) *If V belongs to $\mathcal{A}_{loc}(\mathbb{F})$ (respectively to $\mathcal{A}(\mathbb{F})$), then $U \in \mathcal{A}_{loc}(\mathbb{G})$ (respectively $U \in \mathcal{A}(\mathbb{G})$) and*

$$U^{p,\mathbb{G}} = I_{\llbracket \tau, +\infty \rrbracket} (1-Z_-)^{-1} \cdot (I_{\{\tilde{Z}<1\}} \cdot V)^{p,\mathbb{F}}. \quad (3.6)$$

(c) *Suppose furthermore that τ is finite almost surely. Then, $I_{\llbracket \tau, +\infty \rrbracket} \cdot V \in \mathcal{A}_{loc}(\mathbb{G})$ if and only if $(1-\tilde{Z}) \cdot V \in \mathcal{A}_{loc}(\mathbb{F})$.*

(d) *Suppose furthermore that τ is finite almost surely, and V is $\mathbb{F}$ -predictable process. Then, for any nonnegative and $\mathbb{F}$ -predictable process φ , $\varphi I_{\llbracket \tau, +\infty \rrbracket} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{G})$ if and only if $(1-Z_-)\varphi \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$.*

Proof The proof of the lemma is given in three parts. In the first part we prove both assertions (a) and (b), while in the second and the third parts we focus on assertions (c) and (d) respectively.

1) Let V be an $\mathbb{F}$ -adapted process with finite variation. Then, we obtain

$$\text{Var}(U) = (1-Z)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot \text{Var}(V).$$

Therefore, since $1 - \tilde{Z}_t = P(\tau < t | \mathcal{F}_t) \leq 1 - Z_t$, for any bounded and $\mathbb{F}$ -optional process ϕ such that $\phi \cdot \text{Var}(V) \in \mathcal{A}^+(\mathbb{F})$, we obtain

$$\begin{aligned} E[(\phi \cdot \text{Var}(U))_\infty] &= E\left(\int_0^\infty \frac{\phi_t I_{\{t > \tau\}}}{1 - Z_t} d\text{Var}(V)_t\right) \\ &= E\left(\int_0^\infty \frac{\phi_t P(\tau < t | \mathcal{F}_t)}{1 - Z_t} I_{\{Z_t < 1\}} d\text{Var}(V)_t\right) \\ &\leq E[(\phi \cdot \text{Var}(V))_\infty]. \end{aligned} \quad (3.7)$$

As a result, by taking $\phi = I_{[0, \sigma]}$ in (3.7), for an $\mathbb{F}$ -stopping time σ such that $\text{Var}(V)^{\sigma-} \in \mathcal{A}^+(\mathbb{F})$, we get $E[\text{Var}(U)_{\sigma-}] \leq E[\text{Var}(V)_{\sigma-}]$. This proves that the process U has a finite variation and hence is well defined as well. Being $\mathbb{G}$ -adapted for U is obvious, while being càdlàg follows immediately from (3.7). This ends the proof of assertion (a).

To prove assertion (b), we assume that $V \in \mathcal{A}_{loc}(\mathbb{F})$ and consider $(\vartheta_n)_{n \geq 1}$, a sequence of $\mathbb{F}$ -stopping times that increases to $+\infty$ such that $\text{Var}(V)^{\vartheta_n} \in \mathcal{A}^+(\mathbb{F})$. Then, by choosing $\phi = I_{[0, \vartheta_n]}$ in (3.7), we conclude that U belongs to $\mathcal{A}_{loc}(\mathbb{G})$ whenever V does under $\mathbb{F}$. For the case when $V \in \mathcal{A}(\mathbb{G})$, it is enough to take $\phi = 1$ in (3.7), and conclude that $U \in \mathcal{A}(\mathbb{G})$. To prove (3.6), for any $n \geq 1$, we put

$$U_n := (1 - Z)^{-1} I_{[\tau, +\infty]} I_{\{\tilde{Z} \leq 1 - \frac{1}{n}\}} \cdot V = \left(1 - \tilde{Z}\right)^{-1} I_{[\tau, +\infty]} I_{\{\tilde{Z} \leq 1 - \frac{1}{n}\}} \cdot V, \quad n \geq 1.$$

Then, thanks to (3.1), we derive

$$\begin{aligned} U^{p, \mathbb{G}} &= \lim_{n \rightarrow +\infty} (U_n)^{p, \mathbb{G}} \\ &= \lim_{n \rightarrow +\infty} (1 - Z_-)^{-1} I_{[\tau, +\infty]} \cdot \left(I_{\{\tilde{Z} \leq 1 - \frac{1}{n}\}} \cdot V\right)^{p, \mathbb{F}}. \end{aligned}$$

This clearly implies (3.6).

2) It is easy to see that it is enough to prove the assertion for the case when V is nondecreasing. Thus, suppose that V is nondecreasing. It obvious that $(1 - \tilde{Z}) \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$ implies $I_{[\tau, +\infty]} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{G})$. Hence, for the rest of this part, we focus on proving the reverse. Suppose $I_{[\tau, +\infty]} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{G})$. Then, there exists a sequence $\mathbb{G}$ -stopping times that increases to infinity and $(I_{[\tau, +\infty]} \cdot V)^{\sigma_n^{\mathbb{G}}} \in \mathcal{A}^+(\mathbb{G})$. Thanks to Proposition A.2-(c), we obtain a sequence of $\mathbb{F}$ -stopping times, $(\sigma_n^{\mathbb{F}})_{n \geq 1}$, that increases to infinity and $\sigma_n^{\mathbb{G}} \vee \tau = \tau \vee \sigma_n^{\mathbb{F}}$. Therefore, we get $(I_{[\tau, +\infty]} \cdot V)^{\sigma_n^{\mathbb{G}}} \equiv (I_{[\tau, +\infty]} \cdot V)^{\sigma_n^{\mathbb{F}}}$ and hence

$$E\left((1 - \tilde{Z}) \cdot V_{\sigma_n^{\mathbb{F}}}\right) = E\left(I_{[\tau, +\infty]} \cdot V_{\sigma_n^{\mathbb{F}}}\right) = E\left(I_{[\tau, +\infty]} \cdot V_{\sigma_n^{\mathbb{G}}}\right) < +\infty. \quad (3.8)$$

This proves that the process $(1 - \tilde{Z}) \cdot V$ belongs to $\mathcal{A}_{loc}^+(\mathbb{F})$, and the proof of assertion (c) is achieved.

3) The proof of assertion (d) follows all the steps of the proof of assertion (c), except (3.8) which takes the form of

$$E((1 - Z_-) \cdot V_{\sigma_n^{\mathbb{F}}}) = E(I_{\llbracket \tau, +\infty \rrbracket} \cdot V_{\sigma_n^{\mathbb{F}}}) = E(I_{\llbracket \tau, +\infty \rrbracket} \cdot V_{\sigma_n^{\mathbb{G}}}) < +\infty$$

instead due to the predictability of V . This ends the proof of assertion (d) and the proof of the lemma as well. $\square$

3.2 Construction of Deflators

Herein, we start by introducing a deflator-candidate as follows.

Proposition 3.3 *Suppose that $\tau \in \mathcal{H}$ and consider the $\mathbb{G}$ -local martingale*

$$\widehat{m} := I_{\llbracket \tau, +\infty \rrbracket} \cdot m + (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot \langle m \rangle^{\mathbb{F}}, \quad (3.9)$$

and the two processes

$$\kappa := \frac{1 - Z_-}{(1 - Z_-)^2 + \Delta \langle m \rangle^{\mathbb{F}}} I_{\llbracket \tau, +\infty \rrbracket}, \quad W^{\mathbb{G}} := \frac{\kappa}{1 - \widetilde{Z}} I_{\llbracket \tau, +\infty \rrbracket} \cdot ([m, m] + \langle m \rangle^{\mathbb{F}}). \quad (3.10)$$

Then the following assertions hold.

