Uniform result for solutions of an equation with boundary singularity.
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UNIFORM RESULT FOR SOLUTIONS OF AN EQUATION WITH BOUNDARY SINGULARITY.

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ABSTRACT. We give a blow-up analysis for a Brezis-Merle’s type problem, with singularity and with Dirichlet condition. An application, we have a proof of a compactness result for this problem.

Keywords: blow-up, boundary, singularity, a priori estimate, Lipschitz condition.

1. INTRODUCTION AND MAIN RESULTS

We set \( \Delta = \partial_{11} + \partial_{22} \) on open set \( \Omega \) of \( \mathbb{R}^2 \) with a smooth boundary.

We consider the following equation:

\[
(P_\beta) \begin{cases}
-\Delta u = |x|^{2\beta}Ve^u & \text{in } \Omega \subset \mathbb{R}^2, \\
u = 0 & \text{in } \partial\Omega.
\end{cases}
\]

Here:

\( \beta \in (0, 1), \ 0 \in \partial\Omega \)

and,

\( u \in W_0^{1,1}(\Omega), \ |x|^{2\beta}e^u \in L^1(\Omega) \) and \( 0 \leq V \leq b \).

The above equation was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1-15], we can find some existence and compactness results.

Among other results, we can see in [6] the following important Theorem. (as in [6], we use the first eigenfunction \( \varphi_1 \) and the expansion of \( \varphi_1 \) near 0 to bound uniformly \( \int_K |x|^{2\beta}Ve^u \) for all compact \( K \) of \( \Omega \), because \( 0 < \beta < 1, |x-x_0|^2 \leq C|x-x_0|^{2\beta} \) and we use Holder inequality in \( \int u_i\varphi_1 \leq \left( \int u_i^\gamma \varphi_1^4 \right)^{1/8} \left( \int \varphi_1^{4/7} \right)^{8/7} \leq C(\int |x-x_0|^2\varphi_1Ve^u)^{1/8}, \) here \( x_0 = 0 \).

**Theorem (Brezis-Merle [6]).** If \( (u_i), \) and \( (V_i), \) are two sequences of functions relative to the problem \( P_0) \) with, \( 0 < a \leq V_i \leq b < +\infty, \) then,
\[ \sup_K u_i \leq c, \]

with \( c \) depending on \( a, b, K \) and \( \Omega \).

One can find in [6] an interior estimate if we assume \( a = 0 \), but we need an assumption on the integral of \( e^{u_i} \), namely, we have:

**Theorem B** (Brezis-Merle [6]). For \( (u_i)_i \) and \( (V_i)_i \) two sequences of functions relative to the problem \((P_0)\) with,

\[ 0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C, \]

then it holds;

\[ \sup_K u_i \leq c, \]

with \( c \) depending on \( b, C, K \) and \( \Omega \).

When \( a = 0 \), the boundedness of \( \int_\Omega e^{u_i} \) is a necessary condition to work on the problem \((P_\beta)\) as showed in [6] by the following counterexample.

**Theorem C** (Brezis-Merle [6]). There are two sequences \( (u_i)_i \) and \( (V_i)_i \) of the problem \((P_0)\) with,

\[ 0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C, \]

such that,

\[ \sup_\Omega u_i \to +\infty. \]

In the regular case \((\beta = 0)\) this equation has many properties:

Note that for the problem \((P_0)\), by using the Pohozaev identity, we can prove that \( \int_\Omega e^{u_i} \) is uniformly bounded when \( 0 < a \leq V_i \leq b < +\infty \) and \( ||\nabla V_i||_{L^\infty} \leq A \) and \( \Omega \) starshaped, when \( a = 0 \) and \( \nabla \log V_i \) is uniformly bounded, we can bound uniformly \( \int_\Omega V_i e^{u_i} \). In [14], Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped.

In [9], Chen-Li have proved that if \( a = 0 \) and \( \nabla \log V_i \) is uniformly bounded, then the functions are uniformly bounded near the boundary.

In [9], Chen-Li have proved that if \( a = 0 \) and \( \int_\Omega e^{u_i} \) is uniformly bounded and \( \nabla \log V_i \) is uniformly bounded, then we have the compactness result directly. Ma-Wei in [14], extend this result in the case where \( a > 0 \).
If we assume $V$ more regular, we can have another type of estimates, a $\sup + \inf$ type inequalities. It was proved by Shafrir see [15], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with $A$ the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see [5].

