Effective junction conditions for degenerate parabolic equations

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CYRIL IMBERT AND VINH DUC NGUYEN

Abstract. We are interested in the study of parabolic equations on *a multi-dimensional junction*, i.e. the union of a finite number of copies of a half-hyperplane of dimension $d+1$ whose boundaries are identified. The common boundary is referred to as the *junction hyperplane*. The parabolic equations on the half-hyperplanes are in non-divergence form, fully non-linear and possibly degenerate, and they do degenerate and are quasi-convex along the junction hyperplane. More precisely, along the junction hyperplane the non-linearities do not depend on second order derivatives and their sublevel sets with respect to the gradient variable are convex. The parabolic equations are supplemented with a non-linear boundary condition of Neumann type, referred to as a *generalized junction condition*, which is compatible with the maximum principle. Our main result asserts that imposing a generalized junction condition in a weak sense reduces to imposing an *effective one* in a strong sense. This result extends the one obtained by Imbert and Monneau for Hamilton-Jacobi equations on networks and multi-dimensional junctions. We give two applications of this result. On the one hand, we give the first complete answer to an open question about these equations: we prove in the two-domain case that the vanishing viscosity limit associated with quasi-convex Hamilton-Jacobi equations coincides with the maximal Ishii solution identified by Barles, Briani and Chasseigne (2012). On the other hand, we give a short and simple PDE proof of a large deviation result of Boué, Dupuis and Ellis (2000).

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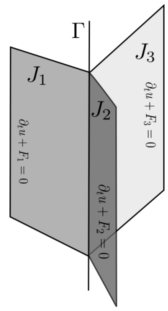

FIGURE 1. A parabolic equation posed on a multi-dimensional junction. Here there are 3 branches ($N = 3$) and the tangential dimension is 1 ($d = 1$). We did not illustrate the junction condition $L = 0$ on the junction hyperplane Γ (which is a line in this example).

1. INTRODUCTION

1.1. **Degenerate parabolic equations on junctions.** *Multi-dimensional junctions* [15, 17, 25] are union of half-spaces whose boundaries are identified – see Figure 1. Precisely:

$$J = \bigcup_{i=1}^{N} J_i \quad \text{with} \quad \begin{cases} J_i = \{x = (x', x_i) : x' \in \mathbb{R}^d, x_i \geq 0\} \simeq \mathbb{R}^{d+1}_+ \\ J_i \cap J_j = \Gamma \simeq \mathbb{R}^d \quad \text{for} \quad i \neq j. \end{cases}$$

Given $T \in [0, +\infty]$, we consider a general degenerate parabolic equation posed on a junction,

$$\begin{cases} u_t + F_i(t, x, Du, D^2u) = 0 & (t, x) \in (0, T) \times J_i^*, i = 1, \dots, N, \\ L(-u_t, \partial_1 u, \dots, \partial_N u, t, x', D'u) = 0 & (t, x) \in (0, T) \times \Gamma \end{cases} \tag{1.1}$$

where J_i^* denotes $J_i \setminus \Gamma$, u_t denotes the time derivative, Du and D^2u respectively denote the gradient and the Hessian of u with respect to x, and for $x' \in \Gamma$, $\partial_i u(x')$ denotes the derivative of $u_i(x) = u|_{J_i}(x)$ with respect to x_i at $x_i = 0$ (recall $x = (x', x_i)$) and $D'u$ denotes the derivative with respect to x'.

Example 1.1. The case $N = 1$ corresponds to the study of a degenerate parabolic equation posed on a half-space, subject to a non-linear boundary condition (dynamic or not). Example 1.11 illustrates how the main theorem can be applied in this special case. The case $N = 2$ corresponds to the *two-domain case*: a degenerate parabolic equation has coefficients which are continuous on either part of a hyperplane (or a smooth interface); the generalized junction condition can be thought as a transmission condition. Theorem 1.12 is an application of the main theorem with $N = 2$.

We make the following assumptions on each F_i.

Assumption (F).

(F1). The function F_i is continuous and degenerate elliptic.

(F2). For all $R > 0$, there exists $C_{i,R} > 0$ such that for all $y = (y', y_i)$, all $p \in \mathbb{R}^{d+1}$, all $B \in \mathbb{S}_{d+1}(\mathbb{R})$ and all $\lambda > 0$

$$\left. \begin{array}{r} s \in (0,T), |y_i| \le 1 \\ |y'| + |B| \le R \end{array} \right\} \Rightarrow F_i(s, y, p, B + \lambda e_{d+1} \otimes e_{d+1}) \ge F_i(s, y, p, B) - C_{i,R} \lambda |y_i|^2.$$

(F3). For all $R > 0$,

$$\lim_{|p| \to +\infty} \inf_{t \in (0,T), |x|+|B| \le R} F_i(t, x, p, B) = +\infty.$$

(F4). There exists $H_i : (0,T) \times \Gamma \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ continuous such that
- for all $(t, x', p', p_i, B) \in (0,T) \times \Gamma \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{S}_{d+1}(\mathbb{R})$, $F_i(t, (x', 0), (p', p_i), B) = H_i(t, x', p', p_i)$;
- for all $t \in (0,T)$, $x' \in \Gamma$, for all $\lambda \in \mathbb{R}$, the set $\{p = (p', p_i) \in \mathbb{R}^{d+1} : H_i(t, x', p) \le \lambda\}$ is convex.

In the assumption above, $\mathbb{S}_{d+1}(\mathbb{R})$ denotes the set of real-valued $(d+1) \times (d+1)$ symmetric matrices and e_{d+1} denotes the unit vector orthogonal to Γ and pointing inside J_i. We recall that $F_i(t, x, p, A)$ is degenerate elliptic if it is non-increasing with respect to A (using the classical partial order on $\mathbb{S}_{d+1}(\mathbb{R})$). The function H_i appearing in (F) is referred to as the *Hamiltonian* from the branch J_i.

Example 1.2 (First order case)**.** The first example we give is the one coming from [18, 17]. It reduces to deal with $F_i(t, x, p, B) = H_i(t, x, p)$ for any $x \in J_i$ (and not only $x = (x', 0) \in \Gamma$) and $p \in \mathbb{R}^d$ with H_i continuous, coercive in p uniformly in x, i.e. satisfying

$$\lim_{|p| \to +\infty} \inf_{(t,x) \in (0,T) \times J} H_i(t, x, p) = +\infty$$

and quasi-convex in p, i.e. the sublevel sets $\{p \in \mathbb{R}^{d+1} : H_i(t, x, p) \le \lambda\}$ are convex for all $\lambda \in \mathbb{R}$ and $(t, x) \in (0,T) \times \Gamma$.

Example 1.3 (The model case)**.** Our results apply to the model case where $F_i(t, x, p, B) = H_i(t, x, p) - \mathrm{Trace}(\sigma_i(x) \sigma_i^T(x) B)$ with H_i is as in Example 1.2 and where the $(d+1) \times m$ real matrix σ_i is such that $\sigma_i \equiv 0$ on Γ and the $(d+1)$-th line σ_i^{d+1} of σ_i satisfies $|\sigma_i^{d+1}(y)| \le c_i |y_{d+1}|$. Remark that this latter condition holds true if $\sigma_i \equiv 0$ on Γ and σ_i is Lipschitz continuous.

As far as the junction function L is concerned, we make the following assumption.

Assumption (L).

(L1). The function L is continuous.
(L2). The function $L(p_0, \dots, p_N, t, x', p')$ is non-increasing in p_i for $i = 0, \dots, N$.
(L3). $\forall i, p_i < q_i \Rightarrow L(p_0, \dots, p_N, t, x', p') > L(q_0, \dots, q_N, t, x', p')$.
(L4). $\inf_{t,x',p'} L(p_0, \dots, p_N, t, x', p') \to +\infty$ as $\min_{i=1,\dots,N} p_i \to -\infty$.
(L5). $\sup_{t,x',p'} L(p_0, \dots, p_N, t, x', p') \to -\infty$ as $\max_{i=0,\dots,N} p_i \to +\infty$.

Example 1.4 (Kirchoff conditions)**.** A model for L is

$$L(p_0, \dots, p_N) = -\sum_{i=1}^{N} \beta_i p_i$$

with $\beta_i > 0$ for all i. Such a condition is called a Kirchoff condition.

Example 1.5 (Flux-limited junction conditions)**.** A second important example of junction functions L is the one related to flux-limited solutions [18, 17]. Given a *flux limiter* A,

$$\begin{cases} A : (0,T) \times \Gamma \times \mathbb{R}^d \to \mathbb{R} \text{ continuous} \\ \text{for all } (t,x') \in (0,T) \times \Gamma, \lambda \in \mathbb{R}, \{p' \in \mathbb{R}^d : A(t,x',p') \le \lambda\} \text{ convex} \end{cases}$$

we consider the associated junction function L_A defined by

$$L_A(p_0, \dots, p_N, t, x', p') = -p_0 + \max(A(t,x',p'), \max_i H_i^-(t,x',p',p_i)) \tag{1.2}$$

where $H_i^-(t,x,p',p_i)$ denotes the non-increasing part of $p_i \mapsto H_i(t,x',p',p_i)$ [17]: if $p_i \mapsto H_i(t,x',p',p_i)$ reaches its minimum at $\pi_i^0(t,x',p')$, which is the minimal minimizer, then

$$H_i^-(t,x',p',p_i) = \begin{cases} H_i(t,x',p',p_i) & \text{if } p_i \le \pi_i^0(t,x',p') \\ H_i(t,x',p',\pi_i^0(t,x',p')) & \text{if } p_i \ge \pi_i^0(t,x',p'). \end{cases}$$

Remark 1.6. The flux-limited function F_A defined in [18, 17] corresponds to

$$\begin{aligned} F_A(p_1, \dots, p_N, t, x', p') &= \max(A(t,x',p'), \max_i H_i^-(t,x',p',p_i)) \\ &= L_A(p_0, p_1, \dots, p_N, t, x', p') + p_0. \end{aligned}$$

The appropriate notion of weak solutions for Hamilton-Jacobi equations is the one of viscosity solutions, introduced by Crandall and Lions [13] – see also [11]. It is explained in [18, 17] that two notions of viscosity solutions are needed in the study of Hamilton-Jacobi equations on networks, depending on the type of junction conditions we impose. We will see that it is also the case for the degenerate parabolic equations we consider in this work. For general junction functions L in (1.1), the junction condition has to be understood in the following weak sense: either the junction condition $L = 0$ or one of the equations $u_t + F_i = 0$ is satisfied. We refer to such viscosity solutions as *relaxed solutions* – see Definition 2.2 below. But for the special junction conditions L_A given by (1.2), relaxed solutions satisfy the junction condition in a stronger sense: the junction condition $L_A = 0$ is indeed satisfied (Proposition 2.10). Such viscosity solutions are referred to as *flux-limited solutions* – see Definition 2.8 below.

1.2. **Main result.** The main result of this article is about *equivalent classes of generalized junction conditions*. Roughly speaking, we prove that imposing a general junction condition amounts to imposing an effective one. This effective junction condition corresponds to some L_A given in (1.2) for some flux limiter $A = A_L$. This flux limiter only depends on the junction function L and the Hamiltonians H_i. Moreover, the *effective junction condition* L_{A_L} is satisfied in a strong sense: if the relaxed solution u is continuously differentiable in time and space up to the junction hyperplane Γ, then the boundary condition $L = 0$ on Γ can be lost (see the discussion above and Definition 2.2) but $L_{A_L} = 0$ on Γ is indeed satisfied in the classical sense.