- 1) The nondecreasing and $\mathbb{G}$ -predictable process $W^{\mathbb{G}}$ belongs to $\mathcal{A}_{loc}^+(\mathbb{G})$.
- 2) The $\mathbb{G}$ -local martingale

$$L^{\mathbb{G}} := \kappa \cdot \widehat{m} + W^{\mathbb{G}} - (W^{\mathbb{G}})^{p, \mathbb{G}}, \quad (3.11)$$

satisfies the following properties:

- (2-a) $\mathcal{E}(L^{\mathbb{G}}) > 0$ (or equivalently $1 + \Delta L^{\mathbb{G}} > 0$) and $I_{\llbracket 0, \tau \rrbracket} \cdot L^{\mathbb{G}} = 0$.
- (2-b) For any $M \in \mathcal{M}_{0, loc}(\mathbb{F})$, we have

$$[L^{\mathbb{G}}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad \left(\text{i.e. } \langle L^{\mathbb{G}}, \widehat{M} \rangle^{\mathbb{G}} \text{ exists} \right), \quad (3.12)$$

where $\widehat{M}$ is defined in (2.4).

Proof Thanks to Lemma 2.6-(b), it is easy to check that the three $\mathbb{G}$ -predictable processes, κ , $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$, and $\kappa^{-1} I_{\llbracket \tau, +\infty \rrbracket}$ are $\mathbb{G}$ -locally bounded. By combining this fact with $[m, m] + \langle m \rangle^{\mathbb{F}} \in \mathcal{A}_{loc}^+(\mathbb{F})$ and Lemma 3.2-(b), we conclude that $(1 - \widetilde{Z})^{-1} I_{\llbracket \tau, +\infty \rrbracket} \cdot ([m, m] + \langle m \rangle^{\mathbb{F}}) \in \mathcal{A}_{loc}^+(\mathbb{G})$, and subsequently assertion (1) holds. Thus, the process $L^{\mathbb{G}}$ —given in (3.11)—is a well defined $\mathbb{G}$ -local martingale. The rest of this proof focuses on proving the properties (2-a) and (2-b). To this end, by combining Lemma 3.1-(b) and $\Delta m = \widetilde{Z} - Z_-$, on $\llbracket \tau, +\infty \rrbracket$ we calculate

$$\begin{aligned} \frac{\Delta L^{\mathbb{G}}}{\kappa} &= \Delta m + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{1 - Z_-} + \frac{(\Delta m)^2 + \Delta \langle m \rangle^{\mathbb{F}}}{1 - \widetilde{Z}} - {}^{p, \mathbb{G}} \left(\frac{(\Delta m)^2 + \Delta \langle m \rangle^{\mathbb{F}}}{1 - \widetilde{Z}} \right) \\ &= \frac{1 - Z_-}{1 - \widetilde{Z}} \Delta m - \frac{\Delta \langle m \rangle^{\mathbb{F}}}{1 - Z_-} + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{1 - \widetilde{Z}} + \frac{{}^{p, \mathbb{F}} \left((\Delta m)^2 I_{\{\widetilde{Z}=1\}} \right)}{1 - Z_-} + \frac{\Delta \langle m \rangle^{\mathbb{F}}}{1 - Z_-} {}^{p, \mathbb{F}} \left(I_{\{\widetilde{Z}=1\}} \right) \\ &= \frac{\Delta m}{\kappa(1 - \widetilde{Z})} + \frac{{}^{p, \mathbb{F}} \left(I_{\{\widetilde{Z}=1\}} \right)}{\kappa} \end{aligned}$$

This implies that, on $\llbracket \tau, +\infty \llbracket$,

$$1 + \Delta L^{\mathbb{G}} = \frac{1 - Z_-}{1 - \widetilde{Z}} + {}^{p, \mathbb{F}} \left(I_{\{\widetilde{Z}=1\}} \right) > 0.$$

This proves the property (2-a). In order to prove the property (2-b), we consider a quasi-left-continuous $\mathbb{F}$ -local martingale M . Then, it obvious that this quasi-left-continuous assumption implies that $\langle m, M \rangle^{\mathbb{F}}$ is continuous, $[X, M] \equiv 0$ for any $\mathbb{G}$ -predictable process with finite variation, and $\kappa(1 - Z_-) \cdot [M, Y] = [M - M^\tau, Y]$ for any $\mathbb{G}$ -semimartingale Y . As a result, we derive

$$\begin{aligned} [L^{\mathbb{G}}, \widehat{M}] &= [L^{\mathbb{G}}, M] = (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \llbracket} \cdot [m, M] + [W^{\mathbb{G}}, M] \\ &= \frac{1}{1 - Z_-} I_{\llbracket \tau, +\infty \llbracket} \cdot [m, M] + \frac{\Delta m}{(1 - Z_-)(1 - \widetilde{Z})} I_{\llbracket \tau, +\infty \llbracket} \cdot [m, M] \\ &= \frac{1}{1 - \widetilde{Z}} I_{\llbracket \tau, +\infty \llbracket} \cdot [m, M]. \end{aligned} \quad (3.13)$$

Therefore, since $[m, M] \in \mathcal{A}_{loc}(\mathbb{F})$, the property (2-b) follows immediately from combining the above equality and Lemma 3.2-(b). This ends the proof of the proposition. $\square$

Now we are in the stage of constructing a deflator for a large class of $\mathbb{F}$ -processes that are already risk-neutralized.

Theorem 3.4 *Let $\tau \in \mathcal{H}$ is finite almost surely and $L^{\mathbb{G}}$ be defined by (3.11). Then, for any quasi-left-continuous $\mathbb{F}$ -local martingale, M , such that $\{\widetilde{Z} = 1 > Z_-\} \cap \{\Delta M \neq 0\}$ is evanescent, $\mathcal{E}(L^{\mathbb{G}})(M - M^\tau)$ is a $\mathbb{G}$ -local martingale.*

Proof Let M be a quasi-left-continuous $\mathbb{F}$ -local martingale satisfying

$$\{\widetilde{Z} = 1 > Z_-\} \cap \{\Delta M \neq 0\} = \emptyset.$$

Therefore, by combining (3.13) and Lemma 3.2-(b), we obtain

$$\begin{aligned} M - M^\tau + \langle L^{\mathbb{G}}, M - M^\tau \rangle^{\mathbb{G}} &= M - M^\tau + \langle L^{\mathbb{G}}, M \rangle^{\mathbb{G}} \\ &= M - M^\tau + (1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \llbracket} \cdot \langle m, M \rangle^{\mathbb{F}} \\ &= \widehat{M} \in \mathcal{M}_{loc}(\mathbb{G}). \end{aligned}$$

This ends the proof of the theorem. $\square$

As a consequence of this theorem, we describe a class of $\mathbb{F}$ -quasi-left-continuous processes for which the NUPBR property is preserved for the part after τ .

Corollary 3.5 *Suppose that $\tau \in \mathcal{H}$ is finite almost surely, and that S is $\mathbb{F}$ -quasi-left-continuous satisfying $NUPBR(\mathbb{F})$ and $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\}$ is evanescent. Then, $S - S^\tau$ satisfies $NUPBR(\mathbb{G})$.*

Proof The proof follows immediately from a combination of Theorem 3.4, Proposition A.1 (see the appendix), and the fact that

$$\{\tilde{Z} = 1 > Z_-\} = \{\tilde{Z}^Q = 1 > Z_-^Q\} \quad \text{for any } Q \sim P, \quad (3.14)$$

where $\tilde{Z}_t^Q := Q(\tau \geq t | \mathcal{F}_t)$ and $Z_t^Q := Q(\tau > t | \mathcal{F}_t)$. This last fact is an immediate application of Theorem 86 of [9] by taking on the one hand $X = I_{\{\tilde{Z}=0\}}$ and $Y = I_{\{\tilde{Z}^Q=0\}}$ and in the other hand $X = I_{\{Z_-=0\}}$ and $Y = I_{\{Z_-^Q=0\}}$. $\square$

Our last result, in this section, extends Theorem 3.4 to the case where S is local martingale and is orthogonal to $m^{(1)}$ that is associated to (τ, S) via Theorem 2.13. In contrast to the previous case, the deflator for this case depends also on S .

