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REGULARIZATION BY NOISE: A MCKEAN-VLASOV CASE

P. E. CHAUDRU DE RAYNAL

Abstract. In this paper, we prove pathwise uniqueness for stochastic systems of McKean-Vlasov type with singular drift, even in the measure argument, and uniformly non-degenerate Lipschitz diffusion matrix. This work extends to the McKean-Vlasov setting the earlier results obtained by Zvonkin [Zvo74], Veretennikov [Ver80], Krylov and Röckner [KR05]. We prove that the noise prevents the ill-posedness coming from the singularity of the drift, even in the measure direction.

Our proof relies on regularization properties of the associated PDE, which is stated on the space $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, where $T$ is a positive number, $d$ denotes the dimension of the equation and $\mathcal{P}_2(\mathbb{R}^d)$ is the space of probability measures on $\mathbb{R}^d$ with finite second order moment. In particular, a smoothing effect in the measure direction is exhibited. Our approach is based on a parametrix expansion of the transition density of the McKean-Vlasov process.

1 Introduction

Let $\mathcal{M}_d(\mathbb{R})$ be the set of $d \times d$ matrices with real coefficients and $\mathcal{P}_2(\mathbb{R}^d)$ be the space of probability measures $\nu$ on $\mathbb{R}^d$ such that $\int x^2 d\nu(x) < +\infty$. For any random variable $Z$, let us denote by $[Z]$ its law and for any measurable function $\varphi$ let us write $\int \varphi d\nu$ with its dual notation: $\langle \varphi, \nu \rangle$.

For a positive number $T$, five measurable functions $b, \sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \times \mathcal{M}_d(\mathbb{R})$, $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i$ in $\{1, 2\}$ and $(B_t, t \geq 0)$ a standard $d$-dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$, we consider, for $t < s$ in $[0, T]^2$ and $\mu$ in $\mathcal{P}_2(\mathbb{R}^d)$, the non-linear (in a McKean-Vlasov sense) system

$$X_{s,t}^{t,\mu} = X_t + \int_t^s b(r, X_{r,t}^{t,\mu}, \langle \varphi_1, [X_{r,t}^{t,\mu}] \rangle) dr + \int_t^s \sigma(r, X_{r,t}^{t,\mu}, \langle \varphi_2, [X_{r,t}^{t,\mu}] \rangle) dB_r, \quad X_t \sim \mu. \quad (1.1)$$

This sort of system arises as the limit of system of interacting players. This happens as follows. Suppose that your are given a large number of players with symmetric dynamic and whose positions depend on the positions of the other players in a mean field way. Then, when the number of players tends to infinity, there is a propagation of chaos phenomenon so that the limit dynamic of each player does not depend on the position of the others anymore, but only on their statistical distribution. This obviously comes from the law of large numbers. This equation is thus said to be non-linear, since the dynamic of the player depends on its own law. We refer to the note of Szmitan’s lecture at Saint-Flour [Szn91] for an overview on the topic.

As done in [Szn91], the proof of existence and uniqueness for this equation relies on classical fixed point argument and so, on Lipschitz property of the coefficients of the equation, the Lipschitz regularity being understand w.r.t. the Wasserstein metric in the case of the measure argument.

Key words and phrases. McKean-Vlasov processes; smoothing effect, non-linear PDE, regularisation by noise.

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We aim at proving the strong well-posedness (which means that the solutions are adapted to the filtration generated by the Brownian motion and are almost surely indistinguishable) of such a system when the drift is not a smooth function of the space and law argument. Namely, the drift \( b \) is asking to be bounded in time and space and Lipschitz with respect the third argument but the mapping \( \varphi_1 \) being only Hölder-continuous. Roughly speaking, this means that when rewriting the drift of (1.1) as

\[
B : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (t, x, \nu) \mapsto B(t, x, \nu) = b(t, x, \langle \varphi_1, \nu \rangle) \in \mathbb{R}^d,
\]

the drift function \( B \) is only assumed to be bounded in time and space and Holder continuous in the measure argument (for the Wasserstein distance).

In order to have well-posedness, we expect that the regularization by noise phenomenon (see [Zvo74, Ver80, KR05]) still holds in the McKean-Vlasov case. We show that this phenomenon indeed occurs, even in the measure space, which was unexpected because the noise does not act in that direction. This makes it possible the proof of well posedness.

Our strategy is an adaptation to our framework of the Zvonkin transformation [Zvo74] and relies on smoothing property of a well-chosen PDE associated to the system (1.1). The non-linear (in a McKean-Vlasov sense) framework leads to a particular class of PDE that can be seen as the linear version of the so called Master Equation coming from Mean field games theory introduced independently by Lasry ad Lions [LL06b, LL06a, LL07] and by Huang, Caines and Malhameé [HMC06].

This PDE has been recently studied from a probabilistic point of view in the independent works of Buckdahn, Li, Peng and Rainer in [BLPR14] (linear version) and of Crisan, Chassagneux and Delarue in [CCD14] (non-linear version). Its main particularity comes from the fact that it is stated on the space \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\) so that it involves derivatives in the measure direction.

The smoothing properties of such a PDE are investigated by using a Feynman-Kac representation of the solution of the PDE and then a parametrix expansion (see [MS67]) of the transition density of the solution of (1.1). Our strategy lead us to investigate for all \( t < s \) in \([0, T]^2\) the regularity of the mapping \( \mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \langle [X^t_s, \mu], \phi \rangle \in \mathbb{R} \) for some Hölder continuous function \( \phi : \mathbb{R}^d \to \mathbb{R} \) (where \( X \) is a solution of (1.1)). In particular, a smoothing effect in the measure direction is exhibited.

**Organization of this paper.** Our paper is organized as follows: we present below our main assumptions and result. Then, we give in Section 2 the mathematical background and our strategy of proof. Especially, we state in this section the PDE associated to (1.1). Since the PDE is stated on the Cartesian space \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\), we give some notions on differentiation of function along a measure. Then, we establish the smoothing properties of the PDE associated to (1.1), which is a key result in the proof of our main Theorem.

Then, we investigate the smoothing property of the PDE. Such investigation is done under regularized framework. It is based on a parametrix expansion of the transition density of (1.1) and is presented as follows. In Section 4 we give estimates on the transition density of (1.1) and on the mapping \( v : \mathcal{P}_2(\mathbb{R}^d) \ni \mu \mapsto \langle [X^t_s, \mu], \phi \rangle \in \mathbb{R} \) for some Hölder continuous function \( \phi : \mathbb{R}^d \to \mathbb{R} \). This permits to estimate the solution of the PDE. Then, estimates on \( v \) are proven in Section 5 and estimates on the transition density of (1.1) are proven in Section 6. As said before, this last follows from a parametrix representation of the transition density of 1.1 and its estimation. An auxiliary results on the parametrix constant is finally given in Section B.
Assumptions and notations

Some notations. For any function \( f : E \times F \times G \to \mathbb{R}^N \), we denote by \( \partial_1 \) (resp. \( \partial_2 \) and \( \partial_3 \)) the differentiation w.r.t. the first (resp. second and third) variable. When we add a subscript in the operator \( \partial_j \), it stands for the variable on which the differentiation operator acts. We recall that the law of a random variable \( X \) is denoted by \([X]\). The superscript "\( * \)" stands for the transpose, the canonical Euclidean inner product on \( \mathbb{R}^d \) is denoted by "\( \cdot \)". We denote by \( \mathcal{M}_d(\mathbb{R}) \) the set of \( d \times d \) matrices with real coefficients and the trace of a matrix \( M \) in \( \mathcal{M}_d(\mathbb{R}) \) is denoted by \( \text{Tr}(M) = \sum_{j=1}^d M_{j,j} \). We let \( C, C', c, c', \tilde{C}, \tilde{C}', \ldots \) be some positive constant depending only on known parameters in (HE), given just below, that may change from line to line and from an equation to another and we add a subscript \( T \) in the constants \( C \) if it depends also on the length of the interval.

Assumptions (HE). We say that assumptions (HE) hold if the following assumptions are satisfied:

(HE1) regularity of the drift: there exists a positive constant \( C_b \) such that \( \|b\|_{\infty} < C_b \). Moreover for all \( (t, x) \in [0, T] \times \mathbb{R}^d \), the mapping \( b(t, x, \cdot) : \mathbb{R} \ni w \mapsto b(t, x, w) \) is differentiable and \( \|\partial_3 b\|_{\infty} < C'_b \). Finally, the mapping \( \varphi_1 : \mathbb{R}^d \ni x \mapsto \varphi_1(x) \) is supposed to be \( \alpha_1 \)-Hölder for some \( 0 < \alpha_1 \leq 1 \).

(HE2) regularity of the diffusion matrix: there exists a positive constant \( C_\sigma \) such that for all \( t \) in \([0, T]\),
\[
\forall x, x' \in \mathbb{R}^d, \; w, w' \in \mathbb{R}, \; |\sigma(t, x, w) - \sigma(t, x', w')| \leq C_\sigma \left( |x - x'| + |w - w'| \right).
\]
Moreover for all \( (t, x) \) in \([0, T] \times \mathbb{R}^d \), the mapping \( \sigma(t, x, \cdot) : \mathbb{R} \ni w \mapsto \sigma(t, x, w) \) is differentiable, \( \|\partial_3 \sigma\|_{\infty} < C'_\sigma \) and there exists a positive constant \( C''_\sigma \) such that for all \( t \) in \([0, T]\) and \( w \) in \( \mathbb{R} \),
\[
\forall x, x' \in \mathbb{R}^d, \; |\partial_3 \sigma(t, x, w) - \partial_3 \sigma(t, x', w)| \leq C''_\sigma |x - x'|^{\alpha'_3}.
\]
Finally, the mapping \( \varphi_2 : \mathbb{R}^d \ni x \mapsto \varphi_2(x) \) is supposed to be Lipschitz.

(HE3) uniform ellipticity of \( \sigma \sigma^* \): the function \( \sigma \sigma^* \) satisfies the uniform ellipticity hypothesis:
\[
\exists \Lambda > 1, \; \forall \zeta \in \mathbb{R}^d, \; \Lambda^{-1} |\zeta|^2 \leq [\sigma \sigma^*(t, x, w) \zeta] \cdot \zeta \leq \Lambda |\zeta|^2,
\]
for all \( (t, x, w) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \).

Main result

We can now state our main result:

Theorem 1.1. Under assumptions (HE), the system (1.1) admits a unique strong solution.

Remark 1. We emphasize that the same arguments should lead to the same result if the dependence of the coefficients w.r.t. the law are of the form \( b(t, x, \nu) = \langle \nu_t, b(t, x, \cdot) \rangle \) and \( \sigma(t, x, \nu) = \langle \nu_t, \sigma(t, x, \cdot) \rangle \), where \( (t, x) \) lies in \([0, T] \times \mathbb{R}^d \) and where \( (\nu_t)_{0 \leq t \leq T} \) is a family of probability measures in \( \mathcal{P}_2(\mathbb{R}^d) \).
2 Mathematical background and strategy of proof

2.1 The Zvonkin transformation

For usual differential equations, there is no hope to restore the well-posedness outside the Lipschitz framework, at least in the classical sense (see [DL89] for some work in that direction). Nevertheless, when the differential system is perturbed by noise (which is the case here) there is a phenomenon, called regularization by noise, that allows to recover the well-posedness (in a strong sense). When the SDE is linear (in a Mckean-Vlasov sense), this has been studied first by Zvonkin [Zvo74] and then generalized by several authors e.g. [Ver80, KR05, Zha11] and [Fla11] for a survey. All these results rely on smoothing properties of elliptic and linear partial second order differential operator, and so, on the non-degeneracy of the diffusion matrix $\sigma\sigma^*$. Let us briefly explain why.

The strategy to recover the Lipschitz property consists in exhibiting a Zvonkin-like transformation of the equation. Let us forget for the moment the dependence of the solution of (1.1) w.r.t. its own law in order to illustrate the main argument. If we denote by $\mathcal{A}$ the generator of (1.1), the idea is to obtain a priori estimates on the solution of the PDE

$$\partial_t u + \mathcal{A}u = b, \quad \text{on } [0, T) \times \mathbb{R}^d, \quad u_T = 0_{\mathbb{R}^d}, \quad (2.1)$$

when $b$ and $\sigma$ are smooth functions, but depending only on regularity of $b, \sigma$ assumed in (HE).

