Reforming the international system of units: On our way to redefine the base units solely from fundamental constants and beyond

Christian J. Bordé

Reforming the international system of units: On our way to redefine the base units solely from fundamental constants and beyond

Ch.J. Bordé*

Laboratoire de Physique des Lasers, UMR 7538 CNRS, Université Paris Nord,

99 avenue J.-B. Clément, 93430 Villetaneuse, France

LNE-SYRTE, UMR 8630 CNRS, Observatoire de Paris

61 avenue de l'Observatoire, 75 014 Paris, France

Abstract

Thanks to considerable progress in quantum technologies, the trend today is to redefine all SI base units from fundamental constants and we discuss strategies to achieve this goal. We first outline the present situation of each of the seven base units and examine the choice of fundamental constants which can reasonably be fixed. A critical issue is how we should redefine the unit of mass in the context of modern relativistic quantum theory. At the microscopic level the link of mass with proper time as conjugate variables in the quantum phase $S/\hbar$ is well established. This link strongly suggests that we should fix the value of Planck's constant h, thus defining mass through a de Broglie-Compton frequency mc^2/h. This frequency can be accurately measured for atomic and molecular species by atom interferometry. The main difficulty is then to bridge the gap with the macroscopic scale for which phases are usually scrambled by decoherence and where all mechanical quantities are built from the classical action S only without connexion to a quantum phase. Two ways are now being explored to make this connexion: either the electric kilogram which uses recent progress in quantum electrical metrology or atom interferometry combined with the Avogadro number determination using a silicon sphere. Consequences for a new definition of the unit will be explored as the two methods hopefully converge towards an accurate value of Planck's constant. Another important choice is the electric charge connecting electrical and mechanical units: we could keep Planck's charge and vacuum properties μ_0 and Z_0, which is the case today or shift to a fixed electric charge e which seems to be the favourite choice for to-morrow. We recall

*Membre de l'Institut, Académie des Sciences, President of the Conférence Générale des Poids et Mesures (CGPM) in 1999, 2003, 2007 and 2011

that temperature and time are linked through Boltzmann's constant and there is a general agreement to fix that constant after suitable measurements. Finally the unit of time is looking for a new more universal and accurate definition based on Bohr frequencies corresponding to higher and higher frequency clocks. A last challenge is to produce a unified framework for fundamental metrology in which all base quantities and relevant fundamental constants appear naturally and consistently. We suggest a generalized 5D framework in which both gravito-inertial and electromagnetic interactions have a natural geometrical signification and in which all measurements can be reduced to phase determinations by optical or matter-wave interferometry.

1 Introduction: from the French Revolution to Max Planck.

The metric system was born during the French Revolution with the idea of settling a universal system of units, open to every people, in every time. At that time, the dimensions of the Earth, the properties of water appeared as a universal basis, but some time later James Clerk Maxwell judged them less universal than the properties of the molecules themselves. The next step was taken by George Johnstone-Stoney, then by Max Planck, showing that a deeper aim was to found the system of units only on a set of fundamental constants originating from theoretical physics. As a consequence, a long divorce began between the practical requirements of instrumental metrology and the dreams of theoretical physicists. Might they marry again? This has now become possible thanks to a set of recent discoveries and new technologies: laser measurements of length, Josephson effect, quantum Hall effect, cold atoms, atom interferometry, optical clocks, optical frequency measurements.... So a strong tendency to tie the base units to fundamental constants is rising again, and the debate is open as to the relevance, the opportunity and the formulation of new definitions.

Since the very beginning of this adventure the French Academy of Sciences has had a leading role in the development of ideas and the settling of the metric system. An Academy committee on Science and Metrology is still working on this theme today. We outline in this paper some of the questions under consideration in this committee and their underlying physical grounds. Our purpose is to offer a logical analysis of the system of units and to explore possible paths towards a consistent and unified system with an original perspective. The path taken here builds on the fact that, thanks to modern quantum technologies, any measurement can be reduced to a dimensionless phase measurement thanks to optical or matter-wave interferometry and we shall try to follow this simple guiding line. We shall finally show how one could progress even further on the path of a synthetic framework for fundamental metrology based upon pure geometry in five dimensions. The reader who does not wish to enter into these mathematical considerations can skip the last paragraph and Appendix 2. The conclusion emphasizes the role of quantum mechanics and uncertainty relations in the new metrology.

2 Present status and evolution of the international system of units, the SI.

The SI (11th CGPM, 1960) comprises seven base units which are all more or less concerned by the process of evolution mentioned above:

- the metre has already been given a new definition from the time unit and the velocity of light in 1983 (see next section);

- the kilogram is still defined today by an artefact of iridium/platinum alloy but as we shall see in detail it could find a new definition from Planck's constant in a near future;

- the SI ampere is defined through a property of the vacuum, specifically its magnetic permeability $\mu_0 = 4\pi.10^{-7}$ H/m (9th CGPM, 1948), but the electrical units have *de facto* already gained their independence from the SI ampere, by adopting conventional values for Josephson and von Klitzing constants and the natural temptation today is to adopt the value of the electric charge e in order to freeze the numerical values of these constants;

- the kelvin is defined through the triple point of water, whereas fixing Boltzmann's constant k_{B} would be more satisfactory;

- the candela, unit of luminous intensity, is nothing else but a physiological unit derived from an energy flux, hence redundant with the other base units. Furthermore, it does not take into account the coherence properties, spectral content and spatial mode content of the source. Hence we shall not give any further consideration to this pseudo-base unit;

- the mole (added to the SI by the 14th CGPM in 1971) is defined from the mass of the carbon atom by a dimensionless number, the Avogadro number. A better determination of this number should give an alternative option in which it would be fixed to redefine the mass unit from the mass of an atom or of the electron. The tendency today is simply to retain this numerical value as a conventional number of entities;

- the second, unit of time, was originally defined as the fraction 1/86 400 of the "mean solar day". The exact definition of the "mean solar day" was left to astronomers. However, observations have shown that this definition was not satisfactory owing to irregularities in the Earth rotation. To give more accuracy to the definition, the 11th CGPM (1960) approved a definition given by the International Astronomical Union based on the tropical year 1900. But experimental work had already shown that an atomic standard of time, based on a transition between two energy levels of an atom or a molecule, could be realized and reproduced much more accurately. Considering that a more precise definition of the unit of time was essential for science and technology, the 13th CGPM (1967/68) replaced the definition of the second by the following (source BIPM) :

The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom. It follows that the hyperfine splitting in the ground state of the caesium 133 atom is exactly 9 192 631 770 hertz.

This definition refers to a caesium atom at rest at a temperature of 0 K. This implies that the corresponding frequency should be corrected from Doppler shifts and from the shifts coming from all sources of ambient radiation (CCTF 1999).

The second should soon be better defined by an optical clock or even by a much higher frequency clock (nuclear transition or matter-antimatter annihilation process). Ideally, physicists would have dreamed of an atomic hydrogen clock; it would have allowed to tie the time unit to the Rydberg constant and possibly, some day, to the electron mass. But this choice could be behind us today.

One should emphasize that the unit of time refers to proper time[1]. Proper time, as we shall discuss in detail, is associated with the internal evolution of a massive object such as an atom and is the measurable quantity from which the time coordinate is more or less artificially constructed. The time unit definition should thus have referred to the atom Bohr frequency and not to the radiation frequency. Among other things no mention is made of the recoil shift.

So we are facing ill-assorted definitions piled up along the years, without any global consistency. The direct connexion between the definition of a base unit from a fundamental constant, the practical working out of it, and a main scientific discovery is well illustrated by the case of the metre and its new definition issued from the technological progress of laser sources. This represents the archetype of the path to be followed for the other units.

3 The example of the metre

The metre is the best-known example of a base unit for which a new definition was based upon a fundamental constant, the velocity of light in vacuum c, thanks to progress in optics and especially laser physics during the second half of the XXth century.

The coordinates of space and time are naturally connected by Lorentz transformations within the conceptual frame of the theory of relativity, and the velocity of light takes place as a factor of conversion in these symmetry transformations. The existence of a symmetry is a first situation which allows to create an association between two units and hence to reduce the number of independent units.

A second favourable condition is the existence of mature technologies to

[1] We should carefully distinguish two different meanings of time: on the one hand, time and position mix as coordinates and this refers to the concept of time coordinate for an event in space-time, which is only one component of a 4-vector; on the other hand, time is the evolution parameter of a composite system and this refers to the proper time of this system and it is a Lorentz scalar (see below).

implement this symmetry. Relativity uses clocks and rods to define time and space coordinates. The rods of relativity are totally based on the propagation properties of light waves, either in the form of light pulses or of continuous beams whose frequency can now be locked to atomic clocks. It was possible to redefine the length unit from the time unit, because modern optics allowed not only the measurement of the speed of light generated by superstable lasers with a relative uncertainty lower than the best length measurements, but also because today the same techniques allow the new definition of the metre to be realized in an easy and daily way.

