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Abstract: Within the context of an acoustic scattering problem involving small heterogeneities, ones can use the matched asymptotic expansion method to build an approximate model. The key step of this method is the matching procedure between the near-field expansion and the far-field one. We control this step by estimating the difference between these two expansions and a so-called matching function in the intermediate zone. These estimates are based on the valuation of the rest of the radial expansion of the regular solution to wave equation which is, up to our knowledge, not given in the literature. We thus provide a proof of convergence which uses Mellin transform and the fundamental theorem on singularities for the wave equation.

Key-words: Acoustic wave propagation, matched asymptotic expansion method, scattering problem, Mellin transform, Singularity theory

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Modélisation asymptotique d'un problème de diffraction acoustique par petits obstacles: justification mathématique

Résumé : Dans le contexte d'un problème de diffraction acoustique par de petites hétérogénéités, on peut utiliser la méthode des développements asymptotique raccordés pour construire un modèle approché. L'étape clé de cette méthode est l'étape de raccord entre le développement asymptotique en champ proche et le développement asymptotique en champ lointain. Nous contrôlons cette étape en évaluant la différence entre ces deux développements et une fonction dite de raccord dans la zone intermédiaire. Ces évaluations sont basées sur l'estimation du reste du développement radial de la solution régulière de l'équation des ondes qui est, à notre connaissance, non donnée dans la littérature. Nous fournissons ainsi une preuve de convergence qui utilise la transformée de Mellin et le théorème fondamental des singularités pour l'équation des ondes.

Mots-clés : Équation des ondes acoustique, méthode des développements asymptotiques raccordés, problème de diffraction, transformée de Mellin, théorie des singularités

The numerical simulation of waves is used in many applications and today, the capability of supercomputers makes it possible to consider complex and thus realistic configurations. However, some highly contrasted propagation media are still difficult to account for and there is a need in constructing approximate models which are easier to solve. By easier we mean systems of equations which use less computational resources while providing accurate solutions. The case of propagation media which include very small defects is a good illustration of this kind of problems. Regarding the phenomenon of wave propagation, very small defects correspond to a scattering problem in which the size of the obstacle is very small as compared to the characteristic wavelengths of the problem. Finite elements are generally involved in the numerical approximation since they can be based on tetrahedral grids which are easily adaptable to the geometry of the defects. When tackling this problem straightforwardly with finite elements, it is mandatory to use a refined mesh in the vicinity of the obstacle. If not, the numerical waves are not able to capture the response of the defects. But refined meshes can be very tricky to generate and they induce extra computational costs due to a large number of degrees of freedom. In case of materials including networks of very small defects, the computational costs of the numerical method quickly become prohibitive accentuating the need of approximate solutions. Moreover, in case of time dependent problems, the time discretization becomes a possible issue since it is necessary to decrease the time step to keep stability and a too small time-step may cause dispersion errors. Local-time stepping can be a solution as it has been shown in [13] and [5] but in case of multiple obstacle problems, the numerical method is not obvious to implement and does not seem fully convenient in an industrial setting. Another option consists in using approximate systems of equations which avoid to mesh the small obstacles and for that purpose, asymptotic models can be considered by taking the size of the obstacle as a small parameter. This idea has been investigated by several authors [9], [17], [3], [18], [19], [8] in the context of stationnary problems and to the best of our knowledge, the first attempt for time-dependent problems is in [13].

The derivation of asymptotic problems for scattering problems is a tricky task since it involves two different asymptotic developments defined as a far-field expansion and a near-field expansion. These two expansions need to be matched at the end to form the approximate model. Two asymptotic expansions are required just because the waves can not be represented in the same way near and far from the obstacle. To illustrate this point, let us consider an obstacle that is a small sphere centred at the origin with a near-zero radius ε. The scattering problem reads then as:

$$\left\{\begin{array}{c}\Delta u_\varepsilon(\mathbf{x},t)-\dfrac{1}{c^2}\partial_t^2 u_\varepsilon(\mathbf{x},t)=0, \quad \text{on} \quad \mathbb{R}^3\backslash B_\varepsilon, \quad \forall t\geq 0,\\ u_\varepsilon(\mathbf{x},t)=0, \quad \text{on} \quad \partial B_\varepsilon,\\ u_\varepsilon(\mathbf{x},0)=v_0(\mathbf{x}), \quad \partial_t u_\varepsilon(\mathbf{x},0)=v_1(\mathbf{x}), \quad \text{on} \quad \mathbb{R}^3\backslash B_\varepsilon,\end{array}\right. \tag{1}$$

where $\Delta=\partial_x^2+\partial_y^2+\partial_z^2$; B_ε is the ball of radius ε centered at the origin; v_0, v_1 are supported in $B_{r_\star}:=\{\mathbf{x}\in\mathbb{R}^3:|\mathbf{x}|\leq r_\star\}$ and $r_\star$ is a positive real.
Far from the obstacle, the solution is given by the far-field expansion

$$u_\varepsilon(\mathbf{x},t)\sim\sum_{i=0}^{+\infty}u_i(\mathbf{x},t)\varepsilon^i \tag{2}$$

where $\mathbf{x}=(r,\theta,\varphi)$ is the spatial variable in spherical coordinates and $t\geq 0$ denotes the time variable. The first term u_0 is a regular solution of the wave equation defined in the whole space (i.e. without obstacle). The set of u_i with $i>0$ are singular solutions which are non-defined at the origin, recalling that the origin is the centre of the small obstacle. The terms u_i are finite sums of multipoles that are explicitly known up to their magnitudes. In the neighbourhood of the origin, the solution u formally reads as the near-field expansion :

$$u(\mathbf{X}\varepsilon, t) = U_\varepsilon(\mathbf{X}, t) \sim \sum_{i=0}^{+\infty} U_i(\mathbf{X}, t)\varepsilon^i, \tag{3}$$

where $\mathbf{X} = (R, \theta, \varphi) = \frac{\mathbf{x}}{\varepsilon}$ is a dimensionless variable in spherical coordinates and $t \geq 0$ still denotes the time variable. The two first terms U_0 and U_1 of this expansion are solutions to the Laplacian equation whereas the others U_i for all $i > 1$ satisfy the nested Laplace equations given by

$$\Delta U_{i+2}(\mathbf{X}, t) = -\frac{\partial_t^2}{c^2} U_i(\mathbf{X}, t). \tag{4}$$

The terms U_i are finite sums of growing functions (satisfying $\lim\limits_{R\to+\infty} \max\limits_{|X|=R} |U_i(\mathbf{X}, t)| = +\infty$). Then the connection between these two developments is made thanks to a matching procedure which requires the determination of the behaviour of u_i close to the origin and of U_i close to infinity. In practice, for all i, we develop u_i in the vicinity of the origin as

$$u_i(r, \theta, \varphi, t) \sim \sum_{p=-\infty}^{+\infty} u_{i,p}(\theta, \varphi, t) r^p, \tag{5}$$

and U_i is expanded in the neighbourhood of infinity as

$$U_i(R, \theta, \varphi, t) \sim \sum_{p=-\infty}^{+\infty} U_{i,p}(\theta, \varphi, t) R^p, \tag{6}$$

with $R = \frac{r}{\varepsilon}$. Identifying u_ε and U_ε in the transition zone and using the expansions (2), (3), (5) and (6) we obtain the following matching relation

$$\begin{cases} u_{i,p}(\theta, \varphi, t) = U_{i+p,p}(\theta, \varphi, t) & \forall i \in \mathbb{Z}, \quad p \in \mathbb{Z}, \\ U_{i,p}(\theta, \varphi, t) = u_{i-p,p}(\theta, \varphi, t) & \forall i \in \mathbb{Z}, \quad p \in \mathbb{Z}. \end{cases} \tag{7}$$

This step is very often made formally, see for instance [6]. However, if ones would like to obtain error estimates validating the approximate model, it is necessary to justify these formal computations.