Theorem 3.6 *Consider $\tau \in \mathcal{H}$ is finite almost surely, $L^{\mathbb{G}}$ defined in (3.11), and S is quasi-left-continuous $\mathbb{F}$ -local martingale such that $\langle S, m^{(1)} \rangle^{\mathbb{F}} \equiv 0$, where $m^{(1)}$ is the $\mathbb{F}$ -martingale associated to (τ, S) via Theorem 2.13. Then, there exists a $\mathbb{G}$ -local martingale $L^{(1)}$ such that $\Delta L^{(1)} \geq 0$ and $\mathcal{E}(L^{\mathbb{G}} + L^{(1)})(S - S^\tau)$ is a $\mathbb{G}$ -local martingale.*

The $\mathbb{G}$ -local martingale, $L^{(1)}$, will be defined explicitly, while it is technically involved. Thus, its description as well as the proof of the theorem are postponed to the next section.

4 Proof of Theorems 2.13, 2.17 and 3.6

This section focuses on the proofs of Theorems 2.13, 2.17 and 3.6. All three theorems are based essentially on the key $\mathbb{F}$ -local martingale $m^{(1)}$ that we start by describing. To this end, we recall some notations on semimartingale predictable characteristics and related decompositions for m and S .

To the process S , we associate its random measure of jumps $\mu(dt, dx) := \sum_{u>0} I_{\{\Delta S_u \neq 0\}} \delta_{(u, \Delta S_u)}(dt, dx)$. For any nonnegative product-measurable functional $H(t, \omega, x)$, we define the process $H \star \mu$ and a σ -finite measure M_μ^P on the measurable space $(\Omega \times \mathbb{R}^+ \times \mathbb{R}^d, \mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(\mathbb{R}^d))$ by

$$H \star \mu_t := \int_0^t \int_{\mathbb{R}^d} H(u, x) \mu(du, dx) \quad \text{and} \quad M_\mu^P(H) := E[H \star \mu_\infty] = \int H dM_\mu^P. \quad (4.1)$$

Throughout the rest of the paper, for any filtration $\mathbb{H}$, we denote

$$\tilde{\mathcal{O}}(\mathbb{H}) := \mathcal{O}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d), \quad \tilde{\mathcal{P}}(\mathbb{H}) := \mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d),$$

and $M_\mu^P(W | \tilde{\mathcal{P}}(\mathbb{H}))$, for a nonnegative or bounded functional W , is the unique $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional Y satisfying $M_\mu^P(YU) = M_\mu^P(WU)$ for any bounded and $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional U . In the following, we will use two types of stochastic integration with respect to the random measure μ . The following defines their sets of integrands.

Definition 4.1 1) We call $W \in \mathcal{G}_{loc}^1(\mu, \mathbb{H})$ (respectively $W \in \mathcal{H}_{loc}^1(\mu, \mathbb{H})$) if it is $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable (respectively $\tilde{\mathcal{O}}(\mathbb{H})$ -measurable) and $\sqrt{W^2} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{H})$.
 2) For any $W \in \mathcal{G}_{loc}^1(\mu, \mathbb{H})$, the process $W \star (\mu - \nu)$ denotes the integral of W with respect to the compensated random measure. This resulting integral is the unique pure jump $\mathbb{H}$ -local martingale whose jumps are $W(\Delta S)I_{\{\Delta S \neq 0\}}$.
 3) If $W \in \mathcal{H}_{loc}^1(\mathbb{H})$ satisfying $M_\mu^P(W|\tilde{\mathcal{P}}(\mathbb{H})) = 0$, then we denote by $W \star \mu$ the unique pure jump $\mathbb{H}$ -local martingale whose jumps are $W(\Delta S)I_{\{\Delta S \neq 0\}}$.

By applying Theorem A.1 of [1] (see also [15, Theorem 3.75, page 103] and to [16, Lemma 4.24, Chap III]), we derive the canonical decomposition of m

$$m = M_0 + m^c + f_m \star (\mu - \nu) + g_m \star \mu + m',$$

where $g_m \in \mathcal{H}_{loc}^1(\mu, \mathbb{F})$, $f_m \in \mathcal{G}_{loc}^1(\mu, \mathbb{F})$, and m' is an $\mathbb{F}$ -local martingale satisfying $[m', S] \equiv 0$. Furthermore, f_m is given by

$$f_m := M_\mu^P(\Delta m | \tilde{\mathcal{P}}(\mathbb{F})), \text{ and one has } M_\mu^P(\tilde{Z} | \tilde{\mathcal{P}}(\mathbb{F})) = Z_- + f_m. \quad (4.2)$$

4.1 The Explicit Description of $m^{(1)}$

Now, we are in the stage of constructing the process $m^{(1)}$.

Lemma 4.2 *The following hold.*

(a) *We have $\{Z_- + f_m = 1\} \subset \{\tilde{Z} = 1\}$, M_μ^P -a.e. or equivalently*

$$\{Z_- + f_m = 1\} \cap \{\Delta S \neq 0\} \subset \{\tilde{Z} = 1\} \cap \{\Delta S \neq 0\}. \quad (4.3)$$

(b) *The functional $f_m I_{\{Z_- + f_m = 1 > Z_-\}}$ belongs to $\mathcal{G}_{loc}^1(\mu, \mathbb{F})$, and the $\mathbb{F}$ -local martingale*

$$m^{(1)} := f_m I_{\{Z_- + f_m = 1 > Z_-\}} \star (\mu - \nu), \quad (4.4)$$

is quasi-left-continuous, pure jump, $m_0^{(1)} = 0$, and satisfies (2.10).

Proof Recall that we always have

$$E[W \star \mu_\infty] = E[M_\mu^P(W|\tilde{\mathcal{P}}(\mathbb{F})) \star \nu_\infty],$$

for any non-negative $\tilde{\mathcal{O}}(\mathbb{F})$ -measurable functional W . Thus, since $M_\mu^P(\tilde{Z}|\tilde{\mathcal{P}}(\mathbb{F})) = Z_- + f_m$, we derive

$$E[(1 - \tilde{Z})I_{\{Z_- + f_m = 1\}} \star \mu_\infty] = E[(1 - Z_- - f_m)I_{\{Z_- + f_m = 1\}} \star \nu_\infty] = 0.$$

This proves that $(1 - \tilde{Z})I_{\{Z_- + f_m = 1\}} \star \mu_\infty$ is a null random variable, or equivalently that $\{Z_- + f_m = 1\} \subset \{\tilde{Z} = 1\}$ M_μ^P -a.e. This proves the first assertion.

It is obvious that $f_m I_{\{Z_- + f_m = 1 > Z_-\}}$ belongs to $\mathcal{G}_{loc}^1(\mu, \mathbb{F})$ since f_m possesses the same property. It is also obvious that $m^{(1)}$ is quasi-left-continuous as S , and $m_0^{(1)} = 0$. Furthermore, we have

$$\Delta m^{(1)} = f_m(\Delta S)I_{\{Z_- + f_m(\Delta S) = 1 > Z_-\}}I_{\{\Delta S \neq 0\}} = (1 - Z_-)I_{\{Z_- + f_m(\Delta S) = 1 > Z_-\}}I_{\{\Delta S \neq 0\}}.$$

This implies (2.10), and the proof of the lemma is achieved. $\square$

4.2 Proof of Theorem 2.13

Recall that μ is the random measure of the jumps of S , ν is its $\mathbb{F}$ -compensator random measure, and the functional f_m is defined in (4.2). Put

$$\begin{aligned}\mu^{\mathbb{G}}(dt, dx) &:= I_{\llbracket \tau, +\infty \llbracket}(t) \mu(dt, dx), \\ \nu^{\mathbb{G}}(dt, dx) &:= I_{\llbracket \tau, +\infty \llbracket}(t) \left(1 - \frac{f_m(x, t)}{1 - Z_{t-}}\right) \nu(dt, dx).\end{aligned}\quad (4.5)$$

It is easy to check that $\nu^{\mathbb{G}}$ is the random measure compensator under $\mathbb{G}$ of $\mu^{\mathbb{G}}$. The canonical decomposition of $S - S^\tau$ under $\mathbb{G}$ is given by

$$\begin{aligned}S - S^\tau &= \widehat{S^c} + h \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + b I_{\llbracket \tau, +\infty \llbracket} \cdot A - \frac{c\beta_m}{1 - Z_-} I_{\llbracket \tau, +\infty \llbracket} \cdot A \\ &\quad - h \frac{f_m}{1 - Z_-} I_{\llbracket \tau, +\infty \llbracket} \star \nu + (x - h) \star \mu^{\mathbb{G}},\end{aligned}$$

where $\widehat{S^c}$ is defined by (2.4).