It is precisely optical interferometry and especially the work of Albert A. Michelson (Nobel Prize winner in 1907) that allow us to go from the nanometric length which is the wavelength linked to an atomic transition, to a macroscopic length at the metre level. Michelson interferometers can measure the tiniest length variations (10^{-23}) induced by gravitational waves over distances ranging between hundreds of kilometres on earth and millions of kilometres in spatial projects such as LISA. Any length measurement can thus be reduced to a phase measurement i.e. to the determination of an invariant number.

This evolution started in 1960 when the metre was redefined from the radiation of the krypton lamp. The birth of lasers, in 1959, helped to carry on steadfastly in that direction. Above all it was the discovery of sub-Doppler spectroscopic methods, and particularly of saturated absorption spectroscopy in 1969 [30, 29] which turned lasers into sources of stable and reproducible optical frequencies. The other revolution was the technique of the MIM diodes introduced by Ali Javan that led to measure the frequency of these light sources directly from the caesium clock. From then on, the velocity of light could be measured with a sufficiently small uncertainty, and so the CGPM in 1983 fixed its value linking the metre to the second. All the above implies a procedure to put the definition into practice (*mise-en-pratique*), using wavelengths of lasers locked to recommended atomic or molecular transitions.

Finally this redefinition was possible because there was a theoretical background universally accepted to describe the propagation of light in real interferometers.

To extend this approach let us investigate to what extent a similar situation can be met for the other units and what fundamental constants are available for each of them. A detailed discussion, partly reproduced in Appendix 1, is given in references [1, 14].

4 The dimensioned and dimensionless fundamental constants and their place in present physics:

The fundamental constants we are referring to, come out of the major theories of modern physics: relativity theory, quantum mechanics, statistical mechanics, field theories, ...Consequently they rely on our models and representations of the physical world.

What set of fundamental constants must we choose in the end ? They belong to two very distinct categories. On the one hand, we have what can be called conversion constants. Such constants are used to connect together quantities originally believed to be of a different nature, but later understood to refer to the same physical entity. A famous example is the equivalence between heat and work which led to the mechanical equivalent of the calorie: 4.18 joules. The conversion constants have the dimension of the ratio between the linked units. They can be given a fixed numerical value, and the number of independent units is thus reduced. Several constants play this role unequivocally: such was the case with the velocity of light, and it is still the case with Planck and Boltzmann constants as discussed later. In other cases we will have a choice to make between several constants of the same nature: it will be the case of the electric charge for instance. On the other hand, nature forces on us another sort of constants: the value of non-dimensional ratios: such are, for example, the coupling constants linked to the fundamental interactions. The best known are the fine structure constant describing the coupling of matter with the electromagnetic field:

$$\alpha = \frac{\mu_0 c e^2}{4\pi\hbar} \tag{1}$$

and its gravitational analog

$$\alpha_G = G m_e^2/\hbar c \tag{2}$$

involving the gravitation constant G and the electron mass m_e.

The value of these coupling constants cannot be discussed, and remains independent of the system of units. It is a constraint to be taken into account in our choices.

5 The kilogram and the mole, determining Avogadro number with the silicon sphere:

Since 1889 (1rst CGPM) the mass unit has been the mass of the international prototype, a platinum-iridium alloy cylinder baptised $\mathfrak{K}$ and kept in a vault of the Pavillon de Breteuil with 6 copies. After the three intercomparisons made in 1889, 1946/53 and 1989/92, there is now a general agreement on the idea that the mass of the standard prototype, constant by definition, has in fact drifted by several 10 or so micrograms (i.e. some 10^{-8} in relative value). This situation in which the electrons and other elementary particles of the universe have a mass value changing with time, when the piece of metal in the vault in Sèvres has not, is quite embarassing. So, every effort must be done to modify the definition (recommendation of the 21rst CGPM). It would be much more satisfactory and justified to start from the mass of microscopic particles (electron or atom) *a priori* quite reproducible, and then to climb up the macroscopic scale. But if masses can be easily compared both at the macroscopic and at the atomic scales, the connection between these two scales is quite difficult. To make

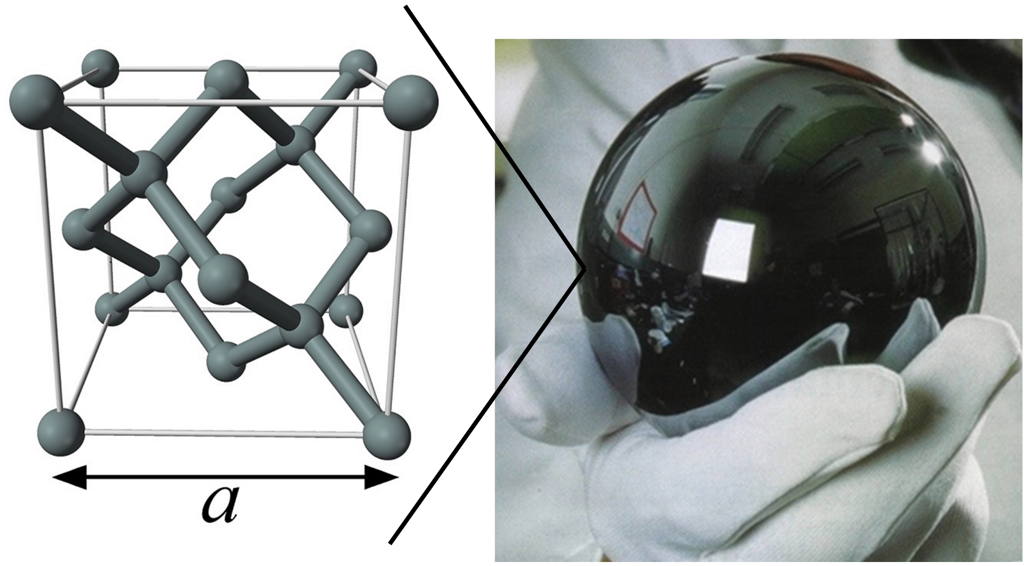


Figure 1: Starting with a silicon monocrystal purified by the floating zone method, several nearly perfect spheres with masses $\sim$ 1 kg were made (surface defects below some tens of nanometres) then, thanks to mass spectrometry, X and optical interferometries, the size a of the cell d_{220}, the density $\rho = V/m$ and the molar mass M were determined. The cubic crystal cell corresponding to eight atoms it was possible to obtain the Avogadro constant from the formula $N_A = 8M/(\rho a^3)$.

this connection we need to make an object with a known number of atoms and whose mass could be compared to that of the standard kilogram. This amounts to determining the Avogadro number $\mathfrak{N}_{\mathrm{A}}$ which defines the mole. The mole is a quantity of microscopic objects defined as a conventional number of identical objects. This number (of course without dimension) has been arbitrarily chosen equal to the number of supposedly isolated atoms, at rest and in their fundamental state, contained in 0.012 kg of carbon 12. Consequently it is, up to a numerical factor 0.012, the ratio of the mass of the standard prototype to the mass of a carbon atom. Avogadro's constant N_{A} generally refers to that same number per mole, and it is expressed in mol^{-1}. This number and this constant are just another way of expressing the mass of a carbon atom, or its 12th part, which is the unified atomic mass unit m_u.

An international program (the XRCD program for X-ray crystal density program) has been developed to determine the Avogadro number from the knowledge of a silicon sphere studied under "every angle": physical characteristics of dimension, mass, volume, cell parameter, isotopic composition, surface state etc...(see figure 1). The International Avogadro Coordination project is refining

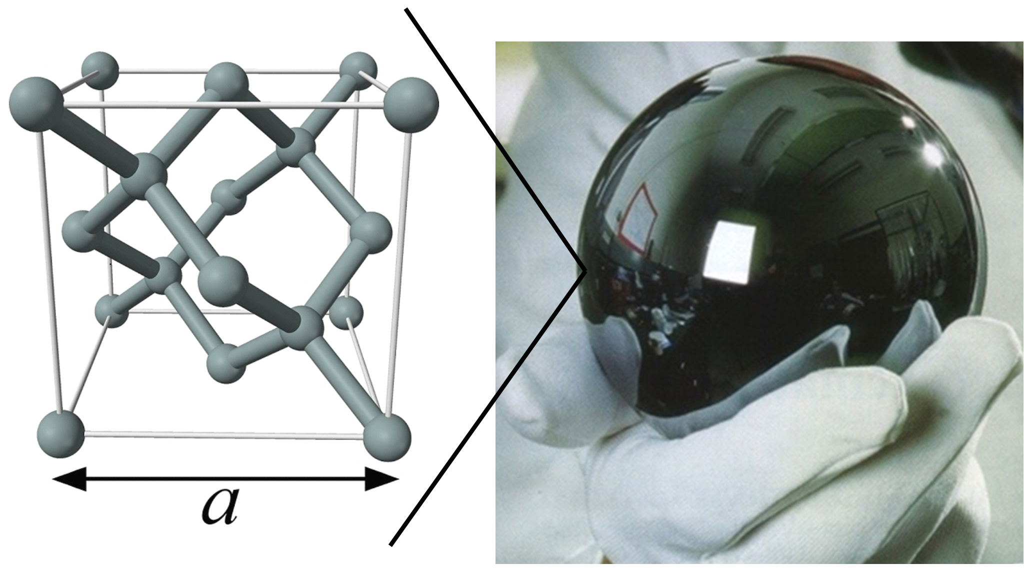


Figure 2: Starting with a silicon monocrystal purified by the floating zone method, several nearly perfect spheres with masses ~ 1 kg were made (surface defects below some tens of nanometres) then, thanks to mass spectrometry, X and optical interferometries, the size a of the cell d_{220}, the density $\rho = V/m$ and the molar mass M were determined. The cubic crystal cell corresponding to eight atoms it was possible to obtain the Avogadro constant from the formula $N_A = 8M/(\rho a^3)$.

the application of the XRCD method to isotopically enriched ^{28}Si spheres with the goal of reaching a 1.5 10^{-8} relative standard uncertainty [21]. The Avogadro constant N_A based on these measurements is presently 6.022 140 76(19) 10^{23} mol^{-1}[20]. This program has already faced and overcome many difficulties, and one day it should eventually reach the goal of fixing the Avogadro number with an accuracy that allows a redefinition of the kilogram.