Practically speaking, it is mandatory to obtain a spatial expansion of the regular solution in the vicinity of the origin which reads

$$u(\mathbf{x}, t) = \sum_{p=-\infty}^{N} r^p u_p(\theta, \varphi, t) + \mathfrak{r}^N(\mathbf{x}, t), \tag{8}$$

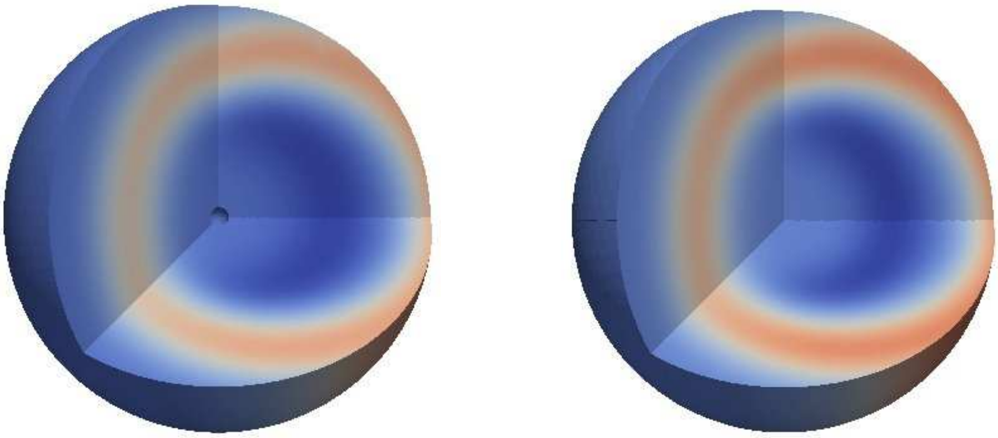
where $t \geq 0$ denotes the time variable, (r, θ, φ) are the spherical coordinates of $\mathbf{x} \in \mathbb{R}^3$ and

$$\begin{cases} \max\limits_{t\leq T} |\mathfrak{r}^N(\mathbf{x}, t)| = \underset{r\to 0}{O}(r^{N+1}), \\ \max\limits_{t\leq T} |\partial_r \mathfrak{r}^N(\mathbf{x}, t)| = \underset{r\to 0}{O}(r^N). \end{cases} \tag{9}$$

This has been supposed to hold true in [13].

Then the scattering problem (1) is replaced by the second order far-field expansion

$$u_{\varepsilon,2}(\mathbf{x}, t) := u_0(\mathbf{x}, t) + \varepsilon u_1(\mathbf{x}, t) + \varepsilon^2 u_2(\mathbf{x}, t), \tag{10}$$



(a) D.G. method (b) M.A.E. method

Figure 1: Comparison at time $t = 1.3$s

	D.G. Method	M.A.E. Method
CPU Time	18 400s	6 100s
Degrees of freedom	2 500 000	1 800 000
Mesh	Locally refined	Uniform

Table 1: Some informations about the numerical simulations

with u_0 the regular solution to wave equation and

$$\left\{ \begin{aligned} u_1(\mathbf{x}, t) &= -\frac{u_0(\mathbf{0}, t - r/c)}{r}, \\ u_2(\mathbf{x}, t) &= -\frac{\partial_t u_0(\mathbf{0}, t - r/c)}{rc}. \end{aligned} \right. \tag{11}$$

Its solution is computed without considering the obstacle anymore. To illustrate the efficiency of this approach, we have performed two numerical experiments. The first one uses a Discontinuous Galerkin approximation coupled with a leap-frog scheme and it requires meshing the obstacle. The computational domain is the unit ball and we set an absorbing boundary condition on its exterior boundary. The second one computes the solution given by the Matched Asymptotic Expansion (MAE.) set on the whole unit ball without the obstacle. Table 1 explicit some numerical features on the mesh and the CPU time while Figure 1 provides a snapshot of both the solutions which are very similar. These two experiments illustrate the good performance of MAE and it is worth noting that we provide an example where solving MAE with a numerical method is in fact unnecessary because the first iterate u_0 can be obtained analytically. The given CPU time for MAE can thus be even more smaller each time analytic u_0 can be computed. and it is the case for regular-shaped obstacles like spheres or ellipsoids.
The purpose of this paper is thus to prove that the solution to the scattering problem admits the expansion (8) and after truncation, the rest $\mathfrak{r}^N$ satisfies (9). Our main task consists in proving $\max\limits_{t \le T} |\mathfrak{r}^N(\mathbf{x}, t)| = \underset{r \to 0}{O}(r^{N+1})$, and the second estimation (9) directly follows. In [13], the expansion

series of the wave equation solution have been simplified by using suitable bases. In particular, it is possible to get the same expansion both for regular and singular solutions to the wave equation (see pages 74 and 75 in [13]). It is thus sufficient to prove (9) for regular solutions[1]. We thus consider the following problem : find a C^∞ function $u : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ solution to the wave equation

$$\begin{cases} \Delta u(\mathbf{x},t) = \dfrac{1}{c^2}\partial_t^2 u(\mathbf{x},t), \\ u(\mathbf{x},0) = v_0(\mathbf{x}) \quad \text{and} \quad \partial_t u(\mathbf{x},0) = v_1(\mathbf{x}), \end{cases} \tag{12}$$

with $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$; v_0, v_1 supported in $B_{r_\star} := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leq r_\star\}$ and $r_\star$ is a positive real. In this paper, we propose a proof of :

Theorem 1. *The regular solution to the wave equation can be represented by the series expansion:*

$$u(\mathbf{x},t) = \sum_{p=0}^{N} r^p\, u_p(\theta,\varphi,t) + \mathfrak{r}^N(\mathbf{x},t), \tag{13}$$

where $\mathfrak{u}^N$ *satisfies*

$$\max_{t\leq T} |\mathfrak{r}^N(\mathbf{x},t)| = \underset{r\to 0}{O}(r^{N+1}). \tag{14}$$

This paper is planned as follows. In a first part, we remind some properties of the solution u to the wave equation (12) in the free space, focusing in particular on its regular expansion series and its asymptotic development in the vicinity of the origin. Then, we start rephrasing Theorem 1 into Theorem 2 to get a simplest proof. Then, in a second part, we obtain the equation that is satisfied by the projectors involved in the expansion of the solution. Next we transform this equation by using the formalism of Kondratiev spaces[10]) thanks to the Mellin transform (see [1]). Then, in a third part, we show that the problem provides an example of application to the fundamental theorem of singularities which can be found for example in [1]. This theorem allows us to get an estimate of $\mathfrak{r}^N$ on the unit sphere which can be extended to the whole space $\mathbb{R}^3$ but the resulting estimate is not optimal. Fortunately, it is possible to use properties of the Mellin transform to adjust the order of approximation and we end up with the expected estimate (14) in the last part.

1 Already known properties of the wave equation

In this part, we prepare the proof of Theorem 1 by establishing some preliminary results. More particularly, we clarify the support and the $L^2(\mathbb{R}^3)$ estimate of both the solution u and its time derivative. We then give the expansion of the solution as an explicit series and give an equivalent way of writing u.

1.1 Support and energy conservation

Property 1. *The regular solution to the wave equation* (12) *has its support in* $B_{r_\star+ct}$. *Moreover, for any* $t \geq 0$, *the function* $\mathbf{x} \mapsto u(\mathbf{x},t)$ *belongs to* $L^2(\mathbb{R}^3)$ *and satisfies*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} |u(\mathbf{x},t)|^2 d\mathbf{x} \leq \frac{(r_\star+ct)^2}{c^2\pi^2}\Big(\int_{\mathbb{R}^3} \big|v_1(\mathbf{x})\big|^2 + c^2\int_{\mathbb{R}^3}\big|\nabla v_0(\mathbf{x})\big|^2\Big), \quad \forall t \geq 0, \\ \displaystyle\int_{\mathbb{R}^3} \big|\partial_t u(\mathbf{x},t)\big|^2 d\mathbf{x} \leq \int_{\mathbb{R}^3}\big|v_1(\mathbf{x})\big|^2 + c^2\int_{\mathbb{R}^3}\big|\nabla v_0(\mathbf{x})\big|^2, \quad \forall t \geq 0. \end{cases} \tag{15}$$

[1] note that regular solutions do not involve negative values of p into (8)

Remark 1. *These estimates show that the solution u and its time derivatives are in $L^2(\mathbb{R}^3)$. Moreover, we note that $\|\partial_t u(\mathbf{x},t)\|_{L^2(\mathbb{R}^3)}$ is bounded independently of the time which partly illustrates the energy conservation.*