In this paper we look to the behavior of the blow-up points on the boundary and we have a proof of compactness result when we assume $V_i$ uniformly Lipschitzian.

Here, we write an extension of Brezis-Merle Problem (see [6]) is:

**Problem.** Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$, with, $0 \leq V_i \leq b$ for some positive constant $b$. Also, we consider a sequence of solutions $(u_i)$ of $(P)$ relatively to $(V_i)$ such that,

$$\int_{\Omega} |x|^{2\beta} e^{u_i} dx \leq C,$$

is it possible to have:

$$||u_i||_{L^\infty} \leq C = C(V, \beta, \Omega, C)?$$

Here, we give a caracterization of the behavior of the blow-up points on the boundary and also a proof of the compactness theorem when $V_i$ are uniformly Lipschitzian. For the behavior of the blow-up points on the boundary, the following condition is enough,

$$0 \leq V_i \leq b,$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary.

But for the proof of the compactness result (for the Brezis-Merle type problem) we assume that:

$$||\nabla V_i||_{L^\infty} \leq A.$$

We have the following caracterization of the behavior of the blow-up points on the boundary.

**Theorem 1.1.** Assume that $\max_{\Omega} u_i \to +\infty$, where $(u_i)$ are solutions of the probleme $(P)$ with:

$$0 \leq V_i \leq b, \text{ and } \int_{\Omega} |x|^{2\beta} e^{u_i} dx \leq C, \forall \ i,$$
then; after passing to a subsequence, there is a function \( u \), there is a number \( N \in \mathbb{N} \) and \( N \) points \( x_1, x_2, \ldots, x_N \in \partial \Omega \), such that,

\[
\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^{N} \alpha_j \delta_{x_j}, \quad \alpha_j \geq 4\pi \text{ weakly in the sense of measure } L^1(\partial \Omega).
\]

and,

\[
u_i \to u \text{ in } C^1_{loc}(\bar{\Omega} - \{x_1, \ldots, x_N\}).
\]

In the following theorem, we have a new proof for the global a priori estimate which concern the problem \( (P) \).

**Theorem 1.2.** Assume that \( (u_i) \) are solutions of \( (P) \) relative to \( (V_i) \) with the following conditions:

\[
\beta \in (0, 1), \quad 0 \in \partial \Omega,
\]

and,

\[
0 \leq V_i \leq b, \quad ||\nabla V_i||_{L^\infty} \leq A, \quad \int_{\Omega} |x|^{2\beta} e^{u_i} \leq C,
\]

We have,

\[
||u_i||_{L^\infty} \leq c(b, \beta, A, C, \Omega),
\]

2. **Proof of the theorems**

**Proof of theorem 1.1:**

We have,

Since \( \int_{\Omega} |x|^{2\beta} e^{u_i} \leq C \), we have, by the Brezis-Merle result see [6], \( e^{k u_i} \in L^1(\Omega) \) for \( k > 2 \) and the elliptic estimates imply that

\[
u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).
\]

We denote by \( \partial_\nu u_i \) the inner normal derivative of \( u_i \). By the maximum principle, \( \partial_\nu u_i \geq 0 \).

By the Stokes formula, we obtain
\[ \int_{\partial \Omega} \partial_{\nu} u_{i} d\sigma \leq C. \]

Thus, using the weak convergence in the space of Radon measures, we have the existence of a positive Radon measure \( \mu \) such that

\[ \int_{\partial \Omega} \partial_{\nu} u_{i} d\sigma \leq C, \]

Without loss of generality, we can assume that \( \partial_{\nu} u_{i} > 0 \). Thus, using the weak convergence in the space of Radon measures, we have the existence of a positive Radon measure \( \mu \) such that,

\[ \int_{\partial \Omega} \partial_{\nu} u_{i} \varphi d\sigma \rightarrow \mu(\varphi), \quad \forall \ var \in C^{0}(\partial \Omega). \]