The proof of the theorem is based on the following

Lemma 4.3 *($S - S^\tau$) satisfies NUPBR($\mathbb{G}$) if and only if there exist an $\mathbb{F}$ -predictable process $\beta^{\mathbb{F}}$, and a positive $\tilde{\mathcal{P}}(\mathbb{F})$ -functional $f^{\mathbb{F}}$, such that*

$$(\beta^{\mathbb{F}})^{tr} c\beta^{\mathbb{F}} I_{\llbracket \tau, +\infty \llbracket} \cdot A \text{ and } \sqrt{(f^{\mathbb{F}} - 1)^2 \star \mu^{\mathbb{G}}} \text{ belong to } \mathcal{A}_{loc}^+(\mathbb{G}). \quad (4.6)$$

and $P \otimes A - a.e.$ on $\{Z_- < 1\}$, we have

$$\varphi^{\mathbb{F}} := \int |x f^{\mathbb{F}}(x) (1 - Z_- - f_m(x)) - h(x)| F(dx) < +\infty, \quad (4.7)$$

and

$$b + c \left(\beta^{\mathbb{F}} - \frac{\beta_m}{1 - Z_-} \right) + \int \left[x f^{\mathbb{F}}(x) \left(1 - \frac{f_m(x)}{1 - Z_-} \right) - h(x) \right] F(dx) \equiv 0. \quad (4.8)$$

Proof ($S - S^\tau$) satisfies NUPBR($\mathbb{G}$) if and only if there exist a real-valued $\mathbb{G}$ -predictable process $\Phi^{\mathbb{G}}$, and a real-valued $\mathbb{G}$ -local martingale $N^{\mathbb{G}}$ that can be chosen as

$$N^{\mathbb{G}} := \beta^{\mathbb{G}} \cdot \widehat{S^c} + (f^{\mathbb{G}} - 1) \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}),$$

such that $0 < \Phi^{\mathbb{G}} \leq 1$ and $\mathcal{E}(N^{\mathbb{G}}) > 0$ and $[\Phi^{\mathbb{G}} \cdot (S - S^\tau)] \mathcal{E}(N^{\mathbb{G}})$ is a $\mathbb{G}$ -local martingale, which is equivalent to

$$\Phi^{\mathbb{G}} \cdot (S - S^\tau) + [N^{\mathbb{G}}, \Phi^{\mathbb{G}} \cdot (S - S^\tau)] \text{ is a } \mathbb{G}\text{-local martingale.}$$

Thanks to Itô's formula, this is equivalent to

$$\Phi^{\mathbb{G}} |x f^{\mathbb{G}}(x) - h(x)| \star \mu^{\mathbb{G}} \in \mathcal{A}_{loc}^+(\mathbb{G}), \quad (4.9)$$

and $P \otimes A$ -a.e. on $\llbracket \tau, +\infty \rrbracket$ we have

$$0 \equiv b + c \left(\beta^{\mathbb{G}} - \frac{\beta_m}{1 - Z_-} \right) + \int \left[x f^{\mathbb{G}}(x) \left(1 - \frac{f_m(x)}{1 - Z_-} \right) - h(x) \right] F(dx) \quad (4.10)$$

Then, Lemma A.2 guarantees the existence of two $\mathbb{F}$ -predictable processes $\Phi^{\mathbb{F}}$ and $\beta^{\mathbb{F}}$, and a $\tilde{\mathcal{P}}(\mathbb{F})$ -functional $f^{\mathbb{F}}$, such that $0 < \Phi^{\mathbb{F}} \leq 1$, $0 < f^{\mathbb{F}}$ and

$$\Phi^{\mathbb{F}} = \Phi^{\mathbb{G}}, \quad \beta^{\mathbb{F}} = \beta^{\mathbb{G}}, \quad f^{\mathbb{F}} = f^{\mathbb{G}} \quad \text{on } \llbracket \tau, +\infty \rrbracket.$$

Then, (4.6) follows, and (4.9) becomes $\Phi^{\mathbb{F}} |x f^{\mathbb{F}}(x) - h(x)| \star \mu^{\mathbb{G}} \in \mathcal{A}_{loc}^+(\mathbb{G})$. Thus, from a direct application of Proposition B.1-(b), we deduce that (4.7) follows immediately. Furthermore, (4.8) follows from a combination of this property and (4.10) in which we substitute $\beta^{\mathbb{F}}$ and $f^{\mathbb{F}}$ to $\beta^{\mathbb{G}}$ and $f^{\mathbb{G}}$ respectively and we take the $\mathbb{F}$ -predictable projection of the resulting equation afterwards. This ends the proof of the lemma. $\square$

Proof of Theorem 2.13 The proof of the theorem will be achieved in two steps where we prove (a) $\implies$ (b) and the reverse sense respectively. Throughout this part, we put

$$S^{(1)} := x f_m I_{\{Z_- < 1 = Z_- + f_m\}} \star \mu = [S, m^{(1)}]. \quad (4.11)$$

1) Proof of (a) $\implies$ (b): Suppose that $S - S^\tau$ satisfies NUPBR($\mathbb{G}$). Then, thanks to Lemma 4.3, we deduce the existence of the $\mathbb{F}$ -predictable pair $(\beta^{\mathbb{F}}, f^{\mathbb{F}})$ satisfying $f^{\mathbb{F}} > 0$, and (4.6), (4.7) and (4.8) hold. Then, fix $\delta \in (0, 1)$, and put

$$\bar{S} := (1 - Z_-) \cdot S - S^{(1)} \quad \text{and} \quad \Gamma := \{Z_- \leq 1 - \delta \text{ \& } 1 = Z_- + f_m\}.$$

Now, we put $\psi := 1 - Z_- - f_m := 1 - M_\mu^P(\tilde{Z} | \tilde{\mathcal{P}}(\mathbb{F}))$, and consider the following

$$\begin{aligned} \beta &:= (\beta^{\mathbb{F}} - \frac{\beta_m}{1 - Z_-}) I_{\{Z_- \leq 1 - \delta\}}, \\ f &:= f^{\mathbb{F}}(x) \left(1 - \frac{f_m(x)}{1 - Z_-} \right) I_{\{\psi > 0 \text{ \& } Z_- \leq 1 - \delta\}} + I_{\{\psi = 0 \text{ or } Z_- \geq 1 - \delta\}}, \end{aligned}$$

and we assume for a while that

$$\beta \in L(S^c, \mathbb{F}) \quad \text{and} \quad (f - 1) \in \mathcal{G}_{loc}^1(\mu, \mathbb{F}). \quad (4.12)$$

Then, the process $N := \beta \cdot S^c + (f - 1) \star (\mu - \nu)$ is a well defined $\mathbb{F}$ -local martingale, and it is easy to check, using Itô's formula, that $(\xi I_{\{Z_- \leq 1 - \delta\}} \cdot \bar{S}) \mathcal{E}(N)$ is a local martingale due to (4.7) and (4.8), where $\xi := (1 + \varphi^{\mathbb{F}} I_{\{Z_- \leq 1 - \delta\}})^{-1}$ and $\varphi^{\mathbb{F}}$ is given by (4.7). This proves that $I_{\{Z_- \leq 1 - \delta\}} \cdot \bar{S}$ satisfies NUPBR($\mathbb{F}$) as long as (4.12) holds. The remaining proof in this part will focus on proving this assumption.