Proof. The solution to the wave equation is a function of $\mathcal{C}^\infty$. It tests the following energy identity

$$\int_{\mathbb{R}^3} \big|\partial_t u(\mathbf{x},t)\big|^2 + c^2 \int_{\mathbb{R}^3} \big|\nabla u(\mathbf{x},t)\big|^2 d\mathbf{x} \;=\; \int_{\mathbb{R}^3} \big|v_1(\mathbf{x})\big|^2 + c^2 \int_{\mathbb{R}^3} \big|\nabla v_0(\mathbf{x})\big|^2, \quad \forall t \geq 0. \tag{16}$$

That the support of u is in $B_{r_\star + ct}$ is because of (16), together with the supports of v_0 and v_1 which are in $B_{r_\star}$ and the propagation velocity c. Using the Poincaré inequality inside the ball with radius r, that is

$$\|v\|^2_{L^2(\mathbb{R}^3)} \;\leq\; \frac{r^2}{\pi^2}\|\nabla v\|^2_{L^2(\mathbb{R}^3)}, \tag{17}$$

we get

$$\int_{B_{r_\star+ct}} |u(\mathbf{x},t)|^2 d\mathbf{x} \;\leq\; \frac{(r_\star + ct)^2}{\pi^2} \int_{B_{r_\star+ct}} \big|\nabla u(\mathbf{x},t)\big|^2 d\mathbf{x}. \tag{18}$$

The energy identity (16) combined with (18) lead to the expected result, which completes the proof. □

Remark 2. *The time derivatives of higher order can also be estimated by using the same technique than previously. For any integer $p \in \mathbb{N}$, we have*

$$\begin{cases} \Delta\Big(\partial_t^{2p} u\Big)(\mathbf{x},t) = \dfrac{1}{c^2}\partial_t^2\Big(\partial_t^{2p}u\Big)(\mathbf{x},t), \\ \Big(\partial_t^{2p}u\Big)(\mathbf{x},0) = c^{2p}\Delta^p v_0(\mathbf{x}) \quad \textit{and} \quad \partial_t\Big(\partial_t^{2p}u\Big)(\mathbf{x},0) = c^{2p}\Delta^p v_1(\mathbf{x}). \end{cases} \tag{19}$$

It follows from (15) *that*

$$\int_{\mathbb{R}^3} \Big|\partial_t^{2p+1}u(\mathbf{x},t)\Big|^2 d\mathbf{x} \;\leq\; c^{4p}\Big(\int_{\mathbb{R}^3} \big|\Delta^p v_1(\mathbf{x})\big|^2 + c^2 \int_{\mathbb{R}^3} \big|\nabla\Delta^p v_0(\mathbf{x})\big|^2\Big), \quad \forall t \geq 0. \tag{20}$$

Similarly, for any $p \geq 1$

$$\begin{cases} \Delta\Big(\partial_t^{2p-1} u\Big)(\mathbf{x},t) = \dfrac{1}{c^2}\partial_t^2\Big(\partial_t^{2p-1}u\Big)(\mathbf{x},t), \\ \Big(\partial_t^{2p-1}u\Big)(\mathbf{x},0) = c^{2p-2}\Delta^{p-1} v_1(\mathbf{x}) \quad \textit{et} \quad \partial_t\Big(\partial_t^{2p-1}u\Big)(\mathbf{x},0) = c^{2p}\Delta^p v_0(\mathbf{x}). \end{cases} \tag{21}$$

From Eq.(15) *we deduce that for any $p \geq 1$*

$$\int_{\mathbb{R}^3} \Big|\partial_t^{2p}u(\mathbf{x},t)\Big| d\mathbf{x} \leq c^{4p}\Big(\int_{\mathbb{R}^3} \big|\Delta^p v_0(\mathbf{x})\big|^2 d\mathbf{x} + \frac{1}{c^2}\int_{\mathbb{R}^3} \big|\nabla\Delta^{p-1} v_1(\mathbf{x})\big|^2 d\mathbf{x}\Big). \tag{22}$$

We can thus bound the $L^2(\mathbb{R}^3)$ norm of its all order time derivatives with a constant independent of the time, just as was previously observed for the first-order time derivative.

1.2 The solution as a series expansion

In what follows, we remind results that have been established at chapter 1 of [13] and that we will be used to prove Theorem 1. The regular solution u reads for every $t > 0$ as the following series which converges into $L^2(\mathbb{R}^3)$:

$$u(\mathbf{x}, t) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} u_{m,n}(r, t) \times Z_{m,n}(\theta, \varphi). \tag{23}$$

The coefficients $u_{m,n}$ are defined as the projections of u onto the basis of $L^2(\mathbb{S})$ composed of $Z_{m,n}$

$$u_{m,n}(r, t) = \int_{\mathbb{S}} u(r, \theta, \varphi, t) Z_{m,n}(\theta, \varphi) \sin\theta d\theta d\varphi, \tag{24}$$

where $\mathbb{S}$ denotes the unit sphere. The functions $Z_{m,n}$ are given by

$$Z_{m,n}(\theta, \varphi) = \frac{1}{\sqrt{2\pi}} \overline{P_n^{|m|}}(\cos\theta) \exp(im\varphi). \tag{25}$$

To define $Z_{m,n}$, we use the normalized Legendre function $\overline{P_n^m}$ with integer order m and degree n. Let $m \in [0, n]$ be an integer and let $x \in [-1, 1]$ be a real. We have

$$\overline{P_n^m}(x) = \frac{P_n^m(x)}{\|P_n^m(x)\|_{L^2([-1,1])}} = \sqrt{(n + \frac{1}{2})\frac{(n-m)!}{(n+m)!}} \times (1 - x^2)^{\frac{m}{2}} d_x^m P_n(x), \tag{26}$$

where P_n^m denotes the Legendre polynomial with order m and degree n (see p 37 in [15]) and P_n is the Legendre polynomial with degree n and given by

$$P_n(x) = \frac{1}{2^n n!} d_x^n \left[(x^2 - 1)^n\right]. \tag{27}$$

More details on Legendre polynomials are available on p 35 in [15] or p 47 in [8] or p 353 in [7]. The series (23) can be understood as a series whose terms are orthogonal in pairs

- of the time $t \geq 0$ with values on $L^2(\mathbb{R}^3)$,
- of the time $t \geq 0$ and of the radial space variable $r > 0$ with values on $L^2(\mathbb{S})$, with $\mathbb{S}$ the unit sphere which can be parametrized by the angular space variables (θ, φ).

Property 2. *The functions $Z_{m,n}$ satisfy*

1. $\|Z_{m,n}\|_{L^2(\mathbb{S})} = 1$,
2. $Z_{m,n}(0, \varphi) = 0, \quad$ *if* $m \neq 0$, $\forall \varphi \in [0, 2\pi]$,
3. $Z_{0,n}(0, \varphi) = \sqrt{\frac{n + 1/2}{2\pi}}, \quad$ *for* $m = 0$, $\forall \varphi \in [0, 2\pi]$.

Proof. 1. By construction, the basis functions of $\mathbb{S}$ define an orthonormal basis of $\mathbb{S}$. Indeed, by définition (see (25)), we have

$$\|Z_{m,n}\|^2_{L^2(\mathbb{S})} = \frac{1}{2\pi} \int_0^{2\pi} |\exp(im\varphi)|^2 d\varphi \int_0^{\pi} |\overline{P_n^{|m|}}(\cos\theta)|^2 \sin\theta d\theta. \tag{28}$$

Let us note that the corresponding Legendre polynomials are orthogonal and satisfy for any $m \in [0, n]$

$$\int_{-1}^{1} P_n^m(x)^2 dx = \frac{(n+m)!}{(n+1/2)(n-m)!}. \tag{29}$$

We refer to as p 37 in [15]. By applying the change of variable $x = \cos\theta$, we thus find the normalized Legendre polynomials which satisfy

$$\int_0^{\pi} \overline{P_n^{|m|}}(\cos\theta)^2 \sin\theta d\theta = 1. \tag{30}$$

This ends the proof.

2. By definition (see (26)), $\overline{P_n^m}(1) = 0$, $\forall m \neq 0$.
3. By definition (see (25) and (26)), we have

$$Z_{0,n}(0,0) = \frac{\overline{P_n^0}(1)}{\sqrt{2\pi}} = \sqrt{\frac{n+1/2}{2\pi}}, \tag{31}$$

because $P_n(1) = 1$. Indeed, following [15], page 35, we have

$$(x^2-1)P_n'(x) = nxP_n(x) - nP_{n-1}(x), \tag{32}$$

and by using this recurrence relation for $x = 1$, we get $P_n(1) = P_{n-1}(1)$. It is then sufficient to see that $P_0(1) = 1$ for concluding.