This allows to consider a sequence $(u^n)_{n \geq 0}$ of classical solutions of the PDE (2.1) along a sequence of mollified coefficients $((\sigma^n\sigma^*)_n, b_n)_{n \geq 0}$. Then, applying Itô’s formula on $X_t - u^n(t, X_t)$ we can get rid of the drift of the equation by letting the regularization procedure tend to the infinity and recover an SDE whose coefficients have Lipschitz constants uniformly on the regularization procedure, so that the estimates pass through the limit.

When these constants can be chosen as small as $T$ is small (which follows from the vanishing boundary condition of (2.1)), we then recover existence and uniqueness on small time intervals. If in addition the constants do not degenerate with the time, we can iterate the procedure and then recover existence and uniqueness on $\mathbb{R}^+$.

The smoothing properties of the PDE (2.1), are, in fact, the crucial points. It is well known that such smoothing properties are related to the noise propagation in the associated SDE through all the directions of the space. Hence, two issues arise from the non linear framework studied here: how the operator $\mathcal{A}$ looks like in our Mckean-Vlasov case, and how to regularize in the measure direction since the noise does not act in that direction.

2.2 PDE on space of probability measure

Roughly speaking, in order to make our previous arguments work in the Mckean-Vlasov framework, we have to find a PDE that reflects the Markov structure of the underlying process. Here, the Markov property has to be understood on the space $[0, T] \times \mathbb{R}^d \times P_2(\mathbb{R}^d)$, so that it seems natural to consider a PDE on this space.

The PDE associated to the Markov structure of process (1.1) has been recently studied independently by Buckdahn, Li, Peng and Rainer in [BLPR14] and Crisan, Chassagneux and Delarue in [CCD14]. It
is called the Master Equation and appears naturally when considering mean-field games. What follows is essentially inspired by the second work [CCD14], from which we adopted some of the notations.

Before considering a PDE on the space $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, let us give some notions of differentiation along a probability measure. The one used here has been introduced by Lions during his lecture at the Collège de France and can be found in Cardaliaguet’s note [Car10]. The strategy of Lions consists in lifting the function $V : \mathcal{P}_2(\mathbb{R}^d) \ni \nu \mapsto V(\nu) \in \mathbb{R}$ to a function $V : L^2(\Omega, \mathcal{F}, \mathbb{P}) \ni Z \mapsto V(Z) \in \mathbb{R}$, $Z$ being a random variable of law $\nu$. We can then take advantage of the Hilbert structure of the $L^2$ space and then define, in the Frechet sense, the mapping $DV$. Thanks to Riezs’ representation Theorem, we can identify $DV(Z)$ as $DV(\nu)(Z)$. We then call the derivative of $V$ w.r.t. the law, and we denote by $\partial_\nu V(\nu)$, the mapping in $L^2(\mathbb{R}^d, \nu; \mathbb{R}^d)$:

$$\partial_\nu V(\nu) : \mathbb{R}^d \ni z \mapsto \partial_\nu V(\nu)(z) \in \mathbb{R}^d.$$  

Let us emphasize that, in our case, the law interaction appears as the action of the law on some function $\varphi_i : V(\nu) = \int \varphi_i(x)d\nu(x)$. Using the lifting argument described above we get that for any random variable $X$ and $H$ in $L^2(\Omega)$:

$$V([X + \epsilon H]) = \mathbb{E}[\varphi(X + \epsilon H)] = \mathbb{E}[\varphi(X)] + \epsilon\mathbb{E}[\varphi'(X) \cdot H] + o(\epsilon),$$

so that

$$\partial_\nu \langle \varphi_i, \nu \rangle : \mathbb{R}^d \ni z \mapsto \varphi_i'(z) \in \mathbb{R}^d.$$ 

Finally, let us just notice that this definition justifies the choice of the space $\mathcal{P}_2(\mathbb{R}^d)$ for the initial data in (1.1).

We can now state the PDE of interest. As said before, this PDE is the linear version of the master equation studied in [CCD14]. It has the following form:

$$\begin{cases}
(\partial_t + A)u(t, x, \mu) = b(t, x, \langle \mu, \varphi_1 \rangle), \text{ on } [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d). \\
u(T, x, \mu) = 0_{\mathbb{R}^d}.
\end{cases}$$

where $a := \sigma\sigma^*$ and where, for any smooth enough function $\psi : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d$

$$A\psi(t, x, \mu) = \frac{1}{2} \text{Tr} \left[a(t, x, \langle \mu, \varphi_1 \rangle)\partial^2_x \psi(t, x, \mu) + b(t, x, \langle \mu, \varphi_1 \rangle)\partial_x \psi(t, x, \mu) \right] + \int b(t, x, \langle \mu, \varphi_1 \rangle)\partial_\mu \psi(t, x, \mu)(z)d\mu(z) + \frac{1}{2} \int \text{Tr} \left[a(t, x, \langle \mu, \varphi_2 \rangle)\partial_z (\partial_\mu \psi(t, x, \mu)(z)) \right] d\mu(z).$$

When the coefficients are smooth, it follows from [CCD14] that such a PDE admits a classical solution $u$. We refer to the aforementioned paper for more explanations on the meaning of classical solution and especially on the question about the regularity of $\partial_\mu u$ as an element of $L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ w.r.t. the random variable $z$ and $\mu$.

### 2.3 Smoothing properties of the PDE

It thus remains to exhibit the smoothing property of the PDE. This is done under the following assumptions.
Assumptions (HE). We say that assumptions (HE) hold if Assumptions (HE) are satisfied with assumption (HE2) replaced by

(HE2) regularity of the diffusion matrix: there exists a positive constant $C_\sigma$ such that for all $t$ in $[0,T]$, for all $w$ in $\mathbb{R}$

$$\forall x, x' \in \mathbb{R}^d, \ |\sigma(t,x,w) - \sigma(t,x',w)| \leq C_\sigma |x - x'|^{\gamma_0},$$

for some $0 < \gamma_0 \leq 1$. Moreover for all $(t,x)$ in $[0,T] \times \mathbb{R}^d$, the mapping $\sigma(t,x,\cdot) : \mathbb{R} \ni w \mapsto \sigma(t,x,w)$ is differentiable, $||\partial_3 \sigma||_\infty < C'_\sigma$ and there exists a positive constant $C''_\sigma$ such that for all $t$ in $[0,T]$ and $w$ in $\mathbb{R}$,

$$\forall x, x' \in \mathbb{R}^d, \ |\partial_3 \sigma(t,x,w) - \partial_3 \sigma(t,x',w)| \leq C''_\sigma |x - x'|^{\gamma'_0},$$

for some $0 < \gamma'_0 \leq 1$. Finally, the mapping $\varphi_2 : \mathbb{R}^d \ni x \mapsto \varphi_2(x) \in \mathbb{R}$ is supposed to be $\alpha_2$-H"older continuous, $0 < \alpha_2 \leq 1$.

We let the diffusion coefficient $a$ be only a Hölder-continuous function w.r.t. the space and law variable. We emphasize that (HE) implies (HE).

Since we do not need to solve the whole systems in order to apply the Zvonkin’s transformation, we can regularize the coefficients of the equation and then exhibit a Lipschitz bound on the regularized solution and its space derivative depending only on known parameters in (HE).

In our context, it is indeed possible to mollify the coefficients $b, a$ and the functions $\varphi_i$, $i = 1, 2$, and to obtain a sequence of smooth coefficients $(b_n, a_n)_{n \geq 1}$ (say bounded and infinitely differentiable with bounded derivatives of all order), and functions $(\varphi^{(n)}_1, \varphi^{(n)}_2)_{n \geq 1}$ (infinitely differentiable with bounded derivatives of all order greater than 1) that converges uniformly to $b, a$ and $\varphi_i$, $i = 1, 2$ and we can suppose that the following assumptions hold:

Assumptions (HER). We say that assumptions (HER) hold if assumptions (HE) hold true and $b, \sigma, \varphi_1, \varphi_2$ are infinitely differentiable functions with bounded derivatives of all order, greater than 1 for the functions $\varphi_1$ and $\varphi_2$.

For the sake of clarity, we denote by $u^n$ the solution of the regularized version of the solution of (2.2) and by $A^n$ the regularized version of the operator $A$ throughout the remainder of this section. We can now state the main result about the smoothing properties of (2.2).

**Theorem 2.1.** Under hypothesis (HER), the regularized system of PDEs (2.2) admits a unique solution in the sense defined in [CCD14] and there exists a positive $T_{2.1}$, a positive constant $C_{2.1}$ and a positive number $\delta_{2.1}$ depending only on known parameters in (HE), such that, for all $(t,x,\mu)$ in $[0,T] \times \mathbb{R}^d \times P_2(\mathbb{R}^d)$ and all $n$ in $\mathbb{N}^*$ the mapping $u^n$ that solves the regularized system of PDEs (2.2) satisfies:

$$|\partial_\mu u^n(t,x,\mu)(z)| + |\partial_\mu (\partial_x u^n(t,x,\mu))(z)| + |\partial_x u^n(t,x,\mu)| + |\partial_x^2 u^n(t,x,\mu)| \leq C_{2.1} T^{\delta_{2.1}},$$

for all $z$ in $\mathbb{R}^d$ and for $T$ less than $T_{2.1}$.

The smoothness of the solution in space is not new. This phenomenon is well-known and follows from the ellipticity assumption on $a$. What is more unexpected is that there are bounds obtained
uniformly on the regularization procedure on the measure derivative. Indeed, the coefficients of the
PDE are not differentiable w.r.t. the argument $\mu$. It is clear that any differentiation of $u^n$ w.r.t. $\mu$
should involve the differentiation of the source term $b$ w.r.t. this argument, and so, by the chain rule,
the derivative of the mapping $P_2(\mathbb{R}^d) \ni \nu \mapsto \langle \varphi^n_i, \nu \rangle$, which is given by $(\varphi^n_i)'$, so that the estimate
should depend on the regularization procedure.

Nevertheless, it appears that for all $s < t \in [0, T]^2$ the derivative of the mapping $\mu \mapsto \langle \varphi_i, [X^{t, \mu}_s] \rangle$
can be estimated in term of known parameters in $(HE)$ (in fact, combining an additional estimate on $\partial_2 \partial_\mu u$
together with Arzelà-Ascoli Theorem, we are able to show that the estimate on $\partial_\mu u$ holds for
the mild solution of (2.2)).

We hence have a smoothing property in the measure space without any action of the Laplacian
in that direction. This follows from the fact that the function $\varphi_i$ is integrated against the law of the
process so that there still is a Gaussian convolution of the initial data “$\mu$” at any time $s > t$. Therefore,
we recover the spatial smoothing.

To the best of our knowledges, this result is new, especially since we do not add any noise on the
probability measures space. Concerning this last aspect, we refer to the works [CDL14] and [CDLL15]
where Mean Field Games with common noise are investigated. Roughly speaking, the Authors show
that common noise on the original system of interacting players translates into McKean-Vlasov system
with random law (the family of probability measures of the underlying stochastic process having now a
stochastic dynamic). This perturbation then allows them to recover existence and uniqueness of Nash
equilibrium.

Also, in the same spirit as us, David R. Baños studies in [Bn15] the Malliavin differentiability of
processes having the same dynamic as (1.1) with Lipschitz coefficients. Although he does not consider
explicitly a regularization phenomenon in the measure direction (the functions $\varphi_i$ are continuously differ-
entiables with bounded Lipschitz derivatives), he shows that the space regularization phenomenon
still holds so that the mapping $x \mapsto \langle [X^{t, \delta_s}_s], \phi \rangle \in \mathbb{R}$, $\phi$ in $L_2([X^{t, \delta_s}_s])$ is weakly differentiable for any
$s > t$ thanks to a stochastic perturbation approach of Bismut type.

Finally, let us emphasize that Remark 1 also applies for Theorem 2.1. It seems that the main
structural assumption that allows to recover the space smoothing property in the measure direction
comes from the particular “polynomial” dependence of the coefficients w.r.t. the measure.