Since $\beta_m^{tr} c \beta_m \cdot A \in \mathcal{A}_{loc}^+(\mathbb{F})$ and $(\beta^{\mathbb{F}})^{tr} c \beta^{\mathbb{F}} I_{\llbracket \tau, +\infty \rrbracket} \cdot A \in \mathcal{A}_{loc}^+(\mathbb{G})$, then Lemma

3.2-(c) leads to $\beta^{tr} c \beta \cdot A \in \mathcal{A}_{loc}^+(\mathbb{F})$, or equivalently $\beta \in L(S^c, \mathbb{F})$.

By putting $\Sigma_1 := \{\psi > 0 \text{ \& } Z_- \leq 1 - \delta\}$, we calculate

$$f - 1 = (f^{\mathbb{F}} - 1) \left(1 - \frac{f_m(x)}{1 - Z_-} \right) I_{\Sigma_1} - \frac{f_m(x)}{1 - Z_-} I_{\Sigma_1} =: W_1 + W_2.$$

Then, we put $\bar{\mu} := (1 - \tilde{Z}) \cdot \mu$ and we obtain, in the one hand

$$W_1^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu \leq \delta^{-2} (f^{\mathbb{F}} - 1)^2 (1 - Z_- - f_m) I_{\{Z_- \leq 1 - \delta\}} I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu$$

where $(f^{\mathbb{F}} - 1)^2 (1 - Z_- - f_m) I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} I_{\{Z_- \leq 1 - \delta\}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F})$ (which is equivalent to $(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} I_{\{Z_- \leq 1 - \delta\}} \star \bar{\mu} \in \mathcal{A}_{loc}^+(\mathbb{F})$). In the other hand,

$$|W_1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu = \delta^{-1} |f^{\mathbb{F}} - 1| (1 - Z_- - f_m) I_{\{Z_- \leq 1 - \delta\}} I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu$$

where $|f^{\mathbb{F}} - 1| (1 - Z_- - f_m) I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} I_{\{Z_- \leq 1 - \delta\}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F})$ (this is equivalent to $|f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} I_{\{Z_- \leq 1 - \delta\}} \star \bar{\mu} \in \mathcal{A}_{loc}^+(\mathbb{F})$). Thus, by combining these two remarks and

$$\sqrt{W_1^2 \star \mu} \leq \sqrt{W_1^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu} + |W_1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu,$$

we conclude that $\sqrt{W_1^2 \star \mu} \in \mathcal{A}_{loc}^+(\mathbb{F})$. Similarly we notice that

$$W_2^2 \star \mu \leq \delta^{-2} W_3 \in \mathcal{A}_{loc}^+(\mathbb{F}),$$

where $W_3 := f_m^2 \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F})$ satisfies $EW_3(\sigma) \leq E[m, m]_\sigma$ for any $\mathbb{F}$ -stopping time σ . This proves that $(f - 1) \in \mathcal{G}_{loc}^1(\mu, \mathbb{F})$, and the proof of (a) $\implies$ (b) is completed.

2) Proof of (b) $\implies$ (a): Suppose that for any $\delta \in (0, 1)$, $I_{\{Z_- \leq 1 - \delta\}} \cdot \bar{S}$ satisfies NUPBR($\mathbb{F}$). Then, there exists a pair (β, f) satisfying $f > 0$,

$$\beta^{tr} c \beta \cdot A \in \mathcal{A}_{loc}^+(\mathbb{F}), \quad \sqrt{(f - 1)^2 \star \mu} \in \mathcal{A}_{loc}^+(\mathbb{F}),$$

and

$$\int |xf(x) - h(x)| F(dx) < +\infty \quad P \otimes A \text{ on } \{1 - Z_- \geq \delta\}, \quad (4.13)$$

and $P \otimes A$ -a.e. on $\{1 - Z_- \geq \delta\}$ (recall that $\psi = 1 - Z_- - f_m$)

$$b + c\beta + \int [xf(x) I_{\{\psi > 0\}} - h(x)] F(dx) = 0 \quad (4.14)$$

Now we start constructing a σ -martingale density for $I_{\{Z_- \leq 1 - \delta\}} \cdot (S - S^\tau)$ as follows. Consider

$$\beta^{\mathbb{G}} := \left(\beta + \frac{\beta_m}{1 - Z_-} \right) I_{\llbracket \tau, +\infty \rrbracket}, \quad f^{\mathbb{G}} := \frac{f}{1 - f_m(x)/(1 - Z_-)} I_{\llbracket \tau, +\infty \rrbracket} + I_{\llbracket 0, \tau \rrbracket},$$

and assume for a while that

$$\beta^{\mathbb{G}} \in L(\widehat{S}^c, \mathbb{G}) \quad \text{and} \quad (f^{\mathbb{G}} - 1) \in \mathcal{G}_{loc}^1(\mu^{\mathbb{G}}, \mathbb{G}). \quad (4.15)$$

Then, using Itô and (4.13)–(4.14) afterwards, we can easily prove that the process $(\phi^{\mathbb{G}} \bullet (S - S^\tau)) \mathcal{E}(N^{\mathbb{G}})$ is a $\mathbb{G}$ -local martingale, where

$$\phi^{\mathbb{G}} := \left(1 + \int |xf(x)I_{\{Z_- + f_m < 1\}} - h(x)|F(dx)I_{\llbracket \tau, +\infty \rrbracket} \right)^{-1}.$$

Thus assertion (a) follows immediately once (4.15) is proved. This will be the main goal of the rest of the proof of this part.

Since $\beta^T c \beta \cdot A$ and $\beta_m^T c \beta_m \cdot A$ belong to $\mathcal{A}_{loc}^+(\mathbb{F}) \subset \mathcal{A}_{loc}^+(\mathbb{G})$ and $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$ is $\mathbb{G}$ -locally bounded, we deduce that $\beta^{\mathbb{G}} \in L(\widehat{S}^c, \mathbb{G})$. To prove the second property in (4.15), we start by calculating on $\llbracket \tau, +\infty \rrbracket$,

$$f^{\mathbb{G}} - 1 = \frac{f - 1}{1 - f_m/(1 - Z_-)} + \frac{f_m}{1 - Z_- - f_m} =: g_1 + g_2.$$

Since $\sqrt{(f - 1)^2 \star \mu} \in \mathcal{A}_{loc}^+(\mathbb{F})$, we deduce that —due to Proposition B.1–(a)— that

$$[(f - 1)^2 I_{\{|f-1| \leq \alpha\}} + |f - 1| I_{\{|f-1| > \alpha\}}] \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F}).$$