6 Mass concept from proper time: Relativity and Quantum Mechanics.

In fact, the notion of mass does not boil down to that of a quantity of matter and, if a redefinition of the mass unit from the mass of a reference elementary particle goes the right way, it does not reduce the number of independent units. However, there is a possibility, as in the case of the metre, to link the mass unit to the time unit. Indeed the theory of relativity allows us to identify the mass m of an object with its internal energy, according to the well-known relation $E = mc^2$. What is more, Louis de Broglie, in his famous Note in 1923 [31], teaches us that this energy can be linked to the proper time τ of the object to produce the phase of an internal oscillation. The product $mc^2\tau$ of these two quantities is an action, which must be related to an elementary action, Planck's constant h, to give the phase without dimension of that oscillation $mc^2\tau/h$ (see Appendix 2). In other words the quantity mc^2/h is a frequency which we shall call de Broglie-Compton frequency (dBC)[2]. This frequency can be indirectly measured in the case of microscopic particles such as atoms or molecules by modern techniques of atomic interferometry in which de Broglie waves are precisely made to interfere. The first experiments of this type were performed by measuring the recoil frequency shift which occurs when laser light is absorbed or emitted by molecules in saturation spectroscopy (Hall and Bordé, 1973 [10]). They were followed by cold atom interferometry [19] using Bordé-Ramsey interferometers [7, 9, 13]. Today this measurement is done with a relative uncertainty less than 10^{-8}(Biraben et al. [17, 18]). From that point, by simply multiplying with Avogadro number $\mathfrak{N}_A$, we can have access to the de Broglie-Compton frequency of the kilogram from that of the atomic mass unit m_u,

$$\nu_{dBC} = \frac{M_{\mathfrak{K}} c^2}{h} = 1000\mathfrak{N}_A \left(\frac{m_\mathrm{u} c^2}{h} \right) \tag{3}$$

and so link the mass unit to the time unit. Then the mass unit would be defined by fixing that de Broglie-Compton frequency, which amounts to fix Planck's constant. Such was the recommendation made by the working group of the Académie des Sciences to the CIPM in 2005. The definition of the unit of mass would essentially look like:

[2] There is presently no physical clock at the de Broglie-Compton frequency although it appears quite possible in the future through stimulated absorption and emission of photon pairs in the creation/annihilation process of electron-positron pairs.

"The kilogram is the unit of mass, it is the mass of a body whose de Broglie-Compton frequency is equal to $(299\ 792458)^2/(6.6260693.10^{-34})$ hertz exactly". This definition has the effect of fixing the value of the Planck constant, h, to be $6.6260693.10^{-34}$ joule second exactly.

This definition currently meets several criticisms: besides being an unfamiliar concept involving too large a number, it is a quantum-mechanical concept used in a range where its validity may be questioned because of decoherence among other things[3]. There is certainly no physical clock at such a high frequency, which thus appears as fictitious. Even at the single atom level the connection between the de Broglie-Compton frequency, which is measured indirectly, and a real clock frequency has been the subject of recent controversy. We shall see in the generalized 5D approach that the overall action and hence the overall phase cancels along the classical trajectory. Interference fringes result only from the phase added by a coupling of modes with different wave vectors or frequencies. A real atomic clock is generated at the Bohr frequency by a superposition of two internal states b and a and it oscillates at the difference of the two corresponding de Broglie-Compton frequencies on both sides of an interferometer:

$$\nu_{\text{Bohr}} = \frac{m_b c^2}{h} - \frac{m_a c^2}{h} \tag{4}$$

The unit of mass may now be defined from this difference of the de Broglie - Compton frequencies of both states which has a clear physical signification and, if the chosen transition is the atomic transition which defines the unit of time, we make an explicit link between both units:

"The kilogram is the unit of mass, it is the mass of N massive particles without mutual interactions with a mass equal to the mass difference between the two internal states which define the unit of time" where N is a fixed numerical value of $c^2/h\nu_{\text{Bohr}}$ obtained by fixing the value of the Planck constant, h, to be $6.6260693.10^{-34}$ joule second exactly.

We shall come back on this point since, as we will see, another way of measuring the de Broglie-Compton frequency of the kilogram exists; it uses the spectacular progress of quantum electric metrology that we are now going to recall.

7 Quantum electric metrology: Josephson and quantum Hall effects

The electrical units underwent two quantum revolutions at the end of the previous century: the Josephson effect which allows us to realize the volt, and the quantum Hall effect which allows us to carry out the ohm.

[3] A well-defined phase assumes that the object should be in an eigenstate of its internal Hamiltonian. In the case of a collection of atoms this could be realized only with a large Bose-Einstein condensate.

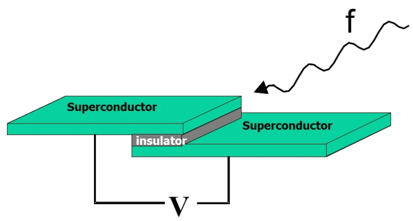


Figure 3: Josephson junction

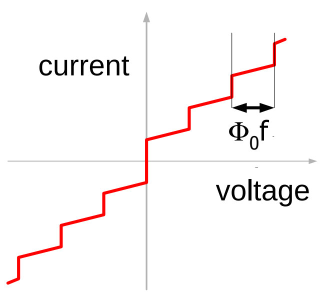
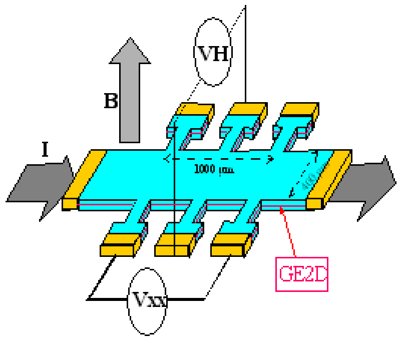


Figure 4: The Josephson effect (Nobel prize 1973) uses the junction comprising a very thin insulating layer sandwiched between two supraconducting plates. When this junction is irradiated by an electromagnetic wave of frequency f, its current-voltage characteristic presents voltage plateaux connected to the frequency f by a simple proportionality relation in which n is an integer characterizing each plateau: $V = nK_{\mathrm{J}}^{-1}f$. The Josephson constant K_{J} is given with an excellent approximation by $2e/h$. The charge $2e$ is that of Cooper pairs of electrons which are able to tunnel across the junction. This effect has a topological nature ($\phi_0 = h/2e$ is a quantum of flux) hence its universal character, independent of the detailed realisation of the junction and verified at the 10^{-10} accuracy level.



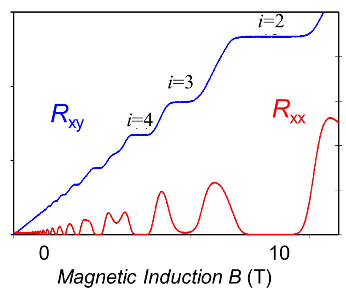


Figure 5: Quantum Hall effect. When a bidimensional gas of electrons in a semiconductor is submitted to a strong magnetic field, the transverse resistance (Hall resistance) exhibits steps quantized by the integer i and von Klitzing (Nobel prize 1985) resistance R_{K} whereas the longitudinal resistance vanishes: $R_{\mathrm{H}} = R_{\mathrm{K}} / i$. Here again the effect has a universal topological nature and is protected by the chiral anomaly introduced by Schwinger, which suggests that $R_{\mathrm{K}} = h/e^2$ with an excellent approximation.

