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Volume of the steady-state space of financial flows in a monetary stock-flow-consistent model

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Abstract

We show that a steady-state stock-flow consistent macro-economic model can be represented as a Constraint Satisfaction Problem (CSP). The set of solutions is a polytope, which volume depends on the constraints applied and reveals the potential fragility of the economic circuit, with no need to study the dynamics. Several methods to compute the volume are compared, inspired by operations research methods and the analysis of metabolic networks, both exact and approximate. We also introduce a random transaction matrix, and study the particular case of linear flows with respect to money stocks.

Keywords: economics, physics and society, constraint satisfaction, monte-carlo, random network, convex polytope, finance

In this article we propose an approach to macro-economic modeling inspired by stock-flow consistent (SFC) models [1] and statistical physics, solving a Constraint Satisfaction Problem (CSP) in a way similar to recent works in the field of metabolic networks [2]. The SFC framework provides accounting identities ensuring that "everything comes from somewhere and everything goes some where" [1, p.38], thanks to budget constraints and behavioral constraints. The formalism of DSGE (Dynamic Stochastic General Equilibrium) is dominant today in macro-economics, partly because the corresponding models can be written in the form of state-space models and estimated in a well-studied statistical framework¹. Their usefulness has been widely debated among economists [4, 5] and physicists [6] because of their inability to predict crises. Many of their hypotheses have been criticized, such as representative rational agents, exogeneity of financial factors, clearing markets where offer always meet demand, etc. . . . Moreover, DSGE models usually do not implement SFC accounting identities.

Most SFC works take place at the macroeconomic aggregate level. Various assets (loans, equities, bonds, . . .) and sectors (households, firms, banks, states, . . .) have been considered in the literature [7]. Models can be more or less detailed, depending on the focus of the study (for example, the production sector can be aggregated or multi-sectoral). The issue of the micro-foundations of SFC models has been tackled with the combination of SFC and agent-based models (ABM). ABM [8, 9] can represent large populations of heterogeneous agents, to explore the influence of networks effects, coordination, bounded rationality and learning. However, they do not usually implement stock-flow consistency. Recent works combine SFC and ABM [10, 11, 12], providing micro-foundations to SFC models, and imposing macro constraints to ABMs. Nevertheless, the computational cost of ABM simulations is high, and theoretical understanding is limited so far. Calibration and validation are known to be difficult problems.

We consider a simplified stock-flow consistent model developed by macro-economists, where the state of the economy is the set of all stocks and flows of money. It is shown that one can compute the set of admissible steady-state configurations of this simple model. In this steady-state solution space, all configurations are equally weighted, thus allowing unusual states of the financial flows to be encompassed. The marginal probabilities of individual configurations can be approximated over the whole solution space.

¹ see [3, §3.2] for a discussion.

Our standpoint is to transpose ideas from the field of metabolic networks where steady-state fluxes have been studied as CSP. These studies were in turn inspired by Von Neumann’s growth model of production economies [13]. The results obtained with metabolic networks were successfully compared to experimental data, as in the Red Blood Cell metabolism or the central metabolism of E.coli [14, 2, 15, 16]. Such system-scale studies reveal some interesting features of metabolisms, for example the cooperation between pathways. It has been shown also that organisms such as E.coli do not necessarily optimize their metabolic fluxes.

The steady-state equilibrium hypothesis is accepted in the field of metabolic networks because of the separation of timescales between metabolic and genetic regulations [17, 18]. In economy, the existence of cycles and their corresponding time constants has been the subject of many theories and debates. Recent empirical works are able to identify the timescale at which some specific phenomena operate [19]. In the case of the model examined in this article, we consider that at the time scale that separates two balance sheets (one year), capital accumulation and output growth are slow and will be considered constant (as noted in [20], the global annual per capita growth rate of production is 0.8 % on average on the 1700-2012 time interval).

The expected benefits of applying these methods in macro-economics include the analysis of fragilities, notably the sensitivity to arbitrary flow constraints, such as shortages. Indeed, the volume of the solution space evoked above is immediately impacted when constraints are added or removed, and can reveal the flexibility or rigidity of financial flows subject to perturbations.

In section 1, we detail the model of financial flows that will be used as a benchmark, and present its background from a macro-economic modeling point of view. Comparisons are made with ABM and econophysics. We present the different methods used to compute exactly and approximately the volume of the steady-state solution space. Then in section 2, the experimental results are explained. Finally, sections 3 and 4 are devoted to discussion and conclusion.

1. Background and methods

1.1. Steady-state solution space in a stock-flow-consistent model

In SFC models agents are grouped by sectors (banks, firms, workers, state, central bank) that are linked by money transfers. For example:

- Banks lend money to firms, which pay interests to the former.
- Banks pay interests on deposits made by workers.
- Firms pay wages to workers.
- Workers buy consumption goods to firms.
- Firms invest in capital goods bought from other firms.

Assets and liabilities at a given instant t in time are summarized in a balance sheet, where positive and negative signs stand for uses and sources of money. The balance sheet in Tab. 1 corresponds to the BMW model, discussed in [1, chap. 7], which will be used in this article. In the BMW model, bank issue loans to finance the investments of the productive sector, while households are both consumers and workers. The state and central bank are omitted. Production firms and the banks make no net profit. The net worth of these sectors is zero. The net wealth equals the total tangible capital K .

The transaction matrix sums up all the flows of funds between sectors within a time interval $[t, t + \Delta t]$. Positive and negative signs stand for inflows and outflows of money. They are balanced using a double-entry book-keeping representation where rows sum to zero since each transaction has a counterparty, and columns sum to zero because of the sector’s budget constraints.

Tab. 2 shows the transaction matrix corresponding to the BMW model with one agent per sector. After [1, chap. 7], we make the hypothesis that demand terms, with the subscript d , equal supply terms, with the subscript s . Notations are summarized in Tab. 3.

	Households	Production Firms	Banks	Σ
Money deposits	+M		-M	0
Loans		-L	+L	0
Fixed capital		+K		+K
Balance (net worth)	$-V_h$	0	0	$-V_h$
Σ	0	0	0	0

Table 1: Balance sheet of the BMW model. M, L, K are the money deposits, loans, and tangible capital. V_h is the net worth of households.