Without loss of generality we assume that this process and $f_m^2 \star \mu$ are integrable. Thanks to Lemma 3.1–(a), there exists a sequence of $\mathbb{G}$ -stopping times $(\tau_n)_{n \geq 1}$ that increase to infinity and $Z^{\tau_n} \leq 1 - 1/n$ on $\llbracket \tau, +\infty \rrbracket$. Then, setting $\Gamma_n(\alpha) := \{|f - 1| \leq \alpha \text{ \& } 1 - Z_- - f_m \geq 1/(2n)\}$, we calculate

$$\begin{aligned} E[g_1^2 I_{\Gamma_n(\alpha)} \star \mu_{\infty}^{\mathbb{G}}] &= E[g_1^2 I_{\Gamma_n(\alpha)} \star \nu_{\infty}^{\mathbb{G}}] \\ &\leq 2n E[(f - 1)^2 I_{\{|f-1| \leq \alpha\}} \star \nu_{\infty}] < +\infty, \end{aligned}$$

and by setting $\tilde{\Omega} := \Omega \times [0, +\infty)$ and using $\{1 - Z_- - f_m < \frac{1}{2n}\} \cap \llbracket 0, \tau_n \rrbracket \subset \{1 - Z_- - f_m < \frac{1}{2n}\} \cap \{Z_- \leq 1 - 1/n\} \subset \{f_m > \frac{1}{2n}\} \subset \{|f_m| > \frac{1}{2n}\}$, we get

$$\begin{aligned} E\left[\sqrt{g_1^2 I_{\{|f-1| \leq \alpha\}} I_{\tilde{\Omega} \setminus \Gamma_n(\alpha)} \star \mu_{\tau_n}^{\mathbb{G}}}\right] &\leq E\left[|g_1| I_{\{|f-1| \leq \alpha\}} I_{\tilde{\Omega} \setminus \Gamma_n(\alpha)} \star \nu_{\tau_n}^{\mathbb{G}}\right], \\ &\leq \alpha E\left[I_{\{|f_m| > \frac{1}{2n}\}} \star \nu_{\infty}\right] \leq 4n^2 \alpha E[f_m^2 \star \nu_{\infty}] < +\infty. \end{aligned}$$

Also we calculate

$$\begin{aligned} E\sqrt{g_1^2 I_{\{|f-1| > \alpha\}} \star \mu_{\infty}^{\mathbb{G}}} &\leq E[|g_1| I_{\{|f-1| > \alpha\}} \star \nu_{\infty}^{\mathbb{G}}] \\ &\leq E[|f - 1| I_{\{|f-1| > \alpha\}} \star \nu_{\infty}] < +\infty. \end{aligned}$$

Therefore, $\sqrt{g_1^2 \star \mu^{\mathbb{G}}} \in \mathcal{A}_{loc}^+(\mathbb{G})$. Similarly, the fact that $\sqrt{g_2^2 \star \mu^{\mathbb{G}}} \in \mathcal{A}_{loc}^+(\mathbb{G})$ follows from

$$\begin{aligned} E[g_2^2 I_{\{1 - Z_- - f_m \geq \frac{1}{2n}\}} \star \mu_{\tau_n}^{\mathbb{G}}] &= E[g_2^2 I_{\{1 - Z_- - f_m \geq \frac{1}{2n}\}} \star \nu_{\tau_n}^{\mathbb{G}}] \\ &\leq 2n^2 E[(f_m)^2 \star \nu_{\infty}] < +\infty \end{aligned}$$

and

$$\begin{aligned} E\sqrt{g_2^2 I_{\{1-Z_- - f_m < \frac{1}{2n}\}} \star \mu_{\tau_n}^{\mathbb{G}}} &\leq E\left[g_2 I_{\{1-Z_- - f_m < \frac{1}{2n}\}} \star \mu_{\tau_n}^{\mathbb{G}}\right] \\ &\leq nE\left[|f_m| I_{\{|f_m| > \frac{1}{2n}\}} \star \nu_{\infty}\right] \leq 2n^2 E[f_m^2 \star \nu_{\infty}] < +\infty. \end{aligned}$$

This ends the proof of this part, and the proof of the theorem is completed. $\square$

4.3 Proof of Theorem 2.17

In virtue of (4.5) and $\llbracket \tau, +\infty \rrbracket \subset \{Z_- < 1\}$, the assumption (2.12) holds iff

$$\begin{aligned} 0 &= E(1 - Z_-)^{-1} I_{\{Z_- + f_m = 1 > Z_-\}} I_{\llbracket \tau, +\infty \rrbracket} \star \nu_{\infty} \\ &= EI_{\{Z_- + f_m = 1 > Z_-\}} \star \nu(\infty) = EI_{\{Z_- + f_m = 1 > Z_-\}} \star \mu_{\infty}. \end{aligned}$$

This implies that $|f_m| I_{\{Z_- + f_m = 1 > Z_-\}} \star \nu$ and $|f_m| I_{\{Z_- + f_m = 1 > Z_-\}} \star \mu$ are null. Thus, we deduce that $m^{(1)} = f_m I_{\{Z_- + f_m = 1 > Z_-\}} \star \mu - f_m I_{\{Z_- + f_m = 1 > Z_-\}} \star \nu$ is also null, and the proof of the theorem follows from Corollary 2.15-(ii). $\square$

4.4 Proof of Theorem 3.6

Put

$$g^{(1)} := \frac{1 - \psi}{1 - f_m(1 - Z_-)^{-1}} I_{\{\psi > 0\}} I_{\llbracket \tau, +\infty \rrbracket}, \quad \psi := M_{\mu}^P(I_{\{\tilde{Z}=1\}} | \tilde{\mathcal{P}}(\mathbb{F})). \quad (4.16)$$

Then, if

$$g^{(1)} \in \mathcal{G}_{loc}^1(\mu^{\mathbb{G}}, \mathbb{G}) \quad \text{and} \quad |x|g^{(1)} \star \mu^{\mathbb{G}} \in \mathcal{A}_{loc}^+(\mathbb{G}), \quad (4.17)$$

then $L^{(1)} := g^{(1)} \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}})$ is a well defined $\mathbb{G}$ -local martingale satisfying the properties of Theorem 3.6. Indeed, due to the quasi-left-continuous of S , we get $\Delta L^{(1)} = g^{(1)}(\Delta S) I_{\{\Delta S \neq 0\}} \geq 0$, and the second property of (4.17) is equivalent to $[S, L^{(1)}] = xg^{(1)} \star \mu^{\mathbb{G}} \in \mathcal{A}_{loc}(\mathbb{G})$. Now, we calculate $\langle S - S^{\tau}, L^{(1)} \rangle^{\mathbb{G}}$:

$$\langle S - S^{\tau}, L^{(1)} \rangle^{\mathbb{G}} = \left(xg^{(1)} \star \mu^{\mathbb{G}} \right)^{p, \mathbb{G}} = xg^{(1)} \star \nu^{\mathbb{G}} = x(1 - \phi) I_{\{\psi > 0\}} I_{\llbracket \tau, +\infty \rrbracket} \star \nu$$

Thus, a combination of this with (3.13) and Lemma 3.2-(b), we obtain

$$\begin{aligned} S - S^{\tau} + \langle S - S^{\tau}, L^{\mathbb{G}} + L^{(1)} \rangle^{\mathbb{G}} \\ = \widehat{S} - I_{\llbracket \tau, +\infty \rrbracket} \cdot \left(I_{\{\tilde{Z}=1\}} \cdot [m, S] \right)^{p, \mathbb{F}} + x(1 - \phi) I_{\{\psi > 0\}} I_{\llbracket \tau, +\infty \rrbracket} \star \nu \\ = \widehat{S} - x(1 - \psi) I_{\{\psi=0\}} I_{\llbracket \tau, +\infty \rrbracket} \star \nu = \widehat{S} \in \mathcal{M}_{loc}(\mathbb{G}). \end{aligned}$$

The second equality follows from $\left(I_{\{\tilde{Z}=1\}} \cdot [m, S] \right)^{p, \mathbb{F}} = (1 - Z_-)x(1 - \psi) \star \nu$, while the last equality follows from $xI_{\{\psi=0\}} \star \nu \equiv 0$ (which comes from the fact that S is orthogonal to $m^{(1)}$ or equivalently $\langle S, m^{(1)} \rangle^{\mathbb{F}} = xI_{\{\psi=0\}} \star \nu = 0$).