3 Proof of the main result

We prove our main result by using a Picard’s iteration. Let $m$ be a positive integer, set $(X_0^\mu)^0 = X_0$
for all $t$ in $[0, T]$ and define $(X_0^\mu)^{m+1}$ as the solution of:

$$(X_0^\mu)^{m+1} = X_0 + \int_0^t b(r, (X_0^\mu)^m, \langle \varphi_1, [(X_0^\mu)^m] \rangle) dr + \int_0^t \sigma(r, (X_0^\mu)^m, \langle \varphi_2, [(X_0^\mu)^m] \rangle) dB_r, \quad X_0 \sim \mu.$$ 

We have now to use an Itô’s Formula that matches our framework, i.e. stated on $[0, T] \times \mathbb{R}^d \times P_2(\mathbb{R}^d)$. This is given in [CCD14] and applying it to

$$(X_0^\mu)^{m+1} = u^n(t, (X_0^\mu)^{m+1}, [(X_0^\mu)^{m+1}]),$$
leads to
\[
(X_\mu^m)^{m+1} = X_0 - \mathbf{u}^n(0, X_0, \mu) - \mathbf{u}^n(t, (X_\mu^m)^{m+1}, [(X_\mu^m)^{m+1}]) \tag{3.1}
\]
\[
+ \int_0^t \sigma(s, (X_\mu^m)^m, [(X_\mu^m)^m]) \left[ 1 - \partial_2 \mathbf{u}^n(s, (X_\mu^m)^{m+1}, [(X_\mu^m)^{m+1}]) \right] dB_s + \mathcal{R}^m_t(n),
\]
where
\[
\mathcal{R}^m_t(n) = \int_0^t b^n(s, (X_\mu^m)^m, \langle \varphi_1, [(X_\mu^m)^m] \rangle) - b(s, (X_\mu^m)^m, \langle \varphi_1, [(X_\mu^m)^m] \rangle)
\]
\[
+ (\mathbf{A}^n - \mathbf{A}) \mathbf{u}^n(s, (X_\mu^m)^{m+1}, [(X_\mu^m)^{m+1}])) ds.
\]

Hence,
\[
|[(X_\mu^m)^{m+1} - (X_\mu^m)^m| \leq |\mathbf{u}^n(t, (X_\mu^m)^{m+1}, [(X_\mu^m)^{m+1}]) - \mathbf{u}^n(t, (X_\mu^m)^m, [(X_\mu^m)^m])|
\]
\[
+ \int_0^t \left\{ \sigma(s, (X_\mu^m)^m, [(X_\mu^m)^m]) \left[ 1 - \partial_2 \mathbf{u}^n(s, (X_\mu^m)^{m+1}, [(X_\mu^m)^{m+1}]) \right]
\]
\[
- \sigma(s, (X_\mu^m)^{m-1}, [(X_\mu^m)^{m-1}]) \left[ 1 - \partial_2 \mathbf{u}^n(s, (X_\mu^m)^m, [(X_\mu^m)^m]) \right] \right\} dB_s
\]
\[
+ |\mathcal{R}^m_t(n)| + |\mathcal{R}^{m-1}_t(n)|.
\]

Let us now emphasize that when \( T < \mathcal{T}_{2.1} \), Theorem 2.1 implies that for all \((t,x)\) in \([0,T] \times \mathbb{R}^d\), all \(Z, Z'\) in \(L_2(\Omega)\):
\[
|\partial_2^0 \mathbf{u}(t,x,[Z]) - \partial_2^0 \mathbf{u}(t,x,[Z'])| \leq \int_0^1 \mathbb{E} \left[ |\partial_2 \partial_2^0 \mathbf{u}(t,x,[Z])| \right] (1-\lambda)Z + \lambda Z' \right] d\lambda 
\leq C_{2.1} T^{\delta_{2.1}} \mathbb{E}[|Z - Z'|].
\]

for \(n = 0, 1\), so that for any \(T\) less than \(\mathcal{T}_{2.1}\), there exists a positive \(\delta\), depending on known parameters in (HE), such that:
\[
\mathbb{E} \sup_{t \leq T} |(X_\mu^m)^{m+1} - (X_\mu^m)^m|^2 \leq C T^\delta \mathbb{E} \sup_{t \leq T} |(X_\mu^m)^{m+1} - (X_\mu^m)^m|^2
\]
\[
+ C \int_0^t \mathbb{E} \sup_{r \leq s} |(X_\mu^m)^m - (X_\mu^m)^{m-1}|^2 ds + 2 \mathbb{E} |\mathcal{R}^m_T(n)|^2 + 2 \mathbb{E} |\mathcal{R}_T^{m-1}(n)|^2,
\]

Since for any \(m\), \(\mathbb{E}|\mathcal{R}_T^m(n)|^2\) tends uniformly to 0 as \(n\) tend to infinity, we can let \(n\) tend to infinity in the right hand side of the equation above and we get that
\[
(1 - C T^\delta) \mathbb{E} \sup_{t \leq T} |(X_\mu^m)^{m+1} - (X_\mu^m)^m|^2 \leq C_T \int_0^t \mathbb{E} \sup_{r \leq s} |(X_\mu^m)^m - (X_\mu^m)^{m-1}|^2 ds.
\]
Finally, we can find a positive $T$ depending on known parameters in (HE) such that for any $T$ less than $T$:

$$\mathbb{E} \sup_{t \leq T} |(X_t^\mu)^{m+1} - (X_t^\mu)^m|^2 \leq C_T \int_0^T \mathbb{E} \sup_{s \leq r} |(X_r^\mu)^m - (X_r^\mu)^{m-1}|^2 \, ds.$$  

By induction, we deduce that for any $T$ less than $T$:

$$\mathbb{E} \sup_{t \leq T} |(X_t^\mu)^{m+1} - (X_t^\mu)^m|^2 \leq \frac{C^{m+1}T^m}{m!} \mathbb{E} \sup_{t \leq T} |(X_t^\mu)^1 - (X_t^\mu)^0|^2 \leq C \frac{C^{m+1}T^m}{m!}.$$  

So that $(X_t^\mu)^m$ converges almost surely to a solution $X_t^\mu$ of (1.1). We deduce the uniqueness part from the previous computations. We hence have existence and uniqueness of a solution on $[0, T]$. We can then iterate the construction and then obtain the result for all $T$ in $\mathbb{R}^+$.

4 Estimation on the solution of the PDE: proof of Theorem 2.1

Let us first reduce the problem. We emphasize that any component of the $d$-dimensional solution of the system of PDEs (2.2) above can be described by the solution of:

$$\begin{cases}
(\partial_t + \mathcal{A})u(t, x, \mu) = \tilde{b}(t, x, \langle \mu, \varphi_1 \rangle), & \text{on } [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \\
u(T, x, \mu) = 0,
\end{cases}$$

where $\tilde{b} : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ plays the role of one of the components of $b$. Hence, we only have to prove the estimates in Theorem 2.1 for the function $u$ defined above. We know from [CCD14] that under (HER) this PDE admits a unique classical solution. Let us now give a suitable representation of this solution.

Under assumptions (HER) it follows from the Sznitman’s note [Szn91] that equation (1.1) admits a unique strong solution. For any $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and $x$ in $\mathbb{R}^d$, we denote its flow as the solution on $[t, T]$ of

$$X_{s, x, \mu} = x + \int_t^s \tilde{b}(r, X_r^{t, x, \mu}, \langle \varphi_1, [X_r^{t, \mu}] \rangle) \, dr + \int_t^s \sigma(r, X_r^{t, x, \mu}, \langle \varphi_2, [X_r^{t, \mu}] \rangle) \, dB_r.$$  

Given the family of marginals $[X^{t, \mu}] := ([X_s^{t, \mu}])_{t \leq s \leq T}$ of the solution of (1.1), we can consider the stochastic system (4.2) as a linear system parametrized by the time dependent parameter $[X^{t, \mu}]$. We can then define for all $(s', y')$ in $[0, T] \times \mathbb{R}^d$ the process $X_{s', y', [X^{t, \mu}]}$ as the solution of (4.2) on $[s', T]$ with starting point $y'$ at time $s'$ and whose coefficients depend on $[X^{t, \mu}]$.

It is then clear, thanks to the well posedness of (4.2) under (HER), that $X_{t, x, [X^{t, \mu}]} = X_{t, x, \mu}$. Finally, from classical theory of linear SDEs, the flow $X_{t, x, [X^{t, \mu}]}$ admits a transition density $p$ which is also parametrized by $[X^{t, \mu}]$.

Since from the arguments of [CCD14] we have that, for all $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R})$, the solution of the PDE (4.1) writes:

$$u(t, x, \mu) = \mathbb{E} \int_t^T \tilde{b}(s, X_s^{t, x, \mu}, \langle \varphi_1, [X_s^{t, \mu}] \rangle) \, ds,$$
we deduce from the previous discussion that
\[
  u(t, x, \mu) = \int_t^T \int_{\mathbb{R}^d} \tilde{b}(s, y, \langle \varphi_1, [X_{s,t}^{t,\mu}] \rangle)p([X_{s,t}^{t,\mu}], t; x, s, y) dy ds.
\]
(4.3)

This is our suitable representation for the solution \( u \) of the PDE (4.1).

In order to keep the notations clear, we only mention the dependence of \( p \) w.r.t. the initial data \((t, \mu)\) of (1.1) in the following and we forget its first argument when the starting time of (1.1) and (4.2) are the same. Hence, for all \((s', y')\) and \((s, y)\) in \([t, T] \times \mathbb{R}^d\): \( p(t, \mu; s', y'; s, y) := p([X_{s,t}^{t,\mu}], s', y; s, y) \) and \( p(\mu; t, y'; s, y) := p(t, \mu; t, y'; s, y) \).

Hence, when differentiating the function \( u \) in the measure direction, we have to differentiate the integrand in the above expression in that direction. In the one hand, we have to estimate a quantity of the form
\[
  \partial_{\mu} \langle \varphi, [X_{s,t}^{t,\mu}] \rangle, \quad s \in (t, T).
\]

Then, in the following, for any \( \alpha \)-Hölder function \( \phi, \alpha \in (0, 1] \), we denote by \( v \) the mapping:
\[
v : (t < s, \mu, \phi) \in [0, T]^2 \times \mathcal{P}_2(\mathbb{R}^d) \mapsto v_s(t, \mu, \phi) = \langle \phi, [X_{s,t}^{t,\mu}] \rangle,
\]
(4.4)
and we prove in Section 5 the following

**Proposition 4.1.** Suppose that assumptions (HE\(R\)) hold, let \( \phi \) be some \( \alpha \)-Hölder function from \( \mathbb{R}^d \) to \( \mathbb{R} \), let \( t \) in \([0, T]\) and let us denote by \( \mu \) the law of the solution of (1.1) at time \( t \). There exist a positive number \( T_{4.1} \) and a positive constant \( C_{4.1} \), depending only on known parameters in (HE\(E\)), such that for all \( z \) in \( \mathbb{R}^d \) and \( s \) in \((t, T] \):
\[
|\partial_{\mu}v_s(t, \mu, \phi)(z)| \leq C(s - t)^{(-1+\alpha)/2},
\]
(4.5)
for all \( T \) less than \( T_{4.1} \).