Definition 1.7 (The effective flux limiter A_L)**.** Let

$$A_0(t, x', p') = \max_{i=1,\dots,N} \min_{p_i \in \mathbb{R}} H_i(t, x', p', p_i) \tag{1.3}$$

and $p_i^0 \geq \pi_i^0(t, x', p')$ be the minimal p_i such that $H_i(t, x', p', p_i) = A_0(t, x', p')$. For all (t, x', p'), the *effective flux limiter* $A_L(t, x', p')$ is defined as follows: if

$$L(A_0(t, x', p'), p_1^0, \dots, p_N^0, t, x', p') \leq 0,$$

then $A_L(t, x', p') = A_0(t, x', p')$, else $A_L(t, x', p')$ is the only real number $\lambda \geq A_0(t, x', p')$ such that there exists $p_i^+ \geq p_i^0$ with

$$H_i(t, x', p', p_i^+) = \lambda \quad \text{and} \quad L(\lambda, p_1^+, \dots, p_N^+, t, x', p') = 0.$$

Remark 1.8. We will give in Section 4 other representations of A_L – see Proposition 4.1. We note that if L satisfies (L) then λ is unique. But the p_i^+ are not (in general) – see the case on the right at the top of Figure 2 in Example 1.11 below.

Theorem 1.9 (Effective junction conditions)**.** *Assume* (F), (L)*. Then $A_L : (0, T) \times \Gamma \times \mathbb{R}^d \to \mathbb{R}$ given in Definition 1.7 is well-defined, continuous, such that*

$$\lim_{|p'| \to +\infty} \inf_{(t,x') \in (0,T) \times \Gamma} A_L(t, x', p') = +\infty$$

and such that any L-relaxed sub-solution (resp. super-solution) of (1.1) *is an A_L-flux-limited sub-solution (resp. super-solution) of* (1.1)*. Moreover, if*

$$\{p' \in \mathbb{R}^d : L(p_0, p_1, \dots, p_N, t, x', p') \leq \lambda\} \text{ is convex}$$

for all $p_0, \dots, p_N, \lambda \in \mathbb{R}$ and all $(t, x') \in (0, T) \times \Gamma$, then

$$\{p' \in \mathbb{R}^d : A_L(t, x', p') \leq \lambda\} \text{ is convex}$$

for all $\lambda \in \mathbb{R}$ and $(t, x') \in (0, T) \times \Gamma$.

Remark 1.10. Applying Theorem 1.9 in the case $N = 1$, *effective boundary conditions* for degenerate parabolic equations posed on a domain are obtained; see Example 1.11 for instance. In the case $N = 2$, we get *effective transmission conditions*; see Theorem 1.12 for instance.

Example 1.11 (The 1D Neumann problem on a half-line)**.** We illustrate our result on the simplest example:

$$\begin{cases} u_t + H(u_x) - x^2 u_{xx} = 0, & x > 0, \\ -u_x = 0, & x = 0 \end{cases}$$

where H is a quasi-convex function (i.e. $\{p \in \mathbb{R} : H(p) \leq \lambda\}$ convex for all $\lambda \in \mathbb{R}$) as illustrated in Figure 2. This example corresponds to the case $N = 1$ (number of branches) $d = 0$ (dimension of the tangential space) and $H_1 = H$ and $L_{\mathrm{Neu}}(-u_t, \partial_1 u) = -\partial_1 u$. In

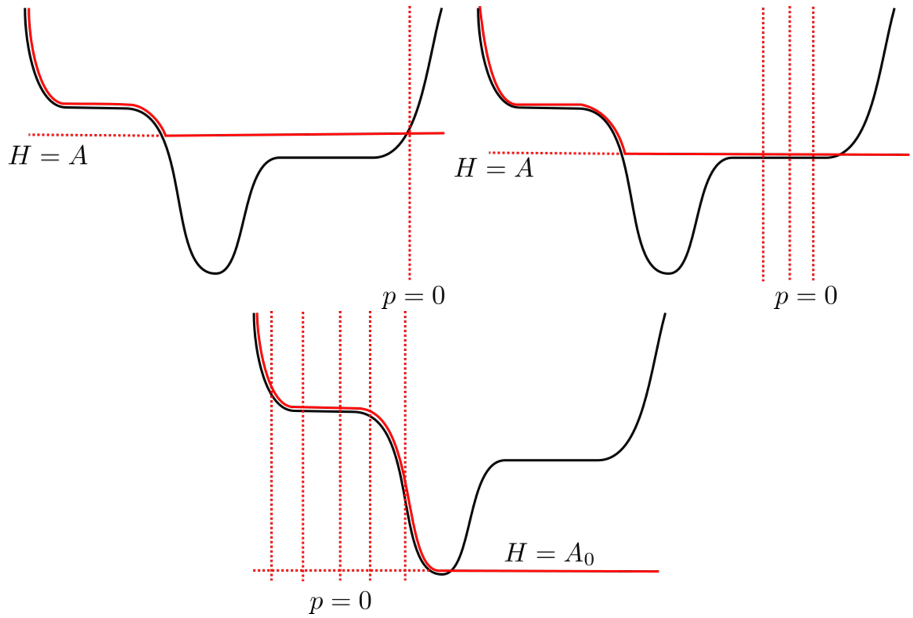

FIGURE 2. This figure illustrates Example 1.11 where $N = 1$ (number of branches) and $d = 0$ (dimension of the tangential space). The effective flux limiter A is determined in each case by looking at the points where the vertical line $\{p = 0\}$ intersects the graph of the Hamiltonian $H = H_1$; the variable p reduces here to p_1 in the general setting.

the three pictures, the plain black curve represents the Hamiltonian $H = H_1$ and the plain red curve represents the effective flux-limited function $F_A(p_1) = \max(A, H^-(p_1))$ associated with the generalized flux function L_{Neu} associated with the Neumann boundary condition. Depending on the position of the graph of H and the vertical line $\{p = 0\}$, the effective flux limiter A associated with the Neumann condition varies. On the left at the top, the line $\{p = 0\}$ intersects the graph of H in its increasing part. On the right at the top, the vertical line $\{p = 0\}$ intersects the graph of H in the non-decreasing part, but on a constant part. This second case illustrates that we exhibit equivalent classes of boundary conditions; indeed, different junction conditions can be equivalent to the same effective one: other vertical lines (corresponding to $u_x = \text{const}$ at $x = 0$) have the same effective boundary condition (because they have the same effective flux limiter). This is also illustrated in the last case: the vertical line $\{p = 0\}$ (and others) intersects the graph of H in its non-increasing part, which implies that the flux limiter coincides with $A_0 = \min H = \max_i(\min H_i)$ for all the vertical lines appearing in this third picture.

1.3. **Comments on the main result.** Our main result, Theorem 1.9, extends the results contained in [18, 17] in two directions: first, we can deal with Kirchoff conditions (see Example 1.4), second we can deal with second order terms (but degenerating along the junction).

As in [18, 17], the effective junction condition result is quite a straightforward consequence of the following important fact (Theorem 3.2): in order to check that a function is a flux-limited sub- and super-solution, it is enough to use a reduced set of test functions φ whose normal derivatives $\partial_i \varphi$ have specific values along Γ. For instance, these normal derivatives are equal to $\pi_i^+(p', A(p'))$ if the Hamiltonian has no constant parts and does not depend on x'. We recall that, roughly speaking, π_i^+ is the inverse function of the non-decreasing part of H_i, see (1.7) below.

The first version of this paper contained a comparison principle for (1.1), under stronger assumptions on F. On the one hand, the proof was quite difficult, relying on the vertex test function introduced in [18, 17], for which C^2 regularity was to be proved in the multi-dimensional setting. On the other hand, new and simpler techniques now emerge to attack this problem, see for instance [7, 16, 23, 24]. In particular, it is explained in [7] that the equations considered in the present work can be handled in the two-domain case. For these two reasons, we decided to restrict ourselves to the core of the work, that is to say the study of effective junction conditions.

1.4. **Comments on assumptions.** Assumptions (F1), (L1), (L2) are natural (if not necessary) when dealing with viscosity solutions of continuous Hamilton-Jacobi equations. In particular, (L2) ensures that the junction condition is compatible with the maximum principle. We recall that our goal is to exhibit effective junction conditions for degenerate parabolic equations. In particular, we want to understand what are the *effective* junction conditions that are imposed at the junction. From this point of view, it is necessary to consider degenerate parabolic equations which actually degenerate along Γ. This is exactly (F4). We also assume that the Hamiltonians have convex sublevel sets, see (F4). This condition can probably be relaxed but until very recent contributions [16, 24, 23] (none of these contributions were not available when the first version of this work appeared), the non-convex case was out of reach. As far as (F3) is concerned, it ensures that the Hamiltonians are coercive, a property which is used repeatedly and is at the core of most proofs. It is used together with (L4) for instance to derive the "weak continuity" of sub-solutions (see Lemma 2.3 below). Condition (F2) is used in an essential way when proving that the set of test functions can be reduced (see the proof of Lemma 3.5 about critical slopes below). Remark that this condition is weaker than the one which is needed in order to prove uniqueness, see [12, Condition (3.14)]. To finish with, (L3) and (L5) are used when proving the main result.

1.5. **An application: the vanishing viscosity limit.** Because we are able to deal with Kirchoff conditions, we are in position to adress an open problem about Hamilton-Jacobi equations from "regional control" problem: the identification of the vanishing viscosity limit.

We study the limit as $\varepsilon \to 0$ of the equation posed in $(0, +\infty) \times \mathbb{R}^{d+1}$

$$\begin{cases} v_t^\varepsilon + \tilde{H}_1(t, x, Dv^\varepsilon) = \varepsilon \Delta v^\varepsilon, & x_{d+1} < 0, t > 0 \\ v_t^\varepsilon + \tilde{H}_2(t, x, Dv^\varepsilon) = \varepsilon \Delta v^\varepsilon, & x_{d+1} > 0, t > 0 \\ v^\varepsilon(0, x) = v_0(x), & x \in \mathbb{R}^{d+1} \end{cases} \tag{1.4}$$

where $x = (x', x_{d+1}) \in \mathbb{R}^{d+1}$. In the previous equation, we do not need to impose any condition since the Laplacian is strong enough to ensure the existence of solutions that are continuously differentiable in the space variable $x \in \mathbb{R}^d$ despite the discontinuity of the first order term. In particular, the following condition holds at $x_{d+1} = 0$,

$$\partial_{x_{d+1}} v^\varepsilon(t, x', 0+) = \partial_{x_{d+1}} v^\varepsilon(t, x', 0-). \tag{1.5}$$

In this specific singular perturbation problem, the limit is identified by remarking that (1.5) is a Kirchoff condition and that consequently we can pass to the limit using relaxed solutions; more precisely, the limit of v^ε corresponds to a relaxed solution associated with this specific generalized junction condition. But the main theorem tells us that the limit thus corresponds to a flux-limited solution associated with a flux limiter A that is explicitly given by a formula (see Definition 1.7). Looking closely at this formula, we can prove that it corresponds to the maximal Ishii solution of the limit equation recently identified by Barles, Briani and Chasseigne [5, 6].