□

In [11], it is established that the basis functions of $\mathbb{S}$ are linked to the Laplace-Beltrami operator. The relation reads as follows.

Property 3. *The functions $Z_{m,n}$ satisfy*

$$\Delta_\Gamma Z_{m,n}(\theta,\varphi) = -n(n+1)Z_{m,n}(\theta,\varphi), \tag{33}$$

where Δ_Γ denotes the Laplace-Beltrami operator which reads in spherical coordinates as follows:

$$\Delta_\Gamma = \frac{1}{\sin\theta}\partial_\theta\Big[\sin\theta\partial_\theta\Big] + \frac{1}{\sin^2\theta}\partial_\varphi^2. \tag{34}$$

We thus end up with an explicit expansion of the solution u and each term of the expansion is well-defined, thanks to the mathematical properties we have provided.

1.3 Relation between two series expansions of the solution

We remind that we aim at proving Theorem 1 which says that the following radial expansion holds:

$$u(\mathbf{x},t) = \sum_{p=0}^{N} u_p(\theta,\varphi,t)r^p + \mathfrak{r}^N(\mathbf{x},t). \tag{35}$$

For that purpose, we found convenient to have a connection with the expansion of the regular solution at a given time t as a convergent series in $L^2(\mathbb{R}^3)$ which reads as:

$$u(\mathbf{x},t) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} u_{m,n}(r,t)Z_{m,n}(\theta,\varphi). \tag{36}$$

Let N be a positive integer. The series (36) can be split as follows:

$$u(\mathbf{x},t) = \mathcal{S}^N(\mathbf{x},t) + \mathfrak{u}^N(\mathbf{x},t), \tag{37}$$

where $\mathcal{S}^N$ is obtained from the truncation of the series sum at order N:

$$\mathcal{S}^N(\mathbf{x},t) = \sum_{n=0}^{N} \sum_{m=-n}^{n} u_{m,n}(r,t) Z_{m,n}(\theta,\varphi), \tag{38}$$

and $\mathfrak{u}^N$ denotes the rest

$$\mathfrak{u}^N(\mathbf{x},t) = \sum_{n=N+1}^{+\infty} \sum_{m=-n}^{n} u_{m,n}(r,t) Z_{m,n}(\theta,\varphi). \tag{39}$$

According to Theorem 1.3 p 57 in [13], we know that the coefficients $u_{m,n}$ admit a Taylor series which satisfies : $u_{m,n,p}(t) = 0$ for any $p < n$. Moreover, we have

$$u_{m,n}(r,t) = \sum_{p=0}^{P} u_{m,n,p}(t) r^p + \underset{r\to 0}{O}(r^{P+1}), \quad \forall P > 0, \tag{40}$$

and in particular

$$u_{m,n}(r,t) = \underset{r\to 0}{O}(r^n). \tag{41}$$

Then, by observing that the number of terms in the truncated sum is finite, we can deduce from (40) the Taylor expansion of $\mathcal{S}^N$ at order N

$$\mathcal{S}^N(\mathbf{x},t) = \sum_{p=0}^{N} \mathcal{S}_p^N(\theta,\varphi,t) r^p + \mathfrak{s}^N(\mathbf{x},t), \tag{42}$$

where $\mathfrak{s}^N(\mathbf{x},t)$ is the Taylor series rest which satisfies

$$\mathfrak{s}^N(\mathbf{x},t) = \underset{r\to 0}{O}(r^{N+1}). \tag{43}$$

Moreover, the coefficients $\mathcal{S}_p^N$ involved in the radial expansion $\mathcal{S}^N$ are given by

$$\mathcal{S}_p^N(\theta,\varphi,t) = \sum_{n=0}^{N} \sum_{m=-n}^{n} u_{m,n,p}(t) Z_{m,n}(\theta,\varphi). \tag{44}$$

Since $u_{m,n,p}(t) = 0$ for any $p < n$, we have that $\mathcal{S}_p^N$ does not depend on N and it is then denoted by u_p and defined by

$$u_p(\theta,\varphi,t) = \sum_{n=0}^{p} \sum_{m=-n}^{n} u_{m,n,p}(t) Z_{m,n}(\theta,\varphi). \tag{45}$$

We have thus broken u down into

$$u(\mathbf{x},t) = \sum_{p=0}^{N} u_p(\theta,\varphi,t) r^p + \mathfrak{u}^N(\mathbf{x},t) + \mathfrak{s}^N(\mathbf{x},t). \tag{46}$$

This result indicates then that Theorem 1 to be proved is equivalent to:

Theorem 2. *The regular solution to the wave equation admits the following representation:*

$$u(\mathbf{x},t) = \sum_{n=0}^{N} \sum_{m=-n}^{n} u_{m,n}(r,t) \times Z_{m,n}(\theta,\varphi) + \mathfrak{u}^N(\mathbf{x},t), \tag{47}$$

where $\mathfrak{u}^N$ *is the rest testing*

$$\max_{t\leq T} |\mathfrak{u}^N(\mathbf{x},t)| = \underset{r\to 0}{O}(r^{N+1}). \tag{48}$$

Remark 3. *The coefficients* u_p *in Theorem 1 are given by formula* (45)*.*

We now move on proving Theorem 2.

2 Separation of variables and Mellin transform

To prepare the proof of Theorem 2, we first define the equation which each $u_{m,n}$ tests. Then, we remind some definitions and properties of the Mellin transform. This section will end up with the recall of the fundamental theorem of the singularities theory proved for example in [1].

2.1 Equation satisfied by $u_{m,n}$

The expansion (23) can be understood as a spectral decomposition of the trace of u onto the sphere with radius r. The coefficients $u_{m,n}$ are then the spectral coefficients $(\theta,\varphi) \mapsto u(r,\theta,\varphi)$ for given r and t

$$u_{m,n}(r,t) = \int_0^{\pi} \int_0^{2\pi} u(r,\theta,\varphi,t) Z_{m,n}(\theta,\varphi) \sin\theta d\theta d\varphi. \tag{49}$$

The partial differential equation

$$\Delta\, u(\mathbf{x},t) = \frac{1}{c^2}\partial_t^2 u(\mathbf{x},t), \quad \text{with} \quad \Delta = \frac{1}{r^2}\partial_r\Big[r^2\partial_r\Big] \;+\; \frac{1}{r^2}\Delta_\Gamma, \tag{50}$$

can be diagonalized by using the spectral coefficients $u_{m,n}$. According to (33) together with the fact that the functions $Z_{m,n}$ are orthogonal in $L^2(\mathbb{S})$, the coefficients $u_{m,n}$ satisfy for any $r > 0$ and $t \geq 0$

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r u_{m,n}(r,t)\right) - \frac{n(n+1)}{r^2}u_{m,n}(r,t) = \frac{1}{c^2}\partial_t^2 u_{m,n}(r,t). \tag{51}$$

Each $u_{m,n}$ is a function of $\mathcal{C}^\infty$ with support in $\{(r,t) : t \geq 0 \quad \text{et} \quad 0 < r < r_\star + ct\}$. We now rewrite (51) as

$$u_{m,n}^{\{2\}}(r,t) + u_{m,n}^{\{1\}}(r,t) - n(n+1)u_{m,n}(r,t) = r^2\frac{1}{c^2}\partial_t^2 u_{m,n}(r,t), \tag{52}$$

with the differential operator $\cdot^{\{\ell\}}$ defined by

$$v^{\{\ell\}}(r) \;=\; \left(r\frac{d}{dr}\right)^{\ell} v(r). \tag{53}$$

On the other hand, since for a given time t, the function $\mathbf{x} \mapsto u(\mathbf{x},t)$ belongs to $\mathcal{C}^\infty$ with a compact support, the following energy is bounded:

$$E_\ell^T = \max_{t\leq T} \int_{\mathbb{R}^3} |\partial_t^\ell u(r,\theta,\varphi,t)|^2 d\mathbf{x}. \tag{54}$$

Following the Parseval equality related to the spectral decomposition, we have that

$$\int_0^{\pi}\int_0^{2\pi} |v(\theta,\varphi)|^2 \sin\theta d\theta d\varphi = \sum_{m,n} |v_{m,n}|^2, \quad \forall v \in L^2(\mathbb{S}), \tag{55}$$

with $v_{m,n} = \int_{\mathbb{S}} v(\theta,\varphi)\sin\theta d\theta d\varphi$. These terms can be displayed with the spectral coefficients:

$$E_\ell^T = \max_{t\le T}\int_0^{+\infty}\sum_{m,n}|\partial_t^\ell u_{m,n}(r,t)|^2. \tag{56}$$

2.2 Using the Mellin variables

Mellin transform is one of the tools for the theory of singularities which allows to give the behavior of a function in the neighborhood of a given point. As the Fourier transform, it changes the differential (53) into an operator of multiplication. One of the original features of this work is to provide an approach which does not require functions in Hilbert spaces (as it is done in [10]) and which applies to time-dependent problems. In this section, we transport the problem into Kondratiev spaces which are weighted spaces suitable for the analysis and this is done by using the Mellin transform.
We begin with some notations and some properties of the Mellin transform. For more details, we refer to as the research report [1] in which proofs can be found and [16].