On the other hand, it is clear that the derivative of \( u \) or \( \partial_x u \) along the measure involves the derivative of the transition density \( p \) or \( \partial_x p \) along the measure. We then have to obtain suitable control of these quantities. Here, these controls are summarized by the following Proposition whose proof is postponed to Section 6:

**Proposition 4.2.** Suppose that assumptions (HE\(R\)) hold, let \( t \) in \([0, T]\) and let us denote by \( \mu \) the law of the solution of (1.1) at time \( t \). Then, for all \( x \) in \( \mathbb{R}^d \), for all \((s, y)\) in \((t, T) \times \mathbb{R}^d \) and all \( z \) in \( \mathbb{R}^d \): \( p(\mu; t, x; s, y) \leq \hat{p}_c(t, x; s, y) \) where \( \hat{p}_c \) is the Gaussian like kernel defined by:
\[
  \hat{p}_c(t, x; s, y) = \frac{c}{(s - t)^{d/2}} \exp \left( -c \frac{|y - x|^2}{(s - s')^2} \right),
\]
(4.6)
where \( c \) depends on known parameters in (HE\(E\)) only. Moreover, there exist two positive constants \( C_{4.2} \) and \( C_{4.2}' \), depending only on known parameters in (HE\(E\)) such that for all \( x \) in \( \mathbb{R}^d \), for all \((s, y)\) in \((t, T) \times \mathbb{R}^d \) and all \( z \) in \( \mathbb{R}^d \)
\[
  \partial_{\mu}p(\mu; t, x; s, y)(z) \leq C_{4.2} \sum_{i=1}^{2} (s - t)^{(\alpha_i - 1)/2} \partial_{\mu}v_{(\alpha_i - 1)/2}(\mu; t, s)(z) \hat{p}_c(t, x; s, y),
\]
(4.7)
\[
  \partial_{\mu}\partial_x p(\mu; t, x; s, y)(z) \leq C_{4.2}' \sum_{i=1}^{2} (s - t)^{\alpha_i/2 - 1} \partial_{\mu}v_{\alpha_i/2 - 1}(\mu; t, s)(z) \hat{p}_c(t, x; s, y),
\]
(4.8)
where we use the abusive notation
\[ \overline{\partial_\mu v}_\gamma^i(\mu; t, s)(z) = \sup_{r \in [t, s]} \{ (r - t)^{-\gamma} |\partial_\mu v_r(t, \mu, \varphi_i)(z)| \}, \]

We have now all the ingredients to complete the proof. Thanks to estimates (4.5) on \( \partial_\mu v \) and (4.7) on \( \partial_\mu p \) we deduce that we can invert the differentiation and integration operators when differentiating the right hand side of (4.3) w.r.t. the measure. Hence, the derivative of \( u \) in the measure direction writes, at any point \( z \) of \( \mathbb{R}^d \):
\[
\partial_\mu u(t, x, \mu)(z) = \int_t^T \int_{\mathbb{R}^d} \partial_3 \tilde{b}(s, y, \langle \varphi_1, [X_t^{t, \mu}] \rangle) \partial_\mu_\mu (\varphi_1, [X_t^{t, \mu}]) p(\mu; t, x, s, y) dy ds \\
+ \int_t^T \int_{\mathbb{R}^d} \tilde{b}(s, y, \langle \varphi_1, [X_t^{t, \mu}] \rangle) \partial_\mu p(\mu; t, x, s, y) p_c(t, x, s, y) dy ds,
\]

and satisfies, thanks to estimates (4.7) and (4.5)
\[
|\partial_\mu u(t, x, \mu)(z)| \\
\leq C \int_t^T \int_{\mathbb{R}^d} |\partial_3 \tilde{b}|_\infty (s - t)^{(-1 + \alpha)/2} p(\mu; t, x, s, y) dy ds \\
+ \int_t^T \int_{\mathbb{R}^d} |\tilde{b}|_\infty C \sum_{i=1}^2 (s - t)^{(-1 + \alpha_i)/2} |\partial_\mu v_i|_{\alpha_i} p_c(t, x, s, y) dy ds.
\]

Therefore, there exist a positive constant \( C' \) and a positive number \( \delta \), depending only on known parameters in (HE), such that:
\[
|\partial_\mu u(t, x, \mu)(z)| \leq C' T^\delta.
\]

Now, we have:
\[
\partial_x u(t, x, \mu) = \int_t^T \int_{\mathbb{R}^d} \tilde{b}(s, y, \langle \varphi_1, [X_t^{t, \mu}] \rangle) \partial_x p(\mu; t, x, s, y) dy ds.
\]

Hence, we can differentiate the mapping \( \partial_x u \) along the measure and the same arguments as above, by using estimates (4.8) on \( \partial_\mu \partial_x p \) instead of (4.7) we obtain that
\[
|\partial_\mu (\partial_x u(t, x, \mu))(z)| \leq C'' T^{\delta'},
\]
for some positive constant \( C'' \) and a positive number \( \delta' \), depending only on known parameters in (HE).

Finally, the estimates on \( \partial_x u \) and \( \partial^2_x u \) can be obtained by classical arguments, when viewing the argument \( \mu \) as a parameter. See e.g. [Fri64]. This concludes the proof of Theorem 2.1.
Differentiation and estimation of $v$

With the notations defined in the previous section, we have that for all $s$ in $(t, T]$,

$$v_s(t, \mu, \phi) = \int_{\mathbb{R}^d} \phi(y) \int_{\mathbb{R}^d} p(\mu; t, x; s, y) d\mu(x) dy,$$

where the function $P$ may be seen as the function

$$P : (t < s, \lambda, \nu, y) \in [0, T]^2 \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to P(t, (\lambda, \nu); s, y) = \int_{\mathbb{R}^d} p(\lambda; t, x; s, y) d\nu(x).$$

With this notation and by using the fact that $p$ is a density we have:

$$\partial_\mu v_s(t, \mu, \phi) = \partial_\mu \int_{\mathbb{R}^d} \phi(y) P(t, \mu, \mu; s, x) dy$$

$$= \left[ \partial_\lambda \int_{\mathbb{R}^d} \phi(y) P(t, \lambda, \mu; s, y) dy \right]_{\lambda=\mu} + \left[ \partial_\nu \int_{\mathbb{R}^d} \phi(y) P(t, \mu, \nu; s, y) dy \right]_{\nu=\mu}$$

$$= \left[ \partial_\lambda \int_{\mathbb{R}^d} \phi(y) P(t, \lambda, \mu; s, y) dy \right]_{\lambda=\mu} + \left( \int_{\mathbb{R}^d} (\phi(y) - \phi(\xi)) \partial_\nu p(t, \mu, \nu; s, y) dy \right)_{\nu=\mu},$$

whatever $\xi$ in $\mathbb{R}^d$. Since by Fubini’s Theorem we have

$$\partial_\lambda \int_{\mathbb{R}^d} \phi(y) P(t, \lambda, \mu; s, y) dy = \partial_\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(y) - \phi(x)) p(\lambda; t, x; s, y) d\mu(x) dy$$

$$+ \partial_\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(x) p(\lambda; t, x; s, y) d\mu(x) dy$$

$$= \partial_\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(y) - \phi(x)) p(\lambda; t, x; s, y) d\mu(x) dy$$

$$+ \partial_\lambda \int_{\mathbb{R}^d} \phi(x) \int_{\mathbb{R}^d} p(\lambda; t, x; s, y) dy d\mu(x)$$

$$= \partial_\lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(y) - \phi(x)) p(\lambda; t, x; s, y) d\mu(x) dy,$$

and since by definition

$$\partial_\nu P(t, \mu, \nu; s, y)(\cdot) = \partial_2 p(\mu; t, \cdot; s, y),$$

we deduce that for all $\xi$ in $\mathbb{R}^d$, the derivative $\partial_\mu v$ taking at any point $z$ in $\mathbb{R}^d$ writes
\[ \partial_{\mu}v_s(t, \mu, \phi)(z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(y) - \phi(x)) \partial_{\mu}p(\mu; t, x; s, y)(z) d\mu(x) dy + \int_{\mathbb{R}^d} (\phi(y) - \phi(\xi)) \partial_{x}p(\mu; t, z; s, y) dy. \]  

(5.1)

So that for any \( z \), by choosing \( \xi = z \) we get

\[ \partial_{\mu}v_s(t, \mu, \phi)(z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(y) - \phi(x)) \partial_{\mu}p(\mu; t, x; s, y)(z) d\mu(x) dy + \int_{\mathbb{R}^d} (\phi(y) - \phi(z)) \partial_{x}p(\mu; t, z; s, y) dy. \]

Thanks to the estimates we have on the transition density \( p \) and its derivatives in the measure direction from Proposition 4.2, Fubini’s Theorem, regularity assumed on \( \phi \) and by using the Gaussian decay of \( \tilde{p}_c \) \(^1\) we obtain the following bound:

\[
|\partial_{\mu}v_s(t, \mu, \phi)(z)| \leq C \left\{ \sum_{i=1}^{2} (s - t)^{(\alpha_i - 1)/2} \frac{\partial^{i}v}{\partial \mu^{i}}(\alpha_i - 1)/2(\mu; t, s)(z) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y - x|^\alpha \tilde{p}_c(t, x; s, y) d\mu(x) dy \right.

+ \left. \int_{\mathbb{R}^d} |y - z|^\alpha (s - t)^{-1/2} \tilde{p}_c(t, z; s, y) dy \right\},

(5.2)

which holds true for any \( \alpha \)-Hölder function \( \phi \). Then, by choosing \( \phi = \varphi_1 \) (and so \( \alpha = \alpha_1 \)), by multiplying both sides by \( (s - t)^{(1-\alpha_1)/2} \), we deduce from a circular argument that there exists a positive time \( T' \) depending only on known parameters in \( (HE) \), such that for all \( T \) less than \( T' \):

\[
(s - t)^{(1-\alpha_1)/2} |\partial_{\mu}v_s(t, \mu, \varphi_1)(z)| \leq C'' \left( (s - t)^{\alpha_2/2} \frac{\partial^{2}v}{\partial \mu^{2}}(\alpha_2 - 1)/2(\mu; t, s)(z) + 1 \right).

(5.3)

By plugging this estimate in (5.2) and by iterating this argument (choosing \( \phi = \varphi_2 \) so that \( \alpha = \alpha_2 \) then multiplying both sides by \( t^{(1-\alpha_2)/2} \) and using a circular argument) we obtain that there exists a positive time \( T'' \), depending only on known parameters in \( (HE) \), such that for all \( T \) less than \( T'' \):

\[
(s - t)^{(1-\alpha_2)/2} |\partial_{\mu}v_s(t, \mu, \varphi_2)(z)| \leq C'''.

(5.4)

Again, by plugging this estimate in (5.3) and then using the resulting estimate together with (5.4) in (5.2), we finally deduce that there exists a positive time \( T \), depending only on known parameters in \( (HE) \), such that for all \( T \) less than \( T \):

\[
(s - t)^{(1-\alpha)/2} |\partial_{\mu}v_s(t, \mu, \phi)(z)| \leq C'''' ,

which concludes the proof of of Proposition 4.1.

---

\(^{1}\text{i.e. the inequality: } \forall \eta > 0, \forall q > 0, \exists C > 0 \text{ s.t. } \forall \sigma > 0, \sigma^q e^{-\eta \sigma} \leq C.\)
6 Estimation of the transition density \( p \)

This section is dedicated to the proof of Proposition 4.2 and so to the study of the transition density \( p \) of the flow (4.2). As we already mentioned, the investigations are done using the classical McKean-Singer parametrix expansion of \( p \). We indeed recall that, under (H\&R), it is clear that for all initial data \((t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), equation (1.1) admits a unique solution \( X_{t,\mu} \). We then suppose that the family of probability measures \([X_{t,\mu}]\) acts as a time dependent parameter in (4.2), so that the unique solution of (4.2) has a classical transition density \( p \) parametrized by the family of probability measures \([X_{t,\mu}]\). Once the law dependence is fixed, we can now use a parametrix expansion for linear processes in order to represent the transition density \( p \).

The parametrix expansion has been introduced by McKean and Singer in [MS67]. It is based on the following observation: in small time, the transition density of a (smooth enough) process should be approximated by the transition density of the same process whose coefficients are constant and fixed at the (final) value of the process. We call frozen transition density this function. Hence, the transition density of interest should be expanded in terms of the frozen transition density.

Usually, the frozen transition density enjoys well known properties, e.g., it has an explicit form or can be estimated by explicit (and nice) functions. This method then requires a good knowledge of the frozen transition density and on the associated frozen process.

Thus, this section is organized as follows: we first introduce in subsection 6.1 the frozen process and its associated transition density and give its explicit expression. Then, we give the estimates on the frozen transition density and its derivative. In subsection 6.2 we show how the transition density of (4.2) can be expanded in terms of the transition density of the frozen process. Hence, we have an explicit form of the transition density \( p \) of (4.2) which can be estimated. These estimations are done in subsection 6.3 and lead to proof of Proposition 4.2. Finally, we suppose throughout this section that \( T < 1 \).