Theorem 1.12 (The vanishing viscosity limit selects the maximal Ishii solution)**.** *Assume*

$$\begin{cases} \tilde{H}_i \text{ continuous} \\ \forall (t, x') \in (0, T) \times \mathbb{R}^d, \forall \lambda \in \mathbb{R}, \{p = (p', p_i) \in \mathbb{R}^{d+1} : \tilde{H}_i(t, x', p) \leq \lambda\} \text{ convex} \\ \lim_{|p| \to +\infty} \inf_{(t,x') \in (0,T) \times \mathbb{R}^d} \tilde{H}_i(t, x', p) = +\infty \end{cases}$$

and v_0 is uniformly continuous in $\mathbb{R}^{d+1}$. Let v^ε be solution of (1.4) such that there exists $C > 0$ (independent of ε) such that $|v^\varepsilon(t, x) - v_0(x)| \leq Ct$ for all $(t, x) \in (0, T) \times J$. Then v^ε converges towards the maximal Ishii solution v of

$$\begin{cases} v_t + \tilde{H}_1(t, x, Dv) = 0, & x_{d+1} < 0, t > 0 \\ v_t + \tilde{H}_2(t, x, Dv) = 0, & x_{d+1} > 0, t > 0 \end{cases} \tag{1.6}$$

subject to the initial condition

$$v(0, x) = v_0(x), \quad x \in \mathbb{R}^{d+1}.$$

Remark 1.13. The function v is associated with the unique flux-limited solution u of the previous Hamilton-Jacobi equation for some flux limiter $A_I^-(t, x', p')$ that was identified in a previous work (see (5.5) in Proposition 5.6 below, corresponding to [17, Proposition 4.1]). The functions v and u satisfy the following equality: $v(t, x', x_{d+1}) = u(t, x', |x_{d+1}|)$, see Theorem 5.8 in Section 5.

1.6. **Review of literature.** Semi-linear uniformly parabolic equations on compact networks were studied in [29, 32, 21, 26] where uniqueness, existence, strong maximum principle among other results were proved to be true.

The first results for Hamilton-Jacobi equations on networks were obtained in [27] for eikonal equations. Some years later, the results were extended in [28, 1, 19]. Many new results were obtained since then, see for instance [18, 17] and references therein.

In [5, 6], the authors study regional control, i.e. control with dynamics and costs which are regular on either side of a hyperplane but with no compatibility or continuity assumption along the hyperplane. They identify the maximal and minimal Ishii solutions as value functions of two different optimal control problems. They also use the vanishing viscosity limit on a 1D example in order to prove that the two Ishii solutions can be different. Moreover, the authors ask about the vanishing viscosity limit in the general case.

In [9], the authors study the vanishing viscosity limit associated with Hamilton-Jacobi equations posed on a junction (the simplest network, see above). The main difference with our results is that the authors impose some compatibility conditions on Hamiltonians. In particular, this allows them to construct viscosity solutions which satisfy Kirchoff conditions in a strong sense. We proceed in a different setting and in a different way: no compatibility conditions on Hamiltonians are imposed, and Kirchoff conditions are understood in a relaxed sense, which is stable under local uniform convergence (and even relaxed semi-limits). We then use Theorem 1.9 to prove that imposing Kirchoff conditions reduce to the study of a flux-limited problem (for which uniqueness holds true).

In his lectures at Collège de France [22], Lions also treats problems related to Hamilton-Jacobi equations with discontinuities. After posting a first version of this paper, Lions and Souganidis [23] wrote a note about a new approach for Hamilton-Jacobi equations posed on junctions with coercive Hamiltonians that are possibly not convex.

We previously mentioned that, since the first version of this paper were posted, Guerand and Monneau studied independently effective non-linear boundary conditions in the non-convex case. On the one hand Guerand [16] studied the case $N = 1$ in the 1D setting, which amounts to studying first order non-convex Hamilton-Jacobi equations with nonlinear boundary conditions of Neumann type. On the other hand Monneau [24] mentioned to us that he studies effective junction conditions for non-convex Hamilton-Jacobi equations posed on multi-dimensional junctions.

As far as effective boundary conditions are concerned, we would like to mention that there are some results for motion of interfaces by Elliott, Giga and Goto [14] and for conservation laws by Andreianov and Sbihi [3, 2, 4].

To finish with, the link between the theory developed in [5, 6] and flux-limited solutions from [18, 17] is explored in [7]. In particular, [7] contains alternative proofs in the two-domain case of the comparison principle from [17] and of the vanishing viscosity limit obtained in the present work.

1.7. **Organization of the paper.** In Section 2, the notions of relaxed and flux-limited solutions are presented and their properties studied. In Section 3, it is proved that in order to check that a function is a flux-limited solution, the set of test functions can be reduced. In Section 4, we prove the main result of this paper, Theorem 1.9. Section 5 is devoted to the study of the vanishing viscosity limit. The last section (Section 6) is devoted to the proof of a known result about large deviations using the main result of this work.

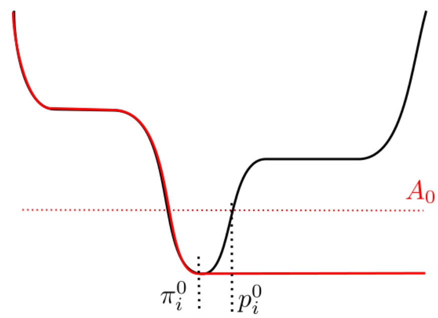

FIGURE 3. Non-increasing part H_i^- of a Hamiltonian H_i. The Hamiltonian is in black, the monotone part in red. The tangent variables (t, x', p') are not shown. In this example, the minimum of H_i is lower than A_0.

1.8. **Notation.** A distance is naturally associated with the junction J: for $x \in J_i$ and $y \in J_j$,

$$d(x,y) = \begin{cases} |x'-y'| + |x_i - y_i| & \text{if } i = j, \\ |x'-y'| + x_i + y_j & \text{if } i \neq j. \end{cases}$$

The open ball $B_r(t_0, x_0)$ centered at $(t_0, x_0) \in \mathbb{R} \times J$ is defined as $(t_0 - r, t_0 + r) \times \{y \in J : d(y, x_0) < r\}$.

The junction hyperplane Γ is the common boundary of J_i: we have $\Gamma = \partial J_i$. We identify Γ with $\mathbb{R}^d$ and we do not write the injection of $\mathbb{R}^d$ into J_i: $x' \mapsto (x', 0)$. For this reason, we write indisctinctively $x = (x', 0) \in \Gamma$ and $x' \in \Gamma$.

The Hamiltonian $H_i(t, x', p', p_i)$ is defined for $x' \in \Gamma$ and $p \in \mathbb{R}^{d+1}$. The minimal minimizer of $p_i \mapsto H_i(t, x', p', p_i)$ is denoted by $\pi_i^0(t, x', p')$. The functions H_i^- and H_i^+ are defined as follows

$$H_i^-(t, x', p', p_i) = \begin{cases} H_i(t, x', p', p_i) & \text{if } p_i \leq \pi_i^0(t, x', p') \\ H_i(t, x', p', \pi_i^0(t, x', p')) & \text{if } p_i \geq \pi_i^0(t, x', p') \end{cases}$$

$$H_i^+(t, x', p', p_i) = \begin{cases} H_i(t, x', p', p_i) & \text{if } p_i \geq \pi_i^0(t, x', p') \\ H_i(t, x', p', \pi_i^0(t, x', p')) & \text{if } p_i \leq \pi_i^0(t, x', p'). \end{cases}$$

For $\lambda \geq \min_{p_i \in \mathbb{R}} H_i(t, x', p', p_i)$, the functions π_i^+ and $\hat{\pi}_i^+$ are defined by

$$\pi_i^+(t, x', p', \lambda) = \inf\{p_i : H_i(t, x', p', p_i) = H_i^+(t, x', p', p_i) = \lambda\}, \tag{1.7}$$

$$\hat{\pi}_i^+(t, x', p', \lambda) = \sup\{p_i : H_i(t, x', p', p_i) = H_i^+(t, x', p', p_i) = \lambda\} \tag{1.8}$$

The function A_0 is defined for $t, x', p' \in \mathbb{R}^d$ by (1.3). We recall that

$$A_0(t, x', p') = \max_{i=1,\ldots,N} \min_{p_i \in \mathbb{R}} H_i(t, x', p', p_i).$$

The functions $p_i^0(t, x', p')$ are defined as

$$p_i^0(t, x', p') = \pi_i^+(t, x', p', A_0(t, x', p')).$$

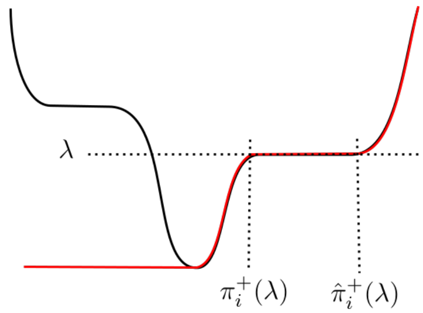

FIGURE 4. Non-decreasing part H_i^+ of a Hamiltonian H_i. The Hamiltonian is in black, the monotone part in red. The tangent variables (t, x', p') are not shown.

2. Relaxed and flux-limited solutions

2.1. **Test functions.** In order to define relaxed and flux-limited solutions, the set of test functions is to be made precise.

Definition 2.1 (Test functions)**.** A function $\phi : (0,T) \times J \to \mathbb{R}$ is a *test function* for (1.1) if it is continuous in $(0,T) \times J$, $\phi|_{(0,T)\times J_i}$ is $C^1_t \cap C^1_x$ and $\phi|_{(0,T)\times J_i^*}$ is C^2_x.

We classically say that a function ϕ touches another function u at a point (t,x) from below (respectively from above) if $u \geq \phi$ (respectively $u \leq \phi$) in a neighbourhood of (t,x) with equality at (t,x).

2.2. **Relaxed solutions.**

Definition 2.2 (L-relaxed solutions)**.** A function $u\colon (0,T) \times J \to \mathbb{R}$ is an *L-relaxed sub-solution* (resp. *L-relaxed super-solution*) of (1.1) if it is upper semi-continuous (resp. lower semi-continuous) and for all test functions ϕ touching u from above (resp. from below) at $(t,x) \in (0,T) \times J_i$, we have

$$\begin{aligned} &\phi_t + F_i(t,x,D\phi,D^2\phi) \leq 0 \text{ at } (t,x) \\ (\text{resp.} \quad &\phi_t + F_i(t,x,D\phi,D^2\phi) \geq 0 \text{ at } (t,x)) \end{aligned}$$

if $x \notin \Gamma$, and

$$\begin{cases} \text{either } \phi_t + H_i(t,x,D\phi) \leq 0 \text{ at } (t,x) \text{ for some } i \in \{1,\dots,N\}, \\ \text{or } L(-\phi_t, \partial_1\phi, \dots, \partial_N\phi, t, x, D'\phi) \leq 0 \text{ at } (t,x) \end{cases}$$

$$\left(\text{resp.} \quad \begin{cases} \text{either } \phi_t + H_i(t,x,D\phi) \geq 0 \text{ at } (t,x) \text{ for some } i \in \{1,\dots,N\}, \\ \text{or } L(-\phi_t, \partial_1\phi, \dots, \partial_N\phi, t, x, D'\phi) \geq 0 \text{ at } (t,x) \end{cases} \right)$$

if $x \in \Gamma$.

The following observation is important for stability and the reduction of the set of test functions. The proof contained in [18] can be easily extended to generalized junction conditions. We give a short proof for the reader's convenience.

Lemma 2.3 ("Weak continuity" of relaxed sub-solutions)**.** *Assume* (F) *and* (L). *Let* $u : (0,T) \times J \to \mathbb{R}$ *be an L-relaxed sub-solution of* (1.1). *Then for all* $i \in \{1,\dots,N\}$, *and* $x = (x',0) \in \Gamma$,

$$u(t,x) = \limsup_{(s,y)\to(t,x), y \in J_i^*} u(s,y).$$

Proof. Let $i \in \{1,\dots,N\}$. Since u is upper semi-continuous, we have for all $(t,x) \in (0,T) \times \Gamma$,

$$u(t,x) \geq \limsup_{(s,y)\to(t,x), y \in J_i^*} u(s,y) =: U_i(t,x).$$

Remark that the function $U_i : (0,T) \times \Gamma \to \mathbb{R}$ is upper semi-continuous. In order to prove that $u = U_i$ in $(0,T) \times \Gamma$, we assume that there exists $(t_*,x_*) \in (0,T) \times \Gamma$ such that

$$u(t_*,x_*) \geq U_i(t_*,x_*) + \delta \tag{2.1}$$

for some $\delta > 0$.