We take an \( x_{0} \in \partial \Omega \) such that, \( \mu(x_{0}) < 4\pi \). Without loss of generality, we can assume that the following curve, \( B(x_{0}, \epsilon) \cap \partial \Omega := I_{\epsilon} \) is an interval. (In this case, it is more simple to construct the following test function \( \eta_{\epsilon} \)). We choose a function \( \eta_{\epsilon} \) such that,

\[ \begin{cases} 
\eta_{\epsilon} \equiv 1, & \text{on } I_{\epsilon}, \ 0 < \epsilon < \delta/2, \\
\eta_{\epsilon} \equiv 0, & \text{outside } I_{2\epsilon}, \\
0 \leq \eta_{\epsilon} \leq 1, \\
\|\nabla \eta_{\epsilon}\|_{L^{\infty}(I_{2\epsilon})} \leq \frac{C_{0}(\Omega, x_{0})}{\epsilon}.
\end{cases} \]

We take \( \tilde{\eta}_{\epsilon} \) such that,

\[ \begin{cases} 
-\Delta \tilde{\eta}_{\epsilon} = 0 & \text{in } \Omega \\
\tilde{\eta}_{\epsilon} = \eta_{\epsilon} & \text{on } \partial \Omega.
\end{cases} \]

We use the following estimate, see [7]

\[ \|\nabla u_{i}\|_{L^{q}} \leq C_{q}, \quad \forall \ i \text{ and } 1 < q < 2. \]

We deduce from the last estimate that, \( (u_{i}) \) converge weakly in \( W_{0}^{1,q}(\Omega) \), almost everywhere to a function \( u \geq 0 \) and \( \int_{\Omega} e^{u} < +\infty \) (by Fatou lemma). Also, \( V_{i} \) weakly converge to a nonnegative function \( V \) in \( L^{\infty} \). The function \( u \) is in \( W_{0}^{1,q}(\Omega) \) solution of:

\[ \begin{cases} 
-\Delta u = |x|^{2\beta} V e^{u} \in L^{1}(\Omega) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \]
As in the corollary 1 of Brezis-Merle result, see [6], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in C^1(\overline{\Omega})$.

We can write,

$$-\Delta((u_i - u)\tilde{\eta}_\epsilon) = |x|^{2\beta} (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon + 2 < \nabla (u_i - u) |\nabla \tilde{\eta}_\epsilon > .$$  \hfill (1)

We use the interior estimate of Brezis-Merle, see [6].

**Step 1:** Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and $u$, we obtain,

$$\int_\Omega |x|^{2\beta} V e^{u} \tilde{\eta}_\epsilon dx = \int_{\partial \Omega} \partial_\nu u \eta \tilde{\eta}_\epsilon \leq 4\epsilon ||\partial_\nu u||_{L^\infty} = C\epsilon$$  \hfill (2)

We have,

\[
\begin{cases}
-\Delta u_i = |x|^{2\beta} V_i e^{u_i} & \text{in } \Omega \\
u_i = 0 & \text{on } \partial \Omega,
\end{cases}
\]

We use the Green formula between $u_i$ and $\tilde{\eta}_\epsilon$ to have:

$$\int_\Omega |x|^{2\beta} V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial \Omega} \partial_\nu u_i \eta \tilde{\eta}_\epsilon d\sigma \to \mu(\eta_\epsilon) \leq \mu(I_{2\epsilon}) \leq 4\pi - \epsilon_0, \ \epsilon_0 > 0$$  \hfill (3)

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_\Omega |x|^{2\beta}|(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon$$  \hfill (4)

**Step 2:** Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{ x \in \Omega, d(x, \partial \Omega) = \epsilon^3 \}$ and $\Omega_{\epsilon^3} = \{ x \in \Omega, d(x, \partial \Omega) \geq \epsilon^3 \}, \epsilon > 0$. Then, for $\epsilon$ small enough, $\Sigma_\epsilon$ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2 \epsilon^3 \leq \mu_L (\Omega - \Omega_{\epsilon^3}) \leq k_1 \epsilon^3$.