6.1 The frozen system

Let \((t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), for any point \( \xi \in \mathbb{R}^d \), we define the frozen flow by the solution of:

\[
\tilde{X}_{s'}^{s,y',\xi}[X_{t,\mu}] = y' + \int_s^{s'} b(r, \xi, (\varphi_1, [X_r^{t,\mu}])) dr + \int_s^{s'} \sigma(r, \xi, (\varphi_2, [X_r^{t,\mu}])) dB_r, \tag{6.1}
\]

for all \((s', y') \in [t, T] \times \mathbb{R}^d\). Under (H\&R), it is clear that this flow exists and is unique, moreover, it has the transition density \( \tilde{p} \) defined for all \((s, y) \) in \([t, T] \times \mathbb{R}^d\) by:

\[
\tilde{p}(t, \mu; s', y'; s, y) = \frac{1}{(2\pi)^{d/2}} \left[ \det(a_{s',s}^\xi(t, \mu)) \right]^{-1/2} \exp \left( -\frac{1}{2} \left[ a_{s',s}^\xi(t, \mu) \right]^{-1/2}(y - y' - m_{s',s}^\xi(t, \mu))^2 \right), \tag{6.2}
\]

where we adopted the same convention of notations for \( \tilde{p} \) as for \( p \) and where

\[
m_{s',s}^\xi(t, \mu) = \int_{s'}^{s} b(r, \xi, (\varphi_1, [X_r^{t,\mu}])) dr, \tag{6.3}
\]

\[
a_{s',s}^\xi(t, \mu) = \int_{s'}^{s} a(r, \xi, (\varphi_2, [X_r^{t,\mu}])) dr. \tag{6.4}
\]
Its generator is given by
\[
\mathcal{L}_{s', y'}^{\xi, \mu} := b(s', \xi, \langle \varphi_1, [X_{s'}^{t, \mu}] \rangle) \partial_{y'} + \frac{1}{2} \text{Tr} \left[ a(s', \xi, \langle \varphi_2, [X_{s'}^{t, \mu}] \rangle) \partial_{y'}^2 \right].
\]
(6.5)

Above, the subscript \((s', y')\) means that the coefficients of the operator are evaluated at time \(s'\) and that the differentiation operator acts on the space variable \(y'\).

This transition density admits Gaussian type bounds, namely, we prove in Section A that:

**Proposition 6.1.** Suppose that hypothesis (H\&R) holds. Let \(t \in [0, T]\) and let \(\mu\) denotes the law of the process (1.1) at time \(t\). Then:

- There exists a positive constant \(C_{6.1}\), depending only on known parameters in (H\&E), such that for all \(s < s' \leq [0, T]^2\), \(\forall y, y' \in \mathbb{R}^d\), \(\forall \xi \in \mathbb{R}^d\), \(\tilde{p}^\xi(t, \mu; s', y'; s, y) \leq C_{6.1} \tilde{p}_c(s', y'; s, y),\)

\[
\tag{6.6}
\label{eq:gaussian_bound}
\]

where \(\tilde{p}_c\) is the Gaussian like kernel defined by (4.6).

- There exist two positive constants \(C_{6.1}'\) and \(C_{6.1}''\), depending on known parameters in (H\&E) only, such that for all \(s \in (t, T]\), for all \(\xi \in \mathbb{R}^d\) and all \(x, y \in \mathbb{R}^d\):

\[
\left| \partial_\mu \tilde{p}^\xi(\mu; t, x; s, y)(z) \right| \leq C_{6.1}' \sum_{i=1}^2 (s - t)^{(\alpha_1 - 1)/2} \partial_\mu v_{(\alpha_1 - 1)/2}(\mu; t, s')(z) \tilde{p}_c(t, x; s, y),
\]

\[
\left| \partial_\mu \partial_x \tilde{p}^\xi(\mu; t, x; s, y)(z) \right| \leq C_{6.1}' \sum_{i=1}^2 (s - t)^{(\alpha_1 - 1)/2} \partial_\mu v_{(\alpha_1 - 1)/2}(\mu; t, s)(z) \tilde{p}_c(t, x; s, y),
\]

\[
\tag{6.7}
\label{eq:bound_on_derivatives1}
\]

\[
\left| \partial_\mu \tilde{p}^\xi(\mu; t, s'; y'; s, y)(z) \right| \leq C_{6.1}'' (s' - s)^{-1/2} \sum_{i=1}^2 (s' - t)^{(\alpha_1 - 1)/2} \partial_\mu v_{(\alpha_1 - 1)/2}(\mu; t, s)(z) \tilde{p}_c(s', y'; s, y),
\]

\[
\left| \partial_\mu \partial_y \tilde{p}^\xi(\mu; t, s'; y'; s, y)(z) \right| \leq C_{6.1}'' (s' - s)^{-1/2} \sum_{i=1}^2 (s' - t)^{(\alpha_1 - 1)/2} \partial_\mu v_{(\alpha_1 - 1)/2}(\mu; t, s')(z) \tilde{p}_c(s', y'; s, y),
\]

\[
\tag{6.8}
\label{eq:bound_on_derivatives2}
\]

for all \(z \in \mathbb{R}^d\).

- For all \(s' < s \in (t, T]\), for all \(y', y \in \mathbb{R}^d\):

\[
\left| \partial_\mu \tilde{p}^\xi(t, \mu; s', y'; s, y)(z) \right| \leq C \sum_{i=1}^2 (s' - t)^{(\alpha_1 - 1)/2} \partial_\mu v_{(\alpha_1 - 1)/2}(\mu; t, s)(z) \tilde{p}_c(s', y'; s, y),
\]

\[
\left| \partial_\mu \partial_y \tilde{p}^\xi(t, \mu; s', y'; s, y)(z) \right| \leq C' (s - s')^{-1/2} \sum_{i=1}^2 (s' - t)^{(\alpha_1 - 1)/2} \partial_\mu v_{(\alpha_1 - 1)/2}(\mu; t, s')(z) \tilde{p}_c(s', y'; s, y),
\]

\[
\left| \partial_\mu \partial_y \partial_y \tilde{p}^\xi(t, \mu; s', y'; s, y)(z) \right| \leq C'' (s' - s')^{-1} \sum_{i=1}^2 (s' - t)^{(\alpha_1 - 1)/2} \partial_\mu v_{(\alpha_1 - 1)/2}(\mu; t, s)(z) \tilde{p}_c(s', y'; s, y),
\]

for all \(z \in \mathbb{R}^d\).

**6.2 The parametrix expansion**

We now give the parametrix representation of the transition density. Proof of such a result is somehow classical, we nevertheless wrote it since it allows to understand the crucial estimates, which will be useful in the sequel.

**Proposition 6.2.** There exists a smoothing kernel \(H : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times [0, T] \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}\) given by

\[
H(t, \mu; s', y'; s, y) = \left( \mathcal{L}^{y,t,\mu} - \mathcal{L} \right)_{s', y'} \tilde{p}^\mu(t, \mu; s', y'; s, y),
\]

(6.9)
such that the transition density \( p \) of the flow \( X \) defined by (4.2) writes:

\[
p(\mu; t, x; s, y) = \tilde{p}^\beta(\mu; t, x; s, y) + \sum_{k=1}^{+\infty} \int_t^s \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; s', y'; s, y) \tilde{p}^\beta(\mu; t, x; s', y') dy' ds',
\]

where \( H^{\otimes k} \) is recursively defined by:

\[
H^{\otimes k+1}(t, \mu; s', y'; s, y) = \int_{s'}^s \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; r, u; s, y) H(t, \mu; s', y'; r, u) du dr,
\]

and \( H^{\otimes 0} = \text{Id} \).

**Proof.** Let \((s, y)\) belong to \([0, T] \times \mathbb{R}^d\), the transition density \( \tilde{p}^\beta(t, \mu; \cdot; \cdot; s, y) \) satisfies the Fokker-Planck equation:

\[
\begin{align*}
\partial_t \tilde{p}^\beta(t, \mu; s', x; s, y) + \mathcal{L}_{s',x} \tilde{p}^\beta(t, \mu; t, x; s, y) &= 0, \quad (s', x) \in [0, s] \times \mathbb{R}^d, \\
\tilde{p}^\beta(t, \mu; s; x; s, y) &= \delta_x(y),
\end{align*}
\]

which can be rewritten as

\[
\begin{align*}
\partial_t \tilde{p}^\beta(t, \mu; s', x; s, y) + \mathcal{L}_{s',x} \tilde{p}^\beta(t, \mu; t, x; s, y) &= (\mathcal{L} - \tilde{\mathcal{L}}_{y,t,\mu})_{s',x} \tilde{p}^\beta(t, \mu; s', x; s, y), \quad (s', x) \in [0, s] \times \mathbb{R}^d, \\
\tilde{p}^\beta(t, \mu; s; x; s, y) &= \delta_x(y).
\end{align*}
\]

Note that \( \tilde{p}^\beta(t, \mu; \cdot; \cdot; s, y) \) is a fundamental solution of this PDE. Therefore \( \tilde{p}^\beta(t, \mu; \cdot; \cdot; s, y) \) writes, for all \((s', x) \in [0, s] \times \mathbb{R}^d\):

\[
\tilde{p}^\beta(t, \mu; s', x; s, y) = p(\mu; t, x; s, y) + \int_{s'}^s \int_{\mathbb{R}^d} (\mathcal{L} - \tilde{\mathcal{L}}_{y,t,\mu})_{s',x} \tilde{p}^\beta(t, \mu; s', y'; s, y) p(\mu; t, x; s', y') dy' ds'.
\]

Hence, by iterating \( N \) times this procedure, we obtain that

\[
p(\mu; t, x; s, y) = \tilde{p}^\beta(\mu; t, x; s, y) + \sum_{k=1}^{N} \int_t^s \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; s', y'; s, y) \tilde{p}^\beta(\mu; t, x; s', y') dy' ds' + \int_t^s \int_{\mathbb{R}^d} H^{\otimes N+1}(t, \mu; s', y'; s, y) p(\mu; t, x; s', y') ds' dy'.
\]

In order to obtain the parametrix expansion of \( p \), depending only on known quantities (i.e. on the smoothing kernel \( H \) defined by (6.9) and on the transition density of the frozen process \( \tilde{p} \)) the idea consists in letting \( N \) tend to infinity.

To this aim, we need a “good” estimate on the approximation error. Here these controls are the estimate (6.6) in Proposition 6.1 and:

**Lemma 6.3.** Under assumption \((\text{HFR})\) the following assertion holds: there exists a positive constant \( C_k^{6.3} \) given by:

\[
C_k^{6.3} = C_k^{6.3} \prod_{r=1}^{k-1} \beta(\gamma_{ar}/2, \gamma_{a}/2),
\]

where \( C_k^{6.3} \) is a positive constant depending only on known parameters in \((\text{HE})\) and \( \beta \) denotes the beta-function and with the convention \( \prod_{\emptyset} = 1 \), such that for all \( s' < s \) in \([t, T]^2\) and \( x, y \) in \( \mathbb{R}^d \):

\[
|H^{\otimes k}(t, \mu; s', x; s, y)| \leq C_k^{6.3} (s - s')^{\gamma_{a}/2 - 1} \hat{p}_c(s', x; s, y).
\]
On the one hand, the first term in the right hand side of (6.12) is controlled by a convolution of two Gaussian functions which is still Gaussian and it is clear from the asymptotic properties of the beta-function (that are recalled in Section B) that the series converges. On the other hand, the $N$th convolution of the kernel $H$ tends uniformly to 0 as $N$ tends to infinity (recall that $T$ is small) and since $p$ is a density, we deduce that the second term in the right hand side of (6.12) tends to 0. Therefore, the density $p$ writes:

$$p(\mu; t, x; s, y) = \tilde{p}^\mu(\mu; t, x; s, y) + \sum_{k \geq 1} \int_t^s \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; s', y'; s, y) \tilde{p}^\mu(\mu; t, x; s', y') dy' ds'. \tag{6.14}$$

Proof of Lemma 6.3. By using classical parametrix arguments (see Chapter 1 of [Fri64]) and by the definition (6.9) of $H$, we deduce that there exist two positive constants $C_{6.3}$ and $c$ depending only on known parameters in $(\mathcal{H})$ such that for all $s' < s \in [t, T]^2$ and $y', y \in \mathbb{R}^d$:

$$|H(t, \mu; s', y'; s, y)| \leq C_{6.3} (s - s')^{\gamma/2 - 1} \hat{p}_c(s', y'; s, y). \tag{6.15}$$