The density theorem [10, Theorem 3.1] can be applied to the restriction of $-u$ to $(0,T)\times\Gamma$ around (t_*,x_*). Roughly speaking, this theorem claims that the proximal subdifferential (which is a subset of the viscosity subdifferential) is nonempty in a dense set. This result even ensures that there exists a point $(t_0,x_0) \in (0,T) \times \Gamma$ such that $(t_0,x_0,-u(t_0,x_0))$ is as close as desired to $(t_*,x_*,-u(t_*,x_*))$ and there exists a viscosity subdifferential of $-u$ at (t_0,x_0). More precisely, for all $\varepsilon > 0$, there exists a C^1 function $\Psi : (0,T) \times \Gamma \to \mathbb{R}$ and $(t_0,x_0) \in (0,T) \times \Gamma$ such that Ψ strictly touches u from above at $(t_0,x_0) \in B_r(t_*,x_*) \cap (0,T) \times \Gamma$ for some $r > 0$ and

$$(t_0,x_0) \in B_\varepsilon(t_*,x_*) \quad \text{and} \quad -u(t_*,x_*) - \varepsilon \leq -u(t_0,x_0) \leq -u(t_*,x_*).$$

In particular, $u(t_0,x_0) \geq u(t_*,x_*)$.

Moreover, since U_i is upper semi-continuous, we can choose ε small enough in order to ensure that $U_i(t_0,x_0) \leq U_i(t_*,x_*) + \delta/2$.

We now get from (2.1) that

$$u(t_0,x_0) \geq \limsup_{(s,y)\to(t_0,x_0), y \in J_i^*} u(s,y) + \delta/2. \tag{2.2}$$

Since the test function strictly touches u at (t_0,x_0), we have $\Psi - u \geq \delta_1 > 0$ in a neighbourhood (with respect to $(0,T) \times J$) of $\partial B_r(t_0,x_0) \subset (0,T) \times \Gamma$. We now consider the test function $\Phi(t,x) = \Psi(t,x') + p_j x_j$ for $x \in J_j$ with $p_j > 0$ if $j \neq i$ and $p_i < 0$. Thanks to (2.2), $|p_i|$ can be chosen arbitrarily large. We now use the coercivity of the F_j (see (F3)) to show that for $\min_j |p_j|$ large enough, Φ touches u from above at (t_0,x_0). But this implies that

$$L(-\partial_t \Phi(t_0,x_0), p_1, \dots, p_N, t_0, x_0', D'\Phi(t_0,x_0)) \leq 0$$

which contradicts (L4) since the $\min_{k=1,\dots,N} p_k = p_i \to -\infty$. The proof is now complete. $\square$

2.3. **Stability and existence.** The following results related to stability of relaxed sub- and super-solutions are expected; even more, relaxed solutions are defined in such a way that they satisfy such stability properties.

In order to state the first stability result, we recall the definition of upper semi-continuous envelope u^* (resp. lower semi-continuous envelope u_*) of a function $u : (0,T) \times J \to \mathbb{R}$,

$$u^*(t,x) = \limsup_{(s,y)\to(t,x)} u(s,y), \quad u_*(t,x) = \liminf_{(s,y)\to(t,x)} u(s,y).$$

Proposition 2.4 (Stability of relaxed solutions - I). *Assume* (F) *and* (L). *If* $(u_\alpha)_\alpha$ *is a family of relaxed sub-solutions (resp. relaxed super-solutions) of* (1.1) *which is locally uniformly bounded from above (resp. from below), then the upper semi-continuous (resp. lower semi-continuous) envelope of* $\sup_\alpha u_\alpha$ *(resp.* $\inf_\alpha u_\alpha$*) is a relaxed sub-solution (resp. relaxed super-solution) of* (1.1).

Proof. We only treat the sub-solution case since the super-solution one is similar. Let u denote the upper semi-continuous envelope of $\sup_\alpha u_\alpha$. Consider a test function ϕ strictly touching u from above at (t,x). There then exist a sequence $(t_n,x_n) \to (t,x)$ and α_n such that ϕ touches u_{α_n} from above at (t_n,x_n). Writing the viscosity inequalities and passing to the limit yields the desired result. □

In order to state the second stability result, we recall the definition of upper semi-limit $\bar{u}$ (resp. lower semi-limit $\underline{u}$) of a family of functions $u^\varepsilon : (0,T) \times J \to \mathbb{R}$, $\varepsilon > 0$,

$$\bar{u}(t,x) = \limsup_{(s,y)\to(t,x),\varepsilon\to 0} u^\varepsilon(s,y), \quad \underline{u}(t,x) = \liminf_{(s,y)\to(t,x),\varepsilon\to 0} u^\varepsilon(s,y).$$

Proposition 2.5 (Stability of relaxed solutions - II). *Assume* (F) *and* (L). *If* $\{u^\varepsilon\}_{\varepsilon>0}$ *is a family of relaxed sub-solutions (resp. relaxed super-solutions) of* (1.1) *which is locally uniformly bounded from above (resp. from below), then the relaxed upper limit (resp. relaxed lower limit) of* $\{u^\varepsilon\}_{\varepsilon>0}$ *is a relaxed sub-solution (resp. relaxed super-solution) of* (1.1).

Proof. We only treat the sub-solution case since the super-solution one is similar. Consider a test function ϕ strictly touching $\bar{u}$ from above at (t,x). We can assume that the contact is strict. There then exist a sequence $(t_k,x_k) \to (t,x)$ and $\varepsilon_k \to 0$ such that ϕ touches u_{n_k} from above at $(t_k,x_k) \to (t,x)$ as $k \to +\infty$. Either there is a subsequence k_p along which $x_{k_p} \in J_i^*$ for some $i \in \{1,\dots,N\}$ or $x_k \in \Gamma$ for large k's. Writing the viscosity inequalities in both cases and passing to the limit yields the desired result. □

The stability properties satisfied by relaxed solutions ensure the existence of discontinuous relaxed solutions.

Theorem 2.6 (Existence of discontinuous relaxed solutions). *Assume* (F) *and* (L) *and consider* u_0 *uniformly continuous. Assume also that for all* $R > 0$,

$$C_R := \sup\{|F_i(t,x,p,A)| : i \in \{1,\dots,N\}, t \in (0,T), x \in J, |p| \le R, |A| \le R\} < +\infty.$$

There exists u *such that its upper semi-continuous (resp. lower semi-continuous) envelope is a relaxed sub-solution (resp. relaxed super-solution) of* (1.1) *such that*

$$u(0,x) = u_0(x) \text{ for } x \in J.$$

Remark 2.7. This theorem states the existence of discontinuous solutions in the sense of Ishii [20].

Proof. In view of the stability results, it is enough to construct a solution for some initial datum u_0 such that $u_0^i = u_0|_{J_i}$ are in $C^{1,1}$. For such u_0's, we can construct barriers in the classical way: $u^\pm(t,x) = u_0(x) \pm Ct$. For $C \geq C_{R_0}$ with $R_0 \geq \|Du_0^i\|_\infty + \|D^2 u_0^i\|_\infty$ for all $i = 1, \dots, N$, the function u^+ is a relaxed super-solution while u^- is a relaxed sub-solution. Indeed, as far as the equations in J_i are concerned, it is classical; as far as the junction condition is concerned, the equation is satisfied up to Γ and thus $u^\pm$ are relaxed semi-solutions on Γ. We then consider W the set of all functions lying below u^+ whose upper semi-continuous envelope is a relaxed sub-solution. Then the supremum of $w \in W$ is in W and it is maximal. Let w denote this maximal element. If the lower semi-continuous envelope is not a relaxed super-solution, there exists a test function ϕ and a point (t,x) such that ϕ touches w_* from below at (t,x) without satisfying the corresponding viscosity inequality. This implies $\phi < (u_+)_*$ in a neighbourdhood of (t,x) and we can prove that ϕ is a relaxed sub-solution in the same neighbourhood. Then we can construct a relaxed sub-solution w_δ which is not below w, contradicting its maximality. □

2.4. **Flux-limited solutions.** It is proved in [18] that, in the special case where $L = L_A$ defined in (1.2) and for first order Hamilton-Jacobi equations, relaxed solutions satisfy the junction condition in a strong sense, which is made precise in the following definition.

Definition 2.8 (Flux-limited solutions)**.** Given a function $A : (0,T) \times \Gamma \times \mathbb{R}^d \to \mathbb{R}$ such that $A \geq A_0$, a function $u\colon (0,T) \times J \to \mathbb{R}$ is a *A-flux-limited sub-solution* (resp. *A-flux-limited super-solution*) of (1.1) if it is upper semi-continuous (resp. lower semi-continuous) and for any test function ϕ in the sense of Definition 2.1 touching u from above (resp. from below) at $(t,x) \in (0,T) \times J_i$, we have

$$\phi_t + F_i(t,x,D\phi,D^2\phi) \leq 0 \text{ at } (t,x)$$
$$\left(\text{resp.} \quad \phi_t + F_i(t,x,D\phi,D^2\phi) \geq 0 \text{ at } (t,x)\right)$$

if $x \notin \Gamma$, and

$$L_A(-\phi_t, \partial_1\phi, \dots, \partial_N\phi, t, x', D'\phi) \leq 0 \text{ at } (t,x)$$
$$\left(\text{resp.} \quad L_A(-\phi_t, \partial_1\phi, \dots, \partial_N\phi, t, x', D'\phi) \geq 0 \text{ at } (t,x)\right)$$

if $x \in \Gamma$.

Remark 2.9. When proving that a function is a sub-solution or a super-solution of (1.1) at one given point of $(0,T) \times \Gamma$, it is enough to consider a reduced set of test functions associated with this specific point. It is thus interesting to consider sub- or super-solution of (1.1) *at only one point of* $(0,T) \times \Gamma$ – see Theorem 3.2 about the reduction of the set of test functions.

The following proposition asserts that L_A-relaxed solutions coincide with A-flux-limited solutions. It was proved in [18, 17] in the case of first order equations. We point out that the multidimensional proof of [17] applies *without any change* to degenerate parabolic equations satisfying (F).

Proposition 2.10 (L_A-relaxed solutions are A-flux-limited solutions – [17])**.** *Assume* (F) *and* (L)*. Then any L_A-relaxed sub-solution (resp. super-solution) of* (1.1) *is an A-flux-limited sub-solution (resp. super-solution) of* (1.1)*.*

3. Reduced set of test functions for flux-limited solutions

In this section, we explain why it is sufficient to consider a reduced set of test functions in order to check that a function is a flux-limited (sub/super)solutions of (1.1). Such a result is used in an essential way when proving Theorem 1.9.

Definition 3.1 (Reduced test functions)**.** Consider a flux limiter $A \geq A_0$ and a point $(t_0, x_0') \in (0,T) \times \Gamma$. A function $\varphi : (0,T) \times J \to \mathbb{R}$ is a *reduced test function* for (1.1) at (t_0, x_0') if there exists a function $\phi \in C^1((0,T) \times \mathbb{R}^d)$ and N functions $\phi_i \in C^1([0,+\infty))$, $i = 1, \dots, N$, such that

$$\forall t \in (0,T), \forall (x', x_i) \in J_i, \qquad \varphi(t, (x', x_i)) = \phi(t, x') + \phi_i(x_i)$$

and, for all $i = 1, \dots, N$, $\phi_i(0) = 0$ and the slope $p_i = \phi_i'(0)$ and the tangential gradient $p' = D'\phi(t_0, x_0')$ satisfy

$$H_i(t_0, x_0', p', p_i) = H_i^+(t_0, x_0', p', p_i) = A(t_0, x_0', p') \tag{3.1}$$

that is to say $p_i \in [\pi_i^+(t_0, x_0', p'), \hat{\pi}_i^+(t_0, x_0', p')]$.