	Households	Production Firms		Banks		Σ
		Current	Capital	Current	Capital	
Consumption	$-C_d$	C_s				0
Investment		I_s	$-I_d$			0
Wages	WB_s	$-WB_d$				0
Depreciation		$-AF_d$	AF_s			0
Interest on loans		$-IL_d$		IL_s		0
Interest on deposits	ID_s			$-ID_d$		0
Change in loans			ΔL		$-\Delta L$	0
Change in deposits	$-\Delta M$				ΔM	0
Σ	0	0	0	0	0	0

Table 2: Transaction matrix of the BMW model. In the stationary case, $\Delta L = \Delta M = 0$.

In the time-dependent case, the balance sheet at time $t + \Delta t$ is obtained adding stocks M, L at time t to changes in deposits and loans ΔM and ΔL that occurred during the time interval $[t, t + \Delta t]$.

Since in this article the stationary steady-state case is considered, the balance sheet and the transaction matrix are constants, and the change terms $\Delta M, \Delta L$ are set to zero.

Furthermore in the BMW model a set of behavioral equations expresses several flows as linear functions of the other variables and of fixed parameters. The demand for consumption in equation Eq. (1) depends on the disposable income and on the money deposit of the household:

$$\begin{aligned}
C_d &= \alpha_0 + \alpha_1 YD + \alpha_2 M \\
&= \alpha_0 + \alpha_1 (WB_s + rM) + \alpha_2 M
\end{aligned} \tag{1}$$

where $\alpha_0, \alpha_1, \alpha_2$ are consumption parameters, and YD is the disposable income. The depreciation of tangible capital is proportional to its stock:

$$AF = \delta K \tag{2}$$

where δ is the rate of depreciation. Interests are proportional to stocks:

$$IL_s = rL \tag{3}$$

$$ID_s = rM \tag{4}$$

where the interest rate r is the same for deposits and loans, for simplicity.

The different constraints (row sums, columns sums, behavioral equations, demand equals supply) can be written in matrix form:

$$S = \{x \text{ s.t. } \xi x = b, \forall i \ x_i \in [0, x_i^{\max}]\} \tag{5}$$

where x_i^{\max} sets the maximum flow for each component, and:

$$x = [C_s \ C_d \ I_s \ I_d \ WB_s \ WB_d \ AF_s \ AF_d \ IL_s \ IL_d \ ID_s \ ID_d]^T \tag{6}$$

and:

$$\xi = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (7)$$

Then:

$$b = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -\alpha_0 - (\alpha_1 r + \alpha_2)M \ -\delta K \ -L \ rM]^T \quad (8)$$

ξ is an $m \times n$ matrix with $m = 14$ and $n = 12$, m being the number of equations, and n the number of unknown flows. The first six rows of ξ represent the row sums constraints of the transaction matrix. The following four rows represent the budget constraints. The last four rows are equivalent to the behavioral equations. Closed form solutions of S are studied in [1, chap. 7].

Constraints can be added or modified: for example, if a firm goes bankrupt, the loan will not be completely repaid, thus equation $IL_s = rL$ may be replaced by $IL_s \leq rL$. Agents may spend less in consumption than what is prescribed by the behavioral equation, which can be written $C_d \leq \alpha_0 + \alpha_1 YD + \alpha_2 M$.

In section 1.2, the scope of the model is extended to the case of multiple agents per sector. In that case, the linear system will be under-determined.

Variable	Label
money deposit	M
capital	K
loans to firms	L
investment	I
interest on loans	IL
wage bill	WB
depreciation allowance	AF
interest on workers deposits	ID
consumption of workers	C

Table 3: Labels associated with the different monetary variables, after [1]. The subscripts d and s stand for demand and supply.

1.2. Many agents per sector

To extend the model of section 1.1 to the case of many agents per sector, the scalars in matrix ξ will be replaced by block matrices. The number of banks, firms and workers are noted nb, nf, nw . Each block will be designed to account not only for the flows and stocks but also for the connectivity between agents.

For example, the first row of matrix ξ , $[1, -1, 0, \dots, 0]$, may be written with blocks instead of scalars, in order to encode the relationship between demand and supply of consumption goods. The positive term, that corresponds to the supply side of consumption, can be replaced by the identity matrix I_{nf} . The negative

	Households			Firms		Banks		Σ
	1	2	3	1	2	1	2	
Money deposits	M_1						$-M_1$	0
		M_2					$-M_2$	0
			M_3				$-M_3$	0
Loans				$-L_1$		L_1		0
					$-L_2$		L_2	0
Fixed capital				K_1				K_1
					K_2			K_2
Balance (net worth)	$-V_{h1}$	$-V_{h2}$	$-V_{h3}$					$-\sum_i V_{hi}$
Σ	0	0	0	0	0	0	0	0

Table 4: Example of balance sheet of the BMW model with many agents $nw = 3$, $nf = 2$, $nb = 2$. Households and firms randomly choose their bank. M_i, L_j, K_k are the individual money deposits, loans, and tangible capital. V_{hi} is the net worth of individual households.

	Households			Production Firms				Banks				Σ	
	1	2	3	Current		Capital		Current		Capital			
				1	2	1	2	1	2	1	2		
Consumption	$-C_{d1}$			C_{d1}									0
		$-C_{d2}$		C_{d2}									0
			$-C_{d3}$		C_{d3}								0
Investment					I_{s1}	$-I_{s1}$							0
				I_{s2}			$-I_{s2}$						0
Wage	WB_{s1}					$-WB_{s1}$							0
		WB_{s2}		$-WB_{s2}$									0
			WB_{s3}	$-WB_{s3}$									0
Depreciation				$-AF_1$		AF_1							0
					$-AF_2$		AF_2						0
Interest on loans				$-IL_1$				IL_1					0
					$-IL_2$				IL_2				0
Interest on deposits	ID_1								$-ID_1$				0
		ID_2							$-ID_2$				0
			ID_3						$-ID_3$				0
Change in loans							ΔL_1			$-\Delta L_1$			0
								ΔL_2			$-\Delta L_2$		0
Change in deposits	$-\Delta M_1$										ΔM_1		0
		$-\Delta M_2$								ΔM_2			0
			$-\Delta M_3$								ΔM_3		0
Σ	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 5: Example of transaction matrix of the BMW model with many agents $nw = 3$, $nf = 2$, $nb = 2$. Households randomly choose firms to buy consumption goods. Firms randomly choose other firms to buy capital goods, and the workers they hire. Interests and changes in loans and deposits are set according to the balance sheet. In the stationary case, $\Delta L = \Delta M = 0$.