Suppose now as an induction hypothesis that for all $s' < s \in [t, T]^2$ and $y' \in \mathbb{R}^d$:

$$|H^{\otimes k}(t, \mu; s', y'; s, y)| \leq C_{k+1}^{6.3} (s - s')^{k \gamma/2 - 1} \hat{p}_c(s', y'; s, y), \tag{6.16}$$

where $C_{k+1}^{6.3}$ is defined by (6.13). Recall that for all integer $k$, $H^{\otimes k+1}$ is recursively defined by

$$H^{\otimes k+1}(t, \mu; s', y'; s, y) = \int_{s'}^s \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; u, y; s, y) H(t, \mu; s', y'; r, u) du dr. \tag{6.17}$$

Hence, by plugging (6.15) and (6.16) in (6.17) and using the Gaussian convolution we obtain that:

$$|H^{\otimes k+1}(t, \mu; s', y'; s, y)| \leq C_{k+1}^{6.3} C_{6.3} \int_{s'}^s (s - r)^{\gamma/2 - 1} (r - s')^{\gamma/2 - 1} dr \hat{p}_c(s', y'; s, y),$$

and by the change of variable $r = (s - s')r' + s'$ we have:

$$|H^{\otimes k+1}(t, \mu; s', y'; s, y)| \leq C_{k+1}^{6.3} C_{k}^{6.3} (s - s')^{(k+1) \gamma/2} \int_0^{1} (1 - r')^{\gamma/2 - 1} (r')^{\gamma/2 - 1} dr' \hat{p}_c(s', y'; s, y)$$

$$= C_{k+1}^{6.3} C_{k}^{6.3} (s - s')^{(k+1) \gamma/2} \beta(\gamma k/2, \gamma a/2) \hat{p}_c(s', y'; s, y)$$

$$= C_{k+1}^{6.3} (s - s')^{(k+1) \gamma/2} \hat{p}_c(s', y'; s, y). \tag*{□}$$

6.3 Differentiation and estimation of the density $p$ along the measure

We are now ready to prove Proposition 4.2. We have that

$$p(\mu; t, x; s, y) = \tilde{p}^\mu(\mu; t, x; s, y) + \sum_{k=1}^{+\infty} \int_t^s \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; s', y'; s, y) \tilde{p}^\mu(\mu; t, x; s', y') dy' ds', \tag*{□}$$
so that the derivative of \( p \) w.r.t. \( \mu \) writes, at any point \( z \) in \( \mathbb{R}^d \):

\[
\partial_\mu p(\mu; t, x; s, y)(z) = \partial_\mu \overline{p}(\mu; t, x; s, y)(z) + \partial_\mu \left( \sum_{k=1}^{\infty} \int_t^s \int_{\mathbb{R}^d} H^{(k)}(t, \mu; s', y'; s, y) \overline{p}(\mu; t, x; s', y') \, dy' \, ds' \right)(z).
\]

(6.18)

Then, in order to invert the integration and differentiation operator, we have to show that for all \( z \),

\[
(s', y') \mapsto \partial_\mu H^{(k)}(t, \mu; s', y'; s, y)(z) \overline{p}(\mu; t, x; s', y') + H^{(k)}(t, \mu; s', y'; s, y) \partial_\mu \overline{p}(\mu; t, x; s', y')(z),
\]

(6.19)

is suitably bounded. More precisely, we have to obtain a Gaussian control on the derivative of the \( k \)th iteration of the smoothing kernel \( \partial_\mu H^{(k)} \) and on \( \partial_\mu \overline{p} \) so that the parametrix expansion still holds, in the same spirit of the proof of Proposition 6.2. These controls are given by the estimates on the frozen transition density in Proposition 6.1 and the following Lemma:

**Lemma 6.4.** Let \( t \in [0, T] \) and let \( \mu \) be the law of the process (1.1) at time \( t \). For all positive integer \( k \), there exists a positive constant \( \tilde{C}_k^{6.4} \), depending only on known parameter in (HE) and given by:

\[
\tilde{C}_k^{6.4} = (C_k^{6.3} + \overline{C}_k^{6.4} \beta((k-1)\tilde{\gamma}_a/2, \tilde{\gamma}_a/2))
\]

where \( \tilde{\gamma}_a = \gamma_a \wedge \gamma'_a \) such that

\[
\left| \partial_\mu H^{(k)}(t, \mu; s', y'; s, y)(z) \right| \leq \tilde{C}_k^{6.4} (s' - s)^{k\gamma_a/2 - 1} \sum_{i=1}^{2} (s' - t)^{(\alpha_i - 1)/2} \partial_{\mu} v_{(\alpha_i - 1)/2}^i(\mu; t, s')(z) \hat{p}_c(s', y'; s, y),
\]

for all \( s' < s \in [0, T]^2 \), \( y', y' \) in \( \mathbb{R}^d \) and \( z \) in \( \mathbb{R}^d \).

From Lemma 6.4 and estimate (6.6) in Proposition 6.1 we have that for all \( k \):

\[
\left| \partial_\mu H^{(k)}(t, \mu; s', y'; s, y)(z) \overline{p}(\mu; t, x; s', y') \right| \leq \tilde{C}_k^{6.4} C_{6.1}^{6.4} (s - s')^{k\gamma_a/2 - 1} \sum_{i=1}^{2} (s' - t)^{(\alpha_i - 1)/2} \partial_{\mu} v_{(\alpha_i - 1)/2}^i(\mu; t, s')(z) \hat{p}_c(t, x; s', y') \hat{p}_c(s', y'; s, y),
\]

and from estimate (6.7) of Proposition 6.1 and Lemma 6.3 we have that for all \( k \):

\[
\left| H^{(k)}(t, \mu; s', y'; s, y) \partial_\mu \overline{p}(\mu; t, x; s', y')(z) \right| \leq C_k^{6.3} C_{6.1}^{6.3} (s - s')^{k\gamma_a/2 - 1} \sum_{i=1}^{2} (s' - t)^{(\alpha_i - 1)/2} \partial_{\mu} v_{(\alpha_i - 1)/2}^i(\mu; t, s')(z) \hat{p}_c(t, x; s', y') \hat{p}_c(s', y'; s, y).
\]

We can hence invert the differentiation and integration operator in the second term in the right hand side of (6.18) and using property of Gaussian convolution we get that there exists a positive constant \( C \) such that
\[ \partial_\mu p(\mu; t, x; s, y)(z) \leq C(s-t)^{(\alpha_1-1)/2} \left( C_{6.1} + \sum_{k \geq 1} \left\{ (C_{6.3} C_{6.1} + C_{6.4} C_{6.1}) (s-t)^{(k\gamma+3)/2} \right\} \right) \times \hat{p}_c(t, x; s, y) \sum_{i=1}^{2} \partial_\mu v^{(\alpha_i-1)/2}(\mu; t, s)(z), \]

so that estimate (4.7) of Proposition 4.2 follows from the asymptotic property of the parametrix constant \( C_{k.4} \) given in Section B.

For the second assertion it is well seen from usual parametrix technique that

\[ \partial_\mu p(\mu; t, x; s, y) = \partial_\mu \bar{p}^{\mu}(\mu; t, x; s, y) + \sum_{k=1}^{+\infty} \int_{t}^{s} \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; s', y'; s, y) \partial_\mu \bar{p}^{\mu}(\mu; t, x; s', y') dy' ds', \]

so that the derivative of \( \partial_\mu p \) w.r.t. \( \mu \) writes, at any point \( z \) in \( \mathbb{R}^d \):

\[ \partial_\mu(\partial_\mu p)(\mu; t, x; s, y)(z) = \partial_\mu(\partial_\mu \bar{p}^{\mu})(\mu; t, x; s, y)(z) + \partial_\mu \left( \sum_{k=1}^{+\infty} \int_{t}^{s} \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; s', y'; s, y) \partial_\mu \bar{p}^{\mu}(\mu; t, x; s', y') dy' ds' \right)(z). \]

We can use the same arguments as above with estimates (6.8) instead of (6.7) in Proposition 6.1 and we obtain that

\[ \partial_\mu \partial_\mu p(\mu; t, x; s, y)(z) \leq C(s-t)^{\alpha_1/2-1} \left( C_{6.1}' + \sum_{k \geq 1} \left\{ (C_{6.3}' C_{6.1} + C_{6.4} C_{6.1}) (s-t)^{(k\gamma+3)/2} \right\} \right) \times \hat{p}_c(t, x; s, y) \sum_{i=1}^{2} \partial_\mu v^{(\alpha_i-1)/2}(\mu; t, s)(z), \]

from which we deduce estimate (4.8) of Proposition 4.2.

**A Proof of Lemmas 6.4 and 6.1**

In order to avoid heavy notations, the proofs are done in the real case \( (d=1) \). We also recall that \( T < 1 \).

**Proof of Lemma 6.4.** Recall that by definition

\[ H(t, \mu; s', y'; s, y) = (b(s', y, v_{s'}(t, \mu, \varphi_1)) - b(s', y', v_{s'}(t, \mu, \varphi_1))) \partial_\mu \bar{p}(\mu; s', y'; s, y) + \frac{1}{2} \left[ (a(s', y, v_{s'}(t, \mu, \varphi_2)) - a(s, y', v_{s'}(t, \mu, \varphi_2))) \partial_\mu \bar{p}(\mu; s', y'; s, y) \right], \]

so that, for any \( z \) in \( \mathbb{R}^d \):
\[ \partial_{\mu}H(t, \mu; s', y'; s, y)(z) \]
\[ = (b(s', y, v_{s'}(t, \mu, \varphi_1)) - b(s', y', v_{s'}(t, \mu, \varphi_1)))\partial_{\mu}\partial_{x}\tilde{p}(\mu; s', y'; s, y)(z) \]
\[ + (\partial_{3}b(s', y, v_{s'}(t, \mu, \varphi_1)) - \partial_{3}b(s', y', v_{s'}(t, \mu, \varphi_1))(\mu; s', y'; s, y)(z) \]
\[ + \frac{1}{2}(a(s', y, v_{s'}(t, \mu, \varphi_2)) - a(s', y', v_{s'}(t, \mu, \varphi_2)))\partial_{\mu}\partial_{x}^{2}\tilde{p}(\mu; s', y'; s, y)(z) \]
\[ + \frac{1}{2}(\partial_{3}a(s', y, v_{s}(t, \mu, \varphi_2)) - \partial_{3}a(s', y', v_{s'}(t, \mu, \varphi_2)))\partial_{\mu}v_{s'}(t, \mu, \varphi_2)(z)\partial_{x}^{2}\tilde{p}(\mu; s', y'; s, y). \]

Hence
\[ \left| \partial_{\mu}H(t, \mu; s', y'; s, y)(z) \right| \]
\[ \leq C \left\{ \left| b \right|_{\infty} \sum_{i=1}^{2} (s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1/2} \right. \]
\[ + \left| \partial_{3}b \right|_{\infty} (s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1/2} \]
\[ + \left| a \right|_{\gamma_{a}}|y - y'|^{\gamma_{a}}(s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1} \]
\[ + \left| \partial_{3}a \right|_{\gamma_{a}}|y - y'|^{\gamma_{a}}(s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1} \left. \right\} \tilde{p}_{c}(s', y'; s, y), \]

where \( \tilde{\gamma} = \gamma_{a} \land \gamma'_{a} \). Therefore, by using the Gaussian decay of \( \tilde{p}_{c} \):
\[ \left| \partial_{\mu}H(t, \mu; s', y'; s, y)(z) \right| \]
\[ \leq C \left\{ \left| b \right|_{\infty} \sum_{i=1}^{2} (s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1/2} \right. \]
\[ + \left| \partial_{3}b \right|_{\infty} (s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1/2} \]
\[ + \left| a \right|_{\gamma_{a}}\sum_{i=1}^{2} (s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1 + \gamma_{a}/2} \]
\[ + \left| \partial_{3}a \right|_{\gamma_{a}}\sum_{i=1}^{2} (s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1 + \gamma_{a}/2} \left. \right\} \tilde{p}_{c}(s', y'; s, y). \]