Theorem 3.2 below generalizes the one contained in [18]. In order to state it, we need to consider the equation on each (open) branch i, i.e. away from the junction hyperplane Γ:

$$u_t + F_i(t, x, Du, D^2u) = 0, \quad (t,x) \in (0,T) \times J_i^*. \tag{3.2}$$

We can now state and prove the following theorem.

Theorem 3.2 (Reducing the set of test functions)**.** *Assume* (F) *and consider a function $A : (0,T) \times \Gamma \times \mathbb{R}^d \to \mathbb{R}$ such that $A \geq A_0$. Given a function $u : (0,T) \times J \to \mathbb{R}$, the following properties hold true.*

i) If, for all $i \in \{1, \dots, N\}$, u is a sub-solution of (3.2) *and for $(t,x) \in (0,T) \times \Gamma$,*

$$u(t,x) = \limsup_{s \to t, y \to x, y \in J_i^*} u(s,y), \tag{3.3}$$

then u is an A_0-flux limited sub-solution of (1.1) *at (t,x).*

ii) If, for all $i \in \{1, \dots, N\}$, u is a sub-solution of (3.2) *satisfying* (3.3) *and if for any reduced test function φ in the sense of Definition 3.1 touching u from above at $(t,x) \in (0,T) \times \Gamma$, we have*

$$\varphi_t(t,x) + A(x', D'\varphi(t,x)) \leq 0,$$

then u is an A-flux-limited sub-solution of (1.1) *at (t,x).*

iii) If, for all $i \in \{1, \dots, N\}$, u is a super-solution of (3.2) and if for any reduced test function φ in the sense of Definition 3.1 touching u from below at $(t, x) \in (0, T) \times \Gamma$ we have

$$\varphi_t(t, x) + A(x, D'\varphi(t, x)) \geq 0,$$

then u is an A-flux-limited super-solution of (1.1) at (t, x).

Remark 3.3. In the previous statement, functions are flux-limited solution of (1.1) at only one point of $(0, T) \times \Gamma$ – see Remark 2.9 above.

Proof. The proof of [18, Theorem 2.7] applies here *without any change* after proving the two lemmas 3.4 and 3.5 about *critical normal slopes.* Indeed, with such technical results in hands, the proof focuses on what happens on Γ and second derivatives do not appear any more. □

Lemma 3.4 (Super-solution property for the critical normal slope on each branch)**.** *Let $i \in \{1, \dots, N\}$ be fixed. Let $u : (0, T) \times J_i \to \mathbb{R}$ be a lower semi-continous super-solution of (3.2). Let ϕ be a test function touching u from below at some point $(t_0, x_0) \in (0, T) \times \Gamma$. We consider*

$$\overline{p_i} = \sup\{\overline{p} \in \mathbb{R} : \exists r > 0, \phi(t, x) + \overline{p} x_i \leq u(t, x) \textit{ for } (t, x) \in B_r(t_0, x_0) \cap (0, T) \times J_i\}.$$

If $\overline{p_i} < +\infty$, then we have

$$\phi_t + H_i(t, x, D'\phi, \partial_i \phi + \overline{p_i}) \quad \geq 0 \quad \textit{at } (t_0, x_0) \quad \textit{with } \overline{p_i} \geq 0.$$

Lemma 3.5 (Sub-solution property for the critical normal slope on each branch)**.** *Let $i \in \{1, \dots, N\}$ be fixed. Let $u : (0, T) \times J_i \to \mathbb{R}$ be a sub-solution of (3.2). Let ϕ be a test function touching u from above at some point $(t_0, x_0) \in (0, T) \times \Gamma$. We consider*

$$\underline{p_i} = \inf\{\overline{p} \in \mathbb{R} : \exists r > 0, \phi(t, x) + \overline{p} x_i \geq u(t, x) \textit{ for } (t, x) \in B_r(t_0, x_0) \cap (0, T) \times J_i\}.$$

If u satisfies

$$u(t_0, x_0) = \limsup_{s \to t_0, y \to x_0, y \in J_i^*} u(s, y), \tag{3.4}$$

then $\underline{p_i} > -\infty$; moreover, we have in this case

$$\phi_t + H_i(t, x, D'\phi, \partial_i \phi + \underline{p_i}) \leq 0 \quad \textit{at } (t_0, x_0) \quad \textit{with } \underline{p_i} \leq 0.$$

We first prove Lemma 3.4.

Proof of Lemma 3.4. The proof follows the same lines of [18, Lemma 2.8].

From the definition of $\overline{p_i}$, for all $\varepsilon > 0$ small enough, there exists $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ such that

$$u(s, y) \geq \phi(s, y) + (\overline{p_i} - \varepsilon) y_i \quad \text{for all } (s, y) \in B_\delta(t_0, x_0) \cap (0, T) \times J_i \tag{3.5}$$

and there exists $(t_\varepsilon, x_\varepsilon) \in B_{\delta/2}(t_0, x_0)$ such that

$$u(t_\varepsilon, x_\varepsilon) < \phi(t_\varepsilon, x_\varepsilon) + (\overline{p_i} + \varepsilon) x_\varepsilon^i.$$

We choose a smooth function $\Psi : \mathbb{R}^{d+2} \to [-1, 0]$ such that

$$\Psi = \begin{cases} 0 & \text{in } B_{1/2}(t_0, x_0) \\ -1 & \text{outside } B_1(t_0, x_0). \end{cases}$$

We define for $(s, y) \in (0, T) \times J_i$,

$$\Phi(s, y) = \phi(s, y) + 2\varepsilon\Psi_\delta(s, y) + (\overline{p_i} + \varepsilon)y_i$$

with $\Psi_\delta(Y) = \delta\Psi\left(\frac{Y}{\delta}\right)$. Remark that for $(s, y) \in \partial(B_\delta(t_0, x_0) \cap (0, T) \times J_i)$, we have $y_i \le \delta$. In particular, $-2\varepsilon\delta + (\overline{p_i} + \varepsilon)y_i \le (\overline{p_i} - \varepsilon)y_i$ for such (s, y). Hence (3.5) implies

$$\begin{cases} \Phi(s, y) = \phi(s, y) - 2\varepsilon\delta + (\overline{p_i} + \varepsilon)y_i \le u(s, y) & \text{for } (s, y) \in \partial(B_\delta(t_0, x_0) \cap (0, T) \times J_i), \\ \Phi(s, x) \le \phi(s, x) \le u(s, x) & \text{for } (s, x) \in (t_0 - \delta, t_0 + \delta) \times \Gamma \end{cases}$$

and

$$\Phi(t_\varepsilon, x_\varepsilon) = \phi(t_\varepsilon, x_\varepsilon) + (\overline{p_i} + \varepsilon)x^i_\varepsilon > u(t_\varepsilon, x_\varepsilon).$$

We conclude that there exists a point $(\overline{t_\varepsilon}, \overline{x_\varepsilon}) \in B_\delta(t_0, x_0) \cap ((0, T) \times J_i^*)$ such that $u - \Phi$ reaches a minimum in $\overline{B_\delta(t_0, x_0)} \cap ([0, T] \times J_i)$. We thus can write the viscosity inequality

$$\Phi_t + F_i(t, x, D\Phi, D^2\Phi) \ge 0 \quad \text{at } (\overline{t_\varepsilon}, \overline{x_\varepsilon})$$

which reads

$$\begin{aligned} &\phi_t(\overline{t_\varepsilon}, \overline{x_\varepsilon}) + 2\varepsilon(\Psi_\delta)_t(\overline{t_\varepsilon}, \overline{x_\varepsilon}) \\ &+F_i(\overline{t_\varepsilon}, \overline{x_\varepsilon}, (D'\phi + 2\varepsilon D'\Psi_\delta)(\overline{t_\varepsilon}, \overline{x_\varepsilon}), \partial_i\phi(\overline{t_\varepsilon}, \overline{x_\varepsilon}) + 2\varepsilon\partial_i\Psi_\delta(\overline{t_\varepsilon}, \overline{x_\varepsilon}) + \overline{p_i} + \varepsilon, D^2\phi + 2\varepsilon D^2\Psi_\delta(\overline{t_\varepsilon}, \overline{x_\varepsilon})) \ge 0. \end{aligned} \tag{3.6}$$

We now send $\varepsilon \to 0$ in the above inequality; recall that $\delta \in (0, \varepsilon)$ and $\Psi_\delta = \delta\Psi(\cdot/\delta)$; in particular,

$$\varepsilon(\Psi_\delta)_t(\overline{t_\varepsilon}, \overline{x_\varepsilon}), \varepsilon D'\Psi_\delta(\overline{t_\varepsilon}, \overline{x_\varepsilon}), \varepsilon\partial_i\Psi_\delta(\overline{t_\varepsilon}, \overline{x_\varepsilon}) \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.7}$$

As far as second derivatives are concerned, we have

$$|\varepsilon D^2\Psi_\delta| \le \|D^2\Psi\|_\infty.$$

In particular,

$$\varepsilon D^2\Psi_\delta(\overline{t_\varepsilon}, \overline{x_\varepsilon})) \to B \in \mathbb{S}_{d+1}(\mathbb{R}) \tag{3.8}$$

along a subsequence. Since $(\overline{t_\varepsilon}, \overline{x_\varepsilon}) \to (t_0, x_0)$, we finally get from (3.6), (3.7) and (3.8) that

$$\phi_t(t_0, x_0) + F_i(t_0, x_0, D'\phi(t_0, x_0), \partial_i\phi(t_0, x_0) + \overline{p}_i, D^2\phi(t_0, x_0) + B) \ge 0$$

which is the desired inequality since $x_0 \in \Gamma$ and F_i satisfies (F4). The proof is now complete. □

We now turn to the proof of Lemma 3.5

Proof of Lemma 3.5. The main difference with the previous lemma is the claim that the critical normal slope is finite. This is the reason why we only explain this point. Here again, we follow closely [18].

Let $p \in (-\infty, 0]$ be such that there exists $r > 0$ such that $\phi + px_i \geq u$ in $B = B_r(t_0, x_0) \cap (0, T) \times J_i$. Remark first that, replacing ϕ with $\phi + (t - t_0)^2 + |x - x_0|^2$ if necessary, we can assume that

$$u(t, x) < \phi(t, x) + px_i \text{ if } (t, x) \neq (t_0, x_0). \tag{3.9}$$

In particular, there exists $\delta > 0$ such that $\phi + px_i \geq u + \delta$ on $\partial B \setminus \Gamma$.

Since u satisfies (3.4), there exists $(t_\varepsilon, x_\varepsilon) \to (t_0, x_0)$ such that $x_\varepsilon \in J_i^*$ and $u(t_0, x_0) = \lim_{\varepsilon \to 0} u(t_\varepsilon, x_\varepsilon)$.