Historically the ampere was the first example, before the metre, of a unit defined from a fundamental constant, the magnetic permeability μ_0 of the vacuum (9th CGPM 1948). The combination of these two definitions fixes all the propagation properties of electromagnetic waves in vacuum : velocity c and impedance $Z_0 = \mu_0 c$. Let us remark that by fixing Planck's constant, an electric charge would be also fixed, the Planck charge given by :

$$q_{\mathrm{P}} = \sqrt{2h/Z_0} \tag{5}$$

In practice the reproducibilities of Josephson and quantum Hall effects (respectively 10^{-10} and 10^{-9} in relative value) reach such a level that today electrical measurements use these effects without any other connection to the definition of the ampere. Were Planck's constant fixed, the electricians would be greatly tempted to fix the electron charge rather than Planck's charge, having in the back of their mind to fix Josephson and von Klitzing's constants. Unfortunately the simple theoretical expressions that link these two constants to e and h have not yet been validated with a high enough accuracy (only 2.10^{-7} for K_J and 3.10^{-8} for R_K in relative value), even if their universality could be demonstrated at a much higher level. Independently from the strong theoretical arguments that lie under these formulas, it is necessary to make sure that possible corrections are low enough for both effects to achieve a reliable realization of $2e/h$ and h/e^2 (see figures 2 to 4).

In the case of the quantum Hall effect such a verification can be made because the ratio of the vacuum impedance to h/e^2 is just the double of the fine structure constant α. The vacuum impedance can be realized thanks to a calculable capacitor (Thomson-Lampard) and the comparison of Z_0/R_K with 2α value obtained by atom interferometry presently sets an uncertainty level around 10^{-8} and will certainly improve beyond 10^{-8}. In the case of the Josephson effect the limit comes from our insufficient knowledge of the proton gyromagnetic ratio. Fortunately, two other verifications will be possible with the metrologic triangle and the watt balance as we shall see below.

In fact, quantum electrical metrology is undergoing a third revolution with the SET (Single Electron Tunnelling) permitting to count electrons one by one. Then Ohm's law becomes an equality between frequencies: the electric potential difference is turned to a Josephson frequency, the electric current to a number of electrons by second, and the electric resistance expressed in terms of von Klitzing resistance has no dimension (see figure 5).

The closure of the metrological triangle will be a real test of the quantum realizations and of the theories that connect K_J and R_K to the fundamental constants of physics. Presently it is done at some 10^{-7} level, but hopefully that limit will reach the 10^{-8} level in the future.

Thus electrical metrology is in profound evolution. In the future it will occupy a key position for the entire metrology, especially thanks to the "electric" kilogram (see below). As for the electrical units, the question can be raised if one should fix the positron charge e or rather Planck's charge q_P? The ratio of both charges being the square root of the fine structure constant, the corresponding uncertainty will be transferred to the non-fixed charge. Some arguments inspired by the recent theories of strings and an easier statement of gauge invariance point to the first choice. Caution towards the formulae giving K_J and R_K speaks for the second choice, which goes back to keep the vacuum impedance fixed as it is now. This choice has in fact already been made by the CIPM and the last recommendations of the CGPM are in favour of fixing the electric charge e.

8 The electric kilogram and the watt balance

If the formulae giving K_J and R_K are considered to be valid, Josephson and quantum Hall effects can be combined to produce an electric power proportional to Planck's constant (figure 6):

$$UI = (h/2e)^2 \,/\, \left(h/e^2\right) f_1 f_2 = \frac{h}{4} f_1 f_2 \qquad (6)$$

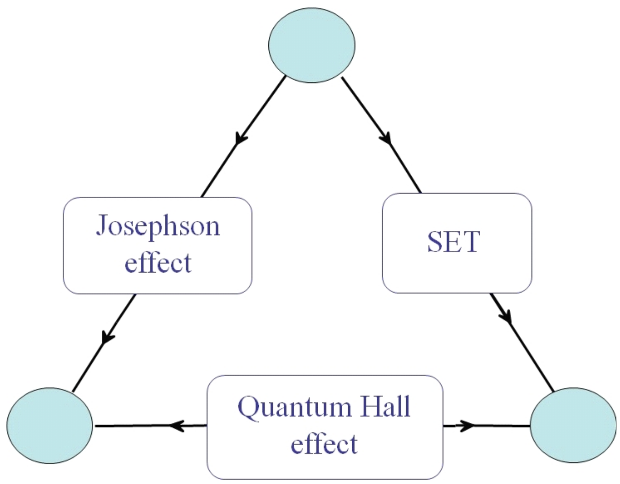


Figure 6: The metrological triangle is the quantum realisation of Ohm' law. If the three effects are described by the canonical formulas with the same constants e and h in the three formulas: $U = \frac{h}{2e} f, I = ef'$, $\frac{R}{R_{\rm K}} = \frac{e^2}{h} R$ one must check that $U = RI$ leads to an equality between frequencies $f = 2(R/R_{\rm K})f'$.

where $f_1$and f_2 are the two Josephson frequencies involved in this measurement.

This opened a new way of measuring the de Broglie-Compton frequency of the kilogram, that of the "electric" kilogram. The "electric kilogram" was born with the watt balance, suggested by Kibble in 1975 [5], which in one step (cryogenic version of the BIPM) or two steps (see figure 6), carries out the direct comparison between a mechanical watt realized by moving a mass in the gravitational field of the earth, and an electric watt given by the combination of Josephson and quantum Hall effects. Such a method demonstrated more than 20 years ago in the USA and in Great Britain that it could reach a level of relative uncertainty consistent with that of the present kilogram, i.e. some 10^{-8}. Two new realizations have been assembled and are under study, one in Switzerland and a newer one in France. Other programs will follow. Within a few years this effort is likely to offer the opportunity to keep track of the evolution of the present kilogram prototype, and later on to give a new definition of this kilogram by fixing its dBC frequency. Clearly there is a competition between two projects to define the mass unit: in the first project Planck's constant is fixed and the watt balance allows to measure masses easily; in the second one Avogadro's number is determined and fixed, and the mass unit is defined from an elementary mass such as the electron mass. However, in that case, the practical realization of a macroscopic mass still has to be done through the realization of a macroscopic object whose number of microscopic entities is known. The first point of view is the most attractive on the conceptual, theoretical and even practical levels, even if the mass unit expression is not easy to grasp for everyone. Anyway both ways towards Planck's constant will need to be fully reconciled.

The CCM[4](2013) therefore *recommends* that the following conditions be met before the CIPM[5] asks CODATA[6] to adjust the values of the fundamental physical constants from which a fixed numerical value of the Planck constant will be adopted,

1. at least three independent experiments, including work from watt balance and XRCD experiments, yield consistent values of the Planck constant with relative standard uncertainties not larger than 5 parts in 10^8,

2. at least one of these results should have a relative standard uncertainty not larger than 2 parts in 10^8,

3. the BIPM prototypes, the BIPM ensemble of reference mass standards used in the watt balance and XRCD experiments have been compared as directly as possible with the international prototype of the kilogram

There has been much progress in the determination of Planck's constant in the recent past. A recent evaluation at NIST resulted in a value published in 2014 with a standard uncertainty of 4.5 parts in 10^8. The last result from NRC has a standard uncertainty of 1.8 parts in 10^8 sufficient to meet condition 2 of CCM. There are a number of other watt balance experiments that will provide independent values.

[4] Comité consultatif pour la masse et les grandeurs apparentées
[5] Comité international des poids et mesures
[6] Committee on Data for Science and Technology

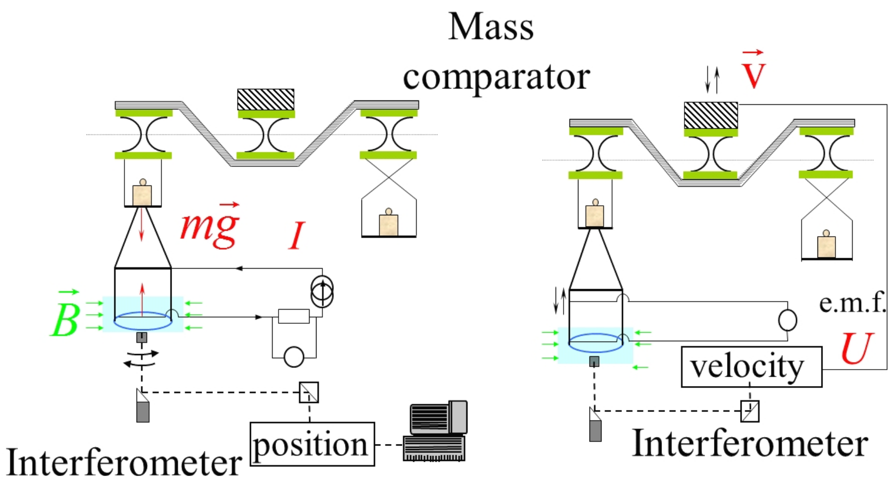


Figure 7: In its classical version, the watt balance is operated in two steps. In the first one, the weight of the kilogram in the gravity field is balanced by the Laplace force exerted on a coil conducting an electric current and placed in a magnetic field. The current I is measured by the combination of Josephson and quantum Hall effects. In the second step the same coil is moved at constant speed v in the same magnetic field and the induced emf U is measured thanks to the Josephson effect. Under these conditions, the properties of the coil and of the magnetic field are common to both modes and the final formula expressing the equality between electrical and mechanical powers $mgv = UI$ involves only times and frequencies for the de Broglie-frequency of the kilogram $\nu_{dBC} = M_{\mathfrak{K}} c^2/h = f_1 f_2 \omega T^2/\left[4\varphi(v/c)\right]$ where the Josephson frequencies of the two steps are f_1 and f_2. The velocity v is measured by optical interferometry and the terrestrial gravity field g by atom interferometry (phase $\varphi = kgT^2$) using laser pulses of pulsation $\omega = kc$ separated by the time interval T.