Rows 1 to 6 of ξ express the row sums constraints of the transaction matrix. Rows 7 to 10 represent the column sums. Rows 11 to 14 are equivalent to the behavioral equations. The last rows correspond to the balance sheet.

Compared to eq. (6), x now includes the individual stocks M_i, L_j, K_k :

$$x = [C_s^T \ C_d^T \ I_s^T \ I_d^T \ WB_s^T \ WB_d^T \ AF_s^T \ AF_d^T \ IL_s^T \ IL_d^T \ ID_s^T \ ID_d^T \ M^T \ L^T \ K^T]^T \quad (12)$$

where x is written as a concatenation of vectors. As an example, $M^T = [M_1, \dots, M_{nh}]^T$. We also write:

$$b = [0_{nf} \ 0_{nf} \ 0_{nf} \ 0_{nf} \ 0_{nb} \ 0_{nb} \ 0_{nh} \ 0_{nf} \ 0_{nf} \ 0_{nb} \ \alpha_0 \cdot 1_{nh} \ 0_{nf} \ 0_{nf} \ 0_{nh} \ M_{\text{tot}} \ M_{\text{tot}} \ 0_{nf} \ 0_{nb}]^T \quad (13)$$

where 0_n and 1_n are vectors, and M_{tot} is the total quantity of deposits.

The dimension of the nullspace of ξ is then determined numerically for a specific set of parameters consistent with [1], and such that $nb < nf < nw = 100$. We find that in the cases examined in Fig. 1, the matrix ξ doesn't have full rank, and that consequently the system $\xi x = b$ is under-determined.

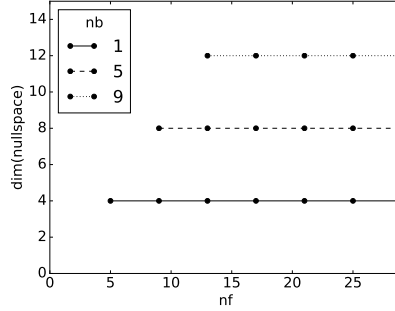


Figure 1: Dimension of the nullspace of ξ determined numerically with $r = 0.03$, $\alpha_1 = 0.75$, $\alpha_2 = 0.1$, $\delta = 0.1$, and $nw = 100$. This dimension is computed only for values of nf identified by the marker \bullet . nb, nf, nw are such that $nb < nf < nw$

The upper bounds x_i^{\max} are set independently for each variable, and are logic consequences of the positivity of the flows, and of the conservation of the total stock of money :

- $\forall i \in [1, nh], M_i \in [0, M_{\text{tot}}]$.
- $\forall i \in [1, nf], L_i \in [0, M_{\text{tot}}]$.
- $\forall i \in [1, nf], K_i \in [0, M_{\text{tot}}]$.
- $\forall i \in [1, nf], AF_i \in [0, \delta M_{\text{tot}}]$.
- $\forall i \in [1, nf], I_i \in [0, \delta M_{\text{tot}}]$.
- $\forall i \in [1, nh], ID_i \in [0, r M_{\text{tot}}]$.
- $\forall i \in [1, nf], IL_i \in [0, r M_{\text{tot}}]$.

A graphical representation of monetary transactions with randomly sampled connections between agents is given in Fig. 2.

Because $\mathbf{A}_{nf,nh}, \mathbf{B}_{nf,nf}, \mathbf{C}_{nf,nh}, \mathbf{D}_{nb,nf}, \mathbf{E}_{nb,nh}$ are realizations of random variables, ξ belongs to an ensemble of random matrices that will be noted $\Xi(nb, nf, nh)$.

When ξ is constant, S defines a convex bounded polytope, that appears in many disciplines, as will be discussed in section 1.3. The solutions to this class of under-specified problems will be examined in sections 1.4 and 1.5. The reader interested in the study of other explicit specifications of flows may look at [21] and

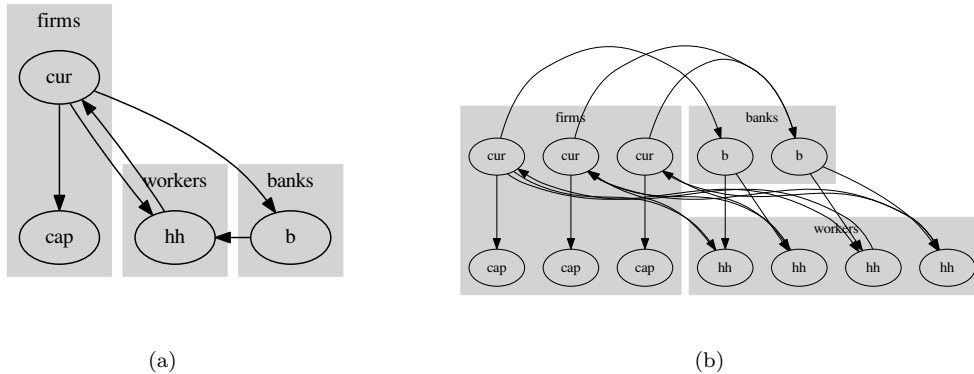


Figure 2: Graph of a subset of monetary transactions. cur, cap, w, b stand for firm's current and capital accounts, worker's deposit accounts, bank's deposit accounts. (a) single agent in each class; (b) random connectivity, with 2 banks, 3 firms, 4 workers.

[22]. Lavoie and Godley examine many increasingly detailed models (open economy,...), as well as dynamic specifications of the flows².