**Remark:** for the unit ball $\bar{B}(0, 1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^3)$.

We write,

$$\int_\Omega \frac{1}{4\pi} < \nabla (u_i - u) |\nabla \tilde{\eta}_\epsilon > |dx = \int_{\Omega_{\epsilon^3}} \frac{1}{4\pi} < \nabla (u_i - u) |\nabla \tilde{\eta}_\epsilon > |dx +$$
\[ + \int_{\Omega - \Omega,3} < \nabla (u_i - u) | \nabla \tilde{\eta}_k > |dx. \]  

(5)

**Step 2.1:** Estimate of \( \int_{\Omega - \Omega,3} < \nabla (u_i - u) | \nabla \tilde{\eta}_k > |dx. \)

First, we know from the elliptic estimates that \( \| \nabla \tilde{\eta}_k \|_{L^{\infty}} \leq C_1/\epsilon^2 \), \( C_1 \) depends on \( \Omega \).

We know that \( (|\nabla u_i|)_i \) is bounded in \( L^q \), \( 1 < q < 2 \), we can extract from this sequence a subsequence which converge weakly to \( h \in L^q \). But, we know that we have locally the uniform convergence to \( |\nabla u| \) (by Brezis-Merle theorem), then, \( h = |\nabla u| \) a.e. Let \( q' \) be the conjugate of \( q \).

We have, \( \forall f \in L^{q'}(\Omega) \)

\[ \int_{\Omega} |\nabla u_i| f dx \to \int_{\Omega} |\nabla u| f dx \]

If we take \( f = 1_{\Omega - \Omega,3} \), we have:

for \( \epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, \ i \geq i_1, \ \int_{\Omega - \Omega,3} |\nabla u_i| \leq \int_{\Omega - \Omega,3} |\nabla u| + \epsilon^3. \)

Then, for \( i \geq i_1(\epsilon) \),

\[ \int_{\Omega - \Omega,3} |\nabla u_i| \leq mes(\Omega - \Omega,3)||\nabla u||_{L^{\infty}} + \epsilon^3 = \epsilon^3(k_1||\nabla u||_{L^{\infty}} + 1). \]

Thus, we obtain,

\[ \int_{\Omega - \Omega,3} < \nabla (u_i - u) | \nabla \tilde{\eta}_k > |dx \leq \epsilon C_1(2k_1||\nabla u||_{L^{\infty}} + 1) \]  

(6)

The constant \( C_1 \) does not depend on \( \epsilon \) but on \( \Omega \).

**Step 2.2:** Estimate of \( \int_{\Omega,3} < \nabla (u_i - u) | \nabla \tilde{\eta}_k > |dx. \)

We know that, \( \Omega, \subset \subset \Omega \), and ( because of Brezis-Merle’s interior estimates) \( u_i \to u \) in \( C^1(\Omega,3) \). We have,

\[ ||\nabla (u_i - u)||_{L^{\infty}(\Omega,3)} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon). \]

We write,
\[
\int_{\Omega_3} |\nabla (u_i - u)| \nabla \tilde{\eta} |dx \leq \|\nabla (u_i - u)\|_{L^\infty(\Omega_3)} \|\nabla \tilde{\eta}\|_{L^\infty} \leq C_1 \epsilon \text{ for } i \geq i_3.
\]

For \( \epsilon > 0 \), we have for \( i \in \mathbb{N}, i \geq \max\{i_1, i_2, i_3\} \),

\[
\int_{\Omega} |\nabla (u_i - u)| \nabla \tilde{\eta} |dx \leq \epsilon C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2) \tag{7}
\]

From (4) and (7), we have, for \( \epsilon > 0 \), there is \( i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max\{i_0, i_1, i_2\} \) such that,

\[
\int_{\Omega} |\Delta [(u_i - u) \tilde{\eta}]|dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1 (2k_1 \|\nabla u\|_{L^\infty} + 2 + C) \tag{8}
\]

We choose \( \epsilon > 0 \) small enough to have a good estimate of (1).

Indeed, we have:

\[
\begin{cases}
-\Delta [(u_i - u) \tilde{\eta}] = g_{i,\epsilon} & \text{in } \Omega, \\
(u_i - u) \tilde{\eta} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

with \( \|g_{i,\epsilon}\|_{L^1(\Omega)} \leq 4\pi - \epsilon_0 \).