We now introduce the following perturbed test function

$$\Psi(t, x) = \phi(t, x) + px_i + \frac{\eta}{x_i}$$

where $\eta = \eta(\varepsilon)$ is a small parameter to be chosen later. Let $(s_\varepsilon, y_\varepsilon)$ realize the infimum of $\Psi - u$ in $\overline{B}$. In particular,

$$(\phi + px_i - u)(s_\varepsilon, y_\varepsilon) \leq \Psi(s_\varepsilon, y_\varepsilon) - u(s_\varepsilon, y_\varepsilon) \leq \Psi(t_\varepsilon, x_\varepsilon) - u(t_\varepsilon, x_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{3.10}$$

as soon as $\eta(\varepsilon) = o(x_\varepsilon^i)$ with $x_\varepsilon = (x'_\varepsilon, x_\varepsilon^i)$. In particular, in view of (3.9), this implies that $(s_\varepsilon, y_\varepsilon) \to (t_0, x_0)$ as $\varepsilon \to 0$. Since u is a sub-solution of (3.2), we know that

$$\phi_t(s_\varepsilon, y_\varepsilon) + F_i(s_\varepsilon, y_\varepsilon, D'\phi(s_\varepsilon, y_\varepsilon), \partial_i \phi(s_\varepsilon, y_\varepsilon) + p - \frac{\eta}{(y_\varepsilon^i)^2}, D^2\phi(s_\varepsilon, y_\varepsilon) + \frac{2\eta}{(y_\varepsilon^i)^3} e_{d+1} \otimes e_{d+1}) \leq 0$$

(where $(e_1, \dots, e_{d+1})$ is an orthonormal basis of $\mathbb{R}^{d+1}$ and e_{d+1} is orthogonal to Γ). Use now (F2) in order to get

$$\phi_t(s_\varepsilon, y_\varepsilon) + F_i(s_\varepsilon, y_\varepsilon, D'\phi(s_\varepsilon, y_\varepsilon), \partial_i \phi(s_\varepsilon, y_\varepsilon) + p - \frac{\eta}{(y_\varepsilon^i)^2}, D^2\phi(s_\varepsilon, y_\varepsilon)) \leq 2C_i \frac{\eta}{y_\varepsilon^i}.$$

Remark now that (3.10) implies

$$\frac{\eta}{y_\varepsilon^i} - \frac{\eta}{x_\varepsilon^i} \leq \big(p(x_\varepsilon^i - y_\varepsilon^i) + (u - \phi)(s_\varepsilon, y_\varepsilon) - (u - \phi)(t_\varepsilon, x_\varepsilon)\big) \to 0 \quad \text{as } \varepsilon \to 0.$$

Recalling that η is chosen so that $\eta / x_\varepsilon^i \to 0$ as $\varepsilon \to 0$, we thus get

$$\frac{\eta}{y_\varepsilon^i} \to 0 \quad \text{as } \varepsilon \to 0.$$

In particular, the coercivity of F_i (see (F3)) implies that $p - \frac{\eta}{(y_\varepsilon^i)^2}$ is bounded as $\varepsilon \to 0$. Hence we can pass to the limit as $\varepsilon \to 0$ in the viscosity inequality and get

$$\phi_t(t_0, x_0) + H_i(t_0, x_0, D'\phi(t_0, x_0), \partial_i \phi(t_0, x_0) + p^0) \leq 0$$

where $p^0 \in (-\infty, 0]$ is any accumulation point of $p - \frac{\eta}{(y_\varepsilon^i)^2}$ as $\varepsilon \to 0$. The previous inequality and (F3) implies in particular that p^0 is bounded from below by a constant C which only depends on $H_i, \phi_t, D\phi$ at (t_0, x_0). Indeed, (F3) implies in particular that

$$\lim_{|p| \to +\infty} \inf_{(t, x') \in (0, T) \times \Gamma} H_i(t, x', p) = +\infty.$$

But this also implies that $p \geq C$ and, in turn, $\underline{p_i} \geq C$. The proof is now complete. □

4. Proof of the main theorem

This section is devoted to the proof of the first main result, Theorem 1.9. Throughout this section, we do not write the (t, x', p') dependence of A_L, π^+, $\hat{\pi}^+$ etc. (see (1.7) and (1.8) for a definition) in order to clarify the presentation and proofs.

The proof of Theorem 1.9 relies on properties and other representations of the effective flux limiter A_L; we gather them in the following preparatory proposition.

Proposition 4.1 (Representations of A_L)**.** *Let A_L be the effective flux limiter given by Definition 1.7.*

i) If $L(A_0, \pi_1^+(A_0), \dots, \pi_N^+(A_0)) \leq 0$ then $A_L = A_0$.

ii) If $L(A_0, \pi_1^+(A_0), \dots, \pi_N^+(A_0)) > 0$ then A_L is well defined: there exists a unique $\lambda^* \in \mathbb{R}$ and there exist $p_i^* \in [\pi_i^+(\lambda^*), \hat{\pi}_i^+(\lambda^*)]$ (not necessarily unique) such that $L(\lambda^*, p_1^*, \dots, p_N^*) = 0$.

iii) If $L(A_0, \pi_1^+(A_0), \dots, \pi_N^+(A_0)) > 0$, then

$$A_L = \sup\{\lambda \geq A_0 : L(\lambda, \pi_1^+(\lambda), \dots, \pi_N^+(\lambda)) > 0\} \tag{4.1}$$

$$A_L = \inf\{\lambda \geq A_0 : L(\lambda, \hat{\pi}_1^+(\lambda), \dots, \hat{\pi}_N^+(\lambda)) < 0\}. \tag{4.2}$$

iv) Moreover, if $L(A_0, \pi_1^+(A_0), \dots, \pi_N^+(A_0)) > 0$, we also have

$$L(A_L, \pi_1^+(A_L), \dots, \pi_N^+(A_L)) \geq 0 \tag{4.3}$$

$$L(A_L, \hat{\pi}_1^+(A_L), \dots, \hat{\pi}_N^+(A_L)) \leq 0. \tag{4.4}$$

Remark 4.2. We point out that $p_i^* \in [\pi_i^+(\lambda), \hat{\pi}_i^+(\lambda^*)$ is equivalent to $p_i^* \geq p_i^0$ and $L(\lambda^*, p_1^*, \dots, p_N^*) = 0$.

Proof. Remark that p_i^0 in Definition 1.7 coincides with $\pi_i^+(A_0)$. In particular, if

$$L(A_0, \pi_1^+(A_0), \dots, \pi_N^+(A_0)) \leq 0$$

then Definition 1.7 says that $A_L = A_0$. This proves i).

We now assume that $L(A_0, \pi_1^+(A_0), \dots, \pi_N^+(A_0)) > 0$. Assumption (L5) implies that there exists $\tilde{\lambda} > A_0$ such that $L(\tilde{\lambda}, \hat{\pi}_1^+(\tilde{\lambda}), \dots, \hat{\pi}_N^+(\tilde{\lambda})) < 0$. In particular, the two following quantities are finite,

$$S := \sup\{\lambda \geq A_0 : L(\lambda, \pi_1^+(\lambda), \dots, \pi_N^+(\lambda)) > 0\}$$
$$I := \inf\{\lambda \geq A_0 : L(\lambda, \hat{\pi}_1^+(\lambda), \dots, \hat{\pi}_N^+(\lambda)) < 0\}.$$

Using that π_i^+ is left continuous and $\hat{\pi}_i^+$ is right continuous, we have

$$L(S, \pi_1^+(S), \dots, \pi_N^+(S)) \geq 0 \tag{4.5}$$

$$L(I, \hat{\pi}_i^+(I), \dots, \hat{\pi}_N^+(I)) \leq 0. \tag{4.6}$$

Proving ii), iii) and iv) (apart from uniqueness in ii)) reduces to proving that $S = I$. Indeed, if $S = I$ then: iii) is proved with $A_L = I = S$; (4.3) and (4.4) are satisfied with $A_L = I = S$; the continuity of L (see (L1)) and the two previous inequalities imply the existence of $p_i^* \in [\pi_i^+(A_L), \hat{\pi}_i^+(A_L)]$ such that $L(A_L, p_1^*, \dots, p_N^*) = 0$.

If $I < S$, then $\hat{\pi}_i^+(I) < \pi_i^+(S)$ for all $i \in \{1, \dots, N\}$; but (L3) and (4.5) then imply that

$$L(I, \hat{\pi}_1^+(I), \dots, \hat{\pi}_N^+(I)) > L(S, \pi_1^+(S), \dots, \pi_N^+(S)) \geq 0$$

which contradicts (4.6). Then $S \leq I$.

If $S < I$ then the definitions of S and I imply that for all $\lambda^* \in]S, I[$,

$$L(\lambda^*, \pi_1^+(\lambda^*), \dots, \pi_N^+(\lambda^*)) \leq 0$$
$$L(\lambda^*, \hat{\pi}_1^+(\lambda^*), \dots, \hat{\pi}_N^+(\lambda^*)) \geq 0.$$

But using the continuity of L (see (L1)), this implies that for all $\lambda^* \in]S, I[$, there exist $p_i^* \in [\pi_i^+(\lambda^*), \hat{\pi}_i^+(\lambda^*)]$, $i = 1, \dots, N$, such that

$$L(\lambda^*, p_1^*, \dots, p_N^*) = 0.$$

But this cannot be true for two different λ^*'s because of (L3). Hence $S = I$. Notice that we can prove in the same way uniqueness in ii). The proof is now complete. □

We now prove the main theorem.

Proof of Theorem 1.9. Let A_L be the effective flux limiter in the sense of Definition 1.7. It is well defined thanks to Proposition 4.1. Since $A_L \geq A_0$, the coercivity is clear: $\lim_{|p| \to +\infty} \inf_{x' \in \Gamma} A(t, x', p') = +\infty$. The proof of the continuity of A_L and the convexity of sublevel sets is the same as in [17, Proof of Theorem 2.13].

We only deal with the sub-solution case since the super-solution case is very similar. If $A_L(t, x', p') = A_0(t, x', p')$, then Lemma 2.3 and Theorem 3.2 imply that any L-relaxed sub-solution of (1.1) is an A_0-flux limited sub-solution of (1.1).

We now consider the case where there exists (t, x', p') such that $A_L(t, x', p') > A_0(t, x', p')$. Let u be an L-relaxed sub-solution of (1.1) and let us prove that it is an $(A_L - \varepsilon)$-flux-limited sub-solution of (1.1) at $(t, x') \in (0, T) \times \Gamma$ for all $\varepsilon > 0$ such that $A_L - \varepsilon > A_0$ (at (t, x', p')). We use here the fact that Theorem 3.2 is local in the sense that it asserts that a function is a flux-limited solution at one given point $(t, x') \in (0, T) \times \Gamma$. In view of Lemma 2.3 and Theorem 3.2, we only have to consider a reduced test function φ touching u from above at $(t, x') \in (0, T) \times \Gamma$. We recall that

$$\varphi(s, y) = \phi(s, y') + \phi_i(y_i)$$

with $\phi_i(0) = 0$ and $\phi_i'(0) \in [\pi_i^+(A_L - \varepsilon), \hat{\pi}_i^+(A_L - \varepsilon)]$. In order to emphasize the interval in which $\phi_i'(0)$ lies, we write $\pi_i^*(A_L - \varepsilon) := \phi_i'(0)$. By definition of relaxed solutions, we have

$$\text{either} \quad L(\lambda, \pi_1^*(A_L - \varepsilon), \dots, \pi_N^*(A_L - \varepsilon)) \leq 0 \tag{4.7}$$
$$\text{or} \quad -\lambda + (A_L - \varepsilon) \leq 0 \tag{4.8}$$

with $\lambda = -\partial_t \phi(t_0, x_0)$.

We claim that (4.8) always holds true. We argue by contradiction by assuming that $(A_L - \varepsilon) > \lambda$. In particular $A_L > \lambda$ and $\pi_i^+(A_L) > \pi_i^*(A_L - \varepsilon)$ for $i = 1, \dots, N$. Using (L3) and (4.7) successively, we have

$$L(A_L, \pi_1^+(A_L), \dots, \pi_N^+(A_L)) < L(\lambda, \pi_1^*(A_L - \varepsilon), \dots, \pi_N^*(A_L - \varepsilon)) \leq 0$$

which contradicts (4.3). The reader may remark that the contradiction cannot be reached without the use of ε.

We now consider

$$L_{A_L-\varepsilon}(-\partial_t\varphi, \partial_1\varphi, \ldots, \partial_N\varphi, x_0', D'\varphi) = \partial_t\phi(t_0, x_0') + \max(A_L - \varepsilon, \max_i H_i^-(\pi_i^*(A_L - \varepsilon))).$$

where the derivatives of φ in the left hand side are computed at (t_0, x_0).