To the best of our knowledge, the properties of random matrices that correspond to SFC models have not been established in the economics literature. This representation calls for a closer examination following many works in the field of complex networks studies.

1.3. Related works in economics, econophysics, and network science

Double-entry book-keeping is used to establish National Accounts as a means to record estimated flows and stocks consistently, for each country, at several levels of aggregation. National accountants build balance sheets, income and product accounts, in order to compare economies, and to analyze the behavior of economies in time, such as growth. Such accounts are also used in macroeconomic modeling by policymakers to calibrate Computable General Equilibrium models. However, models of this type (e.g. DSGE) do not explicitly enforce stock-flow consistency.

Long before the break of the subprime crisis, many works have established ABM as an alternative, in a move to get rid of hypotheses perceived as unjustified. Equilibrium, rationality, and the hypothesis of the representative agent have been criticized by many economists [23, 3].

Even though all ABM do not verify stock-flow consistency, it is an hypothesis used by several authors [9, 24, 10, 11, 12].

ABM are praised for their flexibility, their ability to study large populations of heterogeneous and learning agents that interact in possibly non-linear ways. However, ABM need computer intensive simulations and face the problem of being over-parametrized. Calibration is thus difficult and unstable, all the more since empirical studies and reliable data are not abundant compared to the dimension of the parameter space. Recent works tackle the issue of an efficient exploration of the parameter space [25]. The issue of comparing a model to experimental data is another problem posed to practitioners, that has been addressed by different methods [26]. Under simplifying assumptions such as the aggregation of a subset of agents, theoretical results concerning some macro-economic ABM were recently obtained in a stock-flow-consistent framework, and phase diagrams established, for specific dynamic rules [27, 28].

The importance of the organization of interactions, embodied by networks, has been stressed in economics [29]. Theoretical works have studied their generic properties (supply chain [30], interbank network [31], trade credit [32]). Empirical studies have laid emphasis upon the topology of real economic networks, such as goods market, national inter-firm trading [33], world trade [34], global corporate control among

² see an implementation <https://github.com/kennt/monetary-economics>

transnational corporations [35], firm ownership networks. They must sometime be reconstructed, starting from limited information (of equity investments in the stock market [36], the interbank market [37]). At the level of individuals, detailed topological information about banking, employment, or consumption seems to be lacking.

Such empirical and theoretical material may serve as an input to shape the set of equations and inequalities discussed above.

1.4. Constraint satisfaction problem

As shown in section 1.1, the steady-state solution space associated with eq.(5) is a bounded convex polytope. Without loss of generality, S can be supposed to have full row rank. The polytope is a $n - m$ dimensional object embedded in an n -dimensional space. Properties of convex polytopes such as their different representations are well studied [38]. Computing their volume exactly can be achieved by solving the vertex enumeration problem, which is $\#P$ -hard. Existing implementations, such as *lrs* by Avis et al., allow to solve it in reasonable time when $n - m$ is equal to 10 or below. Exact computation methods are employed in linear programming and operations research to solve classical constraint satisfaction problems such as the map coloring problem, and have real-life applications, for example in resource allocation problems. More solutions can be obtained when relaxing hard constraints.

Approximate methods to determine the solution space were proposed by researchers studying metabolic networks and the metabolic steady-state flux space. These methods allow to sample the solution space, to estimate its volume, and to approximate probabilistic properties of the solutions such as the marginal densities [14, 2, 39, 40, 41, 42]. They can be used to evaluate the sensitivity of the solution space to new constraints. A restriction of this problem known as Flux-Balance-Analysis (FBA) consists in maximizing some objective constraint, which reduces the solution space to a finite set of points [15, 41] or to an hyperface. A parallel may be mentioned with the field of random Constraint Satisfaction Problem (rCSP) that stands at the interface between theoretical computer science and statistical physics, and studies sets of solutions to a large number of random constraints, in a boolean space. Many important results such as phase transitions were developed [43]. Although our main focus is the continuous domain, we can take advantage of this theory.

1.5. Monte-Carlo sampling of the steady-state solution space

In section 1.4 we mentioned the exact computation of the volume of the steady-state solution space using vertex enumeration, when $n - m$ is small. The result obtained is numeric, and not an analytic expression depending on the parameters of the problem in eq. (5).

Another approach proposed in [14, 44], suited for larger problems, is Monte-Carlo sampling in the solution space, using a *hit-and-run* algorithm. The latter needs an initial point inside the convex polytope, which can be found with a relaxation algorithm such as MinOver [45]. Then, sampling from a hypersphere, a direction is selected at random. The half-line defined by the starting point and this direction intersects the boundary in a point. This intersection and the starting point form a segment that can be uniformly sampled to get the next point. The procedure defines a Markov Chain that converges to the uniform distribution over the polytope [46, 47], in nondeterministic polynomial time $\mathcal{O}^*(n^3)$ after appropriate preprocessing. The notation $\mathcal{O}^*(\cdot)$ means that there are logarithmic factors that multiply n^3 , and constants, but are neglected [48].

The *hit-and-run* method provides an estimate of the marginal probability density functions (pdf) denoted $P_i(x)$ for each unknown $i = 1, \dots, N$. Correlations between variables can also be estimated, as well as other quantities that can be approximated with a finite sample of the solution space. They characterise the shape of the polytope and are given by:

$$P_i(x) = Vol(S_i(x))/Vol(S), \quad S_i(x) = \{x \in S \text{ s.t. } x_i = x\} \quad (14)$$

They can also be written as an integral over all stocks and flows. As remarked by [14], the *hit-and-run* method will not permit us to estimate the absolute volume. Instead, approximations for relative volumes

can be obtained, such as $r = Vol(S_1)/Vol(S)$, where S is the same as in eq.(5) while S_1 has additional constraints, such as a lower bound for the variable i .

To sum up, using efficient Monte-Carlo sampling methods such as *hit-and-run*, we will be able to sample medium-sized problems (up to hundreds of variables), to approximate the pdfs, to compute relative volumes. However, we will get no analytic expression of these quantities, nor approximate entropy. Furthermore, the mixing of the Markov Chain should be examined to ensure convergence. In section 2 we use the implementation by Tervonen et al. [49].