2.2.1 Weighted spaces

In what follows, β is a real and p is an integer. The Kondratiev spaces (see for instance [10], [2] and [14]) are defined by

$$\begin{cases} K_\beta^0 = \Big\{v:\mathbb{R}^+\to\mathbb{R} \quad : \quad r^{-\beta-1/2}v\in L^2(\mathbb{R}^+)\Big\}, \\ K_\beta^p \;=\; \Big\{v:\mathbb{R}^+\to\mathbb{R} \quad : \quad v^{\{\ell\}}\in K_\beta^0 \quad \forall \ell\le p\Big\}, \end{cases} \tag{57}$$

where $\cdot^{\{\ell\}}$ denotes the differential operator defined at (53). These spaces are equipped with the Hilbertian norms

$$\begin{cases} \|v\|_{K_\beta^0} = \left(\displaystyle\int_0^{+\infty} r^{-2\beta}\big|v(r)\big|^2\frac{dr}{r}\right)^{1/2}, \\ \|v\|_{K_\beta^p} \;=\; \Big(\displaystyle\sum_{\ell=0}^{p}\|v^{\{\ell\}}\|^2_{K_\beta^0}\Big)^{\frac12}. \end{cases} \tag{58}$$

Proposition 1. *The space $L^2(\mathbb{R}^3)$ can be characterized as a weighted space:*

$$L^2(\mathbb{R}^3) = \{u:\mathbb{R}^3\to\mathbb{R} \quad | \quad u_{m,n}\in K^0_{-3/2}, \forall n\ge 0, m\in[-n,n] \quad \textit{and} \quad \sum_{m,n}\|u_{m,n}\|^2_{K^0_{-3/2}} < +\infty\}. \tag{59}$$

Proof. Let v be in $L^2(\mathbb{R}^3)$, we know from (55) that

$$\int_{\mathbb{R}^3}|v(\mathbf{x})|^2 d\mathbf{x} \;=\; \int_{\mathbb{R}}\int_0^{\pi}\int_0^{2\pi}|v(r,\theta,\varphi)|^2 r^2 dr\sin\theta d\theta d\varphi, \tag{60}$$

$$\;=\; \int_{\mathbb{R}}\sum_{m,n}|v_{m,n}(r,\theta,\varphi)|^2 r^2 dr. \tag{61}$$

By switching sum and integral terms (following Fubini theorem), it holds that

$$\int_{\mathbb{R}^3} |v(\mathbf{x})|^2 d\mathbf{x} = \sum_{m,n} \int_{\mathbb{R}} |v_{m,n}(r,\theta,\varphi)|^2 r^2 dr. \tag{62}$$

We then have to observe that

$$\int_{\mathbb{R}} \left|v(r,\theta,\varphi)\right|^2 r^2 \, dr = \int_{\mathbb{R}} \left|r^{-(-\frac{3}{2})} \, v(r,\theta,\varphi)\right|^2 \frac{dr}{r}, \tag{63}$$

to conclude. □

2.2.2 Mellin transform: definition and properties

Here we recall the definition of the Mellin transform with some properties. We denote by $\lambda \in \mathbb{C}$ the Mellin variable, where $\beta \in \mathbb{R}$ is its real part, $\xi \in \mathbb{R}$ its imaginary part, that is:

$$\lambda = \beta + i\xi. \tag{64}$$

Let $\mathcal{D}(]0,+\infty[)$ be the space of functions with compact support in $]0,+\infty[$

$$\mathcal{D}(]0,+\infty[) \;=\; \Big\{v \; :]0,+\infty[\longrightarrow \mathbb{R} \; : \; \exists (r_-,r_+) \in \mathbb{R}_+^2 \;\text{ with } 0 < r_- < r_+,$$
$$v(r) = 0 \text{ outside } [r_-,r_+]\Big\}. \tag{65}$$

Definition 1. *For any $v \in \mathcal{D}(]0,+\infty[)$, the Mellin transform is defined for any $\lambda \in \mathbb{C}$ by*

$$(\mathcal{M}v)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} r^{-\lambda} v(r) \frac{dr}{r}. \tag{66}$$

Remark 4. *Since $\mathcal{D}(]0,+\infty[)$ is dense into K_β^p, the Mellin transform can be extended on K_β^p.*

Proposition 2. *When $v \in K_\beta^p$, the Mellin transform $\mathcal{M}v$ is defined on a line of the complex plane as follows:*

$$\mathbb{C}_\beta \;:=\; \Big\{\lambda \in \mathbb{C} : \mathcal{R}e(\lambda) = \beta\Big\}. \tag{67}$$

Remark 5. *On the other hand, the Mellin transform is an isomorphism from K_β^p onto $\widehat{K}_\beta^p$ which is defined by:*

$$\widehat{K}_\beta^p = \{\omega : \mathbb{C}_\beta \to \mathbb{C} : \lambda \mapsto \omega^{\{\ell\}}(\lambda) \in L^2(\mathbb{C}_\beta), \quad \forall \ell \leq p\}, \tag{68}$$

where $L^2(\mathbb{C}_\beta)$ denotes the space of square integrable functions on $\mathbb{C}_\beta$.

Property 4 (see [1])**.** *For any $v \in K_\beta^p$, we have $v^{\{p\}} \in K_\beta^0$ and for a.e. $\lambda \in \mathbb{C}_\beta$*

$$(\mathcal{M}v^{\{p\}})(\lambda) \;=\; \lambda^p \big(\mathcal{M}v\big)(\lambda). \tag{69}$$

Property 5 (see [1])**.** *Let $v_q : \mathbb{R}_+ \longrightarrow \mathbb{C}$ be defined by $v_q(r) = r^q v(r)$, with $q \in \mathbb{R}$. If $v \in K_\beta^0$, then*

$$v_q \in K_{\beta+q}^0, \tag{70}$$

and for any $\lambda \in \mathbb{C}_{q+\beta}$

$$(\mathcal{M}v_q)(\lambda) \;=\; (\mathcal{M}v)(\lambda - q). \tag{71}$$

Proposition 3 (see [1]). *For any $v \in K^p_{\beta_0}$, such that $v(r) = 0$ for every $r > \rho_\star > 0$, the function v belongs to K^p_β for any $\beta \leq \beta_0$. The Mellin function is analytical in the half-plane $\mathbb{C}_{]-\infty,\beta_0[} := \{\lambda \in \mathbb{C} : \mathcal{R}e(\lambda) < \beta_0\}$ and for $\mathcal{R}e(\lambda) = \beta < \beta_0$ we have*

$$|\lambda^p (\mathcal{M}v)(\lambda)| \leq \frac{\rho_\star^{\beta_0-\beta}}{\sqrt{2(\beta_0 - \beta)}} \|v\|_{K^p_{\beta_0}}. \tag{72}$$

Definition 2. *Let β_1, β_2 be two real numbers, we introduce $\mathbb{C}_{[\beta_1,\beta_2]}$ as a strip of the complex plane defined by*

$$\mathbb{C}_{[\beta_1,\beta_2]} = \{\lambda \in \mathbb{C} : \mathcal{R}e(\lambda) \in [\beta_1, \beta_2]\}. \tag{73}$$