So, there exists a positive constant \( \tilde{C} \) depending on known parameters in (H\&R) such that:
\[ \left| \partial_{\mu}H(t, \mu; s', y'; s, y)(z) \right| \leq \tilde{C} \sum_{i=1}^{2} (s' - t)^{(\alpha_{1} - 1)/2}\partial_{\mu}v_{i}(\alpha_{1} - 1)/2(\mu; t, s')(z)(s - s')^{-1 + \gamma_{a}/2} \tilde{p}_{c}(s', y'; s, y). \]
Assume now as an induction hypothesis that for all \((s', y')\) in \((t, s) \times \mathbb{R}^d\):

\[
\left| \partial_\mu H^{\otimes k}(t, \mu; s', y'; s, y)(z) \right| \leq C_k^6 2 \sum_{i=1}^{s'} (s' - t)(\alpha_i - 1)/2 \partial_\mu v_{(\alpha_i - 1)/2}(\mu; t, s')(z)(s - s')^{k\gamma_a/2 - 1} \hat{p}_c(s', y'; s, y),
\]

where

\[
\hat{C}_k^6 = (C_k^6 \hat{C} + C C_k^6 \beta)((k - 1)\bar{\gamma}_a/2, \bar{\gamma}_a/2).
\]

We then have

\[
\partial_\mu H^{\otimes k+1}(t, \mu; s', y'; s, y)(z) = \int_{s'}^s \int_{\mathbb{R}^d} \partial_\mu H^{\otimes k}(t, \mu; r, u; s, y)(z) H(t, \mu; s', y'; r, u)dudr
\]

\[
+ \int_{s'}^s \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; r, u; s, y) \partial_\mu H(t, \mu; s', y'; r, u)(z)dudr.
\]

We can bound the first term in the right hand side by using the induction hypothesis above, the estimation (6.17) on \(H\) and the property of the Gaussian convolution:

\[
\left| \int_{s'}^s \int_{\mathbb{R}^d} \partial_\mu H^{\otimes k}(t, \mu; r, u; s, y)(z) H(t, \mu; s', y'; r, u)dudr \right|
\]

\[
\leq C_k^6 \hat{C}_k^6 \int_{s'}^s \sum_{i=1}^{s'} (s' - t)(\alpha_i - 1)/2 \partial_\mu v_{(\alpha_i - 1)/2}(\mu; t, r)(z)(s - r)^{k\gamma_a/2 - 1}(r - s')^{\gamma_a/2 - 1}dr \hat{p}_c(s', y'; s, y)
\]

\[
\leq C_k^6 \hat{C}_k^6 \int_{s'}^s \sum_{i=1}^{s'} (s' - t)(\alpha_i - 1)/2 \partial_\mu v_{(\alpha_i - 1)/2}(\mu; t, s')(z) \hat{p}_c(s', y'; s, y) \int_{s'}^s (s - r)^{k\gamma_a/2 - 1}(r - s')^{\gamma_a/2 - 1}dr.
\]

By the change of variable \(r = (s - s')r' + s'\), one can show that

\[
\int_{s'}^s (s - r)^{k\gamma_a/2 - 1}(r - s')^{\gamma_a/2 - 1}dr \leq (s - s')^{(k+1)\gamma_a/2 - 1}\beta(k\bar{\gamma}_a/2, \bar{\gamma}_a/2),
\]

so that

\[
\left| \int_{s'}^s \int_{\mathbb{R}^d} \partial_\mu H^{\otimes k}(t, \mu; r, u; s, y)(z) H(t, \mu; s', y'; r, u)dudr \right| \leq C_k^6 \hat{C}_k^6 \int_{s'}^s \sum_{i=1}^{s'} (s' - t)(\alpha_i - 1)/2 \partial_\mu v_{(\alpha_i - 1)/2}(\mu; t, s')(z) \hat{p}_c(s', y'; s, y)(s - s')^{(k+1)\gamma_a/2 - 1}\beta(k\bar{\gamma}_a/2, \bar{\gamma}_a/2).
\]

Now, we bound the second term in the right hand side of (A.2). We have from Lemma 6.3:

\[
\left| H^{\otimes k}(t, \mu; r, u; s, y) \right| \leq C_k^6 (s - r)^{k\gamma_a/2 - 1} \hat{p}_c(r; u; s, y),
\]

where
\[ C_k^{6.3} = C_k^{6.3} \prod_{l=1}^{k-1} \beta(\bar{\gamma}_a/2, \bar{\gamma}_a/2). \]

So that, thanks to estimate (A.1):

\[
\left| \int_{s'}^s \int_{\mathbb{R}^d} H^{\otimes k}(t, \mu; r, u; s, y) \partial_{\mu} H (t, \mu; s', y'; r, u)(z) dudr \right| 
\leq \sum_{i=1}^{2} (s' - t)^{(\alpha_i - 1)/2} \bar{\alpha}_i t, s' (z) C_k^{6.3} \tilde{C} \int_{s'}^s (s - r)^{\gamma_a/2 - 1} (r - s')^{\gamma_a/2 - 1} dr.
\]

Hence, plugging (A.3) together with (A.4) in (A.2):

\[
\left| \partial_{\mu} H^{\otimes k+1}(t, \mu; s', y'; s, y)(z) \right| 
\leq (s - s')^{(k+1)\gamma_a/2 - 1} \sum_{i=1}^{2} (s' - t)^{(\alpha_i - 1)/2} \bar{\alpha}_i t, s' (z)(C_k^{6.3} \tilde{C} + C\tilde{C}_k^{6.4}) \beta(k\bar{\gamma}_a/2, \bar{\gamma}_a/2),
\]

and the induction is true since:

\[
\tilde{C}_{k+1}^{6.4} = (C_k^{6.3} \tilde{C} + C\tilde{C}_k^{6.4}) \beta(k\bar{\gamma}_a/2, \bar{\gamma}_a/2).
\]

\(\square\)

**Proof of Lemma 6.1.** We begin with the following Claim.

**Claim A.1.** The following estimates hold:

- \( |\partial_{\mu} m_{t, s, \alpha}^{\xi}(t, \mu)(z)| \leq (s - t)^{(\alpha_1 + 1)/2} |\partial_{\alpha} b|_\infty \bar{\alpha}_{t, s}^{(\alpha_1 - 1)/2}(\mu; t, s)(z), \)
- \( |\partial_{\mu} a_{t, s}^{\xi}(t, \mu)(z)| \leq (s - t)^{(\alpha_2 + 1)/2} |\partial_{\alpha} a|_\infty \bar{\alpha}_{t, s}^{(\alpha_2 - 1)/2}(\mu; t, s)(z), \)

and for all \( s' \in (t, T] :\)

- \( |\partial_{\mu} m_{t, s, \alpha}^{\xi}(t, \mu)(z)| \leq (s - s')(s' - t)^{(\alpha_1 - 1)/2} |\partial_{\alpha} b|_\infty \bar{\alpha}_{t, s}^{(\alpha_1 - 1)/2}(\mu; t, s)(z), \)
- \( |\partial_{\mu} a_{t, s}^{\xi}(t, \mu)(z)| \leq (s - s')(s' - t)^{(\alpha_2 - 1)/2} |\partial_{\alpha} a|_\infty \bar{\alpha}_{t, s}^{(\alpha_2 - 1)/2}(\mu; t, s)(z). \)

And we recall the classical estimate coming from the uniform ellipticity of \( a : \) there exists \( \Lambda > 0 \) such that, for all positive \( \gamma :\)

\[
\frac{1}{[a_{s', s}^{\xi}(t, \mu)]^\gamma} < \Lambda^{-\gamma} (s - s')^{-\gamma}. \tag{A.5}
\]
The derivative of \( \tilde{p} \) evaluated at any point \( z \in \mathbb{R} \) is then given by:
\[
\partial_\mu \tilde{p}^\xi(t, \mu; s', y' ; s, y)(z) = \left( -\frac{1}{2} \partial_\mu \tilde{a}_s^\xi (t, \mu)(z) \right) + \left( \frac{1}{2} \tilde{a}_s^\xi(t, \mu) \right) \left( y - y' - m_{s',s}^\xi(t, \mu) \right) + \partial_\mu m_{s',s}^\xi(t, \mu)(z) \\
\times \left( \frac{y - y' - m_{s',s}^\xi(t, \mu)}{\tilde{a}_s^\xi(t, \mu)} \right) \exp \left( -\frac{1}{2} \left[ \frac{a_{s',s}^\xi(t, \mu)}{\tilde{a}_s^\xi(t, \mu)} \right] \left( y - y' - m_{s',s}^\xi(t, \mu) \right) \right).
\]

Now, by using the Gaussian decay of \( \tilde{p} \), and estimates of Claim A.1, we obtain that, when \( s' = t \):
\[
|\partial_\mu \tilde{p}^\xi(t, \mu; t, y' ; s, y)(z)| \leq C \left( \Lambda^{-1} ||\partial_3 a||_\infty \sup_{r \in [t,s]} \tilde{\partial}_t v^{1/2}(\mu; t, s)(s - t)^{2} \right) + \Lambda^{-1} ||\partial_3 b||_\infty \sup_{r \in [t,s]} \tilde{\partial}_t v^{1/2}(\mu; t, s)(s - t)^{2} \right) \\
\times \frac{1}{\sqrt{2\pi}} \frac{1}{[a_{t,s}^\xi(t, \mu)]^{1/2}} \exp \left( -\frac{1}{2} \left[ \frac{a_{s',s}^\xi(t, \mu)}{a_{t,s}^\xi(t, \mu)} \right] \left( y - y' - m_{t,s}^\xi(t, \mu) \right) \right),
\]
for some positive constant \( C \) and \( c \), with \( c \) strictly less than 1. We now compute the space derivatives: for all \( s' < s \) in \([t, T]^2\)
\[
\partial_s \tilde{p}^\xi(t, \mu; s', y' ; s, y) = \frac{1}{\sqrt{2\pi}} \frac{1}{[a_{s',s}^\xi(t, \mu)]^{1/2}} \exp \left( -\frac{1}{2} \left[ \frac{a_{s',s}^\xi(t, \mu)}{a_{s',s}^\xi(t, \mu)} \right] \left( y - y' - m_{s',s}^\xi(t, \mu) \right) \right),
\]
which gives
\[
|\partial_s \tilde{p}^\xi(t, \mu; s', y' ; s, y)| \leq C(s - t)^{-1/2} \frac{1}{\sqrt{2\pi}} \frac{1}{[a_{s',s}^\xi(t, \mu)]^{1/2}} \exp \left( -\frac{1}{2} \left[ \frac{a_{s',s}^\xi(t, \mu)}{a_{s',s}^\xi(t, \mu)} \right] \left( y - y' - m_{s',s}^\xi(t, \mu) \right) \right),
\]
and,
\[
\partial_x^2 \tilde{p}^\xi(t, \mu; s', x; s, y) = \left( \frac{1}{[a_{s',s}^\xi(t, \mu)]^{2}} \right) \times \frac{1}{\sqrt{2\pi}} \frac{1}{[a_{s',s}^\xi(t, \mu)]^{1/2}} \exp \left( -\frac{1}{2} \left[ \frac{a_{s',s}^\xi(t, \mu)}{a_{s',s}^\xi(t, \mu)} \right] \left( y - y' - m_{s',s}^\xi(t, \mu) \right) \right),
\]
so that
Thus, when \( s' = t \) we have:

\[
|\partial_{\mu} \partial_x \vec{p}^z(t, \mu; t, y'; s, y) |
\leq \left( (s-t)^{(\alpha_1-1)/2} + (s-t)^{(\alpha_2-1)/2} \right) \left\| \partial_3 b \right\|_{\infty} \overrightarrow{\partial_\mu v_1(1-\alpha_1)/2}(\mu; t, s)(z)
\]
\[
+ \left( (s-t)^{\alpha_2/2-1} + 2(s-t)^{(\alpha_2-1)/2} \right) \left\| \partial_3 a \right\|_{\infty} \overrightarrow{\partial_\mu v_2(1-\alpha_2)/2}(\mu; t, s)(z)
\]
\[
\times C A^{-1} \left[ a^z_{t,s}(Xt,\mu) \right] \left( -\frac{1}{2} \left| a^z_{t,s}(Xt,\mu) \right|^{1/2} (y - y' - m^z_{t,s}(Xt,\mu))^* \right|^2 \Bigg) ,
\]

and

\[
|\partial_{\mu} \partial_x \vec{p}^z(t, \mu; s', y'; s, y) |
\leq \left( (s'-t)^{(\alpha_1-1)/2} + (s'-t)^{(\alpha_1-1)/2} \right) \left\| \partial_3 b \right\|_{\infty} \overrightarrow{\partial_\mu v_1(1-\alpha_1)/2}(\mu; t, s)(z)
\]
\[
+ \left( (s'-t)^{\alpha_2/2-1} + 2(s'-t)^{(\alpha_2-1)/2} \right) \left\| \partial_3 a \right\|_{\infty} \overrightarrow{\partial_\mu v_2(1-\alpha_2)/2}(\mu; t, s)(z)
\]
\[
\times C (s-s')^{-1/2} \left[ a^z_{t,s}(Xt,\mu) \right] \left( -\frac{1}{2} \left| a^z_{t,s}(Xt,\mu) \right|^{1/2} (y - y' - m^z_{t,s}(Xt,\mu))^* \right|^2 \Bigg) ,
\]

when \( s' > t \). We conclude with:
\[ \partial_\mu \partial^2_x \bar{p}^\xi(t, \mu; s', y'; s, y)(z) \]
\[ = \left\{ \left( 2\partial_\mu m_{s', s}^\xi([X^t, \mu])(z)(y - y' - m_{s', s}^\xi([X^t, \mu])) - 2(y - y' - m_{s', s}^\xi([X^t, \mu]))^2 \partial_\mu a_{s', s}^\xi(t, \mu)(z) + \partial_\mu a_{s', s}^\xi(t, \mu)(z) \right) \right\} \frac{1}{2\pi} \left( \frac{1}{a_{s', s}^\xi(t, \mu)} \right) \exp \left( -\frac{1}{2} \left[ a_{s', s}^\xi(t, s) \right]^{-1/2} (y - y' - m_{s', s}^\xi(t, \mu))^2 \right). \]