As evoked in section 1.4, researchers have also used the replica method [40], and message passing algorithms [2, 50, 41] to deal with the problem of estimating marginal densities. The computing time of message passing algorithms scales as $\mathcal{O}(n)$ when the factor graph that represents the constraints contains no loop, but without this hypothesis convergence is not guaranteed.

2. Results

The marginal histograms of the variables of the constraint satisfaction problem defined in section 1.2, where each sector is composed of many agents, are represented in Fig. 3. The solutions were sampled as explained in section 1.5, with ξ constant. The histogram are computed over the solutions, selecting one agent in each sector. The effect of averaging over $\Xi(nb, nf, nh)$ will be examined in section 2.2. The total quantity of deposits is constant, and the parameters have values summarized in Tab. 6.

We can first remark that the pdfs in Fig. 3 can be grouped by shape: the pdf of C_d and WB_s that both appear in eq.(1) have an exponential shape. The distribution of ID_s has the same shape as that of M . This is consistent with the linear relation in eq.(4).

The distribution of M is well fitted by a continuous exponential distribution as shown in Fig.4(a). This can be compared with the empirical finding that, for many industrialized countries, the lower part of the distribution of the wealth can be approximated by an exponential or gamma distribution [51, 2.3]. The influence of re-sampling ξ on this result will be examined in section 2.2.

The pdf of the stock of capital K has the same shape as I_d, AF_s, AF_d which are all related by linear equations. The pdfs of L and IL_d have similar shapes, consistently with eq.(3). This last shape will be discussed in section 2.2.

A comparison between supply-side and demand-side can be made. The mean flow of consumption supply is greater than the mean flow of consumption demand, in agreement with the fact that the number of firms is smaller than the number of households. Similarly, the mean wage supply WB_s , which goes to households, is lower than the mean flow WB_d paid by firms. The interest on deposit ID_s paid to households has a lower mean value than the ID_d paid by banks, since the number of banks is smaller than the number of households.

Because the supply C_s is for each firm the sum of the demands of their clients, by a central limit argument we can expect C_s to converge to a normal law when the number of clients grows.

Because of topological effects, the distributions of flows across agents are heterogeneous, as exemplified by the demand for consumption of two agents in Fig.4(b).

The capital to output ratio $\gamma = K_i/Y_i = K_i/(C_{s,i} + I_{s,i})$ can be computed and its histograms over all firms and all solutions to the CSPs is shown by Fig.4(c). Econometric studies at the aggregate level report that the value of this ratio has remained between 2 and 8 for many countries, on the long run (see [52] and the corresponding supplementary material³).

More generally, these remarks show that the model depicted in this article has interesting properties, can be easily modified, but needs more work before being able to reproduce stylized facts. We discuss this topic in section 3.

³<http://piketty.pse.ens.fr/fr/capitalisback>

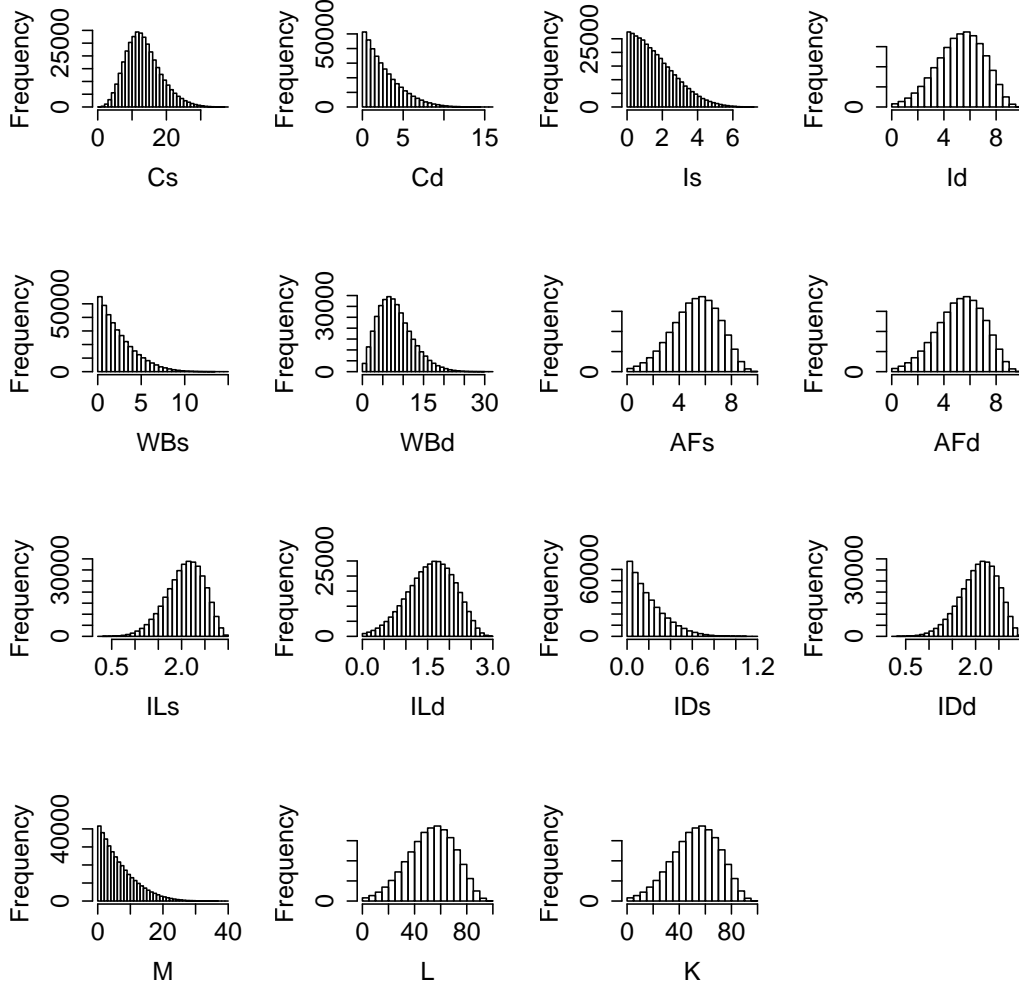


Figure 3: Histograms corresponding to marginal pdf of individual steady-state monetary stocks and flows labelled according to Tab.3, with $n_b = 2, n_f = 3, n_h = 10$. The number of samples solution of the constraint satisfaction problem is $N = 3.10^5$. Subscripts have been dropped for clarity.