Thus, the proof of theorem will achieved as long as we prove (4.17). This is the focus of the remaining part of the proof. By stopping, one can assume, without loss of generality, that $[m, S] \in \mathcal{A}$ and calculate

$$E(|x|g^{(1)} \star \mu_\infty^\mathbb{G}) \leq E\{(1 - Z_-)|x|(1 - \psi) \star \nu_\infty\} \leq E\{Var([S, m])_\infty\} < +\infty.$$

This proves the second property of (4.17). To prove the first property, one recall the $\mathbb{G}$ -local boundedness of $(1 - Z_-)^{-1}I_{\llbracket \tau, +\infty \rrbracket}$ (see Lemma 2.6-(b)) and that of $1 - Z_-$, and conclude $g^{(1)} \star \mu^\mathbb{G} \in \mathcal{A}_{loc}^+(\mathbb{G})$ if and only if $(1 - Z_-)g^{(1)} \star \mu^\mathbb{G} \in \mathcal{A}_{loc}^+(\mathbb{G})$. By assuming $[m, m] \in \mathcal{A}^+(\mathbb{F})$ without loss of generality, we get

$$E\left[(1 - Z_-)g^{(1)} \star \mu_\infty^\mathbb{G}\right] \leq E\left[(1 - Z_-)^2 I_{\{\tilde{Z}=1\}} \star \mu_\infty\right] \leq E[m, m]_\infty < +\infty.$$

This proves that $g^{(1)} \star \mu^\mathbb{G} \in \mathcal{A}_{loc}^+(\mathbb{G})$ which obviously implies that $g^{(1)} \in \mathcal{G}_{loc}^1(\mu^\mathbb{G}, \mathbb{G})$ and the proof of the theorem is completed. $\square$

APPENDIX

A Some Useful Technical Results

Proposition A.1 *Let X be an $\mathbb{H}$ -adapted process. Then, the following assertions are equivalent.*

- (a) *There exists a sequence $(T_n)_{n \geq 1}$ of $\mathbb{H}$ -stopping times that increases to $+\infty$, such that for each $n \geq 1$, there exists a probability Q_n on $(\Omega, \mathbb{H}_{T_n})$ such that $Q_n \sim P$ and X^{T_n} satisfies NUPBR($\mathbb{H}$) under Q_n .*
- (b) *X satisfies NUPBR($\mathbb{H}$).*
- (c) *There exists an $\mathbb{H}$ -predictable process ϕ , such that $0 < \phi \leq 1$ and $(\phi \cdot X)$ satisfies NUPBR($\mathbb{H}$).*

The proof of this proposition can be found in Aksamit et al. [1].

Proposition A.2 *Let $H^\mathbb{G}$ be an $\tilde{\mathcal{P}}(\mathbb{G})$ -measurable functional. Then, the following assertions hold.*

- (a) *There exist two $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional $H^\mathbb{F}$ and $K^\mathbb{F}$ such that*

$$H^\mathbb{G}(\omega, t, x) = H^\mathbb{F}(\omega, t, x)I_{\llbracket 0, \tau \rrbracket} + K^\mathbb{F}(\omega, t, x)I_{\llbracket \tau, +\infty \rrbracket}. \quad (\text{A.1})$$

- (b) *If furthermore $H^\mathbb{G} > 0$ (respectively $H^\mathbb{G} \leq 1$), then we can choose $K^\mathbb{F} > 0$ (respectively $K^\mathbb{F} \leq 1$) in (A.1).*
- (c) *If τ is a finite almost surely honest time and $(\sigma_n^\mathbb{G})_{n \geq 1}$ is a sequence of finite $\mathbb{G}$ -stopping times that increases to infinity, then there exists a sequence of finite $\mathbb{F}$ -stopping times, $(\sigma_n^\mathbb{F})_{n \geq 1}$, that increases to infinity as well and*

$$\max(\sigma_n^\mathbb{G}, \tau) = \max(\sigma_n^\mathbb{F}, \tau), \quad P - a.s. \quad (\text{A.2})$$

- (d) *If $\tau \in \mathcal{H}$ and is finite almost surely, then there exists a sequence of $\mathbb{F}$ -stopping times, $(\sigma_n)_{n \geq 1}$, that increases to infinity almost surely and*

$$\left\{Z_- < 1\right\} \cap \llbracket 0, \sigma_n \rrbracket \subset \left\{1 - Z_- \geq \frac{1}{n}\right\}, \quad \forall n \geq 1. \quad (\text{A.3})$$

Proof The proofs of assertions (a) and (b) follow from mimicking Jeulin's proof [17, Proposition 5.3], and will be omitted herein. The remaining proof contains two parts where we prove assertions (c) and (d) respectively.

1) The proof of assertion (c) relies essentially on the following fact

for any $\mathbb{G}$ -topping time, $\sigma^{\mathbb{G}}$, there exists an $\mathbb{F}$ -stopping time, $\sigma^{\mathbb{F}}$ such that

$$\sigma^{\mathbb{G}} \vee \tau = \sigma^{\mathbb{F}} \vee \tau \quad P - a.s. \quad (\text{A.4})$$

Indeed, if this fact is true, then there exists $\mathbb{F}$ -stopping times, $(\sigma_n)_{n \geq 1}$ such that for any $n \geq 1$, the pair $(\sigma_n^{\mathbb{G}}, \sigma_n)$ satisfies (A.2). Since $\sigma_n^{\mathbb{G}}$ increases with n , by putting $\sigma_n^{\mathbb{F}} := \sup_{1 \leq k \leq n} \tau_k$, we can easily prove that the pair $(\sigma_n^{\mathbb{F}}, \sigma_n^{\mathbb{F}})$ satisfies (A.2) as well. Then, assertion (c) follows immediately from taking the limit in (A.2) and making use of $\tau < +\infty$ P-a.s. which implies that $\sup_{n \geq 1} \tau_n = \lim_{n \rightarrow +\infty} \sigma_n^{\mathbb{F}} = +\infty$ P-a.s. This shows that the proof of assertion (c) is achieved as long as we prove the claim (A.4). This is the main focus of the remaining part of this proof.

By applying the proposition below (which is fully due to Barlow [6]) to the process $Y^{\mathbb{G}} = I_{[\sigma^{\mathbb{G}} \vee \tau, +\infty[}$, we obtain the existence of an $\mathbb{F}$ -progressively measurable process $K^{\mathbb{F}}$ such that

$$Y^{\mathbb{G}} = K^{\mathbb{F}} I_{[\tau, +\infty[}.$$

Then, put

$$\sigma := \inf\{t \geq 0 : K_t^{\mathbb{F}} = 1\}. \quad (\text{A.5})$$

This is an $\mathbb{F}$ -stopping time, and due to $[\sigma^{\mathbb{G}} \vee \tau, +\infty[\subset \{K^{\mathbb{F}} = 1\}$, we get

$$\sigma \leq \tau \vee \sigma^{\mathbb{G}} \quad P - a.s. \quad (\text{A.6})$$

By applying Proposition A.3-(iii) to

$$Y^{nm} := I_{[\sigma^{nm}, +\infty[} := I_{\{\tau < \alpha_{nm} \leq \sigma^{\mathbb{G}} < \beta_{nm}\}} I_{[\sigma^{\mathbb{G}}, +\infty[},$$

we deduce the existence of right continuous $\mathbb{F}$ -optional process K^{nm} that vanishes on $[0, \alpha_{n,m}]$ and satisfies $Y^{nm} = K^{nm} I_{[\tau, +\infty[}$. Thus, again consider

$$\tau^{nm} := \inf\{t \geq 0 : K_t^{nm} = 1\}.$$

Due, to the right continuity of K^{nm} and the fact that it vanishes on $[0, \alpha_{n,m}]$, on the one hand we deduce that

$$\tau^{nm} = \sigma^{\mathbb{G}} \quad \text{on } \{\tau < \alpha_{nm} \leq \sigma^{\mathbb{G}} < \beta_{nm}\}. \quad (\text{A.7})$$

On the other hand, it is easy to see that on $\{\tau < \alpha_{nm} \leq \sigma^{\mathbb{G}} < \beta_{nm}\}$, we have

$$Y^{nm} = Y^{\mathbb{G}}, \quad K^{nm} = K^{\mathbb{F}}, \quad \sigma^{nm} = \sigma^{\mathbb{G}}, \quad \tau^{nm} = \sigma.$$

Thus, by combining this with (A.7), we conclude that

$$\{\tau < \alpha_{nm} \leq \sigma^{\mathbb{G}} < \beta_{nm}\} \subset \{\sigma^{\mathbb{G}} = \sigma\} \quad P - a.s. \quad (\text{A.8})$$

Thanks to (A.10), we have

$$\{\tau < \sigma^{\mathbb{G}}\} = \bigcup_{n,m \geq 1} \{\tau < \alpha_{nm} \leq \sigma^{\mathbb{G}} \beta_{nm}\} \subset \{\sigma = \sigma^{\mathbb{G}}\} \quad P - a.s.$$

Thus, a combination of this with (A.6), the proof of (A.4) follows immediately, and that of assertion (c) as well.