We may imagine future ideal versions of the watt balance working as true matter-wave interferometers analogous to superconducting ring gyros [22, 23]. A superconducting coil directly connected to a Josephson junction works as a Cooper pair interferometer which experiences a phase shift from the change in gravitational potential. The power balance

$$M_{\mathfrak{K}} g v = I U \tag{7}$$

may be written as

$$M_{\mathfrak{K}} c^2 \left(\frac{g v T}{c^2} \right) = N_{2e} h f_{\mathrm{J}} \tag{8}$$

in which $2eN_{2e}/T$ is the intensity of supercurrent, v and T are respectively the velocity and the duration of the coil vertical motion. The Josephson frequency f_{J} thus appears as a measurement of the gravitational shift of the dBC frequency of N_{2e} Cooper pairs sharing the mass $M_{\mathfrak{K}}$.

One can have a fascinating discussion on the question of whether quantum mechanics applies or not when at the macroscopic scale of the kilogram and on the real significance of the appearance of the Planck constant in the watt balance formula. This debate has already started and must continue. Whatever comes out, we are all already persuaded that the mass of a macroscopic object is the sum of that of all its microscopic constituents and of a weak approximately calculable interaction term. This hypothesis is implicit in both possible new definitions of the unit of mass. The concept of mass must be identical at all scales and mass is an additive quantity in the non-relativistic limit. There is no doubt also that, at the atomic scale, mass is directly associated with a frequency via the Planck constant. This frequency can be measured for atoms and molecules even though it is quite a large frequency. As mentioned earlier, measurements of mc^2/h are presently performed with a relative uncertainty much better than 10^{-8}. By additivity the link between a macroscopic mass and a frequency is thus unavoidable. If one accepts to redefine the unit of mass from that of a microscopic particle such as the electron, then the link with the unit of time is *ipso facto* established with a relative uncertainty much better than 10^{-8}.

Both units are *de facto* linked by the Planck constant to better than 10^{-8}. It seems difficult to ignore this link and not to inscribe it in the formulation of the system of units, especially since it leads to a reduction of the number of independent units.

Another extremely important point is that mass is a relativistic invariant. It should thus never be associated with the frequency of a photon field, which transforms as the time component of a 4-vector in reference frame changes. The de Broglie-Compton frequency is a proper frequency, Lorentz scalar, equal by definition to mc^2/h.

Last, in the hypothesis of the mass unit being redefined by fixing Planck's constant, the mole could be redefined separately from the kilogram by fixing Avogadro's number. But should we not keep an exact molecular mass for carbon

12? If the mole is not any more directly connected to 12 grams of carbon, its definition amounts to define an arbitrary number and this number cannot be considered as a fundamental constant of nature. It is only if the mole remains defined by 12 grams of carbon that it rests on a true physical constant, the mass of the carbon atom. This constant has to be determined experimentally if the unit of mass is defined by fixing the Planck constant.

9 Boltzmann's constant and the temperature unit

Statistical mechanics permits to go from probabilities to entropy thanks to another dimensioned fundamental constant, Boltzmann's constant k_{B}. Presently the scale of temperature is arbitrarily fixed by the water triple point, a natural phenomenon of course, but yet very far from fundamental constants.

By analogy with the case of Planck's constant, it seems natural to propose fixing Boltzmann's constant k_{B}. Indeed there is a deep analogy between the two "S's" of physics, which are action and entropy. They provide respectively the phases and the amplitudes for the density operator. The corresponding conjugate variables of energy are time and reciprocal temperature with the two associated fundamental constants: the quantum of action h and the quantum of information k_{B}. Both participate in statistical quantum mechanics through their ratio k_{B}/h. The evolution parameter θ that comes in naturally [7] in the combination of Liouville-von Neumann and Bloch equations for the density operator ρ:

$$i\hbar\frac{\partial\rho}{\partial\theta} = (H - \langle E \rangle)\rho \tag{9}$$

is the complex time:

$$\theta = t + i\hbar\beta/2 = t + i\hbar/2k_{\mathrm{B}}T \tag{10}$$

The link between atom interferometry and the Doppler broadening of line shapes by the thermal motion of atoms is established in reference [16] which brings the connection between phase and temperature measurements. The thermal motion of atoms is responsible for a loss of phase coherence and the Doppler broadening may be seen as a limited visibility of interference fringes.

An interesting analogy may be drawn for the two inaccessible limits that are the velocity of light c and the absolute zero temperature $T = 0$. In both cases the corresponding variable in θ becomes infinite. Internal motion stops and both velocities $d\tau/dt$ (cf Langevin twins) and $u = \sqrt{2k_{\mathrm{B}}T/m} \longrightarrow 0$ (The Doppler width and the black body radiation shift vanish as the thermal decoherence time increases).

To measure Boltzmann's constant several methods, particularly acoustic (propagation of sound in a gas), electrical (Johnson noise) and optical (Doppler width measurements), are presently being studied [25]. They convey the hope of a low enough uncertainty (about 10^{-6}) to consider a new definition of the kelvin

[7] See for example the theory of linear absorption of light by gases and its application to the determination of Boltzmann's constant (reference CRAS 2009).

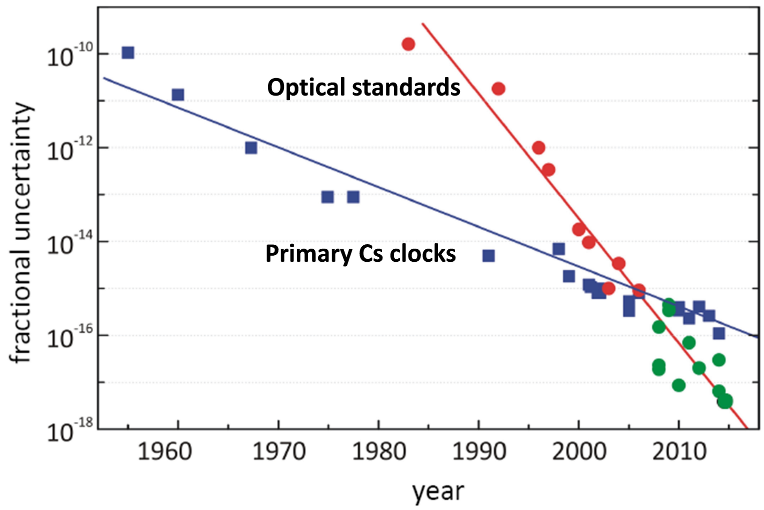


Figure 8: Evolution of the frequency accuracy of atomic clocks (from [27])

from Boltzmann's constant later on. In principle, such a redefinition does not face objections, and so it could be done as soon as two different methods agree at the required accuracy. The Boltzmann constant comes into play at the microscopic level through its ratio to the Planck constant and at the macroscopic level through its product by the Avogadro number. Any future redefinition of the kelvin should take into account one of these associations, according to the future definition of the unit of mass.

10 What about the time unit? Towards a totally unified system?

The measurement of time is the tip top of metrology. The accuracy of atomic clocks has steadily increased by a factor 10 every ten years and this rate has even accelerated recently with the advent of optical clocks [27, 28, 26]. Today their uncertainty reaches 10^{-18}.

Thanks to this very high level of accuracy, time and frequency measurements draw up the measurement of all other quantities. This progress has its roots in the most recent atomic physics with cold atoms and it finds everyday new applications, such as the global positioning satellite system (GPS). The teams of SYRTE at the Paris Observatory and at the Kastler-Brossel laboratory were pioneers in the use of cold atoms to create clocks with atomic fountains. Among new revolutions we can quote the optical clocks which, together with the frequency combs given by the femtosecond lasers (J.L. Hall and T.W. Haensch Nobel prize 2005) will permit to count better and faster, and there is every chance they will take the place of the microwave clocks in the future. The competition runs high between neutral atoms (in free flight or trapped in a light grating to benefit from the Lamb-Dicke effect) and trapped ions. In the end, what part will the space equipments play when it comes to compare clocks and to distribute time? In the future the use of clocks on earth will inevitably be stopped at the level 10^{-17} by the lack of knowledge of the terrestrial gravitational potential. Then an orbital reference clock will be needed. In the future who will be the masters of time ?