Parameter	Value
α_0	0.0
α_1	0.75
α_2	0.1
r	0.03
δ	0.1
M_{tot}	100

Table 6: Values of linear flows parameters, following [1, chap. 7].

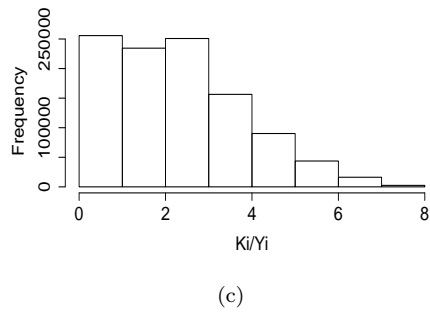
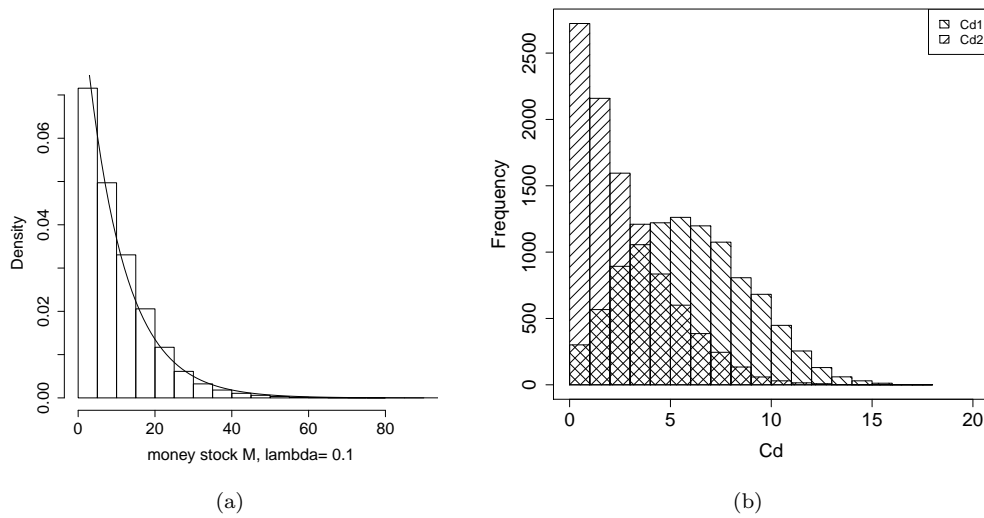


Figure 4: (a) Histogram of money stock M and exponential fit; (b) histogram of C_d for different agents;(b) capital output ratio.

2.1. The influence of knock-outs on systems with many agents

The method of relative volumes devised in section 1.5 is used to study the impact of an external perturbation on the volume of the solution space, for example when a flow is constrained. This is termed *knock-out* in metabolic network analysis, but in the present context can be interpreted as a limitation imposed on monetary variables, such as a credit shortage. Let us define:

$$S_I = \{X \text{ s.t } \xi X = 0, 0 \leq x_i \leq x_i^{\max}, \forall i \in I\} \quad (15)$$

$$S_{I \setminus I_0}^\alpha = \{X \text{ s.t } \xi X = 0, 0 \leq x_i \leq x_i^{\max}, \forall i \in I \setminus I_0, 0 \leq x_i \leq \alpha x_i^{\max}, \forall i \in I_0\} \quad (16)$$

where $X = [x_1, \dots, x_n]$, n is the number of columns of ξ , I is the set indexing the space of variables, and I_0 the set of variables that will be partially knocked-out, by an amount α . Let us also define:

$$r_V = \text{Vol}(S_{I \setminus I_0}^\alpha) / \text{Vol}(S_I) \quad (17)$$

The volume ratio r_V measures the impact of selective variable knock-out on the solution space.

We sample one random topology, as shown in Fig. 2(b), with 2 banks, 3 firms and 10 households. Keeping this topology fixed for the rest of this section, we sample the solution space and summarize the results in Fig. 5, where superposed curves have been removed. All ratios are functions of $1 - \alpha$.

It can be remarked in Fig.5(a) that single flow knock-outs are able to reduce significantly the volume of the steady-state solution space. For example, a 33% knock-out on investment made by one firm (I_S) out of the three defined in this simple model entails a cut by 20% of the volume ratio r_V .

The variables most influenced by group knock-outs in Fig.5(b) appear to be the interest on loans received by banks (IL_s), the investment of firms (I_S), the payment of wages to workers by firms (WB_d). This ranking is partially conserved at the individual and group level, as shown by Fig. 5(a,b).

The variable least influenced by group knock-outs is the demand for consumption goods (C_d). The variables with significant marginal probability on the right of $[0, x_i^{\max}]$ (such as IL_s in Fig.3) undergo a large reduction because the tail is cut off. Conversely, variables with a small tail on the right of $[0, x_i^{\max}]$, such as C_d , show little reduction.

Comparing the left and right panels of Fig. 5, we remark that the influence of group knock-out on volume ratio is larger than single knock-out. This remark is left for further theoretical analysis.

2.2. Random matrix

So far, the matrix ξ was considered constant. This hypothesis is interesting when a particular economic network is examined. However, when the emphasis is put more on the properties of a random ensemble of networks than on a particular instance, ξ must be chosen randomly in $\Xi(nb, nf, nh)$. Consequently, in this section, we consider empirical averages over both the solution space and the set of random matrices $\Xi(nb, nf, nh)$.

For computational reasons, the network size is set to $nb = 2, nf = 3, nh = 6$. Even in this particular case, the set of possible topologies is large, due to its combinatorial nature, but can be sampled exhaustively in reasonable time. Nevertheless, since for each randomly sampled matrix ξ the corresponding solution space S must also be sampled (which scales as $\mathcal{O}^*(n^3)$, as discussed in section 1.5), the number of sampled matrices will be limited to one hundred.