Proposition 4 (see [1]). *Let p be an positive integer, β_1 and β_2 be two real numbers such that $\beta_1 < \beta_2$. If $v \in K^p_{[\beta_1,\beta_2]} := K^p_{\beta_1} \cap K^p_{\beta_2}$, then for any $\beta \in [\beta_1, \beta_2]$, the function v belongs to K^p_β.*

2.2.3 The fundamental theorem

To prove Theorem 2 and thereby giving an estimation of the rest $\mathfrak{u}^N$, we are going to apply the fundamental theorem of the singularities theory, that is:

Theorem 3. *Let $\beta_0 < \beta_1 < \beta_2$ be three real numbers. If $v \in K^0_{\beta_1}$ satisfies*

i) the Mellin transform $\mathcal{M}v : \mathbb{C}_{\beta_1} \to \mathbb{C}$ admits an analytical continuation $\widehat{v}$ in $\mathbb{C}_{]\beta_0,\beta_2[}$;

ii) there exists $\alpha > 0$ such that for any $\lambda = \beta + i\xi \in \mathbb{C}_{]\beta_0,\beta_2[}$ and $|\xi| > 1$

$$|\xi^2\, \widehat{v}(\lambda)| \ \leq \ \alpha; \tag{74}$$

then for any $\beta \in]\beta_0, \beta_2[$, the function $v \in K^1_\beta$, and we get the following estimation

$$|v(r)| \ \leq \ \frac{r^\beta}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left|\widehat{v}(\beta + i\xi)\right| d\xi. \tag{75}$$

We are going to work with this Theorem and check these two hypotheses. For that purpose, we consider the wave equation onto the Kondratiev space via the Mellin transform. Next, we will construct a meromorphic extension which is analytical in a half-plane of the complex plane. All this will validate the first assumption and we will prove next that the extension satisfies the second hypothesis. At last, we will complete the proof of theorem 2 by estimating the rest $\mathfrak{u}^N$.

3 Behavior of $u_{m,n}$ in the neighborhood of the origin for $n \geq 3$ and $-n \leq m \leq n$

Herein we aim at proving that

Proposition 5. *For $n \in \mathbb{N}$, $-n \leq m \leq n$ and $N \in \mathbb{N}$ such that $N \geq 2$ is even, $n \geq 3$ and $N < n$, we have*

$$n(n+1)\, |u_{m,n}(r,t)| \ \leq \ \sqrt{\frac{\pi(r_\star + ct)}{2}}\; r^N\; \frac{\|\partial_t^{N+2} u_{m,n}\|_{K^0_{-3/2}}}{c^{N+2}}. \tag{76}$$

For any given t, the function $\mathbf{x} \mapsto \partial_t^\ell u(\mathbf{x},t)$ belongs to $L^2(\mathbb{R}^3)$, for every integer $\ell \geq 0$. The featuring of $L^2(\mathbb{R}^3)$, given by proposition 1, allows us to conclude that the spectral coefficients $r \mapsto \partial_t^\ell u_{m,n}(r,t)$ are functions of $K^0_{-3/2}$. Since $r \mapsto u_{m,n}(r,t)$ is compactly supported in $[0,\rho_\star]$, we deduce that

$$\partial_t^\ell u_{m,n} \in K^0_\beta, \quad \forall \ell \in \mathbb{N}, \quad \forall \beta \leq -3/2. \tag{77}$$

We can thus apply the Mellin transform to equation (52) and we use properties 4 and 5 to get that for any $t \geq 0$ and for any $\lambda \in \mathbb{C}_{]-\infty,-\frac{3}{2}[}$

$$\alpha_n(\lambda)\ (\mathcal{M}u_{m,n})(\lambda,t) = \hat{\partial}_t^2 (\mathcal{M}u_{m,n})(\lambda-2,t), \tag{78}$$

with $\hat{\partial}_t = \dfrac{\partial_t}{c}$ and for any $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, we have that

$$\alpha_n(\lambda) = \lambda^2 + \lambda - n(n+1). \tag{79}$$

We continue our work by proving that $\lambda \mapsto (\mathcal{M}u_{m,n})(\lambda,t)$ admits a meromorphic continuation $\lambda \mapsto \widehat{u}_{m,n}(\lambda,t)$ in the complex plane. For any $\mathcal{R}e(\lambda) < -3/2$, the meromorphic continuation $\widehat{u}_{m,n}(\lambda,t)$ necessarily coincides with $(\mathcal{M}u_{m,n})(\lambda,t)$

$$\widehat{u}_{m,n}(\lambda,t) = (\mathcal{M}u_{m,n})(\lambda,t), \quad \forall \lambda = \beta + i\xi \in \mathbb{C} \text{ with } \beta < -3/2. \tag{80}$$

For any $\beta \geq -3/2$, we construct the meromorphic continuation by means of formula (78)

$$\widehat{u}_{m,n}(\lambda,t) = \frac{\hat{\partial}_t^{2p} \widehat{u}_{m,n}(\lambda_p,t)}{\displaystyle\prod_{k=0}^{p-1} \alpha_n(\lambda_k)}, \quad \text{with } \lambda_k = \lambda - 2k \quad \text{and} \quad \hat{\partial}_t = \partial_t / c, \tag{81}$$

where $p = p(\lambda) = E(\frac{7}{4} + \frac{\beta}{2})$ denotes the integer such that $\beta - 2p \in [-\frac{7}{2}; -\frac{3}{2}[$. The formula (81) makes it possible to define $\widehat{u}_{m,n}(\cdot,t)$ as a meromorphic function of $\mathbb{C}$. On the other hand, this formula is also valid for any $p \in \mathbb{N}$.

The zeroes of α_n being located in $\lambda = n$ and $\lambda = -n-1$, we can deduce from the formula (81) that $\lambda \mapsto \partial_t^\ell \widehat{u}_{m,n}(\lambda,t), \forall \ell \in \mathbb{N}$, is an analytical function in $\mathbb{C}\backslash\{n+2k\}$, for all $k \in \mathbb{N}$.

Lemma 1. *For any $\lambda = \beta + i\xi \in \mathbb{C}$ such that $-n-1 \leq \beta \leq n$ and $|\xi| \geq 1$, we have that*

$$|\xi^2|\ |\widehat{u}_{m,n}(\lambda,t)| \leq |\hat{\partial}_t^{2n} \widehat{u}_{m,n}(\lambda_n,t)| \quad \forall \beta \leq n \text{ and } |\xi| \geq 1. \tag{82}$$

Proof. By definition (see (81)), since $n \geq p$, $\widehat{u}_{m,n}$ satisfies

$$\widehat{u}_{m,n}(\lambda,t) = \frac{\hat{\partial}_t^{2n} \widehat{u}_{m,n}(\lambda_n,t)}{\displaystyle\prod_{k=0}^{n-1} \alpha_n(\lambda_k)} = \frac{\hat{\partial}_t^{2n} \widehat{u}_{m,n}(\lambda_n,t)}{\alpha_n(\lambda) \displaystyle\prod_{k=1}^{n-1} \alpha_n(\lambda_k)}, \text{ with } \lambda_k := \lambda - 2k. \tag{83}$$

Let β_k be the real part of λ_k where $\lambda_k = \beta_k + i\xi$. For any $\beta_k \in [-n-1, n]$, we have that

$$|\alpha_n(\lambda_k)| \geq |\mathcal{R}e(\alpha_n(\lambda_k))| = n^2 + n - \beta_k^2 - \beta_k + \xi^2 \geq \xi^2 \geq 1, \tag{84}$$

because $n^2+n-\beta_k^2-\beta_k \geq 0$. For any $\beta_k \notin [-n-1,n]$, we have that

$$|\alpha_n(\lambda_k)| \geq |\mathcal{I}m(\alpha_n(\lambda_k))| = |(2\beta_k+1)\xi| \geq |\xi| \geq 1. \tag{85}$$

It follows therefrom that $|\alpha_n(\lambda_k)| \geq 1$ for any $\beta_k \in \mathbb{R}$. On the other hand, since $\beta \in [-n-1,n]$, we obtain that

$$|\alpha_n(\lambda)| = n^2+n-\beta^2-\beta+|\xi|^2 \geq |\xi|^2. \tag{86}$$

We then deduce (82). □

Lemma 2. *For any $\lambda \in \mathbb{C}_{]-2,n[}$ such that $|\xi| \geq 1$*

$$|\xi|^2\ |\widehat{u}_{m,n}(\lambda,t)| \leq \gamma_n(t)\|\hat{\partial}_t^{2n}u(\cdot,t)\|_{L^2(\mathbb{R}^3)}, \tag{87}$$

where γ_n is a continuous function of $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$.