Which gives

\[ \left| \partial_\mu \partial^2_x \bar{p}^\xi(\mu; t, x; s, y)(z) \right| \]
\[ \leq \left| \partial_3 b \right|_{\infty}(s - s')^{-1/2} (s' - t)^{(\alpha_1 - 1)/2} \partial_\mu v^{(1 - \alpha_1)/2}(\mu; t, s)(z) + \left| \partial_3 a \right|_{\infty}((s - s')^{-1} + (s - s')^{-1/2})(s' - t)^{(\alpha_2 - 1)/2} \partial_\mu v^{2(1 - \alpha_2)/2}(\mu; t, s)(z) \]
\[ \times \frac{1}{\sqrt{2\pi}} \frac{1}{a_{s', s}^\xi(t, \mu)} \exp \left( -\frac{1}{2} \left[ a_{s', s}^\xi(t, \mu) \right]^{-1/2} (y - y' - m_{s', s}^\xi(t, \mu))^2 \right). \]

for all \( s' > t. \)

\[ \square \]

**Proof of Claim A.1.** From chain rule we have, for all \( \xi \) in \( \mathbb{R} \) and for all \( s' < s \in [t, T] \)
\[ \partial_\mu m_{s', s}^\xi(t, \mu)(z) = \left( \partial_\mu \int_{s'}^s b(r, \xi, \langle \varphi_1, [X^t, \mu] \rangle) dr \right)(z) \]
\[ = \int_{s'}^s \partial_3 b(r, \xi, \langle \varphi_1, [X^t, \mu] \rangle) \partial_\mu v^\xi(t, \mu, \varphi_1)(z) dr, \]
z in $\mathbb{R}$, so that
\[
|\partial \mu \xi_s(t, \mu)(z)| \leq ||\partial \mu b||_\infty \sup_{r \in [t, s]} \left\{(r - t)^{(1 - \alpha_1)/2} |\partial \mu v_r(t, \mu, \varphi_1)(z)|\right\} \int_s^r (r - t)^{(\alpha_1 - 1)/2} dr
\]
Hence, for all $z$ in $\mathbb{R}$
\[
|\partial \mu \xi_s(t, \mu)(z)| \leq ||\partial \mu b||_\infty \sup_{r \in [t, s]} \left\{(r - t)^{(1 - \alpha_1)/2} |\partial \mu v_r(t, \mu, \varphi_1)(z)|\right\} (s - t)^{(\alpha_1 + 1)/2},
\]
when $s' = t$ and
\[
|\partial \mu \xi_s(t, \mu)(z)| \leq ||\partial \mu b||_\infty \sup_{r \in [t, s]} \left\{(r - t)^{(1 - \alpha_1)/2} |\partial \mu v_r(t, \mu, \varphi_1)(z)|\right\} (s' - s)(s' - t)^{(\alpha_1 - 1)/2},
\]
when $s' > t$. Next we have,
\[
\partial \mu a^{\xi}_{s', s}(t, \mu)(z) = \left(\partial \mu \int_s^{s'} a(r, \xi, \langle \varphi_2, [X_r^{t, \mu}] \rangle)dr\right)(z)
\]
\[
= \int_s^{s'} \partial \mu a(r, \xi, \langle \varphi_2, [X_r^{t, \mu}] \rangle) |\partial \mu v_r(t, \mu, \varphi_2)(z)| dr,
\]
z in $\mathbb{R}$. Therefore
\[
|\partial \mu a^{\xi}_{s', s}(t, \mu)(z)| \leq ||\partial \mu a||_\infty \sup_{r \in [t, s]} \left\{(r - t)^{(1 - \alpha_2)/2} |\partial \mu v_r(t, \mu, \varphi_2)(z)|\right\} \int_t^s (r - t)^{(\alpha_2 - 1)/2} dr
\]
and so
\[
|\partial \mu a^{\xi}_{s', s}(t, \mu)(z)| \leq ||\partial \mu a||_\infty \sup_{r \in [t, s]} \left\{(r - t)^{(1 - \alpha_2)/2} |\partial \mu v_r(t, \mu, \varphi_2)(z)|\right\} (s - t)^{(\alpha_2 + 1)/2},
\]
when $s' = t$ and
\[
|\partial \mu a^{\xi}_{s', s}(t, \mu)(z)| \leq ||\partial \mu a||_\infty \sup_{r \in [t, s]} \left\{(r - t)^{(1 - \alpha_2)/2} |\partial \mu v_r(t, \mu, \varphi_2)(z)|\right\} (s' - s)(s' - t)^{(\alpha_2 - 1)/2},
\]
when $s' > t$.

\[\square\]

**B. Asymptotic properties of the parametrix constants**

**Claim B.1.** There exists a strictly finite and strictly positive integer $K(\gamma_a)$ such that for all $k \geq K(\gamma_a)$:
\[
C_k^{6.3} = C(K(\gamma_a)) C_k^{6.3} \frac{4^k}{\gamma_a!},
\]
where $C(K(\gamma_a)) = 4^{-K(\gamma_a)}(K(\gamma_a)!)^{\gamma_a/2} \prod_{l=1}^{K(\gamma_a)-1} \beta(l\gamma_a/2, \gamma_a/2)$.

**Proof.** Let $K(\gamma_a)$ be equal to $\lceil 2/\gamma_a \rceil$, so that for all $k \geq K(\gamma_a)$
\[
k\gamma_a/2 - 1 \geq 0,
\]
and recall that
\[
\beta(\gamma_a k/2, \gamma_a/2) = \int_0^1 (1 - s)^{k\gamma_a/2}s^{\gamma_a/2 - 1}ds.
\]
For all positive $\epsilon$ strictly less than 1, we have, for all $k \geq K(\gamma_a)$:
\[
\int_0^1 (1-s)^{\gamma_a k/2-1} s^{\gamma_a/2-1} \, ds = \int_0^1 (1-s)^{\gamma_a/2-1} s^{\gamma_a k/2-1} \, ds
\]

(B.1)

\[
= \int_0^{1-\epsilon} (1-s)^{\gamma_a/2-1} s^{\gamma_a k/2-1} \, ds + \int_{1-\epsilon}^1 (1-s)^{\gamma_a/2-1} s^{\gamma_a k/2-1} \, du
\]

\[
\leq \frac{1}{\epsilon^{1-\gamma_a/2}} \int_0^{1-\epsilon} s^{\gamma_a k/2-1} \, ds + \int_{1-\epsilon}^1 (1-s)^{\gamma_a/2-1} \, ds
\]

\[
\leq \frac{1}{\epsilon^{1-\gamma_a/2}} \frac{2}{k\gamma_a} + 2\epsilon^{\gamma_a/2}.
\]

So that, by letting \(\epsilon = 1/k\) we have

\[
\int_0^1 (1-s)^{k/2-1} s^{1/2} \, ds \leq \frac{4}{\gamma_a k^{\gamma_a/2}},
\]

which gives the desired result. \(\square\)

**Claim B.2.** There exists a strictly finite and strictly positive integer \(K := K(\gamma_a, \gamma_a')\) such that for all \(k \geq K\):

\[
\tilde{C}_{k+1}^{6,4} = (C_{k}^{6,3} \tilde{C} + C \tilde{C}_{k}^{6,4}) \beta(k \gamma_a/2, \gamma_a/2) \leq \tilde{C}(\gamma_a, \gamma_a') \left( (k - K) \frac{k^k}{\gamma_a^k(k!)^{\gamma_a/2}} + (4C)^{k+1} \right),
\]

for some positive real \(\kappa \geq 4C_{6,3}\) and some positive constant \(\tilde{C}(\gamma_a, \gamma_a')\).

**Proof.** Let \(K(\gamma_a)\) be equal to \([2/\gamma_a]\), so that For all \(k \geq K(\gamma_a)\)

\[
k \gamma_a/2 - 1 \geq 0,
\]

and let \(k \geq K := K(\gamma_a) \vee K(\gamma_a)\). We know from the proof of Claim B.1:

\[
\beta(k \gamma_a/2, \gamma_a/2) \leq \frac{4}{\gamma_a k^{\gamma_a/2}}.
\]

Hence, from definition of \(C_{k}^{6,3}\) we have that:

\[
\tilde{C}_{k+1}^{6,4} \leq C(K(\gamma_a)) \frac{\tilde{C}_k^k}{\gamma_a^k(k!)^{\gamma_a/2}} + \tilde{C}_{k}^{6,4} \frac{4}{\gamma_a k^\gamma_a/2},
\]

so that

\[
\frac{\tilde{C}_{k+1}^{6,4}}{(k + 1)^{\gamma_a/2}} \leq C(K(\gamma_a)) \frac{\tilde{C}_k^k}{\gamma_a^k((k + 1)!)^{\gamma_a/2}} + \frac{\tilde{C}_{k}^{6,4} 4C}{\gamma_a((k + 1)!)^{\gamma_a/2}}.
\]

Let us define

\[
A_k = \frac{\tilde{C}_{k}^{6,4}}{k^{\gamma_a/2}}, \quad M_{k+1} = \frac{C(K(\gamma_a)) \tilde{C}_k^k}{\gamma_a^k((k + 1)!)^{\gamma_a/2}}, \quad D_{k+1} = \frac{4C}{\gamma_a((k + 1)!)^{\gamma_a/2}}.
\]

hence, we have that

\[
A_{k+1} \leq M_{k+1} + A_k D_{k+1}.
\]
Since $\kappa$ is such that $\kappa \geq 4C_{6.3}$ we obtain:

$$M_kD_{k+1} = \frac{4C_{6.3}}{\gamma_a((k+1)!)^{\gamma_a/2}} C(K(\gamma_a)) \tilde{C}K^{k-1} \leq \frac{\kappa^k \tilde{C} C(K(\gamma_a))}{\gamma_a((k+1)!)^{\gamma_a/2}} = M_{k+1}.$$ 

Therefore, by induction

$$A_{k+1} \leq (k - K)M_{k+1} + \prod_{i=1}^{k+1} \frac{4C_{6.3}}{\gamma_a(i!)^{\gamma_a/2}},$$

which implies that

$$\frac{\tilde{C}_k}{(k + 1)^{\gamma_a/2}} \leq (k - K) \frac{C(K(\gamma_a)) \tilde{C}K^k}{\gamma_a((k+1)!)^{\gamma_a/2}} + \frac{(4C)^{k+1}}{\gamma_a((k+1)!)^{\gamma_a/2}} C(K),$$

where $C(K) = (4C)^{-K} \tilde{C} K!$. Then,

$$\tilde{C}_{k+1} \leq (k - K)C(K(\gamma_a)) \tilde{C} \frac{\kappa^k}{\gamma_a((k+1)!)^{\gamma_a/2}} + C(K) \frac{(4C)^{k+1}}{\gamma_a((k+1)!)^{\gamma_a/2}}.$$ 

□

References


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