As illustrated by Fig. 6, the main observations made at the beginning of section 2 are recovered: the marginal distributions are grouped in the same way, the difference between mean supply and demand is conserved, and the money deposit M still has an continuous exponential shape.

We expect some effects related to topology to be averaged out, such as the heterogeneity between agents, that was exemplified in Fig.4(b). However, to check this with the requested statistical significance, we need to increase the number of sampled matrices, with an acceptable computational load, as will we discussed in section 3.

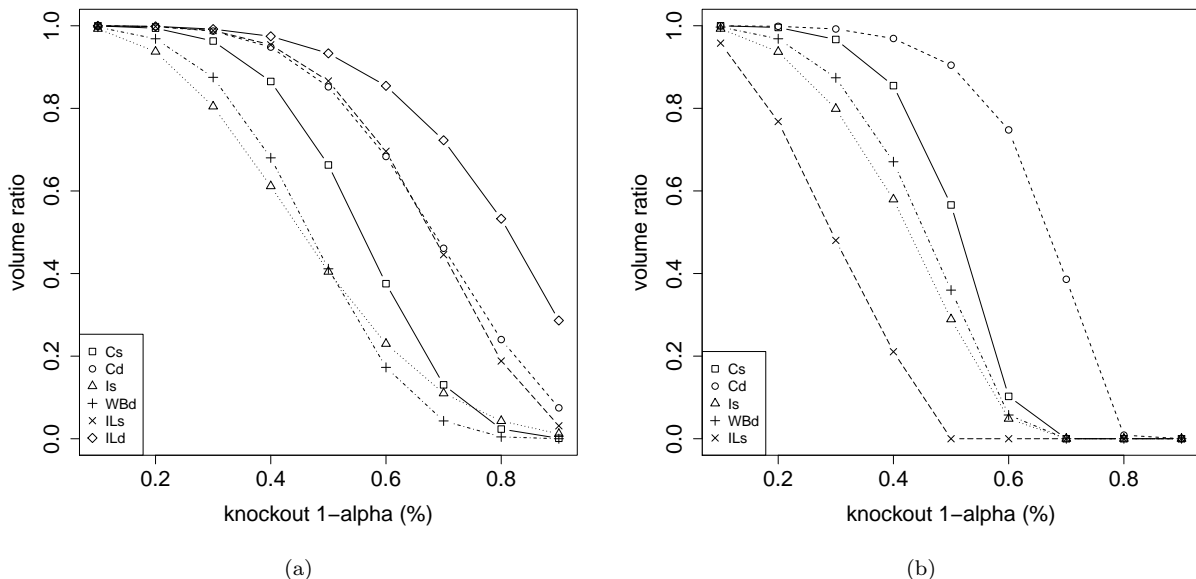


Figure 5: Impact of knock-out on volume ratios, in the case of a single draw of the random connectivity model depicted in sec.1.1, with 2 banks, 3 firms and 10 workers. Volume ratio appear as functions of $(1 - \alpha)$. (left) single flow knock-out; (right) group flow knock-out. Flow labels are explained in Tab. 3. The number of sampled points is $n = 3.10^5$.

3. Discussion

The first point we want to emphasize is that both the model and results in the sections above are preliminary. We do not claim that they can be of any use at the moment regarding economic analysis. More work is needed, in collaboration with economists, to improve their design, to evaluate the empirical stylized facts they can reproduce, and eventually to validate the model [26] using empirical data. In our view, the capacity to account for economic crises can be seen through the rapid decay of the volume ratio in response to modest fluctuations of the parameters, such as an increase in the knock-out factor. This can be seen independently of dynamical considerations.

In section 1.5 we presented an algorithm to compute the volume of the solution space that belongs to the class of Monte-Carlo methods, and illustrates the uniform sampling property of the *hit-and-run* strategy. We didn't discuss convergence issues here, but they will become important as the size of the problem increases. Furthermore, alternative estimation methods exist, for example belief propagation that scales almost linearly in system size, and can be applied when the variables are continuous [50, 53].

None of these algorithms provide litteral expressions that could be compared to results presented in section 2, such as the expression of absolute or relative volumes. For example, the magnitude of volume reduction provoked by knock-outs is an important quantity from a system-scale point of view. Interestingly, some theoretical results have been developed in the statistical physics literature evoked earlier, and should be compared to our numerical experiments, in future works.

Concerning the complexity of the financial SFC model depicted in section 1.1, we started with a very simple setting, with a limited number of transaction types, one agent per class, no state nor central bank. Flows were constrained in magnitude and depended on stocks. Then, the number of agents increased in section 1.2 which gave us a hint of the influence of topology. The accuracy of the model could be increased if random connectivity matrices were sampled in a random ensemble that corresponds to connectivity patterns observed empirically. Seeking inspiration in macro-economic literature (e.g. Lavoie and Godley), we may also add financial constraints on debt ratios at various levels. Transactions not covered in this article can also be added, related to bonds, investments, in an open economy. A major improvement would be to couple