Proof. Since $\lambda \in \mathbb{C}_{]-2,n[}$, $\lambda_n = \lambda - 2n \in \mathbb{C}_{]-2n-2,-n[}$. Then applying Proposition 3 with $\beta_0 = -3/2$, it holds that

$$|\hat{\partial}_t^{2n}\widehat{u}_{m,n}(\lambda_n,t)| \leq \gamma_n(t)\|\hat{\partial}_t^{2n}u_{m,n}(\cdot,t)\|_{K^0_{-3/2}}, \tag{88}$$

with $\gamma_n(t) = \max\left(\frac{\rho_\star^{n-3/2}}{\sqrt{2n-3}}, \frac{\rho_\star^{2n+1/2}}{\sqrt{4n+1}}\right)$ and $\rho_\star = r_\star + ct$. Since $\|\hat{\partial}_t^{2n}u_{m,n}(\cdot,t)\|_{K^0_{-3/2}} \leq \|\hat{\partial}_t^{2n}u(\cdot,t)\|_{L^2(\mathbb{R}^3)}$, (82) implies the estimate (87). □

Proposition 6. *For any $n \geq 3$ and $\beta \in]-\infty,n[$ the function $u_{m,n}(\cdot,t)$ is an element of K^1_β and*

$$\widehat{u}_{m,n}(\lambda,t) = (\mathcal{M}u_{m,n})(\lambda,t), \quad \forall\lambda \in \mathbb{C} \text{ such that } \beta < n. \tag{89}$$

Proof. The function $\lambda \mapsto \widehat{u}_{m,n}(\lambda,t)$ is analytical for any $-2 < \mathcal{R}e(\lambda) < n$ and it satisfies (87) for any $|\xi| > 1$. Theorem 3 implies that $u_{m,n}(\cdot,t) \in K^1_\beta$ for any $\beta \in]-2,n[$. Let us note that $u_{m,n}(\cdot,t) \in K^0_\beta$ for any $\beta \in]-\infty,-3/2]$. □

Lemma 3. *Let $N \geq 2$ be an even integer such that $N < n$. If $\lambda = N + i\xi$, we have*

$$n(n+1)(1+\xi^2)\Big|\widehat{u}_{m,n}(\lambda,t)\Big| \leq \Big|\hat{\partial}_t^{N+2}\widehat{u}_{m,n}(-2+i\xi,t)\Big|. \tag{90}$$

Proof. According to (81) for $p = \frac{N}{2}+1$, we have

$$\widehat{u}_{m,n}(\lambda,t) = \frac{\hat{\partial}_t^{N+2}\widehat{u}_{m,n}(-2+i\xi,t)}{\alpha_n(\lambda)\Big(\prod_{k=1}^{N/2-1}\alpha_n(\lambda_k)\Big)\alpha_n(i\xi)}, \quad \text{with } \lambda_k = N-2k+i\xi. \tag{91}$$

For any $\xi \in \mathbb{R}$, we have

$$|\alpha_n(i\xi)| \geq |\mathcal{R}e(\alpha_n(i\xi))| = n^2+n+\xi^2 \geq n^2+n. \tag{92}$$

Since $1 \leq k \leq N/2-1$, it follows that $\beta_k \in [0,N]$. As a consequence, we get

$$|\alpha_n(\lambda_k)| \geq |\mathcal{R}e(\alpha_n(\lambda_k))| = n^2+n-\beta_k^2-\beta_k+\xi^2 \geq n^2+n-N^2-N+\xi^2 \geq 1, \tag{93}$$

because $n > N$. Likewise, since $\beta = N$, we have

$$|\alpha_n(\lambda)| \geq |\mathcal{R}e(\alpha_n(N+i\xi))| = n^2+n-N^2-N+\xi^2 \geq 1+\xi^2. \tag{94}$$

The expected result follows then (92), (93) and (94). □

Lemma 4. *Let $N \geq 2$ be an even integer such that $N < n$. If $\lambda = N + i\xi$, we have*

$$n(n+1)\left(1+\xi^2\right)\left|\widehat{u}_{m,n}(\lambda,t)\right| \leq \sqrt{\rho_\star}\,\|\hat{\partial}_t^{N+2} u_{m,n}(\cdot,t)\|_{K^0_{-3/2}}. \tag{95}$$

Proof. By applying Proposition 3 with $\beta_0 = -3/2$, we get

$$\left|\hat{\partial}_t^{N+2}\widehat{u}_{m,n}(-2+i\xi,t)\right| \leq \sqrt{\rho_\star}\|\hat{\partial}_t^{N+2} u_{m,n}(\cdot,t)\|_{K^0_{-3/2}}, \text{ with } \rho_\star = r_\star + ct. \tag{96}$$

The conclusion follows (90). ☐

To get estimate (76), it remains to evaluate the following integral

$$\int_{-\infty}^{+\infty}\left|\widehat{u}_{m,n}(N+i\xi)\right|d\xi \leq \frac{\sqrt{\rho_\star}}{n(n+1)}\underbrace{\Big(\int_{-\infty}^{+\infty}\frac{d\xi}{1+\xi^2}\Big)}_{\pi}\left\|\hat{\partial}_t^{N+2}u_{m,n}(\cdot,t)\right\|_{K^0_{-3/2}},$$

and then to apply (75) of Theorem 3.

4 Proof of Theorem 2

This section concludes the proof by finally estimating the rest $\mathfrak{u}^N$. First, we give an estimate of the maximum value of $\mathfrak{u}^N$ by the Laplace-Beltrami operator. Then, we estimate the Laplace-Beltrami operator on the sphere. Finally, we apply an order upgrading to get the estimate of theorem 2.

4.1 Preliminary results

Let $\mathbb{S}$ be the unit sphere. We must remember that any function in $H^2(\mathbb{S})$ is bounded. The goal of this section is to prove a more accurate result for functions having a zero mean.

Lemma 5. *For any $v \in L^2(\mathbb{S})$ such that $\Delta_\Gamma v \in L^2(\mathbb{S})$ and*

$$\int_{\mathbb{S}} v(\theta,\varphi)\sin\theta d\theta d\varphi = 0, \tag{97}$$

we have $v \in L^\infty(\mathbb{S})$ and the following estimate holds:

$$|v(\widehat{\mathbf{y}})| \leq \|\Delta_\Gamma v\|_{L^2(\mathbb{S})}, \quad \forall \widehat{\mathbf{y}} \in \mathbb{S}, \tag{98}$$

that is

$$\|v\|_{L^\infty(\mathbb{S})} \leq \|\Delta_\Gamma v\|_{L^2(\mathbb{S})}. \tag{99}$$

Proof. We set $\widehat{\mathbf{y}} \in \mathbb{S}$ and we introduce a coordinate change that associates $\widehat{\mathbf{x}} \in \mathbb{S}$ with $\widehat{\mathbf{z}} \in \mathbb{S}$

$$\begin{cases} \widehat{\mathbf{z}}_1 = \widehat{\mathbf{x}}\cdot\widehat{\mathbf{y}}_{\perp,1}, \\ \widehat{\mathbf{z}}_2 = \widehat{\mathbf{x}}\cdot\widehat{\mathbf{y}}_{\perp,2}, \\ \widehat{\mathbf{z}}_3 = \widehat{\mathbf{x}}\cdot\widehat{\mathbf{y}}, \end{cases} \tag{100}$$

where $\widehat{\mathbf{y}}_{\perp,1}$ and $\widehat{\mathbf{y}}_{\perp,2}$ are two vectors such that $(\widehat{\mathbf{y}},\widehat{\mathbf{y}}_{\perp,1},\widehat{\mathbf{y}}_{\perp,2})$ are a basis of $\mathbb{R}^3$. The application $R_{\widehat{\mathbf{y}}} : \mathbb{S} \mapsto \mathbb{S}$ which associates $\widehat{\mathbf{x}}$ to $\widehat{\mathbf{z}}$ is an isometry. Let $\widetilde{v} : \mathbb{S} \mapsto \mathbb{S}$ be the function defined by

$$\widetilde{v}(\widehat{\mathbf{z}}) = v(\widehat{\mathbf{x}}), \quad \text{with} \quad \widehat{\mathbf{z}} = R_{\widehat{\mathbf{y}}}(\widehat{\mathbf{x}}). \tag{101}$$