Remark now that $H_i^-(\pi_i^*(A_L - \varepsilon)) = \min_{p_i\in\mathbb{R}} H_i(p_i)$ and in particular $\max_i H_i^-(\pi_i^*(A_L - \varepsilon)) = A_0$. Since $A_L - \varepsilon > A_0$ and $\lambda = -\partial_t\phi(t_0, x_0)$, the previous equality and (4.8) yield

$$L_{A_L-\varepsilon}(-\partial_t\varphi, \partial_1\varphi, \ldots, \partial_N\varphi, x_0', D'\varphi) = -\lambda + A_L(x_0', p_0') - \varepsilon \leq 0$$

which is the desired inequality. The proof is now complete. $\square$

5. The vanishing viscosity limit

This section is devoted to the study of the limit (as $\varepsilon \to 0$) of the solution u^ε of the following Hamilton-Jacobi equation posed on a multi-dimensional junction J,

$$\begin{cases} u_t^\varepsilon + H_i(t, x, Du^\varepsilon) = \varepsilon\Delta u^\varepsilon & (t, x) \in (0, T) \times J_i^*, \\ L(-u_t^\varepsilon, \partial_1 u^\varepsilon, \ldots, \partial_N u^\varepsilon, t, x', D'u^\varepsilon) = 0 & (t, x) \in (0, T) \times \Gamma \end{cases} \tag{5.1}$$

subject to the initial condition

$$u(0, x) = u_0(x), \quad x \in J. \tag{5.2}$$

Notice that this equation is not of the form (1.1) since the diffusion does not degenerate along the junction hyperplane. In particular, Theorem 3.2 does not hold true anymore in this case since it uses the degeneracy along Γ in an essential way. Still, we can consider relaxed solutions as in Definition 2.2, even if we expect solutions to be classical – see Remark 5.2 below. As we shall see it, the solutions u^ε converge towards the solution of

$$\begin{cases} u_t + H_i(t, x, Du) = 0 & (t, x) \in (0, T) \times J_i^*, \\ L(-u_t, \partial_1 u, \ldots, \partial_N u, t, x', D'u) = 0 & (t, x) \in (0, T) \times \Gamma. \end{cases} \tag{5.3}$$

The first result applies to general junction functions L.

Theorem 5.1 (Vanishing viscosity limit)**.** *Assume* (L) *and*

$$\begin{cases} H_i \text{ continuous} \\ \forall (t, x') \in (0, T) \times \Gamma, \lambda \in \mathbb{R}, \quad \{p = (p', p_i) \in \mathbb{R}^{d+1} : H_i(t, x', p) \leq \lambda\} \text{ convex} \\ \lim_{|p|\to+\infty} \inf_{(t,x')\in(0,T)\times\Gamma} H_i(t, x', p) = +\infty. \end{cases}$$

Let u_0 be uniformly continuous in J. Assume there exists a relaxed solution u^ε of (5.1)*,* (5.2) *and a constant C such that $|u^\varepsilon(t, x) - u_0(x)| \leq Ct$ for all $(t, x) \in (0, T) \times J$. Then u^ε converges locally uniformly towards the unique relaxed solution u of* (5.3)*,* (5.2)*.*

Remark 5.2. Even if we will not discuss it, the existence of solutions whose restriction to J_i are $C^{1,1}(J_i) \cap C^2(J_i^*)$ is expected in the case of (5.1). Some results are proved in [30, 31] on compact junctions and some others are announced in [23].

Remark 5.3. As we previously mentioned it, a special case of the theorem is proved in [9].

Proof of Theorem 5.1. By discontinuous stability, the relaxed upper limit $\bar{u}$ of u^ε is an L-relaxed sub-solution of (5.3), i.e. an A_L-flux-limited sub-solution of (5.3) (by Theorem 1.9). The relaxed lower limit $\underline{u}$ is an L-relaxed super-solution of (5.3), i.e. an A_L-flux-limited super-solution of (5.3) (by Theorem 1.9 again). Moreover, the fact that $|u^\varepsilon(t,x) - u_0(x)| \leq Ct$ holds true for all $(t,x) \in (0,T) \times J$ implies that $\bar{u}(0,x) = u_0(x) = \underline{u}(0,x)$ for all $x \in J$. By comparison principle [17, Theorem 1.3], we conclude that $\bar{u} \leq \underline{u}$ which yields the local uniform convergence towards the unique A_L-flux-limited solution of (5.3), (5.2) which coincides with the relaxed solution (by Theorem 1.9). □

Problem (1.4) can be translated into the junction framework as follows,

$$\begin{cases} u^\varepsilon_t + H_i(t,x,Du^\varepsilon) = \varepsilon \Delta u^\varepsilon, & (t,x) \in (0,T) \times J_i^*, \quad i = 1,2 \\ -\partial_1 u^\varepsilon(t,x',0) - \partial_2 u^\varepsilon(t,x',0) = 0, & (t,x') \in (0,T) \times \Gamma \\ u^\varepsilon(0,x) = u_0(x), & x \in J \end{cases}$$

with $H_1(x,p',p_{d+1}) = \tilde{H}_1(x,p',-p_{d+1})$ and $H_2(x,p',p_{d+1}) = \tilde{H}_2(x,p',p_{d+1})$. In view of Theorem 5.1, u^ε converges towards the relaxed solution

$$\begin{cases} u_t + H_i(t,x,Du) = 0, & (t,x) \in (0,T) \times J_i^* \\ u(0,x) = u_0(x), & x \in J \end{cases} \tag{5.4}$$

associated with the generalized flux function

$$L_e(p_0,p_1,p_2,t,x',p') = -p_1 - p_2.$$

Corollary 5.4 (The vanishing viscosity limit for the Kirchoff condition)**.** *The solution u^ε of (5.1), (5.2) converges towards the A_e-flux-limited solution of (5.4) where $A_e(t,x',p')$ is determined as follows: if $p_1^0(t,x',p') + p_2^0(t,x',p') \geq 0$ then $A_e(t,x',p') = A_0(t,x',p')$; else $A_e(t,x',p')$ is the unique $\lambda \geq A_0(t,x',p')$ such that there exist $p_1^{+,e}(t,x',p') \geq p_1^0(t,x',p')$ and $p_2^{+,e}(t,x',p') \geq p_2^0(t,x',p')$ such that*

$$H_i(t,x',p',p_i^{+,e}(t,x',p')) = \lambda \text{ for } i = 1,2, \quad p_1^{+,e}(t,x',p') + p_2^{+,e}(t,x',p') = 0.$$

Remark 5.5. If H_1 and H_2 has no constant parts and $p_1^0(t,x',p') + p_2^0(t,x',p') \leq 0$, then $A_e(t,x',p')$ is the only $A \in \mathbb{R}$ such that $\pi_1^+(t,x',p',A) + \pi_2^+(t,x',p',A) = 0$.

We now recall the result about maximal and minimal Ishii solutions from [17, Proposition 4.1].

Proposition 5.6 (Maximal and minimal Ishii solutions – [17, Proposition 4.1])**.** *The maximal (respectively the minimal) Ishii solution of (1.6) corresponds to the A_I^- (respectively A_I^+) flux-limited solution of (5.4) with*

$$\begin{cases} A_I^+(t,x',p') = \max(A_0(t,x',p'), A^*(t,x',p')) \\ A_I^-(t,x',p') = \begin{cases} A_I^+(t,x',p') & \text{if } \pi_2^0(t,x',p') + \pi_1^0(t,x',p') \leq 0 \\ A_0(t,x',p') & \text{if } \pi_2^0(t,x',p') + \pi_1^0(t,x',p') \geq 0 \end{cases} \end{cases}$$

where

$$A^*(t,x',p') = \max_{p_{d+1}\in I(t,x',p')} \big(\min(H_2(t,x',p',p_{d+1}), H_1(t,x',p',-p_{d+1})\big)$$

and $I(t,x',p') = [\min(-\pi_1^0(t,x',p'), \pi_2^0(t,x',p')), \max(-\pi_1^0(t,x',p'), \pi_2^0(t,x',p'))]$.

Remark 5.7. The functions p_i^0 and π_i^0 are different. The Hamiltonian H_i achieves its minimum at π_i^0 and it reaches the value A_0 at p_i^0. The only case where these functions coincide is when $A_0 = \min_{p_i} H_i(p_i)$ but in general $A_0 \geq \min_{p_i} H_i(p_i)$.

We now prove the following theorem, which is equivalent to Theorem 1.12.

Theorem 5.8 (The vanishing viscosity limit selects the maximal Ishii solution)**.** *Assume*

$$\begin{cases} H_i \text{ continuous} \\ \{p\in\mathbb{R}^{d+1} : H_i(t,x',0,p)\leq \lambda\} \text{ convex for all } \lambda\in\mathbb{R}, \\ \lim_{|p|\to+\infty} \inf_{(t,x')\in(0,T)\times\Gamma} H_i(t,x',p) = +\infty. \end{cases}$$

Then the relaxed solution u^ε *of* (5.1), (5.2) *converges towards the unique* A_I^-*-flux-limited solution of*

$$\begin{cases} u_t + H_i(x,Du) = 0, & x\in J_i^* \\ u(0,x) = u_0(x), & x\in J. \end{cases}$$

Proof. Once again, the tangential variables (t,x',p') are not shown in order to clarify the presentation.

In view of Corollary 5.4, we only have to prove that $A_e = A_I^-$ where A_I^- is given by Proposition 5.6.

If $\pi_1^0+\pi_2^0\geq 0$, then we know on the one hand from Proposition 5.6 that $A_I^- = A_0$ and on the other hand, since $p_1^0+p_2^0 \geq \pi_1^0+\pi_2^0\geq 0$, we know from Corollary 5.4 that $A_e = A_0$. We thus conclude that $A_e = A_0 = A_I^-$ in this case.

We now assume that $\pi_1^0+\pi_2^0\leq 0$. In particular, Proposition 5.6 implies that

$$A_I^- = A_I^+ = \max(A_0, A^*) \tag{5.5}$$

with

$$A^* = \max_{q\in[\pi_2^0,-\pi_1^0]} \min(H_1(-q), H_2(q)).$$

Remark that the function H_2 is non-decreasing on the interval $[\pi_2^0,-\pi_1^0]$ and the function $\tilde H_1(q) = H_1(-q)$ is non-increasing. We are going to distinguish three cases as shown in Figure 5. Either the graphs of H_2 and $\tilde H_1$ do not intersect on the interval $[\pi_2^0,-\pi_1^0]$ and H_2 is above (Case 1), or they do intersect (Case 2), or they do not intersect and $\tilde H_1$ is above (Case 3). To distinguish cases, it is enough to compare the values of $\tilde H_1$ and H_2 at the boundary of the interval.

It is useful to introduce $A_1 = \min_{p_1\in\mathbb{R}} H_1(p_1)$ and $A_2 = \min_{p_2\in\mathbb{R}} H_2(p_2)$. Recall that $A_0 = \max(A_1, A_2)$.