The future possible redefinitions of the second are an open debate. Will the second have, like the metre, a universal definition assorted with a way to put it in practice, plus secondary realizations? This would raise, just as in the case of the metre, the question of a possible variation of the fundamental constants which would modify differently the different retained transitions. The rubidium has better collisional properties than the caesium and its hyperfine transition has been recommended by the CCTF (Consulting Committee for Time and Frequency) as a secondary representation of the time unit. On its side, hydrogen attracts many metrological physicists who would like the definition of the time unit to be based on its transition 1s-2s. That transition was the subject of spectacular intercomparisons (at 10^{-14}) with a cold Caesium fountain. A suitable combination of optical frequencies could also be used to best isolate the Rydberg constant from various corrections. The calculation of the hydrogen spectrum should be carried as far as possible, at the same time keeping in mind the considerable gap that will still separate theory from experience for a long time. Last but not least, between the Rydberg constant and the electron mass m_e we have the fine structure constant which is known only up to $0.7.10^{-9}$ so far, either by measuring the anomalous magnetic moment of the electron or more recently by atom interferometry [17, 18]. Obviously there is still a long way to formally tie the time unit to a fundamental constant, but we must be aware of the implicit link existing between the definition of the time unit and these fundamental constants. In fact this situation is generic: owing to the permanent gap between theory and a naturally very reproducible phenomenon we might never be able to define units in terms of fundamental constants only. At some point we are satisfied with formulas which describe the phenomenon until we discover corrections that are too complex to be evaluated and we have then the choice between having a simple definition from a fundamental theory or the use of a complex but very reproducible experimental procedure. This is the situation for the time unit now. In any case let us recall that the frequency provided by an atomic clock should be corrected not only from the influence of all external fields but also of Doppler and recoil shifts in order to yield a true atom Bohr frequency and that such a Bohr frequency is the difference between two de Broglie-Compton frequencies. Should Planck's constant be fixed to define the unit of mass, the time unit would therefore always be defined by the difference between two masses of an atomic species. It is our choice to select either masses of elementary particles with the advantage of simplicity or masses corresponding to internal states of very complex objects far from fundamental physics but with the possible advantage of a better reproducibility.

As a finishing touch to this quick survey of the base units and their connexion with fundamental constants let us emphasize that a new metrology in which quantum mechanics plays a more and more important part is building up.

Presently, the whole scientific community is still urged to throw light on the different choices aiming at the final decision at the CGPM of 2018.

11 A generalized framework for fundamental metrology: 5D geometry combining space-time and proper time

Beyond the choice of relevant fundamental constants we must give coherence to the new system of units. To obtain a consistent approach to this system we must inscribe it in a unified physical framework for fundamental metrology which contains a proper description of space-time, proper time and mass and includes gravitation and electromagnetism as the main interactions. This goal can be reached on purely geometrical grounds as we show in Appendix 2.

This 5D scheme includes General Relativity with a 4D metric tensor $g^{\mu\nu}$ and an electromagnetic 4-potential A_μ (with $\mu, \nu = 0, 1, 2, 3$) thanks to a metric tensor $G_{\hat{\mu}\hat{\nu}}$ for 5D such that the generalized interval given by:

$$d\sigma^2 = G_{\hat{\mu}\hat{\nu}} d\widehat{x}^{\hat{\mu}} d\widehat{x}^{\widehat{\nu}} \text{with } \hat{\mu}, \hat{\nu} = 0, 1, 2, 3, 4 \tag{11}$$

is an invariant.

This metric tensor in five dimensions $G_{\hat{\mu}\hat{\nu}}$ is written as in Kaluza's theory [32] to include the electromagnetic gauge field potential A_μ:

$$\begin{aligned} G_{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} G_{\mu\nu} & G_{\mu 4} \\ G_{4\nu} & G_{44} \end{pmatrix} = \begin{pmatrix} g_{\mu\nu} - \kappa^2 A_\mu A_\nu & -\kappa A_\mu \\ -\kappa A_\nu & -1 \end{pmatrix} \\ G^{\hat{\mu}\hat{\nu}} &= \begin{pmatrix} G^{\mu\nu} & G^{\mu 4} \\ G^{4\nu} & G^{44} \end{pmatrix} = \begin{pmatrix} g^{\mu\nu} & -\kappa A^\mu \\ -\kappa A^\nu & -1 + \kappa^2 A^\mu A_\mu \end{pmatrix} \end{aligned} \tag{12}$$

where κ is given by the gyromagnetic ratio of the object[8]. It is such that the

equation :

$$G^{\hat{\mu}\hat{\nu}} \widehat{p}_{\hat{\mu}} \widehat{p}_{\hat{\nu}} = 0 \tag{13}$$

[8] In the case of the electron: $\kappa = e/m_e c$, which can also be written as

$$\frac{1}{c}\sqrt{\frac{\alpha}{\alpha_G}}\sqrt{4\pi\varepsilon_0 G}$$

where we have introduced the dilaton field $\sqrt{\alpha/\alpha_G}$ to make the connection with Kaluza's theory [32].

with
$$\widehat{p}_{\hat{\mu}} = (p_\mu, -mc) \tag{14}$$
and $G_{44} = -1$ is equivalent to the usual equation in 4D for a massive particle of mass m and charge q:
$$g^{\mu\nu}\left(p_\mu - qA_\mu\right)\left(p_\nu - qA_\nu\right) = m^2c^2 \tag{15}$$
These last equations give directly the Klein-Gordon equation in 5D for the field φ:
$$\widehat{\Box}\varphi = G^{\hat{\mu}\hat{\nu}}\widehat{\nabla}_{\hat{\mu}}\widehat{\nabla}_{\hat{\nu}}\varphi = 0 \tag{16}$$
where the connection between mechanical quantities and quantum mechanical operators is made as usual through Planck's constant. This equation is analogous to the wave equation for massless particles in 4D.

The phase of this field is given by the 5D-superaction $\widehat{S}$ in units of $\hbar$ and satisfies the Hamilton-Jacobi equation:
$$G^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\widehat{S}\partial_{\hat{\nu}}\widehat{S} = 0 \tag{17}$$

With this geometrical picture we have gathered all quantities concerned by the main base units: space-time, proper time, mass, gravito-inertial and electromagnetic fields in a phase without dimension. Any measurement can then be reduced to a phase measurement through a suitable interferometry experiment since all bases quantities enter the expression of a phase through a comparison with reference quantities of the same nature. This universal link is obtained by fixing Planck's constant. Mass and proper time are entangled concepts which correspond to conjugate variables in classical mechanics and to non-commuting operators in quantum mechanics in complete analogy with momentum and position operators. The photon box of the Einstein-Bohr controversy is a direct illustration of this quantum behaviour and of the non-commuting character of proper time and mass operators:
$$[c\tau_{op}, m_{op}c] = i\hbar \tag{18}$$
Their respective units thus require a joint definition in which the unit of mass is defined from the mass difference of the two levels involved in the definition of the unit of time. A compatible *mise en pratique* requires to associate a quantum clock with a macroscopic mass through a phase measurement either by atom interferometry and atom counting or in the watt balance. The Avogadro number is then obtained directly from the measurement of the de Broglie-Compton frequency of the carbon atom in a recoil experiment.

The proper time acquires a status in quantum mechanics and we may now describe the quantum theory of atomic clocks in general relativity from their internal properties since the phase of atom waves can be corrected from general relativistic effects such as the gravitational red shift [8].

Finally, temperature and time can be combined in a complex time variable in the theory of clocks. This accounts for thermal decoherence through the Doppler shift in atom interferometers. A generalized line shape for the usual Doppler broadening can be derived accordingly [16].

12 Conclusion

Most base quantities of metrology, length, time, mass, electrical quantities, temperature are ultimately measured by optical or matter-wave interferometers. Optics and quantum mechanics play a central role in the description of these devices. As a consequence, future fundamental metrology will deal essentially with phase measurements i.e. invariant numbers. One should also emphasize the non-commuting character of quantities like mass and proper time, which is a reason why Planck's constant has such a special place in the system of units. Base quantities should be quantum observables. Some appear as base quantities with their conjugate partner (e.g. mass and proper time), others do not (e.g. position coordinate and momentum). The quantum-mechanical link between conjugate quantities does not allow any more to leave Planck's constant out of the system of units, which would be the case if the mass unit continued to be defined by the mass of an object, whether macroscopic ($\mathfrak{K}$) or microscopic (atom or electron). We have seen that a natural choice was to couple the definitions for mass and time units. Heisenberg's uncertainty relations will apply in the quantum limit[9]. Measurement theory becomes essential to explore the limits of the new quantum metrology.

A natural 5D theoretical framework for the redefinition of the SI is completely provided by the connection between pure geometry, metric tensor and metrology, that we have outlined. In this way, a clear separation has been made between proper time (observable!) and time coordinate (not observable!) as distinct quantities sharing the same unit. The role of the electromagnetic field is to couple space-time and proper time coordinates through the corresponding off-diagonal components of the metric tensor. The 5D action gathers all phenomena and constants of interest for a fully relativistic quantum metrology in an invariant phase through Planck's constant and this includes the dephasing arising from gravito-inertial fields (e.g. the Sagnac effect or the effect of gravitational waves) as well as those of electromagnetic origin (such as the Aharonov-Bohm or the Aharonov-Casher effect).

Reduce the theory of measurements to the determination of quantum phases was our primary objective and this paper is a first attempt to go in this direction and to unify all aspects of modern quantum metrology. The perspective that we have adopted, incorporates naturally all relevant fundamental constants in a logical scheme with obvious constraints of economy, aesthetics and rigour. The final aim is, of course, to adopt a system of units free of arbitrary and artificial features, in harmony with contemporary physics.