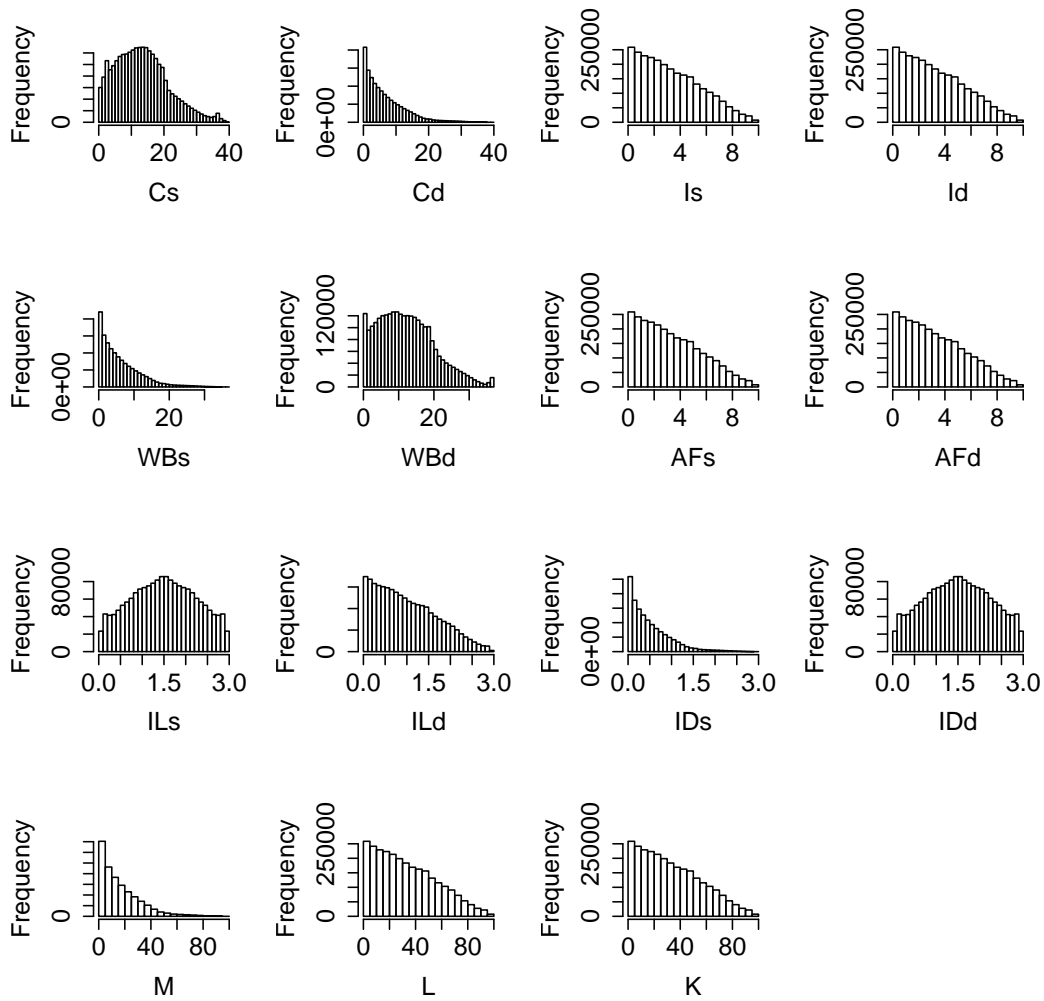


Figure 6: Histograms corresponding to marginal pdf of steady-state monetary stocks and flows when ξ is randomly sampled in $\Xi(nb, nf, nh)$, with $nb = 2$, $nf = 3$, $nh = 6$. 100 random connectivity matrices were generated. For each one, 10000 hit-and-run solutions were sampled.

the financial side to a production model of the economy, in order to compute prices, demand, unemployment and profit [54], but many difficulties can be expected in that direction because of nonlinear relations that transform the linear CSP into a nonlinear one.

The issue of employment can't be addressed directly in this model because all households get a wage from some firm. Unemployed households should be disconnected from firms, and included into another economic circuit, but this is not a feature of the BMW model, nor of the extension proposed here. What can still be studied thanks to the random nature of income in this framework, is the proportion of households whose income is below a given threshold, which could be related, for example, to aggregate demand.

In section 2, following [1, chap. 7], with the hypothesis that flows are linear functions of stocks, we obtained interesting marginal quantities, such as the capital to output ratio. But this raises the issue of choosing the right parameter values. Various strategies can be implemented to address it, such as setting the parameters according to empirical data taken from public statistics, or efficiently exploring the space of parameters [25]. Another point of view is to consider the linear flows parameters as variables defined in specific intervals, and to include them in the sampling scheme used above. Although flows are linear functions of stocks, this problem is also a nonlinear CSP, harder to deal with than a linear one.

On the computational complexity side, a comparison should be made with other classical approaches such as DSGE models and ABM with respect to the number of agents, the sparseness of the network topology, the number of regions or countries, the richness of the financial mechanisms involved. We can remark that in the case of *hit-and-run* sampling, the main cost is polynomial in the system size during sampling. Then in order to compute all the ratios discussed above, it is not necessary to resample: basic thresholding is sufficient, and is linear in system size.

Furthermore, as reported in the random network community [55, 56], sampling network matrices ξ with hard topological constraints raises the issue of bias and efficiency, and will have to be controlled for in future works.

4. Conclusion

We proposed an original strategy to compute macro-economic stocks and flows in a financial economy, inspired by stock-flow consistent models and methods developed in the field of metabolic networks. We show that this approach can be efficiently transposed thanks to approximate Monte-Carlo algorithms designed to solve Constraint Satisfaction Problems. The steady-state in variable space is seen as a polytope included in a large dimensional space, which weighs equally all configurations of the financial stocks and flows that are consistent with the constraints enforced by double-entry accounting.

We proposed a random connectivity extension of the linear flow model BMW by [1], and have obtained a numerical approximation of the probability density of stocks, flows, and the capital to output ratio. The money stock of households was found to be exponential in its lower part, which is reminiscent of standard works in econophysics [51].

Flow knock-outs can be used to model economic phenomena such as credit shortages. We show that different types of constraints have distinct effects on the volume of the solution space, that can be interpreted as characterizing the flexibility of the financial flows. Rapid decay of the volume ratio in response to modest fluctuations of the parameters can be interpreted as crises, independently of dynamical specifications.

Inside the class of SFC models, our approach fills a gap between SFC-ABM on one hand, and aggregate SFC models on the other hand, because the system size can scale up to thousands of agents while preserving the possibility of a theoretical analysis, and heterogeneity among agents. This comes at the cost of a simplification of the model, notably the hypothesis of non-equilibrium stationary state, with linear flows. We discussed many potential improvements concerning the algorithms, the complexity of the model, and the relation to empirical data, and will deal with it in future works.

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Open-source software were used to perform this research: Python, R, the R *hit-and-run* package [49], *lrs*, *graphviz*, *pygraphviz*.

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