In Case 1, we have $H_2(\pi_2^0) = A_0 = A_2 \geq \tilde H_1(\pi_2^0)$. It implies that $\tilde H_1 \leq H_2$ on the interval. In particular $A^* = \tilde H_1(\pi_2^0) \leq A_0$. On the one hand, (5.5) implies that $A_I^- = A_0$. On the

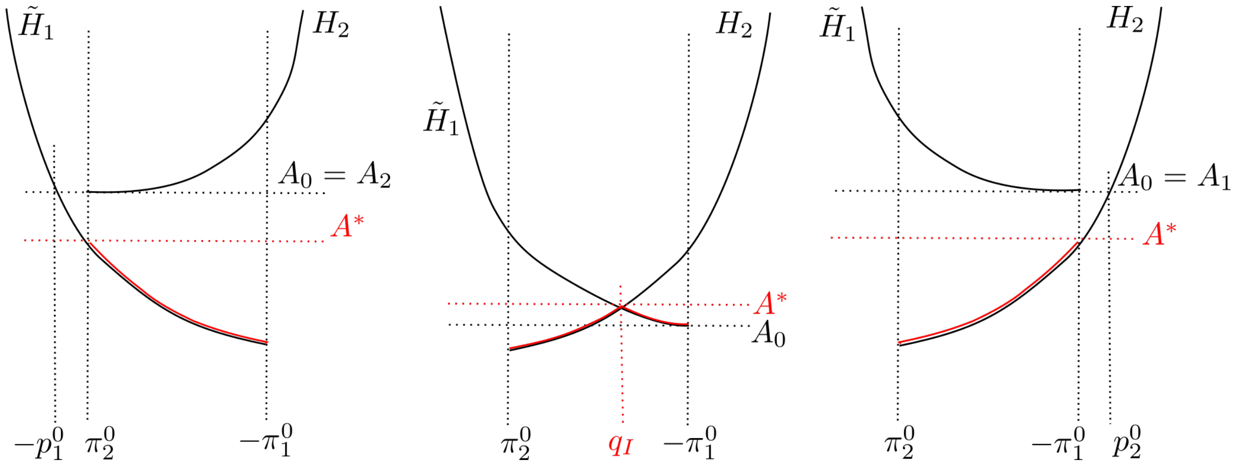

FIGURE 5. Three cases: Case 1 (left), Case 2 (center), Case 3 (right)

other hand, since $A_0 = A_2$, we have $p_2^0 := \pi_2^+(A_0) = \pi_2^+(A_2)$ and $p_1^0 := \pi_1^+(A_0) \geq -p_2^0$ (have a look at the picture). We thus conclude from Corollary 5.4 that $A_e = A_0$. Hence $A_I^- = A_e$ in Case 1.

In Case 2, there exists $q_I \in [\pi_2^0, -\pi_1^0]$ such that $A^* = H_2(q_I) = H_1(-q_I)$ and $A^* \geq A_0$. (5.5) implies that $A_I^- = A^*$. But the fact that $q_I \geq \pi_2^0$ such that $A^* = H_2(q_I)$ implies that $q_I = \pi_2^+(A^*)$; similarly, $-q_I = \pi_1^+(A^*)$; hence $\pi_1^+(A^*) + \pi_2^+(A^*) = 0$ with $A^* \geq A_0$. We thus have from Corollary 5.4 that $A_e = A^*$. Hence $A_I^- = A_e$ in Case 2.

In Case 3, $A_0 = A_1 \geq A^*$. (5.5) implies that $A_I^- = A_0$. We also remark that $-p_1^0 = -\pi_1^0 \leq \pi_2^+(A_0) = p_2^0$ (have a look at the picture). In particular, we have from Corollary 5.4 that $A_e = A_0$. We thus conclude that $A_I^- = A_e$ in Case 3.

The proof is now complete. □

6. A large deviation problem

In [8], the authors study large deviation problems related to diffusion processes whose drift is smooth on either side of a hyperplane. Their proofs rely on probability tools and ideas. Our goal in this section is to propose an analytical/PDE proof. Furthermore, by using the results of previous sections, the rate function is related to the maximal Ishii solution of a Hamilton-Jacobi equation.

Consider the stochastic differential equation in $\mathbb{R}^{d+1}$,

$$dX^\varepsilon(t) = b(X^\varepsilon(t))dt + \varepsilon^{1/2}\sigma(X^\varepsilon(t))dW(t), \quad X^\varepsilon(0) = x_0, \quad 0 \leq t \leq 1 \tag{6.1}$$

with

$$b(x) = \begin{cases} b_1(x) & \text{if } x_{d+1} < 0 \\ b_2(x) & \text{if } x_{d+1} > 0 \end{cases}$$

and

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x_{d+1} < 0 \\ \sigma_2(x) & \text{if } x_{d+1} > 0 \end{cases}$$

In order to introduce the rate function, we have to define first Hamiltonians and Lagrangians. Hamiltonians are defined in [8] by

$$\tilde{H}_i(x,p) = \frac{1}{2}\langle a_i(x)p, p\rangle - b_i(x)p, \quad x, p \in \mathbb{R}^{d+1}$$

with $a_i = \sigma_i \sigma_i^T$. Corresponding Lagrangians $\tilde{L}_1$ and $\tilde{L}_2$ are related to Hamiltonians $\tilde{H}_1$ and $\tilde{H}_2$ by the following formula [8]

$$\tilde{H}_i(x,p) = \sup_{q \in \mathbb{R}^{d+1}} \{-pq - \tilde{L}_i(x,q)\}.$$

Set $\Omega_1 = \mathbb{R}^d \times (-\infty, 0), \Omega_2 = \mathbb{R}^d \times (0, +\infty)$, $\mathcal{H} = \mathbb{R}^d \times \{0\}$.

$$\tilde{L}(x,p) = \begin{cases} \tilde{L}_1(x,p), & x \in \Omega_1, \\ \tilde{L}_2(x,p), & x \in \Omega_2, \\ \tilde{L}_0(x,p), & x \in \mathcal{H}, \end{cases} \tag{6.2}$$

where $\tilde{L}_0$ is defined by

$$\tilde{L}_0(x,p',q) = \inf \left\{ \lambda \tilde{L}_1(x,p',q_1) + (1-\lambda)\tilde{L}_2(x,p',q_2), \begin{cases} \lambda \in [0,1], q_1 \geq 0, q_2 \leq 0, \\ \lambda q_1 + (1-\lambda) q_2 = q \end{cases} \right\}.$$

Call Σ_{x_0} the set of all absolutely continuous function $\phi \in C([0,1], \mathbb{R}^{d+1})$ satisfying $\phi(0) = x_0$. For any $\phi \in \Sigma_{x_0}$, we define the rate function $I_{x_0}(\phi)$ as follows,

$$I_{x_0}(\phi) = \int_0^1 \tilde{L}(\phi(s), \dot{\phi}(s))\, ds \tag{6.3}$$

where $\tilde{L}$ is defined as in (6.2). We first state the Laplace principle as presented in [8]

Definition 6.1. Let $\{Y^\varepsilon(t), \varepsilon > 0, 0 \leq t \leq 1\}$ with $Y^\varepsilon(0) = x_0$ be a family of random variables taking values in a Polish space $\mathcal{Y}$ and let I_{x_0} be a rate function defined as in (6.3). We say that $\{Y^\varepsilon\}$ satisfies a Laplace principle with the rate function I_{x_0} if, for every bounded continuous function h mapping $\mathcal{Y}$ into $\mathbb{R}$, we have

$$\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{E}_{x_0}\big\{ \exp\big[-\frac{h(Y^\varepsilon)}{\varepsilon}\big]\big\} = -\inf_{\phi \in \Sigma_{x_0}} \{h(\phi(1)) + I_{x_0}(\phi)\}. \tag{6.4}$$

In [8], the following large deviation result is proved using probabilistic arguments. We will give a PDE proof.

Theorem 6.2 ([8]). *Assume that*

$$\begin{cases} b_i \text{ is continuous,} \\ \sigma \text{ is continuous and such that } \sigma\sigma^T \geq c\mathcal{I} \text{ with } c > 0, \\ (6.1) \text{ has a unique strong solution} \end{cases}$$

where $\mathcal{I}$ is the identity matrix. Then the family $\{X^\varepsilon, \varepsilon > 0\}$ satisfies the Laplace principle in $C([0,1], \mathbb{R}^{d+1})$ with the rate function I_{x_0} as defined in (6.3).

Proof. Given a function h, let h_ε denote $\exp(\frac{-h}{\varepsilon})$. The function u_ε given by

$$u_\varepsilon(t,x) = \mathbb{E}_x(h_\varepsilon(X^\varepsilon(t)))$$

is a solution of

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \frac{\varepsilon}{2}\,\mathrm{Trace}(a(x)D^2 u_\varepsilon) + b(x)Du_\varepsilon, & t \in (0,1), x \in \Omega_1 \bigcup \Omega_2 \\ \frac{1}{2}\partial_{d+1} u_\varepsilon(t,x',0^+) = \frac{1}{2}\partial_{d+1} u_\varepsilon(t,x',0^-), & x \in \mathcal{H} \\ u_\varepsilon(0,x) = h_\varepsilon(x), & x \in \Omega_1 \bigcup \Omega_2 \end{cases}$$

(where $a = \sigma\sigma^T$) The function $v_\varepsilon = -\varepsilon \ln(u_\varepsilon)$ satisfies

$$\begin{cases} \frac{\partial v_\varepsilon}{\partial t} = \frac{\varepsilon}{2}\,\mathrm{Trace}(a(x)D^2 v_\varepsilon) - \frac{1}{2}\langle a(x)Dv_\varepsilon, Dv_\varepsilon\rangle + b(x)Dv_\varepsilon, & t \in (0,1), x \in \Omega_1 \bigcup \Omega_2 \\ \frac{1}{2}\partial_{d+1} v_\varepsilon(t,x',0^+) = \frac{1}{2}\partial_{d+1} v_\varepsilon(t,x',0), & x \in \mathcal{H} \\ v_\varepsilon(0,x) = h(x), & x \in \Omega_1 \bigcup \Omega_2. \end{cases}$$

Moreover, in view of the definition of u_ε and v_ε, we have

$$v_\varepsilon(t,x) = -\varepsilon \ln \mathbb{E}_x \left\{ \exp\left[\frac{-h(X^\varepsilon(t))}{\varepsilon} \right] \right\}.$$

Hence, our goal is to prove that

$$\lim_{\varepsilon\to 0} v_\varepsilon(1,x) = \inf_{\phi\in\Sigma_x} \{h(\phi(1)) + I_x(\phi)\}$$

where I_x is defined in (6.3).

We know from Theorem 1.12 that v_ε converges locally uniformly towards the maximal Ishii solution U^+ of

$$\begin{cases} \frac{\partial U^+}{\partial t} + \tilde{H}_i(x, DU^+) = 0, & x \in \Omega_i, \quad t \in (0,1) \\ U^+(0,x) = h(x), & x \in \Omega_1 \bigcup \Omega_2. \end{cases} \tag{6.5}$$

It thus remains to prove that

$$U^+(1,x) = \inf_{\phi\in\Sigma_x} \{h(\phi(1)) + I_x(\phi)\}. \tag{6.6}$$

In view of the definition of Lagrangians and Hamiltonians from [8] recalled above, we have

$$\tilde{H}_i(x,p) = \sup_{q\in\mathbb{R}^{d+1}} \{pq - l_i(x,q)\} \quad \text{with} \quad l_i(x,-q) = \tilde{L}_i(x,q),$$

here l_i corresponds to the running costs considered in [18, Section 6]. In view of the definition of $\tilde{L}_0$ recalled above, we have

$$\begin{aligned}\tilde{L}_0(x,q',0) &= \inf\left\{\lambda\tilde{L}_1(x,q',q_1)+(1-\lambda)\tilde{L}_2(x,q',q_2), \begin{cases}0\le\lambda\le 1,\\ q_1\ge 0, q_2\le 0, \lambda q_1+(1-\lambda)q_2=0\end{cases}\right\}\\ &= \inf\left\{\lambda l_1(x,q',v_1)+(1-\lambda)l_2(x,q',v_2), \begin{cases}0\le\lambda\le 1,\\ v_1\le 0, v_2\ge 0, \lambda v_1+(1-\lambda)v_2=0\end{cases}\right\}.\end{aligned}$$

Hence, the formula of U^+ given in [17, 7] coincides with (6.6). The proof is now complete. □

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