[9]One should also keep in mind the uncertainty relation between phase and number of entities i.e. between action and quantity of matter.

13 Appendix 1: What framework for relativistic quantum metrology[10] ?

This framework is naturally the one imposed by the two great physical theories of the 20^{th} century: relativity and quantum mechanics. These two major theories themselves have given birth to quantum field theory, which incorporates all their essential aspects and adds those associated with quantum statistics. The quantum theory of fields allows a unified treatment of fundamental interactions, especially, of electroweak and strong interactions within the standard model. General Relativity is a classical theory, hence gravitation remains apart and is reintegrated into the quantum world only in the recent theories of strings. We do not wish to go that far and we will keep to quantum electrodynamics and to the classical gravitation field. Such a framework is sufficient to build a modern metrology, taking into account an emerging quantum metrology. Of course, quantum physics has been operating for a long time at the atomic level, for example in atomic clocks, but now it also fills the gap between this atomic world and the macroscopic world, thanks to the phenomena of quantum interferences whether concerning photons, electrons, Cooper pairs or more recently atoms in atom interferometers [7].

The main point is to distinguish between a “kinematical” framework associated with fundamental constants having a dimension, such as c, $\hbar$, k_{B}, and a “dynamical” framework where the interactions are described by coupling constants without dimension. The former framework relies on the Statistical Relativistic Quantum Mechanics of free particles, and the latter on the quantum field theory of interactions.

Two possible goals can be pursued:

1 - redefine each unit in terms of a fundamental constant with the same dimension e.g. mass in terms of the mass of an elementary particle

2 - reduce the number of independant units by fixing a fundamental constant having the proper dimension for this reduction e.g. fixing c to connect space and time units or $\hbar$ to connect mass and time units.

The existence of fundamental constants with a dimension is often an indication that we are referring to the same thing with two different names. We recognize this identity as our understanding of the world gets deeper. We should then apply an economy principle (Occam's razor) to our unit system to take this into account and to display this connection.

When abandoning a unit for the sake of another, the first condition (C1) is thus to recognize an equivalence between the quantities measured with these units (e.g. equivalence between heat and mechanical energy and between mass and energy), or a symmetry of nature that connects these quantities in an operation of symmetry (for example a rotation transforming the space coordinates into one another or of a Lorentz transformation mixing the space and time coordinates).

[10] The following discussion is reproduced from references [1, 14]

A second condition (C2) is that a realistic and mature technology of measurement is to be found. For example, notwithstanding the equivalence between mass and energy, in practice the kilogram standard will not be defined by an energy of annihilation, but on the other hand, thanks to the watt balance, it can be tied to its Compton frequency $M_{\mathcal{K}}c^2/h$ by measurements of time and frequency.

A third condition (C3) is connected to the confidence felt for the understanding and the modelization of the phenomenon used to create the link between quantities. For instance, the exact measurement of distances by optical interferometry is never questioned because we believe that we know everything, and in any case, that we know how to calculate everything concerning the propagation of light. That is the reason why redefining the metre ultimately took place without much problem. On the other hand, measuring differences of potential by the Josephson effect or electrical resistances by the quantum Hall effect, still needs support, because despite a 10^{-9} confirmation of their reproducibility, and a good understanding of the universal topological character behind these phenomena, some people still feel uncertain as to whether all possible small parasitical effects have been dealt with. For a physical phenomenon to be used to measure a quantity properly, is directly related to our knowledge of the whole underlying physics. In order to switch to a new definition, this psychological barrier must be overcome, and we must have complete faith in our total understanding of the essentials of the phenomenon. Therefore, through a number of experiments as varied as possible, we must make sure that the measurement results are consistent up to a certain level of accuracy which will be that of the "mise en pratique" and we must convince ourselves that no effect has been neglected at that level. If all of these conditions are fulfilled, the measured constant that linked the units for the two quantities will be fixed e.g. the mechanical equivalent of the calorie or the speed of light.

14 Appendix 2: The status of mass in classical relativistic mechanics: from 4 to 5 dimensions

In special relativity, the total energy E and the momentum components p^1, p^2, p^3 of a particle, transform as the contravariant components of a four-vector

$$p^\mu = (p^0, p^1, p^2, p^3) = (E/c, \overrightarrow{p}) \tag{19}$$

and the covariant components are given by :

$$p_\mu = g_{\mu\nu} p^\nu \tag{20}$$

where $g_{\mu\nu}$ is the metric tensor. In Minkowski space of signature $(+,-,-,-)$:

$$p_\mu = (p_0, p_1, p_2, p_3) = (E/c, -p^1, -p^2, -p^3) \tag{21}$$

These components are conserved quantities when the considered system is invariant under corresponding space-time translations. They will become the generators of space-time translations in the quantum theory. For massive particles of rest mass m, they are connected by the following energy-momentum relation :

$$E^2 = p^2c^2 + m^2c^4 \tag{22}$$

or, in manifestly covariant form,

$$p^\mu p_\mu - m^2c^2 = 0 \tag{23}$$

This equation cannot be considered as a definition of mass since the origin of mass is not in the external motion but rather in an internal motion. It simply relates two relativistic invariants and gives a relativistic expression for the total energy. Thus mass appears as an additional momentum component mc corresponding to internal degrees of freedom of the object and which adds up quadratically with external components of the momentum to yield the total energy squared (Pythagoras' theorem). In the reference frame in which $p = 0$ the squared mass term is responsible for the total energy and mass can thus be seen as stored internal energy just like kinetic energy is a form of external energy. Even when this internal energy is purely kinetic e.g. in the case of a photon in a box, it appears as pure mass m^* for the global system (i.e. the box). This new mass is the relativistic mass of the stored particle:

$$m^*c^2 = \sqrt{p^2c^2 + m^2c^4} \tag{24}$$

The concept of relativistic mass has been criticized in the past but it becomes relevant for embedded systems. We may have a hierarchy of composed objects (e.g. nuclei, atoms, molecules, atomic clock ...) and at each level the mass m^* of the larger object is given by the sum of energies p^0 of the inner particles. It transforms as p^0 with the internal coordinates and is a scalar with respect to the upper level coordinates.

Mass is conserved when the system under consideration is invariant in a proper time translation and will become the generator of such translations in the quantum theory. In the case of atoms, the internal degrees of freedom give rise to a mass which varies with the internal excitation. For example, in the presence of an electromagnetic field inducing transitions between internal energy levels, the mass of atoms becomes time-dependent (Rabi oscillations). It is thus necessary to enlarge the usual framework of dynamics to introduce this new dynamical variable as a fifth component of the energy-momentum vector.

Equation (23) can be written with a five dimensional notation :

$$G^{\hat{\mu}\hat{\nu}}\widehat{p}_{\hat{\mu}}\widehat{p}_{\hat{\nu}} = 0 \text{ with } \hat{\mu}, \hat{\nu} = 0, 1, 2, 3, 4 \tag{25}$$

where $\widehat{p}_{\hat{\mu}} = (p_\mu, p_4 = -mc)$; $G^{\mu\nu} = g^{\mu\nu}$; $G^{\hat{\mu}4} = G^{4\hat{\nu}} = 0$; $G^{44} = G_{44} = -1$

This leads us to consider also the picture in the coordinate space and its extension to five dimensions. As in the previous case, we have a four-vector representing the space-time position of a particle:

$$x^{\mu} = (ct, x, y, z)$$

and in view of the extension to general relativity:

$$dx^{\mu} = (cdt, dx, dy, dz) = (dx^0, dx^1, dx^2, dx^3) \tag{26}$$

The relativistic invariant is, in this case, the elementary interval ds, also expressed with the proper time τ of the particle:

$$ds^2 = dx^{\mu} dx_{\mu} = c^2 dt^2 - d\overrightarrow{x}^2 = c^2 d\tau^2 \tag{27}$$

which is, as that was already the case for mass, equal to zero for light

$$ds^2 = 0 \tag{28}$$

and this defines the usual light cone in space-time.