We can use Theorem 1 of [6] to conclude that there is \( q > 1 \) such that:

\[
\int_{V_{\epsilon}(x_0)} e^{g_{(u_i-u)}}dx \leq \int_{\Omega} e^{g_{(u_i-u)\tilde{\eta}}}dx \leq C(\epsilon, \Omega).
\]

where, \( V_{\epsilon}(x_0) \) is a neighborhood of \( x_0 \) in \( \bar{\Omega} \).

Thus, for each \( x_0 \in \partial \Omega - \{\bar{x}_1, \ldots, \bar{x}_m\} \) there is \( \epsilon_{x_0} > 0, q_1 > 1 \) such that:

\[
\int_{B(x_0, \epsilon_{x_0})} e^{q_1 u_i}dx \leq C, \quad \forall \ i.
\]

By the elliptic estimates, \((u_i \eta)_i\) is uniformly bounded in \( W^{2,q_1}(\Omega) \) and also, in \( C^1(\bar{\Omega}) \).

Finally, we have, for some \( \epsilon > 0 \) small enough,

\[
\|u_i\|_{C^1(\partial B(x_0, \epsilon))} \leq c_3 \quad \forall \ i. \tag{9}
\]

We have proved that, there is a finite number of points \( \bar{x}_1, \ldots, \bar{x}_m \) such that the sequence \((u_i)_i\) is locally uniformly bounded in \( \Omega - \{\bar{x}_1, \ldots, \bar{x}_m\} \).
**Proof of theorem 1.2:**

Without loss of generality, we can assume that 0 is a blow-up point (either, we are in the regular case). Also, by a conformal transformation, we can assume that \( \Omega = B_1^+ \), the half ball, and \( \partial^+ B_1^+ \) is the exterior part, a part which not contain 0 and on which \( u_i \) converge in the \( C^1 \) norm to \( u \). Let us consider \( B_\epsilon^+ \), the half ball with radius \( \epsilon > 0 \).

The Second Pohozaev identity gives (see [14]):

\[
2(1+\beta) \int_{B_\epsilon^+} |x|^{2\beta} V_i e^{u_i} dx + \int_{B_\epsilon^+} < x|\nabla V_i > |x|^{2\beta} V_i e^{u_i} dx + \int_{\partial B_\epsilon^+} < \nu |x > |x|^{2\beta} V_i e^{u_i} d\sigma = \int_{\partial^+ B_1^+} g(\nabla u_i) d\sigma, \tag{10}
\]

with,

\[
g(\nabla u_i) = < \nu |\nabla u_i > < x|\nabla u_i > - < x|\nu > \frac{|\nabla u_i|^2}{2}.
\]

also,

\[
2(1+\beta) \int_{B_\epsilon^+} |x|^{2\beta} V e^{u} dx + \int_{B_\epsilon^+} < x|\nabla V > |x|^{2\beta} V e^{u} dx + \int_{\partial B_\epsilon^+} < \nu |x > |x|^{2\beta} V e^{u} d\sigma = \int_{\partial^+ B_1^+} g(\nabla u) d\sigma, \tag{11}
\]

Thus,

\[
2(1+\beta) \int_{B_\epsilon^+} |x|^{2\beta} V_i e^{u_i} dx - 2(1+\beta) \int_{B_\epsilon^+} |x|^{2\beta} V e^{u} dx +
\]

\[
+ \int_{B_\epsilon^+} < x|\nabla V_i > |x|^{2\beta} V_i e^{u_i} dx - \int_{B_\epsilon^+} < x|\nabla V > |x|^{2\beta} V e^{u} dx + o(1) =
\]

\[
= \int_{\partial^+ B_1^+} g(\nabla u_i) - g(\nabla u) d\sigma = o(1),
\]

First, we tend \( i \) to infinity after \( \epsilon \) to 0, we obtain:

\[
\lim_{\epsilon \to 0 \leftarrow i \to +\infty} 2(1+\beta) \int_{B_\epsilon^+} |x|^{2\beta} V_i e^{u_i} dx = 0, \tag{12}
\]

But,

\[
\int_{B_\epsilon^+} |x|^{2\beta} V_i e^{u_i} dx = \int_{\partial B_\epsilon^+} \partial_{\nu} u_i + o(\epsilon) + o(1) \to \alpha_1 > 0.
\]
A contradiction.

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