We remark that the function $\widetilde{v}$ satisfies

$$\int_{\mathbb{S}} \widetilde{v} = \int_{\mathbb{S}} v = 0, \quad \|\Delta_\Gamma \widetilde{v}\|_{L^2(\mathbb{S})} = \|\Delta_\Gamma v\|_{L^2(\mathbb{S})} \quad \text{and} \quad \widetilde{v}(\widehat{\mathbf{z}}_{\text{top}}) = v(\widehat{\mathbf{y}}), \tag{102}$$

where $\widehat{\mathbf{z}}_{\text{top}} = (0,0,1)$ denotes the North pole of the sphere. Using the spectral decomposition of Δ_Γ, the function $\widetilde{v}$ can be written as

$$\widetilde{v} = \sum_{m,n} \widetilde{v}_{m,n} Z_{m,n}, \tag{103}$$

with $\widetilde{v}_{m,n} = \displaystyle\int_{\mathbb{S}} \widetilde{v} Z_{m,n}$. Following Property 2, we have $Z_{m,n}(0,0) = 0$ for any $m \neq 0$. Moreover, since $Z_{0,0}$ is proportional to the constant function and $\displaystyle\int_{\mathbb{S}} \widetilde{v} = 0$, we obtain $\widetilde{v}_{0,0} = 0$. It follows that

$$\widetilde{v}(\widehat{\mathbf{z}}_{\text{top}}) = \sum_{n=1}^{+\infty} \widetilde{v}_{0,n} Z_{0,n}(\widehat{\mathbf{z}}_{\text{top}}), \tag{104}$$

with $Z_{0,n}(\widehat{\mathbf{z}}_{\text{top}}) = \sqrt{\dfrac{(n+1/2)}{2\pi}}$ (see Property 2). We then apply Cauchy Schwarz inequality

$$|\widetilde{v}(\widehat{\mathbf{z}}_{\text{top}})| \leq \Big(\sum_{n=1}^{+\infty} \frac{n+1/2}{2\pi n^2 (n+1)^2} \Big)^{1/2} \Big(\sum_{n=1}^{+\infty} n^2 (n+1)^2 |\widetilde{v}_{0,n}|^2 \Big)^{1/2}. \tag{105}$$

We then note that

$$\left\{ \begin{array}{l} \displaystyle\sum_{n=1}^{+\infty} n^2 (n+1)^2 |\widetilde{v}_{0,n}|^2 \leq \sum_{m,n} n^2 (n+1)^2 |\widetilde{v}_{m,n}|^2 = \|\Delta_\Gamma \widetilde{v}\|^2_{L^2(\mathbb{S})}, \\ \displaystyle\Big(\sum_{n>0} \frac{n+1/2}{2\pi n^2 (n+1)^2} \Big)^{1/2} \leq 1. \end{array} \right. \tag{106}$$

We conclude that

$$|\widetilde{v}(\widehat{\mathbf{z}}_{\text{top}})| \leq \|\Delta_\Gamma \widetilde{v}\|_{L^2(\mathbb{S})}, \tag{107}$$

and we complete the proof thanks to (102). □

4.2 Estimate of the Laplace-Beltrami operator on the sphere

Lemma 6. *For any $N \in \mathbb{N}$ such that $N \geq 2$ is even, we have*

$$\|\Delta_\Gamma \mathfrak{u}^N(r,\cdot,t)\|_{L^2(\mathbb{S})} \leq \sqrt{\frac{\pi(r_\star + ct)}{2}}\, r^N \, \|\hat{\partial}_t^{N+2} u\|_{L^2(\mathbb{R}^3)}. \tag{108}$$

Proof. According to Parseval equality (55), we get

$$\|\Delta_\Gamma \mathfrak{u}^N(r,\cdot,t)\|^2_{L^2(\mathbb{S})} = \sum_{n=N+1}^{+\infty} \sum_{m=-n}^{n} n^2 (n+1)^2 |u_{m,n}(r,t)|^2. \tag{109}$$

From Proposition 5 we deduce that

$$\|\Delta_\Gamma \mathfrak{u}^N(r,\cdot,t)\|^2_{L^2(\mathbb{S})} \leq \frac{\pi(r_\star + ct)}{2}\, r^{2N} \sum_{n=N+1}^{+\infty} \sum_{m=-n}^{n} \|\hat{\partial}_t^{N+2} u_{m,n}\|^2_{K^0_{-3/2}}. \tag{110}$$

We apply Parseval theorem once again to get

$$\sum_{n=N+1}^{+\infty} \sum_{m=-n}^{n} \|\hat{\partial}_t^{N+2} u_{m,n}\|^2_{K^0_{-3/2}} \leq \|\hat{\partial}_t^{N+2} u\|^2_{L^2(\mathbb{R}^3)}. \tag{111}$$

This ends the proof. □

Remark 6. *The norm involved in the right term of equation* (108) *can be bounded by a constant independent of the time thanks to remark 2.*

We end up with applying Lemma 5. Since the function $\mathfrak{u}^N(r, \cdot, t)$ is zero mean on the unit sphere, we have for every $\mathbf{x}$ on the sphere with radius r

$$\mathfrak{u}^N(\mathbf{x}, t) \leq \sqrt{\frac{\pi(r_\star + ct)}{2}}\, r^N \, \|\hat{\partial}_t^{N+2} u\|_{L^2(\mathbb{R}^3)}. \tag{112}$$

This estimate holding true for any $r > 0$, we deduce a non-optimal estimate of $\mathfrak{u}^N$ for any even integer N set in the following proposition.

Proposition 7. *For any $T > 0$ and for any even integer N with $N \geq 2$, we have*

$$\max_{t \leq T} |\mathfrak{u}^N(\mathbf{x}, t)| = \underset{r \to 0}{O}(r^N). \tag{113}$$

To get the estimate of Theorem 2, we perform an order upgrading.
Let N be a given integer and let P be an even integer such that $P > N$. By definition of $\mathfrak{u}^N$ (see (39)), we have

$$\mathfrak{u}^N(\mathbf{x}, t) = \sum_{n=N+1}^{P} \sum_{m=-n}^{n} u_{m,n}(r, t) Z_{m,n}(\theta, \varphi) + \mathfrak{u}^P(\mathbf{x}, t). \tag{114}$$

According to Proposition 7 and (41), we know that

$$\begin{cases} \max\limits_{t \leq T} \Big| \displaystyle\sum_{n=N+1}^{P} \sum_{m=-n}^{n} u_{m,n}(r, t) Z_{m,n}(\theta, \varphi)\Big| = \underset{r \to 0}{O}(r^{N+1}), \\ \max\limits_{t \leq T} |\mathfrak{u}^P(\mathbf{x}, t)| = \underset{r \to 0}{O}(r^P) = \underset{r \to 0}{O}(r^{N+1}). \end{cases} \tag{115}$$

It follows that

$$\max_{t \leq T} |\mathfrak{u}^N(\mathbf{x}, t)| = \underset{r \to 0}{O}(r^{N+1}). \tag{116}$$

5 Conclusion and perspectives

We have thus illustrated how the Kondratiev theory is useful to justify the asymptotic behavior of an acoustic wave in the neighborhood of the point. In particular, the technique highlights the different roles played by the time and the space variables into the asymptotic expansion representing the solution to the wave equation. The time variable indeed acts more like a parameter than as a variable. In our opinion, this paper is new essentially because it is based on the singularity theory in the time domain which is less developed than in the stationary case. There are very few works dealing with singular perturbations for time-dependent problems and the best of our knowledge, only [12] addressed this kind of issue in two dimensions by applying

different techniques. It is then interesting to observe that even though the separation of variables is often opposed to the Kondratiev theory, our approach clarifies that these two techniques can be mixed together in order to simplify the proof of convergence.
As far as the possible continuations of this work, we would like to consider other models to provide fast computing methods for the propagation of waves in different media including small defects. We indeed believe that our approach can be extended to Maxwell equations and to elastic wave equations, by considering in particular the theory developed in [4] which needs to be applied in the time domain. It would be also very relevant to deal with anisotropic and heterogeneous media but in these cases, the corresponding asymptotic expansion is absolutely non trivial.

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