For massive particles proper time and interval are non-zero and equation (27) defines again an hyperboloid. As in the energy-momentum picture we may enlarge our space-time with the additional dimension $s = c\tau$

$$d\widehat{x}^{\widehat{\mu}} = (cdt, dx, dy, dz, cd\tau) = (dx^0, dx^1, dx^2, dx^3, dx^4) \tag{29}$$

and introduce a generalized light cone for massive particles[11]

$$d\sigma^2 = G_{\hat{\mu}\hat{\nu}} d\widehat{x}^{\hat{\mu}} d\widehat{x}^{\widehat{\nu}} = c^2 dt^2 - d\overrightarrow{x}^2 - c^2 d\tau^2 = 0 \tag{30}$$

As pointed out in the case of mass, proper time is not defined by this equation from other coordinates but is rather a true evolution parameter representative of the internal evolution of the object. It coincides only numerically with the time coordinate in the frame of the object through the relation:

$$cd\tau = \sqrt{G_{00}} dx^0 \tag{31}$$

Finally, if we combine momenta and coordinates to form a mixed scalar product, we obtain a new relativistic invariant which is the differential of the action. In 4D:

$$dS = -p_{\mu} dx^{\mu} \tag{32}$$

and in 5D we shall therefore introduce the superaction:

$$\widehat{S} = -\int \widehat{p}_{\hat{\mu}} d\widehat{x}^{\hat{\mu}} \tag{33}$$

[11] In this picture, anti-particles have a negative mass and propagate backwards on the fifth axis as first pointed out by Feynman. Still, their relativistic mass m^* is positive and hence they follow the same trajectories as particles in gravitational fields as one can check from the equations of motion [15].

equivalent to

$$\widehat{p}_{\hat{\mu}} = -\frac{\partial \widehat{S}}{\partial \widehat{x}^{\hat{\mu}}} \quad \text{with } \hat{\mu} = 0, 1, 2, 3, 4 \tag{34}$$

If this is substituted in

$$G^{\hat{\mu}\hat{\nu}} \widehat{p}_{\hat{\mu}} \widehat{p}_{\hat{\mu}} = 0 \tag{35}$$

we obtain the Hamilton-Jacobi equation in 5D

$$G^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} \widehat{S} \partial_{\hat{\nu}} \widehat{S} = 0 \tag{36}$$

which has the same form as the eikonal equation for light in 4D. It is already this striking analogy which pushed Louis de Broglie to identify action and the phase of a matter wave in the 4D case. We shall follow the same track for a quantum approach in our 5D case.

What is the link between the three previous invariants given above? As in optics, the direction of propagation of a particle is determined by the momentum vector tangent to the trajectory. The 5D momentum can therefore be written in the form:

$$\widehat{p}^{\hat{\mu}} = d\widehat{x}^{\hat{\mu}} / d\lambda \tag{37}$$

where λ is an affine parameter varying along the ray. This is consistent with the invariance of these quantities for uniform motion.

In 4D the canonical 4-momentum is:

$$p_\mu = mc \frac{g_{\mu\nu} dx^\nu}{\sqrt{g_{\mu\nu} dx^\mu dx^\nu}} = mc g_{\mu\nu} u^\nu \tag{38}$$

where $u^\nu = dx^\nu / d\tau$ is the normalized 4-velocity with $d\tau = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ given by (27).

We observe that $d\lambda$ can always be written as the ratio of a time to a mass:

$$d\lambda = \frac{d\tau}{m} = \frac{dt}{m^*} = \frac{d\theta}{M} = ... \tag{39}$$

where τ is the proper time of individual particles (e.g. atoms in a clock or in a molecule), t is the time coordinate of the composed object (clock, interferometer or molecule) and θ its proper time; m, m^*, M are respectively the mass, the relativistic mass of individual particles and their contribution to the scalar mass of the device or composed object.

From :

$$d\sigma^2 = G_{\hat{\mu}\hat{\nu}} d\widehat{x}^{\hat{\mu}} d\widehat{x}^{\hat{\nu}} = 0 \tag{40}$$

we infer that in 5D

$$d\widehat{S} = 0 \tag{41}$$

wheras in 4D

$$dS = -p_\mu dx^\mu = -mc^2 d\tau \tag{42}$$

As a consequence the quantum mechanical phase also cancels along the classical trajectory in 5D. The particle is naturally associated with the position where all phases cancel to generate a constructive interference.

The previous 5D scheme can be extended to General Relativity with a 4D metric tensor $g^{\mu\nu}$ and an electromagnetic 4-potential A_μ [15] with the metric tensor given in the main text.

14.1 Acknowledgements

This paper is adapted from an article in "La lettre de l'Académie des sciences" n°20 (2007) which was kindly translated to English by Mrs Marie-Claire Céreuil. The author is also grateful to a number of colleagues for constructive comments and especially Dr. Olivier Darrigol, Dr. Nadine de Courtenay, Dr. Franck Pereira and Pr. Marc Himbert.

References

[1] Christian J. Bordé, *Métrologie fondamentale: unités de base et constantes fondamentales,* C.R. Physique **5**, 813-820 (2004).

[2] B. N. Taylor and P. J. Mohr, *The role of fundamental constants in the international system of units (SI): present and future*, IEEE Trans. on Inst. and Meas., **50**, 563-567 (2001).

[3] J.W.G. Wignall, *Proposal for an absolute, atomic definition of mass*, Phys. Rev. Letters, **68**, 5-8 (1992).

[4] F. Piquemal et al., *Fundamental electrical standards and the quantum metrological triangle*, in [24], 857-879.

[5] B. P. Kibble, *A measurement of the gyromagnetic ratio of the proton by the strong field method*, in Atomic an fundamental constants 5, ed J.H. Sanders and A. H. Wapstra, Plenum Press, New York 5 (1975) pp. 545-551.

[6] B. N. Taylor and P. J. Mohr, *On the redefinition of the kilogram*, Metrologia, **36**, 63-64 (1999).

[7] Christian J. Bordé, *Atomic interferometry with internal state labelling*, Phys. Lett., **A140**, 10-12 (1989).

[8] Christian J. Bordé, *Atomic clocks and inertial sensors*, Metrologia, 39 (5) 435-463 (2002).

[9] *Atom Interferometry*, ed. P. Berman, Academic Press (1997).

[10] Ch. J. Bordé and J. L. Hall, *Direct resolution of the recoil doublets using saturated absorption techniques*, Bull. Amer. Phys. Soc., **19**, 1196 (1974).

[11] J. L. Hall, Ch. J. Bordé and K. Uehara, *Direct optical resolution of the recoil effect using saturated absorption spectroscopy*, Phys. Rev. Letters, **37**, 1339-1342 (1976).

[12] T. Heupel, M. Mei, M. Niering, B.Gross, M. Weitz, T. W. Hänsch and Ch. J. Bordé, *Hydrogen atom interferometer with short light pulses*, Europhysics Lett., **57**, 158-163 (2002).

[13] Christian J. Bordé, *Matter-wave interferometers: a synthetic approach*, in [9].

[14] Christian J. Bordé, *Base units of the SI, fundamental constants and modern quantum physics,* Phil. Trans. Roy. Soc., **363**, 2177-2202 (2005) p. 2182.

[15] Christian J. Bordé, *Atom interferometry using internal excitation: Foundations and recent theory*, International School of Physics "Enrico Fermi" - COURSE CLXXXVIII Atom Interferometry (2014) 143-170.

[16] Christian J. Bordé, *On the theory of linear absorption line shapes in gases*, C. R. Physique 10, 866–882 (2009).

[17] M. Cadoret et al., *Combination of Bloch Oscillations with a Ramsey-Bordé Interferometer: New Determination of the Fine Structure Constant,* PRL 101, 230801 (2008).

[18] R. Bouchendira et al., *New Determination of the Fine Structure Constant and Test of the Quantum Electrodynamics,* PRL 106, 080801 (2011).

[19] B. Young., M. Kasevich and S. Chu, *Precision atom interferometry with light pulses,* in [9]

[20] R. D. Vocke Jr, S. A. Rabb and G. C. Turk, *Absolute silicon molar mass measurements, the Avogadro constant and the redefinition of the kilogram*, Metrologia **51** (2014) 361–375

[21] B. Andreas et al 2011 *Counting the atoms in a (28)Si crystal for a new kilogram definition,* Metrologia 48 S1–13

[22] J. E. Zimmerman and J. E. Mercereau, *Compton wavelength of superconducting electrons,* Phys. Rev. Lett. **14**, 887-888 (1965).

[23] W. H. Parker and M. B. Simmonds, *Measurement of* h/m_e *using rotating superconductors,* in Precision Measurement and Fundamental Constants, NBS Special Publication 243-247 (1971).

[24] Christian J. Bordé and Jean Kovalevsky, *Special issue on Fundamental metrology*, Comptes Rendus de l'Académie des Sciences **5** (Oct 2004) 789.

[25] Christian J. Bordé and Marc E. Himbert, *Experimental determination of Boltzmann's constant,* Special issue of the Comptes Rendus de l'Académie des Sciences **10** (Nov 2009) 813.

[26] Christophe Salomon, *The measurement of time,* Special issue of the Comptes Rendus de l'Académie des Sciences **16** (2015).

[27] F. Riehle, *Towards a redefinition of the Second based on optical atomic clocks*, in [26] 506-515

[28] M. Abgrall et al., *Atomic fountains and optical clocks at SYRTE: status and perspectives,* in [26]

[29] Christian J. Bordé, *Spectroscopie d'absorption saturée de diverses molécules au moyen des lasers à gaz carbonique et à protoxyde d'azote*, C.R. Acad. Sc. Paris, **271B**, 371-374 (1970).

[30] R.L. Barger and J.L. Hall, *Pressure shift and broadening of methane line at 3.39 μ studied by laser-saturated molecular absorption*, Phys. Rev. Lett., **22**, 4 (1969).

[31] Louis de Broglie, *Ondes et quanta*, Comptes Rendus de l'Académie des Sciences **2** (Sept 1923) 507.

[32] T. Kaluza, *Zum Unitätsproblem der Physik*, Sitz. Preuss. Akad. Wiss. Phys. Math. K1 (1921) 966.