2) Here, we prove assertion (d). Since $\tau \in \mathcal{H}$, then $(1 - Z_-)^{-1} I_{\llbracket \tau, +\infty \rrbracket}$ is locally bounded due to Lemma 2.6-(b). Thus, on the one hand, there exists a sequence of $\mathbb{G}$ -stopping times, $(\sigma_n^{\mathbb{G}})_{n \geq 1}$ that increases to infinity almost surely and

$$\llbracket \tau, +\infty \rrbracket \cap \llbracket 0, \sigma_n^{\mathbb{G}} \rrbracket \subset \left\{ 1 - Z_- \geq 1/n \right\}. \quad (\text{A.9})$$

On the other hand, thanks to assertion (c), there exists a sequence of $\mathbb{F}$ -stopping times, $(\sigma_n)_{n \geq 1}$ that increases to infinity almost surely and satisfies (A.2). Then, by inserting this in (A.9), we get

$$\llbracket \tau, +\infty \rrbracket \cap \llbracket 0, \sigma_n \rrbracket \subset \left\{ 1 - Z_- \geq 1/n \right\}.$$

By taking the $\mathbb{F}$ -predictable projection on both side in the above inclusion, we get

$$(1 - Z_-) I_{\llbracket 0, \sigma_n \rrbracket} \subset I_{\left\{ 1 - Z_- \geq 1/n \right\}}.$$

Then, it is easy to see that this implies (A.3), and the proof of assertion (c) is achieved and that of the proposition as well. $\square$

Proposition A.3 *Suppose that τ is a honest time. Then, the following assertions hold.*

(i) *There exists two double sequences of $\mathbb{F}$ -stopping times $(\alpha_{n,m})_{n,m \geq 1}$ and $(\beta_{n,m})_{n,m \geq 1}$ such that $\alpha_{n,m} \leq \beta_{n,m}$ P -a.s. for all $n, m \geq 1$, and*

$$\llbracket \tau, +\infty \rrbracket \subset \bigcup_{n,m \geq 1} \llbracket \alpha_{n,m}, \beta_{n,m} \rrbracket. \quad (\text{A.10})$$

(ii) *For any $\mathbb{G}$ -optional process $Y^{\mathbb{G}}$, there exists an $\mathbb{F}$ -progressively measurable process $K^{\mathbb{F}}$ such that*

$$Y^{\mathbb{G}} I_{\llbracket \tau, +\infty \rrbracket} = K^{\mathbb{F}} I_{\llbracket \tau, +\infty \rrbracket}. \quad (\text{A.11})$$

(iii) *For any $\mathbb{G}$ -optional càdlàg process $Y^{\mathbb{G}}$ such that $Y^{\mathbb{G}} = 0$ on $\llbracket 0, \alpha_{n,m} \rrbracket$ and constant on $\llbracket \beta_{n,m}, +\infty \rrbracket$, there exists an $\mathbb{F}$ -progressively measurable process $K^{\mathbb{F}}$ that is càdlàg and (A.11) holds.*

Proof For the proof we refer the reader to [6]. In fact, assertion (i) is exactly Lemma 4.1-(iv) in [6], while the assertion (ii) is a combination of Proposition 4.3 and Lemma 4.4-(ii) of the same paper.

B $\mathbb{G}$ -local Integrability Involving Random Measure

This subsection connects the $\mathbb{G}$ -localisation and the $\mathbb{F}$ -localisation for the part after τ . This completes the analysis of [1], where the part up to τ is fully discussed. There is a major difference between the current results and those of [1], which lies in the fact that for the case up to τ we loose information after an $\mathbb{F}$ -stopping when we pass to $\mathbb{F}$. However, for the part after τ , as long as τ is finite, we pass from $\mathbb{G}$ -localisation to $\mathbb{F}$ -localisation without any loss of information.

Proposition B.1 *Suppose that $\tau \in \mathcal{H}$ is finite. Then, the following properties hold.*

(a) *Let $\alpha > 0$ and f be a $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional. Then, $\sqrt{(f-1)^2 \star \mu}$ belongs to $\mathcal{A}_{loc}^+(\mathbb{H})$ if and only if*

$$[(f-1)^2 I_{\{|f-1| \leq \alpha\}} + |f-1| I_{\{|f-1| > \alpha\}}] \star \mu \in \mathcal{A}_{loc}^+(\mathbb{H}).$$

(b) *Let $\Phi^{\mathbb{G}}$ a $\mathbb{G}$ -predictable process and k a nonnegative and $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional such that $0 < \Phi^{\mathbb{G}} \leq 1$ and $\Phi^{\mathbb{G}} k \star \mu_{\mathbb{G}}$ belongs to $\mathcal{A}_{loc}^+(\mathbb{G})$. Then, $P \otimes A$ -a.e.*

$$\int k(x) (1 - Z_- - f_m(x)) F(dx) < +\infty \quad \text{on} \quad \{Z_- < 1\}. \quad (\text{B.1})$$

(c) *Let f be a $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable and positive functional, and $\bar{\mu} := (1 - \tilde{Z}) \cdot \mu$. Then $\sqrt{(f-1)^2 I_{\tau, +\infty}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$ if and only if $\sqrt{(f-1)^2 I_{\{1-Z_- \geq \delta\}}} \star \bar{\mu} \in \mathcal{A}_{loc}^+(\mathbb{F})$ for any $\delta > 0$.*

Proof (a) Assertion (a) is borrowed from [1] (see Proposition C.3-(a)).

(b) This assertion follows directly from a combination of Lemmas ?? and A.2.

(c) Thanks to assertion (a), we have $\sqrt{(f-1)^2 I_{\tau, +\infty}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$ if and only if

$$W^{\mathbb{G}} := [(f-1)^2 I_{\{|f-1| \leq \alpha\}} + |f-1| I_{\{|f-1| > \alpha\}}] \star \mu^{\mathbb{G}} \in \mathcal{A}_{loc}^+(\mathbb{G}).$$

This is equivalent to $[(f-1)^2 I_{\{|f-1| \leq \alpha\}} + |f-1| I_{\{|f-1| > \alpha\}}] \star \nu^{\mathbb{G}} \in \mathcal{A}_{loc}^+(\mathbb{G})$, which, in turn, it is equivalent to

$$\varphi I_{\tau, +\infty} \star A \in \mathcal{A}_{loc}^+(\mathbb{G}),$$

$$\varphi := \int [(f-1)^2 I_{\{|f-1| \leq \alpha\}} + |f-1| I_{\{|f-1| > \alpha\}}] (1 - Z_- - f_m) F(dx).$$

Then, due to Lemma 3.2-(b), we deduce that for any $\delta > 0$, $\varphi I_{\{1-Z_- \geq \delta\}} \star A \in \mathcal{A}_{loc}^+(\mathbb{F})$. Then, again thanks to assertion (a), we conclude that

$$\sqrt{(f-1) I_{\{1-Z_- \geq \delta\}}} \star \bar{\mu} \in \mathcal{A}_{loc}^+(\mathbb{F}), \quad \text{where } \bar{\mu} := (1 - \tilde{Z}) \cdot \mu.$$

This proves assertion (c), and the proof of the proposition is completed. $\square$

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